Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

Generalized Caputo Fractional Proportional Differential Equations and Inclusion Involving Slit-Strips and Riemann-Stieltjes Integral Boundary Conditions

Wafa Shammakh[1*], Hanan A. Alyami[2], Hadeel Z. Alzumi[1]

[1]Department of Mathematics and Statistics, Faculty of Science, University of Jeddah, P.O. Box 13151, Jeddah 21493, Saudi Arabia
[2]Mathematics Department, Deanship of Preparatory Year, Najran University, P.O. Box 1988, Najran 11001, Saudi Arabia
E-mail: wmshammakh@uj.edu.sa

Received: 14 December 2023; **Revised:** 23 February 2024; **Accepted:** 26 February 2024

Abstract: The main purpose of this study is to investigate the existence and uniqueness of solutions to a nonlocal boundary value problem. This newly defined class involves nonlinear fractional differential equations of general proportional fractional derivative and integral with respect to another function. Additionally, the inclusion case results associated to our problem are discussed. Our analysis relies on fixed point theorems and fractional calculus techniques. By giving examples, the obtained results are clearly illustrated.

***Keywords*:** generalzied fractional proportional derivatives, integral boundary conditions, fractional differential equations and inclusions, fixed-point theorems

MSC: 34A08, 34A60

Abbreviation

CFPD	Caputo fractional proportional derivative
FDs	fractional derivatives
${}^{C}\mathfrak{D}^{r}$	the Caputo fractional derivative
GPF	Generalized proportional fractional

1. Introduction

In various fields of engineering and research, including chemistry, physics, electrodynamics, aerodynamics of complex media, polymer rheology, control of dynamical systems, etc., fractional calculus [1, 2] is used more frequently. Recently, new fractional derivatives (FDs) have been discovered that interpolate the Riemann Liouville, Caputo, Hilfer, and Hadamard as well as their generalization. For further information, see [3–6].

Jarad et al. introduced a novel generalized fractional calculus in their work [7], delineating a special case of proportional derivatives characterized by a fractional operator kernel involving exponential functions. This novel approach establishes

DOI: https://doi.org/10.37256/cm.5220244100

a distinct class of fractional operators derived from modified conformable derivatives, encompassing Riemann-Liouville and Caputo fractional derivatives and integrals as specific instances.

These generalized fractional derivatives offer a powerful framework for characterizing the memory and hereditary properties exhibited by diverse materials and processes [8–10]. Numerous researchers have directed their focus towards establishing existence results for solutions in both initial value and boundary value problems involving generalized fractional differential equations. Noteworthy contributions include studies by Ahmad et al. [11–16]. Specifically, Ahmad et al. [17] delved into the existence and uniqueness of solutions concerning fractional differential equations accompanied by nonlocal multi-point and integral boundary conditions:

$$ {}^{C}\mathfrak{D}^{p}({}^{C}\mathfrak{D}^{q}\tilde{\mathrm{x}}(\mathrm{t})+\mathfrak{h}(\mathrm{t},\tilde{\mathrm{x}}(\mathrm{t})))=\mathfrak{z}(\mathrm{t},\tilde{\mathrm{x}}(\mathrm{t})),\quad \mathrm{t}\in[0,1],0<\mathrm{p},\mathrm{q}\geq 1, $$

$$ \tilde{\mathrm{x}}(0)=\sum_{j=1}^{m}\beta_j\tilde{\mathrm{x}}(\sigma_j), $$

$$ b\tilde{\mathrm{x}}(1)=a\int_0^1\tilde{\mathrm{x}}(s)d\mathscr{H}(s)+\sum_{i=1}^{n}\Xi_i\int_{\xi_i}^{\eta_i}\tilde{\mathrm{x}}(s)ds, $$

The notation ${}^{C}\mathfrak{D}^{r}$ represents the Caputo fractional derivative of order $(r=\mathrm{p},\mathrm{q})$. Functions $\mathfrak{h}$ and $\mathfrak{z}$ are given as continuous functions. Additionally, consider the conditions where $0<\sigma_j<\xi_i<\eta_i<1, a,b\in\mathbb{R}, \Xi_i,\beta_j\in\mathbb{R}$ for $i=1,2,...,n$ and $j=1,2,...,m$. The function $\mathscr{H}(.)$ is characterized as a function of bounded variation.

In [18], the authors researched the existence and uniqueness results for a Langevin differential equation with nonlocal boundary conditions:

$$ {}_{p}^{C}\mathfrak{D}_{a}^{\alpha,\rho,v}({}_{p}^{C}\mathfrak{D}_{a}^{\beta,\rho,v}+\lambda)\hat{z}(t)=\mathfrak{f}(t,\hat{z}(t)),\quad t\in[c,d], $$

$$ \hat{z}(a)=0,\hat{z}(b)=\xi\hat{z}(\eta), $$

where $\mathfrak{f}:[c,d]\times\mathbb{R}$ is given nonlinear term, $\gamma,\xi\in\mathbb{R},0<\alpha,\beta\leq 1$, and $\rho\in(0,1]$. Note that $v(t)$ considers a strictly increasing continuous function on $[c,d]$. The notation ${}_{p}^{C}\mathfrak{D}_{a}^{i,\rho,v}$ indicates for the Caputo fractional proportional derivative (CFPD) with respect to the function v of order $(i=\alpha,\beta)$. Inspired by the aforementioned research, the focus of this paper is to investigate the existence and uniqueness of solutions for a specific nonlinear differential equation. This equation features two distinct nonlinear terms and is accompanied by Caputo fractional proportional type slit-strips and Riemann-Stieltjes boundary conditions.

$$ {}^{C}\mathfrak{D}^{\alpha_1,\rho,\vartheta}({}^{C}\mathfrak{D}^{\alpha_2,\rho,\vartheta}x(t)+h(t,x(t)))=f(t,x(t)),\quad t\in[a,b], \tag{1} $$

$$ \begin{cases} x(a)=A\theta[x], \\ \displaystyle\sum_{i=1}^{m}\delta_i x(\xi_i)=\frac{1}{\rho^{\gamma}\Gamma(\gamma)}\Bigg[a_1\int_a^{\eta_1}e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(\tau))}(\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1}x(\tau)\dot{\vartheta}(\tau)d\tau \\ \qquad\qquad +a_2\displaystyle\int_{\eta_2}^{b}e^{\frac{\rho-1}{\rho}(\vartheta(b)-\vartheta(\tau))}(\vartheta(b)-\vartheta(\tau))^{\gamma-1}x(\tau)\dot{\vartheta}(\tau)d\tau\Bigg], \end{cases} \tag{2} $$

where ${}^{c}\mathfrak{D}^{\alpha_1,\rho,\vartheta}$ and ${}^{c}\mathfrak{D}^{\alpha_2,\rho,\vartheta}$ denote the Caputo fractional proportional derivative (CPFD) with respect to the function ϑ of order $0 < \alpha_i \leq 1$ $(i = 1,2)$, where $\gamma > 0$. Additionally, $\vartheta(t)$ is a strictly increasing function defined on the interval $[a,b]$. Functions $h : [a,b] \times \mathbb{R} \to \mathbb{R}$ and $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ are continuous. Furthermore, $\rho \in (0,1]$, $a < \eta_1 < \xi_1 < \xi_2 < ... < \xi_m < \eta_2 < b$, and δ_i $(i = 1,2,\cdots,m)$ represent real constants. $A \in \mathbb{R}$, and $\theta[x] = \int_a^b x(s)dE(s)$ represents the Riemann-Stieltjes integral with respect to the function $E : [a,b] \longrightarrow \mathbb{R}$, where $x(t) \in C([a,b],\mathbb{R})$.

The boundary conditions in the aforementioned problem involve two types of integral conditions. Firstly, slit-strips conditions refer to contributions from finite strips of arbitrary lengths positioned at (a,η_1) and (η_2,b) on the interval $[a,b]$. These conditions are linked to the values of the unknown functions at nonlocal points ξ_i $(\eta_1 < \xi_1 < \xi_2 < ... < \xi_m < \eta_2)$, located anywhere within the apertures (slits), which form the boundaries of the strips. Examples of these boundary conditions include scattering by slits [19–21], and silicon strip detectors used in scanned multi-slit X-ray imaging [22]. Secondly, integral conditions of Riemann-Stieltjes type play a fundamental role in blood flow problems by providing a flexible mechanism to handle changes in the geometry of blood vessels [23]. These conditions are also useful in regularizing ill-posed problems [24].

Additionally, our study extends to the multivalued analogue of the boundary value problem (1)–(2). In particular, we delve into the existence of solutions concerning the following inclusion problem:

$$ {}^{C}\mathfrak{D}^{\alpha_1,\rho,\vartheta}({}^{C}\mathfrak{D}^{\alpha_2,\rho,\vartheta}x(t) + h(t,x(t))) \in F(t,x(t)), \quad t \in [a,b], \tag{3} $$

$$ \begin{cases} x(a) = A\theta[x], \\ \sum_{i=1}^{m} \delta_i x(\xi_i) = \dfrac{1}{\rho^{\gamma}\Gamma(\gamma)} \Bigg[a_1 \displaystyle\int_a^{\eta_1} e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(\tau))} (\vartheta(\eta_1) - \vartheta(\tau))^{\gamma-1} x(\tau)\dot{\vartheta}(\tau)\, d\tau \\ \qquad + a_2 \displaystyle\int_{\eta_2}^{b} e^{\frac{\rho-1}{\rho}(\vartheta(b)-\vartheta(\tau))} (\vartheta(b) - \vartheta(\tau))^{\gamma-1} x(\tau)\dot{\vartheta}(\tau)\, d\tau \Bigg], \end{cases} \tag{4} $$

the function $F : [a,b] \times \mathbb{R} \to \mathbf{P}(\mathbb{R})$ represents a multivalued map, where $\mathbf{P}(\mathbb{R})$ denotes the collection of all nonempty subsets of $\mathbb{R}$. The remaining variables maintain the same definitions as outlined in the boundary value problem (1)–(2). This paper is organized as follows. In Section 2, we recall some basic definitions and preliminary results related to our work. Section 3 contains the existence and uniqueness results for the boundary value problem (1)–(2). The technique for solving the given problem is converted to a fixed point problem. Then, by applying fixed point theorems, the main results are obtained. The existence results for multivalued problem (3)–(4) are presented in Section 4. Moreover, we give examples for the main results.

2. Preliminaries

In this section, we demonstrate the basic notions and related preliminaries concerning fractional calculus [9, 10], and some fixed points results which are used throughout the paper [25, 26].

Definition 1. *For* $\mathrm{p} \in (0,1]$, $\mathrm{z} \in \mathbb{C}$ *with* $\mathrm{Re(z)} > 0$, $\Pi \in \mathscr{C}([a,b],\mathbb{R})$ *satisfying* $\Pi'(t) > 0$, *we define the Riemann Liouville fractional proportional integral of* $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ *with respect to* Π *as*

$$ \left({}_{a}I^{\mathrm{z},\mathrm{p},\Pi} f\right)(t) = \frac{1}{\mathrm{p}^{\mathrm{z}}\Gamma(\mathrm{z})} \int_a^t e^{\frac{\mathrm{p}-1}{\mathrm{p}}(\Pi(t)-\Pi(\mathrm{s}))} (\upsilon(t) - \Pi(\mathrm{s}))^{\mathrm{z}-1} f(\mathrm{s})\Pi'(\mathrm{s})\, d\mathrm{s}. $$

Definition 2. *For* $\mathrm{p}>0$, $\mathrm{z}\in\mathbb{C}$ *with* $\mathrm{Re}(\mathrm{z})\geq 0$, *and* $\Pi\in\mathscr{C}([a,b],\mathbb{R})$ *satisfying* $\Pi'(t)>0$, *we define the left fractional derivative of* $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ *with respect to* υ *as*

$$\left({}_aD^{\mathrm{z},\mathrm{p},\Pi}f\right)(t)={}^{n,\mathrm{p},\Pi}D\,{}_aI^{n-\mathrm{z},\mathrm{p},\Pi}f(t)$$

$$=\frac{D_t^{n,\mathrm{p},\Pi}}{\mathrm{p}^{n-\mathrm{z}}\Gamma(n-\mathrm{z})}\int_a^t e^{\frac{\mathrm{p}-1}{\mathrm{p}}(\Pi(t)-\Pi(\mathrm{s}))}(\Pi(t)-\Pi(\mathrm{s}))^{n-\mathrm{z}-1}f(\mathrm{s})\Pi'(\mathrm{s})\,d\mathrm{s},$$

where $n=[\mathrm{Re}(\mathrm{z})]+1$,
and

$$D^{n,\mathrm{p},\Pi}=\underbrace{D^{\mathrm{p},\Pi}D^{\mathrm{p},\Pi}\ldots D^{\mathrm{p},\Pi}}_{n\text{-times}}.$$

Definition 3. *For* $\mathrm{p}\in(0,1]$, $\mathrm{z}\in\mathbb{C}$ *with* $\mathrm{Re}(\mathrm{z})\geq 0$, $\Pi\in\mathscr{C}([a,b],\mathbb{R})$ *satisfying* $\Pi'(t)>0$, *we define the left derivative of Caputo type starting at a by*

$$\left({}_a^CD^{\mathrm{z},\mathrm{p},\Pi}f\right)(t)={}_aI^{n-\mathrm{z},\mathrm{p},\Pi}(D^{n,\mathrm{p},\Pi}f)(t)$$

$$=\frac{1}{\mathrm{p}^{n-\mathrm{z}}\Gamma(n-\mathrm{z})}\int_a^t e^{\frac{\mathrm{p}-1}{\mathrm{p}}(\Pi(t)-\Pi(\mathrm{s}))}(\Pi(t)-\Pi(\mathrm{s}))^{n-\mathrm{z}-1}(D^{n,\mathrm{p},\Pi}f)(\mathrm{s})\Pi'(\mathrm{s})\,d\mathrm{s}.$$

where $n=[\mathrm{Re}(\mathrm{z})]+1$.

Proposition 1. *Let* $\mathrm{z},\mathrm{b}\in\mathbb{C}$ *satisfy* $\mathrm{Re}(\mathrm{z})\geq 0$ *and* $\mathrm{Re}(\mathrm{b})>0$. *Considering any* $\mathrm{p}>0$ *and setting* $n=\lfloor\mathrm{Re}(\mathrm{z})\rfloor+1$, *the following assertions hold:*

1.
$$\left({}_aI^{\mathrm{z},\mathrm{p},\Pi}e^{\frac{\mathrm{p}-1}{\mathrm{p}}\Pi(x)}(\Pi(x)-\Pi(a))^{\mathrm{b}-1}\right)(t)=\frac{\Gamma(\mathrm{b})}{\Gamma(\mathrm{b}+\mathrm{z})\mathrm{p}^{\mathrm{z}}}e^{\frac{\mathrm{p}-1}{\mathrm{p}}\Pi(t)}(\Pi(t)-\Pi(a))^{\mathrm{z}+\mathrm{b}-1}.$$

2. *If* $\mathrm{Re}(\mathrm{b})>n$, *then*
$$\left({}_a^CD^{\mathrm{z},\mathrm{p},\Pi}e^{\frac{\mathrm{p}-1}{\mathrm{p}}\Pi(x)}(\Pi(x)-\Pi(a))^{\mathrm{b}-1}\right)(\mathrm{t})=\frac{\mathrm{p}^{\mathrm{z}}\Gamma(\mathrm{b})}{\Gamma(\mathrm{b}-\mathrm{z})}e^{\frac{\mathrm{p}-1}{\mathrm{p}}\Pi(t)}(\Pi(t)-\Pi(a))^{\mathrm{b}-1-\mathrm{z}}.$$

3. *For* $k=0,1,\ldots,n-1$, *we have*
$$\left({}_a^CD^{\mathrm{z},\mathrm{p},\Pi}e^{\frac{\mathrm{p}-1}{\mathrm{p}}\Pi(x)}\left(\Pi(x)-\Pi(a)^k\right)\right)(t)=0.$$

   In particular, $\left({}_a^CD^{\mathrm{z},\mathrm{p}}e^{\frac{\mathrm{p}-1}{\mathrm{p}}\Pi(x)}\right)(t)=0$.

4.
$${}_a^CD^{\mathrm{z},\mathrm{p},\Pi}\left({}_aI^{\mathrm{z},\mathrm{p},\Pi}\psi\right)(t)=\psi(t);$$

5.

$$ {}_a^C D^{\mathrm{z,p,\Pi}} \left({}_a I^{\mathrm{z,p,\Pi}} \psi \right)(t) = \psi(t). $$

Theorem 1. *If* $\mathrm{p} \in (0,1]$, $Re(\mathrm{z}) > 0$, *and* $Re(\mathrm{b}) > 0$. *Then, for* $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ *is continuous functions and defined for* $t \geq a$, *we have*

$$ {}_a I^{\mathrm{z,p,\Pi}} \left({}_a I^{\mathrm{b,p,\Pi}} f \right)(t) = {}_a I^{\mathrm{b,p,\Pi}} \left({}_a I^{\mathrm{z,p,\Pi}} f \right)(t) $$

$$ = \left({}_a I^{\mathrm{z,p,\Pi}} f \right)(t). $$

Theorem 2. *Let* $0 \leq m < \lfloor \mathrm{Re(z)} \rfloor + 1$ *and* $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ *be integrable function in each interval* $[a,t], t > a$, *then*

$$ D^{m,\mathrm{p},\Pi} \left({}_a I^{\mathrm{z,p,\Pi}} f \right)(t) = \left({}_a I^{\mathrm{z}-m,\mathrm{p},\Pi} f \right)(t). $$

Corollary 1. *Let* $0 < \mathrm{Re(b)} < \mathrm{Re(z)}$, $m - 1 < \mathrm{Re(b)} \leq m$ *and* $f : [a,b] \times \mathbb{R} \to \mathbb{R}$. *Then we have*

$$ {}_a D^{\mathrm{b,p,\Pi}} {}_a I^{\mathrm{z,p,\Pi}} f(t) = {}_a I^{\mathrm{z}-\mathrm{b,p,\Pi}} f(t). $$

Theorem 3. *For* $\mathrm{p} > 0$, $n = \lfloor \mathrm{Re(z)} \rfloor + 1$, *and* $f : [a,b] \times \mathbb{R} \to \mathbb{R}$, *we have*

$$ {}_a I^{\mathrm{z,p,\Pi}} \left({}_a^C D^{\mathrm{z,p,\Pi}} f \right)(t) = f(t) - \sum_{k=0}^{n-1} \frac{\left(D^{k,\mathrm{p},\Pi} f \right)(a)}{\mathrm{p}^k k!} (\Pi(t) - \Pi(a))^k e^{\frac{\mathrm{p}-1}{\mathrm{p}} (\Pi(t) - \Pi(a))}. $$

In the next, the following fixed point theorems play a crucial role in deriving our main results.

Theorem 4 (Leray-Schauder Nonlinear Alternative [25])**.** *Let* $\mathbf{E}$ *be a Banach space,* $\mathscr{C}$ *a closed, convex subset of* $\mathbf{E}$, $\mathbf{U}$ *an open subset of* $\mathscr{C}$, *and* $0 \in \mathbf{U}$. *Suppose that* $\mathbf{F} : \bar{\mathbf{U}} \longrightarrow \mathscr{C}$ *is a continuous, compact map (meaning* $\mathbf{F}(\bar{\mathbf{U}})$ *is relatively compact in* $\mathscr{C}$*). Then either:*

(i) $\mathbf{F}$ *has a fixed point in* $\bar{\mathbf{U}}$, *or*

(ii) There exists a $u \in \partial \mathbf{U}$ *(the boundary of* $\mathbf{U}$ *in* $\mathscr{C}$*) and* $\kappa \in (0,1)$ *with* $u = \kappa \mathbf{F}(u)$.

Theorem 5 (Krasnoselskii's Fixed Point Theorem [27])**.** *Let* $\mathscr{X}$ *be a nonempty subset of a Banach space* $\mathscr{Y}$ *that is bounded, closed, and convex. Suppose* φ_1 *and* φ_2 *are operators mapping* $\mathscr{X}$ *into* $\mathscr{Y}$ *satisfying the following conditions:*

(i) For any $x_1, x_2 \in \mathscr{X}$, $\varphi_1 x_1 + \varphi_2 x_2 \in \mathscr{X}$;

(ii) φ_1 *is both compact and continuous;*

(iii) φ_2 *is a contraction mapping.*

Then, there exists $x_3 \in \mathscr{X}$ *such that* $x_3 = \varphi_1 x_3 + \varphi_2 x_3$.

The following Remark explains that using GPF operators allows to unify the different fractional integrals and, consequently, to solve some initial and boundary value problems with different types of fractional integrals and derivatives in an unified way.

Remark 1. *Some of the special cases from equation* (1) *are listed below:*

1. *Let* $h(\mathrm{t},x(\mathrm{t}))=\lambda x(\mathrm{t})$*, where* λ *is a constant.*

$$^{C}\mathfrak{D}^{\mathrm{z,p},\Pi}(^{C}\mathfrak{D}^{\mathrm{b,p},\Pi}+\lambda)x(\mathrm{t})=f(\mathrm{t},x(\mathrm{t})). \tag{5}$$

(i) If $\Pi(\mathrm{t})=\mathrm{t}$*, equation* (5) *reduces to a Langevin equation with two Caputo fractional proportional derivatives*

$$^{C}\mathfrak{D}^{\mathrm{z,p}}(^{C}\mathfrak{D}^{\mathrm{b,p}}+\lambda)x(\mathrm{t})=f(\mathrm{t},x(\mathrm{t})). \tag{6}$$

(ii) For $\mathrm{p}=1$*, equation* (5) *reduces to a Langevin equation with* ϑ*-Hilfer fractional derivatives.*

$$^{C}\mathfrak{D}^{\mathrm{z,p}}(^{C}\mathfrak{D}^{\mathrm{b,p}}+\lambda)x(\mathrm{t})=f(\mathrm{t},x(\mathrm{t})). \tag{7}$$

(iii) For $\mathrm{p}=1$*,* $\Pi(\mathrm{t})=\mathrm{t}$*, equation* (5) *reduces to a Langevin equation with two Caputo fractional derivatives.*

$$^{C}\mathfrak{D}^{\mathrm{z}}(^{C}\mathfrak{D}^{\mathrm{b}}+\lambda)x(\mathrm{t})=f(\mathrm{t},x(\mathrm{t})). \tag{8}$$

(iv) For $\mathrm{p}=1$*,* $\Pi(\mathrm{t})=\ln\mathrm{t}$*, equation* (5) *reduces to a LE with two Caputo-Hadamard fractional derivatives.*

$$^{CH}\mathfrak{D}^{\mathrm{z}}(^{CH}\mathfrak{D}^{\mathrm{b}}+\lambda)x(\mathrm{t})=f(\mathrm{t},x(\mathrm{t})). \tag{9}$$

(v) For $\mathrm{p}=1,\Pi(\mathrm{t})=\mathrm{t},\mathrm{z}\to 1,\mathrm{b}\to 1$*, then equation* (5) *reduces to the equation of motion with non-linear damping.*

$$\mathfrak{D}(\mathfrak{D}+\lambda)x(\mathrm{t})=f(\mathrm{t},x(\mathrm{t})). \tag{10}$$

2. *(i) For* $\mathrm{p}=1$*,* $\Pi=\mathrm{t}$*, the boundary conditions mentioned in equation* (2) *transform into the Riemann-Liovville type of slit-strips boundary conditions.*

   (ii) If $\Pi=\gamma=1$*,* $\Pi=t$*, the boundary conditions stated in equation* (2) *simplify to the conventional slit-strips boundary conditions.*

The following lemma plays a pivotal role in converting the problem (1) into a fixed point problem.

Lemma 1. *Let* $\rho\in(0,1]$*, and* $f,h\in\mathscr{C}([a,b],\mathbb{R})$*. The unique solution of the linear fractional differential equation*

$$^{C}\mathfrak{D}^{\alpha_1,\rho,\vartheta}\left(^{C}\mathfrak{D}^{\alpha_2,\rho,\vartheta}x+y\right)(t)=g(t),\quad 0<\alpha_1,\alpha_2\leq 1,\quad t\in[a,b]. \tag{11}$$

supplemented with the boundary conditions (2) *yields*

$$x(t) = I^{\alpha_1+\alpha_2,\rho,\vartheta} g(t) - I^{\alpha_2,\rho,\vartheta} y(t) + e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_1(t)$$

$$\left[\frac{a_1}{\rho^{\gamma}\Gamma(\gamma)}\int_a^{\eta_1} e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(\tau))}(\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1}\left(I^{\alpha_1+\alpha_2,\rho,\vartheta} g(\tau) - I^{\alpha_2,\rho,\vartheta} y(\tau)\right)\vartheta'(\tau)d\tau\right.$$

$$+\frac{a_2}{\rho^{\gamma}\Gamma(\gamma)}\int_{\eta_2}^{b} e^{\frac{\rho-1}{\rho}(\vartheta(b)-\vartheta(\tau))}(\vartheta(b)-\vartheta(\tau))^{\gamma-1}\left(I^{\alpha_1+\alpha_2,\rho,\vartheta} g(\tau) - I^{\alpha_2,\rho,\vartheta} y(\tau)\right)\vartheta'(\tau) \tag{12}$$

$$\left.-\sum_{i=1}^{m}\delta_i I^{\alpha_1+\alpha_2,\rho,\vartheta} g(\xi_i) + \sum_{i=1}^{m}\delta_i I^{\alpha_2,\rho,\vartheta} y(\xi_i)\right] - e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_2(t)$$

$$\left[A\int_a^b \left(I^{\alpha_1+\alpha_2,\rho,\vartheta} g(\tau) - I^{\alpha_2,\rho,\vartheta} y(\tau)\right) dE(\tau)\right],$$

where

$$\mu_1(t) = \frac{1}{k}\left(\sigma_2 - \frac{\sigma_1(\vartheta(b)-\vartheta(a))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\right),$$

$$\mu_2(t) = \frac{1}{k}\left(\sigma_4 - \frac{\sigma_3(\vartheta(b)-\vartheta(a))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\right),$$

$$\sigma_1 = 1 - A\int_a^b e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(a))} dE(\tau),$$

$$\sigma_2 = \frac{-A}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\int_a^b ((\vartheta(\tau)-\vartheta(a))^{\alpha_2}\, e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(a))} dE(\tau), \tag{13}$$

$$\sigma_3 = \sum_{i=1}^{m}\delta_i e^{\frac{\rho-1}{\rho}(\vartheta(\xi_i)-\vartheta(a))} - \frac{a_1 e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(a))}}{\rho^{\gamma}\Gamma(\gamma+1)}(\vartheta(\eta_1)-\vartheta(a))^{\gamma} - \frac{a_2 e^{\frac{\rho-1}{\rho}(\vartheta(b)-\vartheta(a))}}{\rho^{\gamma}\Gamma(\gamma+1)}(\vartheta(b)-\vartheta(\eta_2))^{\gamma},$$

$$\sigma_4 = \sum_{i=1}^{m}\delta_i \frac{(\vartheta(\xi_i)-\vartheta(a))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)} e^{\frac{\rho-1}{\rho}(\vartheta(\xi_i)-\vartheta(a))} - \frac{a_1 e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(a))}}{\rho^{\gamma+\alpha_2}}\frac{(\vartheta(\eta_1)-\vartheta(a))^{\gamma+\alpha_2}}{\Gamma(\alpha_2+\gamma+1)}$$

$$-\frac{a_2 e^{\frac{\rho-1}{\rho}(\vartheta(b)-\vartheta(a))}}{\rho^{\gamma}\Gamma(\gamma)} w_2,$$

and

$$k = \sigma_2\sigma_3 - \sigma_1\sigma_4 \neq 0. \tag{14}$$

Proof. Applying $I^{\alpha_1,\rho,\vartheta}$ and $I^{\alpha_2,\rho,\vartheta}$ to (11), using Theorem 3 and the relation 1 in Proposition 1, we get

$$x(t) = I^{\alpha_1+\alpha_2,\rho} g(t) - I^{\alpha_2,\rho} y(t) + c_0 e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))} + c_1 \frac{(\vartheta(t)-\vartheta(a))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)} e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}, \tag{15}$$

for some $c_0, c_1 \in \mathbb{R}$. Using the boundary condition (2) in (15), we derive

$$\begin{aligned} &c_0\left[1 - A\int_a^b e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(a))} dE(\tau)\right] - c_1 \frac{A}{\rho^{\alpha_2}\Gamma(\alpha_2+1)} \int_a^b (\vartheta(\tau)-\vartheta(a))^{\alpha_2} \\ &e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(a))} dE(\tau) = A\int_a^b \left(I^{\alpha_1+\alpha_2,\rho,\vartheta} g(\tau) - I^{\alpha_2,\rho,\vartheta} y(\tau)\right) dE(\tau). \end{aligned} \tag{16}$$

$$\begin{aligned} &c_0\Big(\sum_{i=1}^{m} \delta_i e^{\frac{\rho-1}{\rho}(\vartheta(\xi_i)-\vartheta(a))} - \frac{a_1}{\rho^{\gamma}\Gamma(\gamma+1)} e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(a))} (\vartheta(\eta_1)-\vartheta(a))^{\gamma} \\ &- \frac{a_2}{\rho^{\gamma}\Gamma(\gamma)} \int_{\eta_2}^{b} e^{\frac{\rho-1}{\rho}(\vartheta(b)-\vartheta(a))} (\vartheta(b)-\vartheta(\tau))^{\gamma-1} \vartheta'(\tau) d\tau\Big) \\ &+ c_1 \left(\sum_{i=1}^{m} \delta_i \frac{(\vartheta(\xi_i)-\vartheta(a))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)} e^{\frac{\rho-1}{\rho}(\vartheta(\xi_i)-\vartheta(a))} - \frac{a_1}{\rho^{\gamma+\alpha_2}\Gamma(\alpha_2+1)\Gamma(\gamma)} \right. \\ &\int_a^{\eta_1} e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(a))} (\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1} (\vartheta(\tau)-\vartheta(a))^{\alpha_2} \vartheta'(\tau) d\tau \\ &\left. - \frac{a_2}{\rho^{\gamma+\alpha_2}\Gamma(\alpha_2+1)\Gamma(\gamma)} \int_{\eta_2}^{b} e^{\frac{\rho-1}{\rho}(\vartheta(b)-\vartheta(a))} (\vartheta(b)-\vartheta(\tau))^{\gamma-1} (\vartheta(\tau)-\vartheta(a))^{\alpha_2} \vartheta'(\tau) d\tau\right) \\ &= \frac{a_1}{\rho^{\gamma}\Gamma(\gamma)} \int_a^{\eta_1} e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(\tau))} (\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1} \left(I^{\alpha_1+\alpha_2,\rho,\vartheta} g(\tau) - I^{\alpha_2,\rho,\vartheta} y(t)\right) \vartheta'(\tau) d\tau \\ &+ \frac{a_2}{\rho^{\gamma}\Gamma(\gamma)} \int_{\eta_2}^{b} e^{\frac{\rho-1}{\rho}(\vartheta(b)-\vartheta(\tau))} (\vartheta(b)-\vartheta(\tau))^{\gamma-1} \left(I^{\alpha_1+\alpha_2,\rho,\vartheta} g(\tau) - I^{\alpha_2,\rho,\vartheta} y(\tau)\right) \vartheta'(\tau) d\tau \\ &- \sum_{i=1}^{m} \delta_i I^{\alpha_1+\alpha_2,\rho,\vartheta} g(\xi_i) + \sum_{i=1}^{m} \delta_i I^{\alpha_2,\rho,\vartheta} y(\xi_i), \end{aligned} \tag{17}$$

Utilizing the notations (13) within (16) and (17) correspondingly, results in the following system of equations

$$\begin{cases} \sigma_1 c_0 + \sigma_2 c_1 = \sigma_5, \\ \sigma_3 c_0 + \sigma_4 c_1 = \sigma_6, \end{cases} \tag{18}$$

where

$$\sigma_5 = A\int_a^b \left(I^{\alpha_1+\alpha_2,\rho,\vartheta}g(\tau) - I^{\alpha_2,\rho,\vartheta}y(\tau)\right)dE(\tau),$$

$$\sigma_6 = \frac{a_1}{\rho^\gamma\Gamma(\gamma)}\int_a^{\eta_1} e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(\tau))}(\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1}\left(I^{\alpha_1+\alpha_2,\rho,\vartheta}g(\tau) - I^{\alpha_2,\rho,\vartheta}y(\tau)\right)\vartheta'(\tau)d\tau$$

$$+\frac{a_2}{\rho^\gamma\Gamma(\gamma)}\int_{\eta_2}^b e^{\frac{\rho-1}{\rho}(\vartheta(b)-\vartheta(\tau))}(\vartheta(b)-\vartheta(\tau))^{\gamma-1}\left(I^{\alpha_1+\alpha_2,\rho,\vartheta}g(\tau) - I^{\alpha_2,\rho,\vartheta}y(\tau)\right)\vartheta'(\tau)d\tau$$

$$-\sum_{i=1}^m \delta_i I^{\alpha_1+\alpha_2,\rho,\vartheta}g(\xi_i) + \sum_{i=1}^m \delta_i I^{\alpha_2,\rho,\vartheta}y(\xi_i).$$

By solving the system (18) with respect to c_0 and c_1, we obtain

$$c_0 = \frac{\sigma_2\sigma_6 - \sigma_4\sigma_5}{k}, c_1 = \frac{\sigma_3\sigma_5 - \sigma_1\sigma_6}{k},$$

where k is given by (14). Substituting the values of c_0 and c_1 in (15) together with the notations (13), we get the solution (12). □

3. Existence and uniqueness results for single valued case

In this section, we deal with the existence and uniqueness of solutions to the problem (1)–(2) using certain fixed-point theorems. Based on Lemma 1, we convert the boundary value problem (1)–(2) into $Gx = x$ where the operator $G : \mathscr{C}([a,b],\mathbb{R}) \longrightarrow \mathscr{C}([a,b],\mathbb{R})$ is defined as follows

$$Gx(t) = I^{\alpha_1+\alpha_2,\rho,\vartheta}f(t,x(t)) - I^{\alpha_2,\rho,\vartheta}h(t,x(t)) + e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_1(t)$$

$$\Big[\frac{a_1}{\rho^\gamma\Gamma(\gamma)}\int_a^{\eta_1} e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(\tau))}(\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1}\left(I^{\alpha_1+\alpha_2,\rho,\vartheta}f(\tau,x(\tau)) - I^{\alpha_2,\rho,\vartheta}h(\tau,x(\tau))\right)$$

$$\vartheta'(t)d\tau + \frac{a_2}{\rho^\gamma\Gamma(\gamma)}\int_{\eta_2}^b e^{\frac{\rho-1}{\rho}(\vartheta(b)-\vartheta(\tau))}(\vartheta(b)-\vartheta(\tau))^{\gamma-1}\left(I^{\alpha_1+\alpha_2,\rho,\vartheta}f(\tau,x(\tau)) - I^{\alpha_2,\rho,\vartheta}h(\tau,x(\tau))\right) \tag{19}$$

$$\vartheta'(\tau)d\tau - \sum_{i=1}^m \delta_i I^{\alpha_1+\alpha_2,\rho,\vartheta}f(\xi_i,x(\xi_i)) + \sum_{i=1}^m \delta_i I^{\alpha_2,\rho,\vartheta}h(\xi_i,x(\xi_i))\Big] - e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}$$

$$\mu_2(t)\Big[A\int_a^b \left(I^{\alpha_1+\alpha_2,\rho,\vartheta}f(\tau,x(\tau)) - I^{\alpha_2,\rho,\vartheta}h(\tau,x(\tau))\right)dE(\tau)\Big].$$

We note that $\mathscr{C}([a,b],\mathbb{R})$ represents the Banach space of all continuous functions $x : [a,b] \to \mathbb{R}$. The norm of this space is defined as $||x|| = \sup\{|x(t)| \mid t \in [a,b]\}$.

For convenience, let us set

$$\Lambda = \frac{(\vartheta(b)-\vartheta(a))^{\alpha_1+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)} + \frac{(\vartheta(b)-\vartheta(a))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)} + \|\mu_1\|\Big[\frac{|a_1|\,(\vartheta(\eta_1)-\vartheta(a))^{\alpha_1+\alpha_2+\gamma}}{\rho^{\alpha_1+\alpha_2+\gamma}\Gamma(\alpha_1+\alpha_2+\gamma+1)}$$

$$+\frac{|a_1|}{\rho^{\gamma+\alpha_2}\Gamma(\gamma+\alpha_2+1)}(\vartheta(\eta_1)-\vartheta(a))^{\gamma+\alpha_2} + \frac{|a_2|}{\rho^{\gamma}\Gamma(\gamma)}(w_1+w_2)$$

$$+\sum_{i=1}^{m}|\delta_i|\left(\frac{(\vartheta(\xi_i)-\vartheta(a))^{\alpha_1+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)} + \frac{(\vartheta(\xi_i)-\vartheta(a))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\right)\Big] + \|\mu_2\|\Big[|A|\int_a^b\Big(\frac{(\vartheta(\tau)-\vartheta(\alpha))^{\alpha_1+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)} \tag{20}$$

$$+\frac{(\vartheta(\tau)-\vartheta(a))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\Big)dE(\tau)\Big],$$

and

$$\Lambda_1 = \Lambda - \frac{(\vartheta(b)-\vartheta(a))^{\alpha_1+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)} - \frac{(\vartheta(b)-\vartheta(a))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}, \tag{21}$$

where

$$w_1 = \frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)}\int_{\eta_2}^{b}(\vartheta(b)-\vartheta(\tau))^{\gamma-1}(\vartheta(\tau)-\vartheta(a))^{\alpha_1+\alpha_2}\vartheta'(\tau)d\tau,$$

$$w_2 = \frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\int_{\eta_2}^{b}(\vartheta(b)-\vartheta(\tau))^{\gamma-1}(\vartheta(\tau)-\vartheta(a))^{\alpha_2}\vartheta'(\tau)d\tau. \tag{22}$$

Now, we are in a position to state the first main result which relies on the Leray-Schauder Nonlinear Alternative.

Theorem 6. *Let $\rho \in (0,1]$, and assume that $h, f : [a,b]\times\mathbb{R}\to\mathbb{R}$ be continuous functions. In addition, we suppose that:*

(H_1) *there exist two functions $P_1, P_2 \in \mathscr{C}([a,b],\mathbb{R}^+)$, and non-decreasing functions $\varphi_1,\varphi_2 : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, such that $|h(t,x)| \leqslant P_1\varphi_1(\|x\|)$ and $|f(t,x)| \leqslant P_2\varphi_2(\|x\|)$, for all $(t,x)\in[a,b]\times\mathbb{R}$,*

(H_2) *Considering $\varphi = \max\{\varphi_1,\varphi_2\}$ and $P = \max\{P_1,P_2\}$, there exists a positive constant M such that:*

$$\frac{M}{\|P\|\varphi(M)\Lambda} > 1.$$

Then the boundary value problem (1)–(2) *has at least one solution on* $[a,b]$.

Proof. Firstly, we aim to demonstrate that the operator $G : \mathscr{C}([a,b],\mathbb{R}) \to \mathscr{C}([a,b],\mathbb{R})$, as defined by equation (19), maps bounded sets into bounded sets within $\mathscr{C}([a,b],\mathbb{R})$. For a positive number r, we consider a closed ball $B_r = \{x \in \mathscr{C}([a,b],\mathbb{R}) : \|x\| \leqslant r\}$ be bounded set in $\mathscr{C}([a,b],\mathbb{R})$, with $r \geqslant \|P\|\varphi(\|r\|)\Lambda$. Then, in view of assumption (H_1), we have

$$|Gx(t)| = \Big| \frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)} \int_a^t e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(\tau))} (\vartheta(t)-\vartheta(\tau))^{\alpha_1+\alpha_2-1}$$

$$f(\tau,x(\tau))\vartheta'(\tau)d\tau + \frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)} \int_a^t e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(\tau))} (\vartheta(t)-\vartheta(\tau))^{\alpha_2-1} h(\tau,x(\tau))\vartheta'(\tau)d\tau$$

$$+ e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_1(t)\Bigg[\frac{a_1}{\rho^{\gamma}\Gamma(\gamma)} \int_a^{\eta_1} e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(\tau))} (\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1}$$

$$\Big(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)} \int_a^{\tau} e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(s))} (\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1} f(s,x(s))\vartheta'(s)ds$$

$$+ \frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)} \int_a^{\tau} e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(s))} (\vartheta(\tau)-\vartheta(s))^{\alpha_2-1} h(s,x(s))\vartheta'(s)ds\Big)\vartheta'(\tau)d\tau$$

$$+ \frac{a_2}{\rho^{\gamma}\Gamma(\gamma)} \int_{\eta_2}^b e^{\frac{\rho-1}{\rho}(\vartheta(b)-\vartheta(\tau))} (\vartheta(b)-\vartheta(\tau))^{\gamma-1}\Big(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}$$

$$\int_a^{\tau} e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(s))} (\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1} f(s,x(s))\vartheta'(s)ds$$

$$+ \frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)} \int_a^{\tau} e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(s))} (\vartheta(\tau)-\vartheta(s))^{\alpha_2-1} h(s,x(s))\vartheta'(s)ds\Big)\vartheta'(\tau)d\tau$$

$$- \sum_{i=1}^m \delta_i\Big(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)} \int_a^{\xi_i} e^{\frac{\rho-1}{\rho}(\vartheta(\xi_i)-\vartheta(s))} (\vartheta(\xi_i)-\vartheta(s))^{\alpha_1+\alpha_2-1} f(s,x(s))\vartheta'(s)$$

$$ds\Big) + \sum_{i=1}^m \delta_i\Big(\frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)} \int_a^{\xi_i} e^{\frac{\rho-1}{\rho}(\vartheta(\xi_i)-\vartheta(s))} (\vartheta(\xi_i)-\vartheta(s))^{\alpha_2-1} h(s,x(s))\vartheta'(s)ds\Big)\Big]$$

$$+ e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_2(t)\Big[A\int_a^b\Big(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)} \int_a^{\tau} e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(s))}$$

$$(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1} f(s,x(s))\vartheta'(s)ds + \frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)} \int_a^{\tau} e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(s))}$$

$$(\vartheta(\tau)-\vartheta(s)))^{\alpha_2-1} h(s,x(s))\vartheta'(s)ds\Big)dE(\tau)\Big]\Big|$$

$$\leqslant |P_2|\varphi_2(\|x\|)\left\{\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^t(\vartheta(t)-\vartheta(\tau))^{\alpha_1+\alpha_2-1}\vartheta'(\tau)d\tau\right.$$

$$+|\mu_1|\Big[\frac{|a_1|}{\rho^{\gamma}\Gamma(\gamma)}\int_a^{\eta_1}(\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1}\Big(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^{\tau}(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1}$$

$$\vartheta'(s)ds\Big)\vartheta'(\tau)d\tau+\frac{|a_2|}{\rho^{\gamma}\Gamma(\gamma)}\int_{\eta_2}^{b}(\vartheta(b)-\vartheta(\tau))^{\gamma-1}\left(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\right.$$

$$\left.\int_a^{\tau}(\vartheta(\tau)-\vartheta(s))^{\alpha_1-\alpha_2-1}\vartheta'(s)ds\right)\vartheta'(\tau)d\tau+\sum_{i=1}^{m}|\delta_i|\left(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\right.$$

$$\left.\int_a^{\xi_i}(\vartheta(\xi_i)-\vartheta(s))^{\alpha_1+\alpha_2-1}\vartheta'(s)ds\right)\Big]+|\mu_2|\left[|A|\int_a^b\left(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\right.\right.$$

$$\left.\left.\int_a^{\tau}(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2}\vartheta'(s)ds\right)dE(\tau)\right]\Bigg\}+|P_1|\varphi_1(\|x\|)$$

$$\left\{\frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^t(\vartheta(t)-\vartheta(\tau))^{\alpha_2-1}\vartheta'(\tau)d\tau+|\mu_1|\right.$$

$$\Big[\frac{|a_1|}{\rho^{\gamma}\Gamma(\gamma)}\int_a^{\eta_1}(\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1}\left(\frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^{\tau}(\vartheta(\tau)-\vartheta(s))^{\alpha_2-1}\vartheta'(s)ds\right)$$

$$\vartheta'(\tau)d\tau+\frac{|a_2|}{\rho^{\gamma}\Gamma(\gamma)}\int_{\eta_2}^{b}(\vartheta(b)-\vartheta(\tau))^{\gamma-1}\Big(\frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^{\tau}(\vartheta(\tau)-\vartheta(s))^{\alpha_2-1}\vartheta'(s)$$

$$ds\Big)\vartheta'(\tau)d\tau+\sum_{i=1}^{m}|\delta_i|\Big(\frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^{\xi_i}\big(\vartheta(\xi_i)-\vartheta(s)\big)^{\alpha_2-1}\vartheta'(s)ds\Big)\Big]$$

$$\left.+|\mu_2|\left[|A|\int_a^b\left(\frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^{\tau}(\vartheta(\tau)-\vartheta(s))^{\alpha_2-1}\vartheta'(s)ds\right)dE(\tau)\right]\right\}.$$

Then,

$$\|Gx\| \leqslant \|P\|\varphi(\|r\|)\Big\{ \frac{(\vartheta(b)-\vartheta(a))^{\alpha_1+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)} + \frac{(\vartheta(b)-\vartheta(a))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)} + \|\mu_1\|$$

$$\left[\frac{|a_1|\,(\vartheta(\eta_1)-\vartheta(a))^{\alpha_1+\alpha_2+\gamma}}{\rho^{\alpha_1+\alpha_2+\gamma}\Gamma(\alpha_1+\alpha_2+\gamma+1)} + \frac{|a_1|\,(\vartheta(\eta_1)-\vartheta(a))^{\gamma+\alpha_2}}{\rho^{\gamma+\alpha_2}\Gamma(\gamma+\alpha_2+1)} + \frac{|a_2|\,(w_1+w_2)}{\rho^{\gamma}\Gamma(\gamma)} + \sum_{i=1}^{m}|\delta_i| \right.$$

$$\left. \left(\frac{(\vartheta(\xi_i)-\vartheta(a))^{\alpha_1+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)} + \frac{(\vartheta(\xi_i)-\vartheta(a))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)} \right) \right]$$

$$+ \|\mu_2\| \Big[|A| \int_a^b \left(\frac{(\vartheta(\tau)-\vartheta(a))^{\alpha_1+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)} + \frac{(\vartheta(\tau)-\vartheta(a))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)} \right) dE(\tau) \Big] \Big\}.$$

Thus

$$\|Gx\| \leqslant \|P\|\varphi(\|r\|)\Lambda \leq r.$$

This indicates that the set $G(B_r)$ is uniformly bounded. Subsequently, it will be demonstrated that G maps the bounded set B_r into equicontinuous sets in $\mathscr{C}([a,b],\mathbb{R})$. Consider $v_1, v_2 \in [a,b]$. Set

$$Gx(v_1) = I^{\alpha_1+\alpha_2,\rho,\vartheta} f(v_1, x(v_1)) - I^{\alpha_2,\rho,\vartheta} h(v_1, x(v_1))$$

$$Gx(v_2) = e^{\frac{\rho-1}{\rho}(\vartheta(v_2)-\vartheta(a))}\mu_1(v_2) \left[\frac{a_1}{\rho^{\gamma}\Gamma(\gamma)} \int_a^{\eta_1} e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(\tau))} (\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1} \right.$$

$$\left(I^{\alpha+\alpha_2,\rho,\vartheta} f(\tau, x(\tau)) - I^{\alpha_2,\rho,\vartheta} h(\tau, x(\tau)) \right) \vartheta'(\tau) d\tau + \frac{a_2}{\rho^{\gamma}\Gamma(\tau)} \int_{\eta_2}^{b} e^{\frac{\rho-1}{\alpha}(\vartheta(b)-\vartheta(\tau))} (\vartheta(b)-\vartheta(\tau))^{\gamma-1}$$

$$\left. \left(I^{\alpha_1+\alpha_2,\rho,\vartheta} f(\tau, x(\tau) - I^{\alpha_2,\rho,\vartheta} h(\tau, x(\tau) \right) \vartheta'(\tau) d\tau - \sum_{i=1}^{m} \delta_i I^{\alpha_1+\alpha_2,\rho,\vartheta} f(\xi_i, x(\xi_i)) + \sum_{i=1}^{m} \delta_i I^{\alpha_2,\rho,\vartheta} h(\xi_i, x(\xi_i)) \right]$$

$$- e^{\frac{\rho-1}{\rho}(\vartheta(v_2)-\vartheta(a))}\mu_2(v_2) \left[A \int_a^b \left(I^{\alpha_1+\alpha_1,\rho,\vartheta} f(\tau, x(\tau)) - I^{\alpha_2,\rho,\vartheta} h(\tau, x(\tau) \right) dE(\tau) \right].$$

Take $v_1 < v_2$ and $x \in B_r$, then we obtain

$$|Gx(v_2)-Gx(v_1)| \leqslant \frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^{v_1}\left|(\vartheta(v_2)-\vartheta(\tau))^{\alpha_1+\alpha_2-1}-(\vartheta(v_1)-\vartheta(\tau))^{\alpha_1+\alpha_2-1}\right|$$

$$|f(\tau,x(\tau))|\vartheta'(\tau)d\tau+\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_{v_1}^{v_2}(\vartheta(v_2)-\vartheta(\tau))^{\alpha_1+\alpha_2-1}|f(\tau,x(\tau))|\vartheta'(\tau)d\tau$$

$$+\frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^{v_1}\left|(\vartheta(v_2)-\vartheta(\tau))^{\alpha_2-1}-(\vartheta(v_1)-\vartheta(\tau))^{\alpha_2-1}\right||h(\tau,x(\tau))|$$

$$\vartheta'(\tau)d\tau+\frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_{v_1}^{v_2}\left|(\vartheta(v_2)-\vartheta(\tau))^{\alpha_2-1}\right|\mid h(\tau,x(\tau)\mid\vartheta'(\tau)d\tau$$

$$+|\mu_1(v_2)-\mu_1(v_1)|\big[\frac{|a_1|}{\rho^{\gamma}\Gamma(\gamma)}\int_a^{\eta_1}(\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1}\big(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}$$

$$\int_a^{\tau}(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1}|f(s,x(s))|\vartheta'(s)ds+\frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^{\tau}(\vartheta(\tau)-\vartheta(s))^{\alpha_2-1}$$

$$|h(s,x(s))|\vartheta'(s)ds\big)\vartheta'(\tau)d\tau+\frac{|a_2|}{\rho^{\gamma}\Gamma(\gamma)}\int_{\eta_2}^{b}(\vartheta(b)-\vartheta(\tau))^{\gamma-1}\big(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}$$

$$\int_a^{\tau}(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1}|f(s,x(s))|\vartheta'(s)ds+\frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^{\tau}(\vartheta(\tau)-\vartheta(s))^{\alpha_2-1}|h(s,x(s))|$$

$$\vartheta'(s)ds\big)\vartheta'(\tau)d\tau+\sum_{i=1}^{m}|\delta_i|\big(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^{\xi_i}(\vartheta(\xi_i)-\vartheta(s))^{\alpha_1+\alpha_2-1}$$

$$|f(s,x(s))|\vartheta'(s)ds\big)+\sum_{i=1}^{m}|\delta_i|\big(\frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^{\xi_i}(\vartheta(\xi_i)-\vartheta(s))^{\alpha_2-1}|h(s,x(s))|$$

$$\vartheta'(s)ds\big)\big]+|\mu_2(v_2)-\mu_2(v_1)|\left[|A|\int_a^{b}\left(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^{\tau}(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1}\right.\right.$$

$$\left.\left.|f(s,x(s))|\vartheta'(s)ds+\frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^{\tau}(\vartheta(\tau)-\vartheta(s))^{\alpha_2-1}|h(s,x(s))|\vartheta'(s)ds\right)dE(\tau)\right]$$

$$\leqslant \|P\|\varphi(\|x\|)\left\{\frac{2\left(\vartheta\left(v_2\right)-\vartheta(v_1)\right)^{\alpha_1+\alpha_2}+\left|\left(\vartheta\left(v_2\right)-\vartheta(a)\right)^{\alpha_1+\alpha_2}-\left(\vartheta\left(v_1\right)-\vartheta(a)\right)^{\alpha_1+\alpha_2}\right|}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)}\right.$$

$$+\frac{2\left(\vartheta\left(v_2\right)-\vartheta\left(v_1\right)\right)^{\alpha_2}+\left|\left(\vartheta\left(v_2\right)-\vartheta\left(a\right)\right)^{\alpha_2}-\left(\vartheta\left(v_1\right)-\vartheta\left(a\right)\right)^{\alpha_2}\right|}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}$$

$$+\left|\frac{\sigma_1\left(\left(\vartheta\left(v_2\right)-\vartheta(a)\right)^{\alpha_2}-\left(\vartheta\left(v_1\right)-\vartheta(a)\right)^{\alpha_2}\right)}{k\rho^{\alpha_2}\Gamma(\alpha_2+1)}\right|\left[\frac{|a_1|\left(\vartheta\left(\eta_1\right)-\vartheta(a)\right)^{\alpha_1+\alpha_2+\gamma}}{\rho^{\alpha_1+\alpha_2+\gamma}\Gamma(\alpha_1+\alpha_2+\gamma+1)}\right.$$

$$+\frac{|a_1|}{\rho^{\gamma+\alpha_2}\Gamma(\gamma+\alpha_2+1)}\left(\vartheta\left(\eta_1\right)-\vartheta\left(a\right)\right)^{\gamma+\alpha_2}+\frac{|a_2|}{\rho^{\gamma}\Gamma(\gamma)}\left(w_1\ +w_2\right)$$

$$\left.+\sum_{i=1}^{m}|\delta_i|\left(\frac{\left(\vartheta(\xi_i)-\vartheta(a)\right)^{\alpha_1+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)}+\frac{\left(\vartheta\left(\xi_i\right)-\vartheta(a)\right)^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\right)\right]$$

$$+\left|\frac{\sigma_3\left(\vartheta\left(v_2\right)-\vartheta(a)\right)^{\alpha_2}-\left(\vartheta\left(v_1\right)-\vartheta(a)\right)^{\alpha_2}}{k\rho^{\alpha_2}\Gamma(\alpha_2+1)}\right|\left[|A|\int_a^b\left(\frac{\left(\vartheta(\tau)-\vartheta(a)\right)^{\alpha_1+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)}\right.\right.$$

$$\left.\left.+\frac{\left(\vartheta(\tau)-\vartheta(a)\right)^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\right)dE(\tau)\right]\Bigg\}.$$

The right-hand side of the previous inequality approaches zero independently of x in B_r as $v_2-v_1\to 0$. Consequently, G maps bounded set into a set of equicontinuous functions.

Therefore, it follows by the Arzela'-Ascoli theorem that G is completely continuous.

Finally, we will establish the boundness of the set of all solutions to equations $x=\kappa Gx$, for $\kappa\in(0,1)$. Let $x(t)$ be a solution of fractional boundary value problem (1)–(2). Thus, for $t\in[a,b]$,we have

$$\|x\|\leq\|P\|\varphi(\|x\|)\Lambda.$$

By the condition (H_2), we can find a positive number M such that $\|x\|\neq M$. Let us define a set $\Upsilon=\{x\in\mathscr{C}([a,b],\mathbb{R}):\|x\|<M\}$ and note that the operator $G:\bar{\Upsilon}\to\mathscr{C}([a,b],\mathbb{R})$ is continuous and completely continuous. Form the choice of Υ, there is no $x\in\partial\Upsilon$ such that $x=\kappa G(x)$ for some $\kappa\in(0,1)$. In consequence, we deduce by Theorem 4 that the operator G has a fixed point $x\in\bar{\Upsilon}$ which is the solution of the boundary value problem (1)–(2). □

The following existence result is based on Theorem 5.

Theorem 7. *Let $\rho\in(0,1]$ and assume that $h,f:[a,b]\times\mathbb{R}\to\mathbb{R}$ be a continuous functions satisfying the conditions:*

(H_3) *There exists a constant $L>0$ such that $|h(t,x)-h(t,y)|\leqslant L_1|x-y|$, $|f(t,x)-f(t,y)|\leqslant L_2|x-y|$, $L=\max\{L_1,L_2\}$, for each $t\in[a,b],x,y\in\mathbb{R}$.*

(H_4) *$|h(t,x)|\leqslant\lambda_1(t),|f(t,x)|\leqslant\lambda_2(t)$ for all $(t,x)\in[a,b]\times\mathbb{R}$, $\lambda_1,\lambda_2\in\mathscr{C}([a,b],\mathbb{R}^+)$ and $\lambda=\max\{\lambda_1,\lambda_2\}$. If the inequality*

$$\Lambda_1 L<1, \tag{23}$$

is satisfied, where Λ_1 *is determined by* (21)*, then there exists at least one solution to the boundary value problem* (1)–(2) *on the interval* $[a,b]$.

Proof. By the assumption (H_4) and (20), we choose $\bar{r} \geqslant \Lambda||\lambda||$, and considering the closed ball $B_{\bar{r}} = \{x \in \mathscr{C} : \|x\| \leqslant \bar{r}\}$. Introducing the operators G_1 and G_2 on $B_{\bar{r}}$ as follows:

$$(G_1x)(t) = I^{\alpha_1+\alpha_2,\rho,\vartheta} f(t,x(t)) - I^{\alpha_2,\rho,\vartheta} h(t,x(t))$$

$$(G_2x)(t) = e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_1(t)\left[\frac{a_1}{\rho^{\gamma}\Gamma(\gamma)}\int_a^{\eta_1} e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(\tau))}(\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1}\right.$$

$$\left(I^{\alpha+\alpha_2,\rho,\vartheta} f(\tau,x(\tau)) - I^{\alpha_2,\rho,\vartheta} h(\tau,x(\tau))\right)\vartheta'(\tau)d\tau + \frac{a_2}{\rho^{\gamma}\Gamma(\tau)}\int_{\eta_2}^{b} e^{\frac{\rho-1}{\alpha}(\vartheta(b)-\vartheta(\tau))}(\vartheta(b)-\vartheta(\tau))^{\gamma-1}$$

$$\left.\left(I^{\alpha_1+\alpha_2,\rho,\vartheta} f(\tau,x(\tau) - I^{\alpha_2,\rho,\vartheta} h(\tau,x(\tau)\right)\vartheta'(\tau)d\tau - \sum_{i=1}^{m}\delta_i I^{\alpha_1+\alpha_2,\rho,\vartheta} f(\xi_i,x(\xi_i)) + \sum_{i=1}^{m}\delta_i I^{\alpha_2,\rho,\vartheta} h(\xi_i,x(\xi_i))\right]$$

$$-e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_2(t)\left[A\int_a^b\left(I^{\alpha_1+\alpha_1,\rho,\vartheta} f(\tau,x(\tau)) - I^{\alpha_2,\rho,\vartheta} h(\tau,x(\tau)\right)dE(\tau)\right], t \in [a,b].$$

Observe that $Gx = G_1x + G_2x$. Then for $x,y \in B_{\bar{r}}$, taking into consideration that $\left|e^{\frac{p-1}{p}(\vartheta(t)-\vartheta(a))}\right| < 1, \forall t > a$, we find that

$$|G_1x + G_2y| \leqslant \frac{|\lambda_2|}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^t e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(\tau))}(\vartheta(t)-\vartheta(\tau))^{\alpha_1+\alpha_2-1}$$

$$\vartheta'(\tau)d\tau + \frac{|\lambda_1|}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^t e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(\tau))}(\vartheta(t)-\vartheta(\tau))^{\alpha_2-1}\vartheta'(\tau)$$

$$d\tau + e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}|\mu_1|\left[\frac{|a_1|}{\rho^{\gamma}\Gamma(\gamma)}\int_a^{\eta_1} e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(\tau))}(\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1}\right.$$

$$\left(\frac{|\lambda_2|}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^{\tau} e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(s))}(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1}\vartheta'(s)ds\right.$$

$$\left.+\frac{|\lambda_1|}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^{\tau} e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(s))}(\vartheta(\tau)-\vartheta(s))^{\alpha_2-1}\vartheta'(s)ds\right)\vartheta'(\tau)d\tau$$

$$+\frac{|a_2|}{\rho^{\gamma}\Gamma(\gamma)}\int_{\eta_2}^{b} e^{\frac{\rho-1}{\rho}(\vartheta(b)-\vartheta(\tau))}(\vartheta(b)-\vartheta(\tau))^{\gamma-1}\left(\frac{|\lambda_2|}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_{a_1}^{\tau} e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(s))}\right.$$

$$(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1}\vartheta'(s)ds+\frac{|\lambda_1|}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^{\tau}e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(s))}$$

$$(\vartheta(\tau)-\vartheta(s))^{\alpha_2-1}\vartheta'(s)ds\Big)\vartheta'(\tau)d\tau+\sum_{i=1}^{m}\delta_i\left(\frac{|\lambda_2|}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\right.$$

$$\left.\int_a^{\xi_i}e^{\frac{\rho-1}{\rho}(\vartheta(\xi_i)-\vartheta(s))}\left(\vartheta(\xi_i)-\vartheta(s)\right)^{\alpha_1+\alpha_2-1}\vartheta'(s)ds\right)$$

$$+\sum_{i=1}^{m}\delta_i\left(\frac{|\lambda_1|}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^{\xi_i}e^{\frac{\rho-1}{\rho}(\vartheta(\xi_i)-\vartheta(s))}(\vartheta(\xi_i)-\vartheta(s))^{\alpha_2-1}\vartheta'ds\right)\Big]$$

$$+e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}|\mu_2|\left[|A|\int_a^{b}\left(\frac{|\lambda_2|}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^{\tau}e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(s))}\right.\right.$$

$$(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1}\vartheta'(s)ds+\frac{|\lambda_1|}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^{\tau}e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(s))}$$

$$\left.\left.(\vartheta(\tau)-\vartheta(s))^{\alpha_2-1}\vartheta'(s)ds\Big)dE(\tau)\right].\right.$$

Then,

$$\|G_1x+G_2y\|\leqslant\|\lambda\|\left\{\frac{(\vartheta(b)-\vartheta(a))^{\alpha_1+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)}+\frac{(\vartheta(b)-\vartheta(a))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}+\|\mu_1\|\left[\frac{|a_1|\left(\vartheta(\eta_1)-\vartheta(a)\right)^{\alpha_1+\alpha_2+\gamma}}{\rho^{\alpha_1+\alpha_2+\gamma}\Gamma(\alpha_1+\alpha_2+\gamma+1)}\right.\right.$$

$$+\frac{|a_1|}{\rho^{\gamma+\alpha_2}\Gamma(\gamma+\alpha_2+1)}\left(\vartheta(\eta_1)-\vartheta(a)\right)^{\gamma+\alpha_2}$$

$$\left.+\frac{|a_2|}{\rho^{\gamma}\Gamma(\gamma)}(w_1+w_2)+\sum_{i=1}^{m}|\delta_i|\left(\frac{\left(\vartheta(\xi_i)-\vartheta(a)\right)^{\alpha_1+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)}+\frac{\left(\vartheta(\xi_i)-\vartheta(a)\right)^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\right)\right]$$

$$\left.+\|\mu_2\|\left[|A|\int_a^{b}\left(\frac{(\vartheta(\tau)-\vartheta(a))^{\alpha_1+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)}+\frac{(\vartheta(\tau)-\vartheta(a))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\right)dE(\tau)\right]\right\}$$

$$=\|\lambda\|\Lambda\leqslant\bar{r},$$

which proves that $G_1x+G_2y\in B_{\bar{r}}$. Next, we establish that G_2 is a contraction mapping. For $x,y\in\mathbb{R}$ and for each $t\in[a,b]$, we have

$$\|G_2x-G_2y\|\leq e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\|\mu_1\|\Big[\frac{|a_1|}{\rho^{\gamma}\Gamma(\gamma)}\int_a^{\eta_1}e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(\tau))}\left(\vartheta(\eta_1)-\vartheta(\tau)\right)^{\gamma-1}$$

$$\Big(\frac{L_2\|x-y\|}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^\tau e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(s))}(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1}\vartheta'(s)ds$$

$$+\frac{L_1\|x-y\|}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^\tau e^{\frac{\rho}{\rho-1}(\vartheta(\tau)-\vartheta(s))}(\vartheta(\tau)-\vartheta(s))^{\alpha_2-1}\vartheta'(s)ds\Big)\vartheta'(\tau)d\tau+\frac{|a_2|}{\rho^{\gamma}\Gamma(\gamma)}$$

$$\int_{\eta_2}^b e^{\frac{\rho-1}{\rho}(\vartheta(b)-\vartheta(\tau))}(\vartheta(b)-\vartheta(\tau))^{\gamma-1}\left(\frac{L_2\|x-y\|}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^\tau e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(s))}\,(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1}\vartheta'(s)ds\right.$$

$$\left.+\frac{L_1\|x-y\|}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^\tau e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(s))}(\vartheta(\tau)-\vartheta(s))^{\alpha_2-1}\vartheta'(s)ds\Big)\vartheta'(\tau)d\tau\right.$$

$$+\sum_{i=1}^m|\delta_i|\left(\frac{L_2\|x-y\|}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^{\xi_i}e^{\frac{\rho-1}{\rho}(\vartheta(\xi_i)-\vartheta(s))}\,(\vartheta\,(\xi_i)-\vartheta(s))^{\alpha_1+\alpha_2-1}\,\vartheta'(s)ds\right)$$

$$\left.+\sum_{i=1}^m|\delta_i|\left(\frac{L_1\|x-y\|}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^{\xi_i}e^{\frac{\rho-1}{\rho}(\vartheta(\xi_i)-\vartheta(s))}\,(\vartheta\,(\xi_i)-\vartheta(s))^{\alpha_2-1}\,\vartheta'(s)ds\right)\right]+e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}$$

$$\|\mu_2\|\left[|A|\int_a^b\left(\frac{L_2\|x-y\|}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^\tau e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(s))}(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1}\,\vartheta'(s)ds\right.\right.$$

$$\left.\left.+\frac{L_1\|x-y\|}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^\tau e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(s))}(\vartheta(\tau)-\vartheta(s))^{\alpha_2-1}\vartheta'(s)ds\Big)dE(\tau)\right]\right.$$

$$\leqslant L\Bigg\{\|\mu_1\|\Big[\frac{|a_1|}{\rho^{\gamma}\Gamma(\gamma)}\int_a^{\eta_1}(\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1}\left(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^\tau(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1}\right.$$

$$\left.\vartheta'(s)ds+\frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^\tau(\vartheta(\tau)-\vartheta(s))^{\alpha_2-1}\vartheta'(s)ds\right)\vartheta'(\tau)d\tau$$

$$+\frac{|a_2|}{\rho^{\gamma}\Gamma(\gamma)}\int_{\eta_2}^b(\vartheta(b)-\vartheta(\tau))^{\gamma-1}\left(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^\tau(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1}\vartheta'(s)ds\right.$$

$$\left.+\frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^\tau(\vartheta(\tau)-\vartheta(s))^{\alpha_2-1}\vartheta'(s)ds\right)\vartheta'(\tau)d\tau+\sum_{i=1}^m|\delta_i|$$

$$\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^{\xi_i}\left(\vartheta\left(\xi_i\right)-\vartheta(s)\right)^{\alpha_1+\alpha_2-1}\vartheta'(s)ds+\sum_{i=1}^{m}|\delta_i|\frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)}$$

$$\int_a^{\xi_i}\left(\vartheta\left(\xi_i\right)\vartheta(s)\right)^{\alpha_2-1}\vartheta'(s)ds\Bigg]+\|\mu_2\|\left[|A|\int_a^b\left(\frac{1}{p^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2-1)}\int_a^{\tau}(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1}\right.\right.$$

$$\left.\left.\vartheta'(s)ds+\frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^{\tau}(\vartheta(\tau)-\vartheta(s))^{\alpha_2-1}\vartheta'(s)ds\right]dE(\tau)\right\}$$

$$\leqslant L\left\{\|\mu_1\|\left(\frac{|a_1|\left(\vartheta\left(\eta_1\right)-\vartheta(a)\right)^{\alpha_1+\alpha_2+\gamma}}{\rho^{\gamma+\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+\gamma+1)}+\frac{|a_1|}{\rho^{\gamma+\alpha_2}\Gamma(\gamma+\alpha_2+1)}\left(\vartheta\left(\eta_1\right)-\vartheta\left(a\right)\right)^{\gamma+\alpha_2}\right.\right.$$

$$\left.+\frac{|a_2|}{\rho^{\gamma}\Gamma(\gamma)}(w_1+w_2)+\sum_{i=1}^{m}|\delta_i|\left(\frac{\left(\vartheta\left(\xi_i\right)-\vartheta\left(a\right)\right)^{\alpha_1+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)}+\frac{\left(\vartheta\left(\xi_i\right)-\vartheta(a)\right)^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\right)\right.$$

$$\left.+\|\mu_2\|\left[|A|\int_a^b\left(\frac{(\vartheta(\tau)-\vartheta(a))^{\alpha_1+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)}+\frac{(\vartheta(\tau)-\vartheta(a))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\right)dE(\tau)\right\}\|x-y\|\right.$$

$$\leqslant L\Lambda_1\|x-y\|.$$

By using the inequality (23), the mapping G_2 is a contraction. The continuity of f,h implies that the operator G_1 is continuous. Also, G_1 is uniformly bounded on $B_{\bar{r}}$ as

$$\|G_1x\|=\sup_{t\in[a,b]}\left\{\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^t e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(\tau))}(\vartheta(t)-\vartheta(\tau))^{\alpha_1+\alpha_2-1}\right.$$

$$\left.|f(\tau,(\tau))|\vartheta'(\tau)d\tau+\frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^t e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(\tau))}\left(\vartheta(t)-\vartheta(\tau)\right)^{\alpha_2-1}|h(\tau,x(\tau))|\vartheta'(\tau)d\tau\right\}$$

$$\leqslant\|\lambda\|\left(\frac{(\vartheta(b)-\vartheta(a))^{\alpha_1+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)}+\frac{(\vartheta(b)-\vartheta(a))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\right).$$

he compactness of the operator G_1 will be established. Considering (H_3), we proceed to set

$$\sup_{(t,x)\in[a,b]\times B_{\bar{r}}}|h(t,x)|=\bar{h}<\infty,\quad\sup_{(t,x)\in[a,b]\times B_{\bar{r}}}|f(t,x)|=\bar{f}<\infty.$$

Consequently, for $a\leqslant t_1\leqslant t_2\leqslant b$, we have

$$|G_1x(t_2)-G_1x(t_1)| \leqslant \frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^{t_1}((\vartheta(t_2)-\vartheta(\tau))^{\alpha_1+\alpha_2-1}-(\vartheta(t_1)-\vartheta(\tau))^{\alpha_1+\alpha_2-1})$$

$$|f(\tau,x(\tau))|\vartheta'(\tau)d\tau+\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_{t_1}^{t_2}(\vartheta(t_2)-\vartheta(\tau))^{\alpha_1+\alpha_2-1}|f(\tau,x(\tau))|\vartheta'(\tau)d\tau$$

$$+\frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2}\int_a^{t_1}((\vartheta(t_2)-\vartheta(\tau))^{\alpha_2-1}-(\vartheta(t_1)-\vartheta(\tau))^{\alpha_2-1})|h(\tau,x(\tau))|\vartheta'(\tau)d\tau$$

$$+\frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_{t_1}^{t_2}(\vartheta(t_2)-\vartheta(\tau))^{\alpha_2-1}|h(\tau,x(\tau))|\vartheta'(\tau)d\tau$$

$$\leqslant \frac{\bar{f}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)}\Big[2(\vartheta(t_2)-\vartheta(t_1))^{\alpha_1+\alpha_2}+((\vartheta(t_2)-\vartheta(a))^{\alpha_1+\alpha_2}-(\vartheta(t_1)-\vartheta(a))^{\alpha_1+\alpha_2})\Big]$$

$$+\frac{\bar{h}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\Big[2(\vartheta(t_2)-\vartheta(t_1))^{\alpha_2}+((\vartheta(t_2)-\vartheta(a))^{\alpha_2}-(\vartheta(t_1)-\vartheta(a))^{\alpha_2})\Big].$$

The right-hand side of the last inequality's, independent of x, will goes to zero as t_2-t_1 approaches to zero. Thus, G_1 is relatively Compact on $B_{\bar{r}}$. Hence, by the Arzelá-Ascoli theorem, G_1 is compact on $B_{\bar{r}}$. Therefore, all the assumptions of Theorem 5 are satisfied and we deduce the boundary value problem (1)–(2) has at least one solution on $[a,b]$. □

Theorem 8. *Let $\rho \in (0,1]$ and assume that $f,h:[a,b]\times\mathbb{R}\to\mathbb{R}$ are continuous functions satisfying the assumption (H_3). Then the boundary value problem* (1)–(2) *has a unique solution on $[a,b]$ if*

$$L\Lambda < 1, \tag{24}$$

where a constant Λ is given by equation (20)*.*

Proof. Define $M=\max\{M_1,M_2\}$, where M_1 and M_2 are positive numbers such that $\sup_{t\in[a,b]}|h(t,a)|=M_1$ and $\sup_{t\in[a,b]}|f(t,a)|=M_2$. We fix $r\geqslant\frac{M\Lambda}{1-L\Lambda}$. In view of the assumption (H_3), we have

$$\begin{aligned}|h(t,x)|=|h(t,x)-h(t,a)+h(t,a)| &\leqslant |h(t,x)-h(t,a)|+|h(t,a)|\\ &\leqslant L_1\|x\|+M_1\\ &\leqslant L_1r+M_1.\end{aligned}$$

Likewise, it can be derived that $|f(t,x)|\leqslant L_2r+M_2$. Initially, we demonstrate that $GB_r\subset B_r$ where $B_r=\{x\in\mathscr{C}:\|x\|\leqslant r\}$. Then, for any $x\in B_r$, we get

$$\|Gx\| \leqslant (Lr+M)\left\{\frac{(\vartheta(b)-\vartheta(a))^{\alpha_1+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)}+\frac{(\vartheta(b)-\vartheta(a))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}+\|\mu_1\|\right.$$

$$\left[\frac{|a_1|\,(\vartheta(\eta_1)-\vartheta(a))^{\alpha_1+\alpha_2+\gamma}}{\rho^{\alpha_1+\alpha_2+\gamma}\Gamma(\alpha_1+\alpha_2+\gamma+1)}+\frac{|a_1|}{\rho^{\gamma+\alpha_2}\Gamma(\gamma+\alpha_2+1)}\,(\vartheta(\eta_1)-\vartheta(a))^{\gamma+\alpha_2}\right.$$

$$\left.+\frac{|a_2|}{\rho^{\gamma}\Gamma(\gamma)}\,(w_1+w_2)+\sum_{i=1}^{m}|\delta_i|\left(\frac{(\vartheta(\xi_i)-\vartheta(a))^{\alpha_1+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)}+\frac{(\vartheta(\xi_i)-\vartheta(a))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\right)\right]$$

$$\left.+\|\mu_2\|\left[|A|\int_a^b\left(\frac{(\vartheta(\tau)-\vartheta(a))^{\alpha+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}+\frac{(\vartheta(\tau)-\vartheta(a))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}dE(\tau)\right]\right\}=(Lr+M)\Lambda\leqslant r,$$

implying that $GB_r\subset B_r$. Subsequently, for x and y in $\mathbb{R}$ and for every t in $[a,b]$, we get

$$\|Gx-Gy\|\leqslant\frac{L_2\|x-y\|}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^t(\vartheta(t)-\vartheta(\tau))^{\alpha_1+\alpha_2-1}.$$

$$\vartheta'(\tau)d\tau+\frac{L_1\|x-y\|}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^t(\vartheta(t)-\vartheta(\tau))^{\alpha_2-1}\vartheta'(\tau)d\tau$$

$$+\|\mu_1\|\left[\frac{|a_1|}{\rho^{\gamma}\Gamma(\gamma)}\int_a^{\eta_1}(\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1}\left(\frac{L_2\|x-y\|}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\right.\right.$$

$$\left.\int_a^{\tau}(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1}\vartheta'(s)ds+\frac{L_1\|x-y\|}{\rho_2^{\alpha}\Gamma(\alpha_2)}\int_a^{\tau}(\vartheta(\tau)-\vartheta(s))^{\alpha_2-1}\vartheta'(s)ds\right)\vartheta'(\tau)d\tau$$

$$+\frac{|a_2|}{\rho^{\gamma}\Gamma(\gamma)}\int_{\eta_2}^b(\vartheta(b)-\vartheta(\tau))^{\gamma-1}\left(\frac{L_2\|x-y\|}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^{\tau}(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1}\vartheta'(s)ds\right.$$

$$\left.+\frac{L_1\|x-y\|}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^{\tau}(\vartheta(\tau)-\vartheta(s))^{\alpha_2-1}\vartheta'(s)ds\right)\vartheta'(\tau)d\tau$$

$$+\sum_{i=1}^{m}|\delta_i|\left(\frac{L_2\|x-y\|}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^{\xi_i}(\vartheta(\xi_i)-\vartheta(s))^{\alpha_1+\alpha_2-1}\ \vartheta'(s)ds\right)$$

$$\left.+\sum_{i=1}^{m}|\delta_i|\left(\frac{L_1\|x-y\|}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^{\xi_i}(\vartheta(\xi_i)-\vartheta(s))^{\alpha_2-1}\ \vartheta'(s)ds\right]\right.$$

$$+\|\mu_2\|\left[|A|\int_a^b\left(\frac{L_2\|x-y\|}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^\tau(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1}\ \vartheta'(s)ds\right.\right.$$

$$\left.\left.+\frac{L_1\|x-y\|}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^\tau(\vartheta(\tau)-\vartheta(s))^{\alpha_2-1}\vartheta'(s)ds\right)dE(\tau)\right]$$

$$\leqslant L\Lambda\|x-y\|.$$

From the above inequality together with the given condition $L\Lambda<1$, it follows that the operator G is a contraction. By means of the Banach contraction mapping principle, there exists a unique solution for the boundary value problem (1)–(2). □

4. Existence result for inclusion case

In this section, we develop the existence results to comprise the inclusion problem. We establish the existence of solutions for the boundary value problem (3)–(4) by applying the fixed point theorem [28]. We begin to recollect some basic notations for the inclusion case [29]. For a normed space $(X,\|.\|)$, let

$$\mathbf{P}_{\mathrm{cl}}(X)=\{y\in\mathbf{P}(X):y\text{ is closed}\},$$

$$\mathbf{P}_{\mathrm{cp,cv}}(\mathbb{R})=\{y\in\mathbf{P}(\mathbb{R}):y\text{ is compact and convex}\}.$$

Lemma 2. *If $F:X\longrightarrow\mathbf{P}_{cl}(Y)$ exhibits upper semicontinuity, then F constitutes a closed subset of $X\times Y$ in other words, for any sequence $\{x_n\}_{n\in\mathbb{N}}\subset X$ and $\{y_n\}_{n\in\mathbb{N}}\subset Y$, if $n\longrightarrow\infty,\quad x_n\longrightarrow x_*,y_n\longrightarrow y_*$, and $y_n\in F(x_n)$, then $y_*\in F(x_*)$. Conversely, if F is completely continuous and has a closed graph, then it is upper semicontinuous.*

A multivalued map $F:[a,b]\times\mathbb{R}\longrightarrow\mathbf{P}(\mathbb{R})$ is considered Caratheodory when

(i) $t\longrightarrow F(t,x)$ is measurable for each $x\in\mathbb{R}$.

(ii) $x\longrightarrow F(t,x)$ is upper semicontinuous for almost all $t\in[a,b]$.

Furthermore, a Caratheodory function F is called L^1-*Caratheodory* if for each $\alpha>0$, there exists $\varphi_\alpha\in L^1([a,b],\mathbb{R}^+)$ such that

$$\|F(t,x)\|=\sup\{|v|:v\in F(t,x)\}\leq\varphi_\alpha(t),$$

for all $\|x\|_\infty\leq\alpha$ and for a.e. $t\in[a,b]$. For each $y\in\mathscr{C}([a,b],\mathbb{R})$, define the set of selections of F by

$$\mathscr{S}_{F,y}=\left\{v\in L^1([a,b],\mathbb{R}):v(t)\in F(t,y(t)),\text{ for a.e. }t\in[a,b]\right\}.$$

Let (X,d) be a metric space induced from the normed space $(X;|\|)$. We have $H_d:\mathbf{P}(X)\times\mathbf{P}(X)\longrightarrow\mathbb{R}\cup\{\infty\}$ given by

$$H_d(A,B)=\max\left\{\sup_{a\in A}d(a,B),\sup_{b\in B}d(A,b)\right\},$$

where $d(A,b)=\inf_{a\in A}d(a;b)$ and $d(a,B)=\inf_{b\in B}d(a;b)$. So, $(\mathbf{P}_{\mathrm{cl}}(X),H_d)$ is a metric space [26].

Lemma 3. *[30] Let X be a Banach space. Let $F:[a,b]\times\mathbb{R}\longrightarrow \mathbf{P}_{cp,cv}(X)$ be an L^1-Caratheodory multivalued map, and let Θ be a linear continuous mapping from $L^1([a,b],X)$ to $\mathscr{C}([a,b],X)$. Then, the operator*

$$\Theta\circ\mathscr{S}_F:\mathscr{C}([a,b],X)\longrightarrow \mathbf{P}_{cp,cv}(\mathscr{C}([a,b],X)),\quad x\longrightarrow(\Theta\circ\mathscr{S}_F)(x)=\Theta(\mathscr{S}_{F,x})$$

is a closed graph operator in $\mathscr{C}([a,b],X)\times\mathscr{C}([a,b],X)$.

Definition 4. *A function $x\in\mathscr{C}[a,b],\mathbb{R})$ is called a solution of problem* (3)-(4) *if we can find a function $f\in L^1([a,b],\mathbb{R})$ with $f(t)\in F(t,x)$ on $[a,b]$ such that*

$$x(a)=A\theta(x),$$

$$\sum_{i=1}^{m}\delta_i x(\xi_i)=\frac{1}{\rho^{\gamma}\Gamma(\gamma)}\left[a_1\int_a^{\eta_1}e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(\tau))}(\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1}x(\tau)\right.$$

$$\left.\vartheta'(\tau)d\tau+a_2\int_{\eta_2}^{b}e^{\frac{\rho-1}{\rho}(\vartheta(b)-\vartheta(\tau))}(\vartheta(b)-\vartheta(\tau))^{\gamma-1}x(\tau)\vartheta'(\tau)d\tau\right],$$

and

$$x(t)=I^{\alpha_1+\alpha_2,\rho,\vartheta}f(t,x(t))-I^{\alpha_2,\rho,\vartheta}h(t,x(t))+e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_1(t)\left[\frac{a_1}{\rho^{\gamma}\Gamma(\gamma)}\right.$$

$$\int_a^{\eta_1}e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(\tau))}(\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1}\left(I^{\alpha_1+\alpha_2,\rho}f(\tau,x(\tau))-I^{\alpha_2,\rho,\vartheta}h(\tau,x(\tau)\right)\vartheta'(\tau)d\tau$$

$$+\frac{a_2}{\rho^{\gamma}\Gamma(\gamma)}\int_{\eta_2}^{b}e^{\frac{\rho-1}{\rho}(\vartheta(b)-\vartheta(\tau))}(\vartheta(b)-\vartheta(\tau))^{\gamma-1}\left(I^{\alpha_1+\alpha_2,\rho,\vartheta}f(\tau,x(\tau))-I^{\alpha_2,\rho,\vartheta}h(\tau,x(\tau)\right)$$

$$\left.\vartheta'(\tau)d\tau-\sum_{i=1}^{m}\delta_i I^{\alpha_1+\alpha_2,\rho,\vartheta}f(\xi_i,x(\xi_i))+\sum_{i=1}^{m}\delta_i I^{\alpha_2,\rho,\vartheta}h(\xi_i,x(\xi_i))\right]-e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}$$

$$\mu_2(t)\left[A\int_a^{b}\left(I^{\alpha_1+\alpha_2,\rho,\vartheta}f(\tau,x(\tau))-I^{\alpha_2,\rho,\vartheta}h(\tau,x(\tau)\right)dE(\tau)\right].$$

For convenience, we denote

$$\Delta_1=\frac{(\vartheta(b)-\vartheta(a))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}+\|\mu_1\|\left[\frac{|a_1|}{\rho^{\gamma+\alpha_2}\Gamma(\gamma+\alpha_2+1)}(\vartheta(\eta_1)-\vartheta(a))^{\gamma+\alpha_2}\right.$$

$$\left.+\frac{|a_2|}{\rho^{\gamma}\Gamma(\gamma)}(w_2)+\sum_{i=1}^{m}|\delta_i|\frac{(\vartheta(\xi_i)-\vartheta(a))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\right]+\|\mu_2\|\Big[|A|\tag{25}$$

$$\int_a^{b}\frac{(\vartheta(\tau)-\vartheta(a))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}dE(\tau)\Big],$$

$$\Delta_2 = \frac{(\vartheta(b)-\vartheta(a))^{\alpha_1+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)} + \|\mu_1\| \left[\frac{|a_1|}{\rho^{\gamma}\Gamma(\gamma)} \frac{(\vartheta(\eta_1)-\vartheta(a))^{\alpha_1+\alpha_2+\gamma}}{\rho^{\alpha_1+\alpha_2+\gamma}\Gamma(\alpha_1+\alpha_2+\gamma+1)} \right.$$

$$\left. + \frac{|a_2|}{\rho^{\gamma}\Gamma(\gamma)} w_1 + \sum_{i=1}^{m} |\delta_i| \frac{(\vartheta(\xi_i)-\vartheta(a))^{\alpha_1+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)} \right] \tag{26}$$

$$+ |\mu_2| \left[|A| \int_a^b \frac{(\vartheta(\tau)-\vartheta(a))^{\alpha_1+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)} dE(\tau) \right].$$

The following fixed point theorem is used in what follows.

Theorem 9. *[26] Let X and $\bar{X}$ be, respectively, the open and closed subsets of Banach space U, such that $a \in X$; let $\chi_1(x) : \bar{X} \longrightarrow \mathbf{P}_{cp,cv}(U)$ be multivalued and $\chi_2(x) : \bar{X} \to \bar{U}$ be single-valued such that $\chi_1(\bar{X}) + \chi_2(\bar{X})$ is bounded. suppose that*

(a) χ_2 is a contraction with a contraction $n < \left(\frac{1}{2}\right)$,

(b) χ_1 is upper semicontinuous and compact. Then, either

(i) the operator inclusion $\lambda u \in \chi_1 u + \chi_2 u$ has a solution for $\lambda = 1$ or,

(ii) there is an element $x \in \partial X$ such that $\lambda x \in \chi_1 x + \chi_2 x$ for some $\lambda > 1$, where ∂X is boundary of X.

Now, the main theorem of this section is stated as follows.

Theorem 10. *Assume that*

(N1) $F : [a,b] \times \mathbb{R} \to \mathbf{P}_{cp,c}(\mathbb{R})$ is L^1-Caratheoolory.

(N2) There exists a continuous function $\omega \in \mathscr{C}([a,b],\mathbb{R}^+)$ and $\Omega \in \mathscr{C}([a,b],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|x| : x \in F(t,x)\} \leqslant \Omega(t)\omega(\|x\|)$$

for all $(t,x) \in [a,b] \times \mathbb{R}$.

(N3) Let $h : [a,b] \times \mathbb{R} \to \mathbb{R}$ be a continuous functions satisfying

$$|h(s,x) - h(s,\tilde{x})| \leqslant \xi |x - \tilde{x}|, \forall x, \tilde{x} \in \mathbb{R}$$

and $\xi > 0$.

(N4) A number $\tau > 0$ exists such that

$$\frac{\tau}{\|\Omega\| w(\tau)\Lambda} < 1,$$

where Λ is defined by equation (20)*. Then, problem* (3)-(4) *has at least one solution on $[a,b]$ if $\xi\Delta_1 < \left(\frac{1}{2}\right)$.*

Proof. Let $D = \{x \in X : \|x\| < \varepsilon\}$ be an open set in X. Define the multivalued operator $\chi_1 : \bar{D} \to P(X)$ by

$$\chi_1(x) = \Big\{ z \in X : z(t) = I^{\alpha_1+\alpha_2,\rho,\vartheta} f(t,x(t)) + e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_1(t)$$

$$\Big[\frac{a_1}{\rho^{\gamma}\Gamma(\gamma)} \int_a^{\eta_1} e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(\tau))} (\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1} \Big(I^{\alpha_1+\alpha_2,\rho,\vartheta} f(\tau,x(\tau)\Big) \vartheta'(\tau) d\tau$$

$$+ \frac{a_2}{\rho^{\gamma}\Gamma(\gamma)} \int_{\eta_2}^{b} e^{\frac{\rho-1}{\rho}(\vartheta(b)-\vartheta(\tau))} (\vartheta(b)-\vartheta(\tau))^{\gamma-1} \Big(I^{\alpha_1+\alpha_2,\rho,\vartheta} f(\tau,x(\tau)\Big) \vartheta'(\tau) d\tau$$

$$- \sum_{i=1}^{m} \delta_i I^{\alpha_1+\alpha_2,\rho,\vartheta} f(\xi_i,x(\xi_i))\Big] - e^{\frac{\rho-1}{\rho}(\varphi(t)-\vartheta(a))}\mu_2(t) \Big[A \int_a^b \Big(I^{\alpha_1+\alpha_2,\rho,\vartheta} f(\tau,x(\tau)\Big) dE(\tau)]\},$$

and define the single-valued operator $\chi_2 : \bar{D} \to X$ by

$$\chi_2(x) = \Big\{ z \in X : z(t) = -I^{\alpha_2,\rho,\vartheta} h(t,x(t)) + e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_1(t) \Big[\frac{a_1}{\rho^{\gamma}\Gamma(\gamma)}$$

$$\int_a^{\eta_1} e^{\frac{\rho-1}{\rho}(\varphi(\eta_1)-\vartheta(\tau))} (\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1} (-I^{\alpha_2,\rho} h(\tau,x(\tau)) \vartheta'(\tau) d\tau + \frac{a_2}{\rho^{\gamma}\Gamma(\gamma)}$$

$$\int_{\eta_2}^{b} e^{\frac{\rho-1}{\rho}(\varphi(b)-\vartheta(\tau))} (\vartheta(b)-\vartheta(\tau))^{\gamma-1} \Big(-I^{\alpha_2,\rho,\vartheta} h(\tau,x(\tau)\Big) \vartheta'(\tau) d\tau + \sum_{i=1}^{m} \delta_i I^{\alpha_2,\rho,\vartheta} h(\xi_i,x(\xi_i))$$

$$- e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_2(t) \Big[A \int_a^b (-I^{\alpha_2,\rho} h(\tau,x(\tau))) dE(\tau)\Big]\Big\}.$$

Observe that $\chi = \chi_1 + \chi_2$ and it is given by

$$\chi(x) = \{z(t) \in \mathscr{C}([a,b],\mathbb{R}) : z(t) = I^{\alpha_1+\alpha_2,\rho,\vartheta} f(t,x(t)) - I^{\alpha_2,\rho,\vartheta} h(t,x(t)) + e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_1(t)$$

$$\Big[\frac{a_1}{\rho^{\gamma}\Gamma(\gamma)} \int_a^{\eta_1} e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(\tau))} (\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1} \Big(I^{\alpha_1+\alpha_2,\rho,\vartheta} f(\tau,x(\tau)) - I^{\alpha_2,\rho,\vartheta} h(\tau,x(\tau))\Big)$$

$$\vartheta'(t) d\tau + \frac{a_2}{\rho^{\gamma}\Gamma(\gamma)} \int_{\eta_2}^{b} e^{\frac{\rho-1}{\rho}(\vartheta(b)-\vartheta(\tau))} (\vartheta(b)-\vartheta(\tau))^{\gamma-1} \Big(I^{\alpha_1+\alpha_2,\rho,\vartheta} f(\tau,x(\tau)) - I^{\alpha_2,\rho,\vartheta} h(\tau,x(\tau))\Big)$$

$$\vartheta'(\tau) d\tau - \sum_{i=1}^{m} \delta_i I^{\alpha_1+\alpha_2,\rho,\vartheta} f(\xi_i,x(\xi_i)) + \sum_{i=1}^{m} \delta_i I^{\alpha_2,\rho,\vartheta} h(\xi_i,x(\xi_i))\Big] - e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}$$

$$\mu_2(t) \Big[A \int_a^b \Big(I^{\alpha_1+\alpha_2,\rho,\vartheta} f(\tau,x(\tau)) - I^{\alpha_2,\rho,\vartheta} h(\tau,x(\tau))\Big) dE(\tau)\Big], f \in \mathscr{S}_{F,x}\}.$$

Indeed, if $z \in \chi(x)$, then there exists $f \in \mathscr{S}_{F,x}$ such that

$$\mathscr{S}_{F,x} = \left\{ f \in L^1\left([a,b],\mathbb{R}^+\right) : f(t) \in F(t,x(t)), \text{ for } t \in [a,b] \right\}.$$

We will show that the maps χ_1 and χ_2 satisfy the hypotheses of Theorem 9. We divide the proof in several steps.

Step 1.

We claim that χ_2 is a contraction map. Indeed, let $x,\tilde{x} \in \mathbb{R}$, by $(N3)$, we have

$$|\chi_2 x - \chi_2 \tilde{x}| \leqslant \frac{\xi |x-\tilde{x}|}{\rho^{\alpha_2}\Gamma(\alpha_2)} \int_a^t e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(\tau))} (\vartheta(t)-\vartheta(\tau))^{\alpha_2-1}$$

$$\vartheta'(\tau)d\tau + e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))} \|\mu_1\| \Big[\frac{|a_1|}{\rho^{\gamma}\Gamma(\gamma)}$$

$$\int_a^{\eta_1} e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(\tau))} (\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1} \Big(\frac{\xi |x-\tilde{x}|}{\rho_2\Gamma(\alpha_2)} \int_a^{\tau} e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(s))}$$

$$(\vartheta(\tau)-\vartheta(s))^{\alpha_2-1} \vartheta'(s)ds \Big) + \frac{|a_2|}{\rho^{\gamma}\Gamma(\gamma)} \int_{\eta_2}^b e^{\frac{\rho-1}{\rho}(\vartheta(b)-\vartheta(\tau))}$$

$$(\vartheta(b)-\vartheta(\tau))^{\gamma-1} \Big(\frac{\xi |x-\tilde{x}|}{\rho^{\alpha_2}\Gamma(\alpha_2)} \int_a^{\tau} e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(s))} (\vartheta(\tau)-\vartheta(s))^{\alpha_2-1}$$

$$\vartheta'(s)ds \Big) \vartheta'(\tau)d\tau + \sum_{i=1}^{m} |\delta_i| \Big(\frac{\xi |x-\tilde{x}|}{\rho^{\alpha_2}\Gamma(\alpha_2)} \int_a^{\xi_i} e^{\frac{\rho-1}{\rho}(\vartheta(\xi_i)-\vartheta(s))} (\vartheta(\xi_i)-\vartheta(s))^{\alpha_2-1}$$

$$\vartheta'(s)ds \Big) \Big] + \|\mu_2\| e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))} \Big[|A| \int_a^b \Big(\frac{\xi |x-\tilde{x}|}{\rho^{\alpha_2}\Gamma(\alpha_2)}$$

$$\int_a^{\tau} e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(s))} (\vartheta(\tau)-\vartheta(s))^{\alpha_2-1} \vartheta'(s)ds \Big) dE(\tau) \Big]$$

$$\|\chi_2 x - \chi_2 \tilde{x}\| \leqslant \xi \|x-\tilde{x}\| \Big\{ \frac{(\vartheta(b)-\vartheta(a))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)} + \|\mu_1\| \Big[\frac{|a_1|}{\rho^{\gamma+\alpha_2}\Gamma(\gamma+\alpha_2+1)}$$

$$(\vartheta(\eta_1)-\vartheta(a))^{\gamma+\alpha_2} + \frac{|a_2|}{\rho^{\gamma}\Gamma(\gamma)} (w_2) + \sum_{i=1}^{m} |\delta_i| \frac{(\vartheta(\xi_i)-\vartheta(a)^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)} \Big]$$

$$+ \|\mu_2\| \Big[|A| \int_a^b \frac{(\vartheta(\tau)-\vartheta(a))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)} dE(\tau) \Big] \Big\} \leqslant \xi \Delta_1 \|x-\tilde{x}\|,$$

which proves that χ_2 is a contraction map, where w_2 and Δ_1 are defined in (22) and (25).

Step 2.

$\chi_1(x)$ is convex for all $x \in \bar{D}$. Let $z_1, z_2 \in \chi_1(x)$. We select $f_1, f_2 \in \mathscr{S}_{F,x}$ such that, for each $t \in [a,b]$, we obtain

$$z_i(t) = I^{\alpha_1+\alpha_2,\rho} f_i(t,x(t) + e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_1(t)\left[\frac{a_1}{\rho^{\gamma}\Gamma(\gamma)}\int_a^{\eta_1} e^{\frac{\rho-\rho}{\rho}(\vartheta(\eta_1)-\vartheta(\tau))}\right.$$

$$(\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1}\left(I^{\alpha_1+\alpha_2,\rho,\vartheta} f_i(\tau,x(\tau))\right)\vartheta'(\tau)d\tau + \frac{a_2}{\rho^{\gamma}\Gamma(\gamma)}\int_{\eta_2}^{b} e^{\frac{\rho-1}{\rho}(\vartheta(b)-\vartheta(\tau))}$$

$$\left.(\vartheta(b)-\vartheta(\tau))^{\gamma-1}\left(I^{\alpha_1+\alpha_2,\rho,\vartheta} f_i(\tau,x(\tau))\right)\vartheta'(\tau)d\tau - \sum_{i=1}^{m}\delta_i I^{\alpha_1+\alpha_2,\rho,\vartheta} f_i(\xi_i,x(\xi_i))\right]$$

$$- e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_2(t)\left[A\int_a^b\left(I^{\alpha_1+\alpha_2,\rho,\vartheta} f_i(\tau,x(\tau)\right)dE(\tau)\right],\ \text{for } i=1,2.$$

Let $t \in [a,b]$ and $\phi \in [0,1]$. So, we have

$$[\phi z_1 + (1-\phi)z_2](t) = \frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^t e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(\tau))}(\vartheta(t)-\vartheta(\tau))^{\alpha_1+\alpha_2-1}$$

$$(\phi f_1(\tau,x(\tau)) + (1-\phi)f_2(\tau,x(\tau))\,d\tau + e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_1(t)\left[\frac{a_1}{\rho^{\gamma}\Gamma(\gamma)}\int_a^{\eta_1} e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(\tau))}\right.$$

$$(\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1}\left(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^{\tau} e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(s))}(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1}\right.$$

$$\left.(\phi f_1(s,x(s)) + (1-\phi)f_2(s,x(s))\,\vartheta'(s)ds\right)\vartheta'(\tau)d\tau$$

$$-\sum_{i=1}^{m}\delta_i\left(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\right.$$

$$\left.\left.\int_a^{\xi_i} e^{\frac{\rho-1}{\rho}(\vartheta(\xi_i)-\vartheta(s))}\left(\vartheta(\xi_i)-\vartheta(s)\right)^{\alpha_1+\alpha_2-1}(\phi f_1(s,x(s)) + (1-\phi)f_2(s,x(s))\,\vartheta'(s)ds\right)\right]$$

$$- e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_2(t)\left[A\int_a^b\left(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^{\tau} e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(s))}\right.\right.$$

$$\left.\left.(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1}(\phi f_1(s,x(s)) + (1-\phi f_2(s,x(s)))\,\vartheta'(s)dsdE(\tau)\right].$$

Since $\mathscr{S}_{F,x}$ is convex, it follows that $\phi z_1 + (1-\phi)z_2 \in \chi_1(x)$ and then $\chi_1(x)$ is convex-valued.

Step 3.

χ_1 is compact and upper semi-continuous. This will be done in various statements. First, we show that χ_1 maps bounded sets into bounded sets in X. For a positive number ς, let $B_\varsigma = \{u \in X : \|x\| \leqslant \varsigma\}$ be a bounded ball in $\bar{D}$. So, for all $z \in \chi_1, x \in B_\varsigma$, a function f in $\in \mathscr{S}_{F,x}$ exists such that

$$z(t) = I^{\alpha_1+\alpha_2,\rho,\vartheta} f(t,x(t)) + e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_1(t)\left[\frac{a_1}{\rho^\gamma\Gamma(\gamma)}\int_a^{\eta_1} e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(\tau))}\right.$$

$$(\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1}\left(I^{\alpha_1+\alpha_2,\rho,\vartheta} f(\tau,x(\tau)\right)\vartheta'(\tau)d\tau + \frac{a_2}{\rho^\gamma\Gamma(\gamma)}\int_{\eta_2}^b e^{\frac{\rho-1}{\rho}(\vartheta(b)-\vartheta(\tau))}$$

$$\left.(\vartheta(b)-\vartheta(\tau))^{\gamma-1}\left(I^{\alpha_1+\alpha_2,\rho,\vartheta} f(\tau,x(\tau)\right)\vartheta'(\tau)d\tau - \sum_{i=1}^m \delta_i I^{\alpha_1+\alpha_2,\rho,\vartheta} f(\xi_i,x(\xi_i))\right]$$

$$-e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_2(t)\left[A\int_a^b\left(I^{\alpha_1+\alpha_2,\rho,\vartheta} f(\tau,x(\tau)\right)dE(\tau)\right].$$

By using (N_2), for each $t \in [a,b]$, we have

$$|z(t)| \leqslant \frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^t(\vartheta(t)-\vartheta(\tau))^{\alpha_1+\alpha_2-1}|f(\tau,x(\tau))|\vartheta'(\tau)d\tau + |\mu_1(t)|$$

$$\left[\frac{|a_1|}{\rho^\gamma\Gamma(\gamma)}\int_a^{\eta_1}(\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1}\Big(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^\tau(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1}|f(s,x(s))|\vartheta'(s)ds\Big)\right.$$

$$\vartheta'(\tau)d\tau + \frac{|a_2|}{\rho^\gamma\Gamma(\gamma)}\int_{\eta_2}^b(\vartheta(b)-\vartheta(\tau))^{\gamma-1}\Big(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^\tau(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1}$$

$$\left.|f(s,x(s))|\vartheta'(s)ds\Big) + \sum_{i=1}^m|\delta_i|\left(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^{\xi_i}(\vartheta(\xi_i)-\vartheta(s))^{\alpha_1+\alpha_2-1}|f(s,x(s))|\vartheta'(s)ds\right)\right]$$

$$+|\mu_2|\left[|A|\int_a^b\left(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^\tau(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1}|f(s,x(s))|\vartheta'(s)ds\right)dE(\tau)\right]$$

$$\|z\| \leqslant \|\Omega\|\omega(\|x\|)\Big\{\frac{(\vartheta(b)-\vartheta(a))^{\alpha_1+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)} + \|\mu_1\|\left[\frac{|a_1|}{\rho^\gamma\Gamma(\gamma)}\frac{(\vartheta(\eta_1)-\vartheta(a))^{\alpha_1+\alpha_2+\gamma}}{\rho^{\alpha_1+\alpha_2+\gamma}\Gamma(\alpha_1+\alpha_2+\gamma+1)}\right.$$

$$\left.+\frac{|a_2|}{\rho^\gamma\Gamma(\gamma)}(w_1) + \sum_{i=1}^m|\delta_i|\frac{(\vartheta(\xi_i)-\vartheta(a))^{\alpha_1+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)}\right]$$

$$+\|\mu_2\|\Big[|A|\int_a^b\left(\frac{(\vartheta(\tau)-\vartheta(a))^{\alpha_1+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)}\right)dE(\tau)\Big]\Big\},$$

where (w_1) defined in (22), consequently

$$\|z\| \leqslant \|\Omega\|\Delta_2\omega(\|x\|).$$

Then, we prove that χ_1 maps bounded sets into equicontinuous sets, where Δ_2 is defined in (26). Let $u_1,u_2 \in [a,b]$ with $u_1 < u_2$ and $x \in B_\varsigma$, we have

$$|z(u_2)-z(u_1)| \leqslant \frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^{u_1}((\vartheta(u_2)-\vartheta(\tau))^{\alpha_1+\alpha_2-1}-(\vartheta(u_1)-\vartheta(\tau))^{\alpha_1+\alpha_2-1})$$

$$|f(\tau,x(\tau))|\vartheta'(\tau)d\tau+\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_{u_1}^{u_2}(\vartheta(u_2)-\vartheta(\tau))^{\alpha_1+\alpha_2-1}|f(\tau,x(\tau))|\vartheta'(\tau)d\tau$$

$$+|\mu_1(u_2)-\mu_1(u_1)|\Big[\frac{|a_1|}{\rho^{\gamma}\Gamma(\gamma)}\int_a^{\eta_1}(\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1}\Big(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^{\tau}(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1}$$

$$|f(s,x(s))|\vartheta'(s)ds\Big)\vartheta'(\tau)d\tau+\frac{|a_2|}{\rho^{\gamma}\Gamma(\gamma)}\int_{\eta_2}^{b}(\vartheta(b)-\vartheta(\tau)]^{\gamma-1}\Big(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}$$

$$\int_a^{\tau}(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1}|f(s,x(s))|\vartheta'(s)ds\Big)\vartheta'(\tau)d\tau$$

$$+\sum_{i=1}^{m}|\delta_i|\Big(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^{\xi}(\vartheta(\xi_i)-\vartheta(s))^{\alpha_1+\alpha_2-1}|f(s,x(s))|\vartheta'(s)ds\Big)\Big]$$

$$+|\mu_2(u_2)-\mu_2(u_1)|\Big[|A|\int_a^{b}\Big(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^{\tau}(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1}|f(s,x(s))|\vartheta'(s)ds\Big)dE(\tau)\Big]$$

$$\leqslant \frac{2(\vartheta(u_2)-\vartheta(u_1))^{\alpha_1+\alpha_2}+((\vartheta(u_2)-\vartheta(a))^{\alpha_1+\alpha_2}-(\vartheta(u_1)-\vartheta(a))^{\alpha_1+\alpha_2})}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)}$$

$$+|\mu_1(u_2)-\mu_1(u_1)|\Bigg[\frac{|a_1|(\vartheta(\eta_1)-\vartheta(a))^{\alpha_1+\alpha_2+\gamma}}{\rho^{\alpha_1+\alpha_2+\gamma}\Gamma(\alpha_1+\alpha_2+\gamma+1)}+\frac{|a_2|}{\rho^{\gamma}\Gamma(\gamma)}(w_1)+\sum_{i=1}^{m}|\delta_i|$$

$$\frac{(\vartheta(\xi_i)-\vartheta(a))^{\alpha_1+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)}\Bigg]+|\mu_2(u_2)-\mu_2(u_1)|\Bigg[|A|\int_a^{b}\Bigg(\frac{(\vartheta(\tau)-\vartheta(a))^{\alpha_1+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)}+$$

$$\frac{(\vartheta(\tau)-\vartheta(a))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\Bigg)dE(\tau)\Bigg].$$

In the above inequality, the right-hand side tends to zero independently of x in B_ς as $u_2 \to u_1$. Consequently, by the Arzela-Ascoli theorem, it is concluded that $\chi_1 : \bar{D} \to P(X)$ is rendered completely continuous, and thereby, χ_1 is considered completely continuous.

Finally, we show χ_1 has a closed graph. Let $(x_n, z_n) \to (x_*, z_*)$. It is demonstrated that z belongs to $\chi_1(x)$. Since $z_n \in \chi_1(x_n)$, there exists $z_n \in \mathscr{S}_{F,x_n}$ such that for each $t \in [a,b]$, we find that

$$z_n(t) = I^{\alpha_1+\alpha_2,\rho,\vartheta} f_n(t,x(t)) + e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_1(t)\left[\frac{a_1}{\rho^\gamma\Gamma(\gamma)}\int_a^{\eta_1} e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(\tau))}\right.$$

$$(\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1}\left(I^{\alpha_1+\alpha_2,\rho,\vartheta} f_n(\tau,x(\tau))\right)\vartheta'(\tau)d\tau + \frac{a_2}{\rho^\gamma\Gamma(\gamma)}\int_{\eta_2}^{b} e^{\frac{\rho-1}{\rho}(\vartheta(b)-\vartheta(\tau))}(\vartheta(b)-\vartheta(\tau))^{\gamma-1}$$

$$\left.(I^{\alpha_1+\alpha_2,\rho} f_n(\tau,x(\tau))\vartheta'(\tau)d\tau - \sum_{i=1}^{m}\delta_i I^{\alpha_1+\alpha_2,\rho,\vartheta} f_n(\xi_i,x(\xi_i))\right] - e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_2(t)$$

$$\left[A\int_a^b\left(I^{\alpha_1+\alpha_2,\rho,\vartheta} f_n(\tau,x(\tau)\right)dE(\tau)\right].$$

Now, we have to show that there exists $z_* \in \mathscr{S}_{F,x_*}$ such that

$$z_*(t) = I^{\alpha_1+\alpha_2,\rho,\vartheta} f_*(t,x(t)) + e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_1(t)\left[\frac{a_1}{\rho^\gamma\Gamma(\gamma)}\int_a^{\eta_1} e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(\tau))}\right.$$

$$(\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1}\left(I^{\alpha_1+\alpha_2,\rho,\vartheta} f_*(\tau,x(\tau)\right)\vartheta'(\tau)d\tau + \frac{a_2}{\rho^\gamma\Gamma(\gamma)}\int_{\eta_2}^{b} e^{\frac{\rho-1}{\rho}(\vartheta(b)-\vartheta(\tau))}$$

$$\left.(\vartheta(b)-\vartheta(\tau))^{\gamma-1}\left(I^{\alpha_1+\alpha_2,\rho,\vartheta} f_*(\tau,x(\tau)\right)\vartheta'(\tau)d\tau - \sum_{i=1}^{m}\delta_i I^{\alpha_1+\alpha_2,\rho} f_*(\xi_i,x(\xi_i))\right]$$

$$- e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_2(t)\left[A\int_a^b\left(I^{\alpha_1+\alpha_2,\rho,\vartheta} f_*(\tau,x(\tau)\right)dE(\tau)\right].$$

Consider the continuous linear operator $\Theta : L^1([a,b],\mathbb{R}) \longrightarrow X$ given by

$$f \to \Theta(f)(t) = I^{\alpha_1+\alpha_2,\rho,\vartheta} f(t,x(t)) + e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_1(t)\left[\frac{a_1}{\rho^{\gamma}\Gamma(\gamma)}\int_a^{\eta_1} e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(\tau))}\right.$$

$$(\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1}\left(I^{\alpha_1+\alpha_2,\gamma} f(\tau,x(\tau))\right)\vartheta'(\tau)d\tau + \frac{a_2}{\rho^{\gamma}\Gamma(\gamma)}\int_{\eta_2}^{b} e^{\frac{\rho-1}{\rho}(\vartheta(b)-\vartheta(\tau))}$$

$$\left.(\vartheta(b)-\vartheta(\tau))^{\gamma-1}\left(I^{\alpha_1+\alpha_2,\rho,\vartheta} f(\tau,x(\tau))\right)\vartheta'(\tau)d\tau - \sum_{i=1}^{m}\delta_i I^{\alpha_1+\alpha_2,\rho,\vartheta} f(\xi_i,x(\xi_i))\right]$$

$$- e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_2(t)\left[A\int_a^b\left(I^{\alpha_1+\alpha_2,\rho,\vartheta} f(\tau,x(\tau))\right)dE(\tau)\right].$$

Note that

$$\|z_n(t)-z_*(t)\| = \sup_{t\in[a,b]}\left|I^{\alpha_1+\alpha_2,\rho}\left(f_n(t,x(t)-f_*(t,x(t))\right) + e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_1(t)\right.$$

$$\left[\frac{a_1}{\rho^{\gamma}\Gamma(\gamma)}\int_a^{\eta_1} e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(\tau))}(\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1}\left(I^{\alpha_1+\alpha_2,\rho}\left(f_n(\tau,x(\tau)-f_*(\tau,x(\tau))\right)\vartheta'(\tau)d\tau\right.\right.$$

$$+\frac{a_2}{\rho^{\gamma}\Gamma(\gamma)}\int_{\eta_2}^{b} e^{\frac{\rho-1}{\rho}(\vartheta(b)-\vartheta(\tau))}(\vartheta(b)-\vartheta(\tau))^{\gamma-1}\left(I^{\alpha_1+\alpha_2,\rho}\left(f_n(\tau,x(\tau)-f_*(\tau,x(\tau))\right)\right)\vartheta'(\tau)d\tau$$

$$\left.\left.-\sum_{i=1}^{m}\delta_i I^{\alpha_1+\alpha_2,\rho,\vartheta}(f_n(\xi_i,x(\xi_i)-f_*(\xi_i,x(\xi_i))\right] - e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_2(t)$$

$$\left.\left[A\int_a^b\left(I^{\alpha_1+\alpha_2,\rho,\vartheta}\left(f_n(\tau,x(\tau)-f_*(\tau,x(\tau))\right)\right)dE(\tau)\right]\right|,$$

which goes to zero , as $n \longrightarrow \infty$. It follows by Lemma 2 that $\Theta \circ \mathscr{S}_F$ is a closed graph operator. Furthermore, we obtain $z_n(t) \in \Theta(\mathscr{S}_{F,x_n})$. Since $x_n \to x_*$, we have

$$z_*(t) = I^{\alpha_1+\alpha_2,\rho} f_*(t,x(t)) + e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_1(t)\left[\frac{a_1}{\rho^\gamma\Gamma(\gamma)}\int_a^{\eta_1} e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(\tau))}\right.$$

$$(\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1}\left(I^{\alpha_1+\alpha_2,\rho} f_*(\tau,x(\tau))\right)\vartheta'(\tau)d\tau + \frac{a_2}{\rho^\gamma\Gamma(\gamma)}\int_{\eta_2}^{b} e^{\frac{\rho-1}{\rho}(\vartheta(b)-\vartheta(\tau)}$$

$$\left.(\vartheta(b)-\vartheta(\tau))^{\gamma-1}\left(I^{\alpha_1+\alpha_2,\rho,\vartheta} f_*(\tau,x(\tau)\right)\vartheta'(\tau)d\tau - \sum_{i=1}^{m}\delta_i I^{\alpha_1+\alpha_2,\rho,\vartheta} f_*(\xi_i,x(\xi_i))\right]$$

$$-e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_2(t)\left[A\int_a^b\left(I^{\alpha_1+\alpha_2,\rho} f_*(\tau,x(\tau))\right)dE(\tau)\right],$$

for some $z_* \in \mathscr{S}_{F,x_*}$. Hence, χ_1 has a closed graph (and therefore has closed values). Hence, we conclude that χ_1 is compact multivalued map, upper semicontinuous with convex closed values.

Step 4.

There exists an open set $Q \subseteq \mathscr{C}([a,b],\mathbb{R})$ with $x \notin \lambda\chi_2(x)$ for any $\lambda > 1$ and for each $x \in \partial Q$. Take $\lambda > 1$. Let x be a solution of (3)–(4) , then, a function $f \in L^1([a,b],\mathbb{R})$ satisfying $f \in \mathscr{S}_{F,x}$ exists, and for $t \in [a,b]$, the following relationship holds:

$$x(t)=\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^t e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(\tau))}(\vartheta(t)-\vartheta(\tau))^{\alpha_1+\alpha_2-1}f(\tau,x(\tau))\vartheta'(\tau)d\tau$$

$$-\frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^t e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(\tau))}(\vartheta(t)-\vartheta(\tau))^{\alpha_2-1}h(\tau,x(\tau))\vartheta'(\tau)d\tau$$

$$+e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_1(t)\left[\frac{a_1}{\rho^{\gamma}\Gamma(\gamma)}\int_a^{\eta_1} e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(\tau))}(\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1}\right.$$

$$\Big(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^{\tau} e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(s))}(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1}f(s,x(s))\vartheta'(s)ds$$

$$\left.-\frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^{\tau} e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(s))}(\vartheta(\tau)-\vartheta(s))^{\alpha_2-1}h(s,x(s))\vartheta'(s)ds\right)\vartheta'(\tau)d\tau$$

$$+\frac{a_2}{\rho^{\alpha_2}\Gamma(\gamma)}\int_{\eta_2}^b e^{\frac{\rho-1}{\rho}(\vartheta(b)-\vartheta(\tau))}(\vartheta(b)-\vartheta(\tau))^{\gamma-1}\Big(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^{\tau} e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(s))}$$

$$(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1}f(s,x(s))\vartheta'(s)ds-\frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^{\tau} e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(s))}(\vartheta(\tau)-\vartheta(s))^{\alpha_2-1}h(s,x(s))$$

$$\vartheta'(s)ds\Big)\vartheta'(\tau)d\tau-\sum_{i=1}^{m}\delta_i\Big(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^{\xi_i} e^{\frac{\rho-1}{\rho}(\vartheta(\xi_i)-\vartheta(s))}(\vartheta(\xi_i)-\vartheta(s))^{\alpha_1+\alpha_2-1}f(s,x(s))$$

$$\left.\vartheta'(s)ds\Big)+\sum_{i=1}^{m}\delta_i\left(\frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^{\xi_i} e^{\frac{\rho-1}{\rho}(\vartheta(\xi_i)-\vartheta(s))}(\vartheta(\xi_i)-\vartheta(s))^{\alpha_2-1}h(s,x(s))\vartheta'(s)ds\right)\right]$$

$$-e^{\frac{\rho-1}{\rho}(\vartheta(t)-\vartheta(a))}\mu_2(t)\left[A\int_a^b\left(\frac{1}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^{\tau} e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(s))}(\vartheta(\tau)-\vartheta(s))^{\alpha_1+\alpha_2-1}f(s,x(s))\right.\right.$$

$$\left.\left.\vartheta'(s)ds-\frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^{\tau} e^{\frac{\rho-1}{\rho}(\vartheta(\tau)-\vartheta(s))}(\vartheta(\tau)-\vartheta(s))^{\alpha_2-1}h(s,x(s))\vartheta'(s)ds\right)dE(\tau)\right].$$

Therefore,

$$|x(t)| \leqslant \|\Omega\|\omega(\|x\|)\Big\{\frac{(\vartheta(b)-\vartheta(a))^{\alpha_1+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)}+\frac{(\vartheta(b)-\vartheta(a))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}+\|\mu_1\|$$

$$\left[\frac{|a_1|\,(\vartheta(\eta_1)-\vartheta(a))^{\alpha_1+\alpha_2+\gamma}}{\rho^{\alpha_1+\alpha_2+\gamma}\Gamma(\alpha_1+\alpha_2+\gamma+1)}+\frac{|a_1|}{\rho^{\gamma+\alpha_2}\Gamma(\gamma+\alpha_2+1)}\,(\vartheta(\eta_1)-\vartheta(a))^{\gamma+\alpha_2}\right.$$

$$+\frac{|a_2|}{\rho^{\gamma}\Gamma(\gamma)}\,(w_1+w_2)+\sum_{i=1}^{m}|\delta_i|\left(\frac{(\vartheta(\xi_i)-\vartheta(a))^{\alpha_1+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)}\right.$$

$$\left.\left.+\frac{(\vartheta(\xi_i)-\vartheta(a))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\right)\right]+\|\mu_2\|\Big[|A|\int_a^b\left(\frac{(\vartheta(\tau)-\vartheta(a))^{\alpha_1+\alpha_2}}{\rho^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2+1)}-\frac{(\vartheta(\tau)-\vartheta(a))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\right)$$

$$dE(\tau)\Big]\Big\}$$

$$\leqslant \|\Omega\|\omega(\|x\|)\Lambda,$$

which implies

$$\frac{\|x\|}{\|\Omega\|\omega(\|x\|)\Lambda}<1.$$

Using the assumption $(N4)$, there exists $\tau>0$ such that $\|x\|\neq\tau$. Define a set $Q=\{x\in\mathscr{C}([a,b],\mathbb{R}):\|x\|<\tau\}$.

It should be noted that the operator $\chi_1:\bar{Q}\longrightarrow P(X)$ is a compact multivalued map, upper semi-continuous with convex closed values. With the selected Q, the existence of $x\in\partial Q$ satisfying $x\in\lambda\chi(x)$ for some $\lambda>1$ is not possible. As a result, a fixed point $x\in\bar{Q}$ is attained by the operator $\chi(x)$, which serves as a solution to the boundary value problem (3)–(4). □

5. Examples

The following examples illustrate the possibility of applying the research results in numerical simulations.

Example 1. *Consider the following boundary value problem with (CFPDs):*

$$\begin{cases} {}^{c}\mathfrak{D}^{\frac{1}{2},\frac{1}{2},\vartheta}({}^{c}\mathfrak{D}^{\frac{1}{4},\frac{1}{2},\vartheta}x(t)+h(t,x(t)))=f(t,x(t)),t\in[0,1],\\ x(0)=A\theta[x],\\ \sum_{i=1}^{m}\delta_i x(\xi_i)=\frac{1}{\rho^{\gamma}\Gamma(\gamma)}\Big[a_1\int_a^{\eta_1}e^{\frac{\rho-1}{\rho}(\vartheta(\eta_1)-\vartheta(\tau))}(\vartheta(\eta_1)-\vartheta(\tau))^{\gamma-1}x(\tau)\dot{\vartheta}(\tau)\,d\tau\\ +a_2\int_{\eta_2}^{b}e^{\frac{\rho-1}{\rho}(\vartheta(b)-\vartheta(\tau))}(\vartheta(b)-\vartheta(\tau))^{\gamma-1}x(\tau)\dot{\vartheta}(\tau)\,d\tau\Big], \end{cases} \tag{27}$$

where $a=0, b=1, \alpha_1=\frac{1}{2}, \alpha_2=\frac{1}{4}, \rho=\frac{1}{2}, \eta_1=\frac{1}{6}, \eta_2=\frac{5}{6}, \delta_1=1, \delta_2=3, \xi_1=\frac{1}{3}, \xi_2=\frac{1}{2}, a_1=1, a_2=2, \gamma=\frac{1}{4}, A=\frac{1}{4}, w_1=5.186759689, w_2=3.837868325, E(\tau)=\tau^2, \vartheta(\tau)=\tau^2+5$. *With the provided values, we conclude that* $\Lambda \simeq 12.31618167$ *and* $\Lambda_1 \simeq 9.174273532$*(*$\Lambda$ *and* Λ_1 *given by* (20) *and* (21).

Let

$$f(t,x)=\frac{e^{-3t}}{40}\left(\frac{|x|}{3(1+|x|)}+2x\cos x+\frac{5}{3}\right), h(t,x)=\frac{\cos t}{4\sqrt{36+t^2}}\left(\frac{e^{-t}}{10}+\sin x\right). \tag{28}$$

Then

$$|f(t,x)| \leq \frac{e^{-3t}}{40}\left(\frac{1}{3}+2||x||+\frac{5}{3}\right) \leq \frac{e^{-3t}}{20}(1+||x||)=P_2(t)\varphi_2(||x||),$$

$$|h(t,x)| \leq \frac{\cos t}{4\sqrt{36+t^2}}\left(\frac{1}{10}+||x||\right)=P_1(t)\varphi_1(||x||),$$

with

$$P_2(t)=\frac{e^{-3t}}{10} \leq \frac{1}{20},$$

$$P_1(t)=\frac{1}{4\sqrt{36+t^2}} \leq \frac{1}{24}.$$

Clearly

$$\|P_1\|=\frac{1}{24}, \|P_2\|=\frac{1}{20}, P=\max\{P_1,P_2\}=\frac{1}{20}, \varphi=\max\{\varphi_1,\varphi_2\}=1+\|x\|.$$

Based on (H_2)*, we find that* $M > 1.602872575$. *Since all the condition of Theorem 6 are satisfied, the boundary value problem* (27) *has at least one solution on [0,1] with* $f(t,x)$ *and* $h(t,x)$ *given by* (28).

Example 2. *Consider the boundary value problem* (27) *with*

$$\begin{aligned} f(t,x)&=\frac{1}{30}\sin(x)+e^{-2t}\cos(t),\\ h(t,x)&=\frac{e^{-t}}{t^2+40}\frac{|x|}{1+|x|}+\frac{1}{9}. \end{aligned} \tag{29}$$

Then

$$|h(t,x)| \leq \frac{e^{-t}}{t^2+40}+\frac{1}{9}=\lambda_1(t),$$

$$|f(t,x)| \leq \frac{1}{30}+e^{-2t}\cos(t)=\lambda_2(t).$$

Therefore,

$$|h(t,x)-h(t,y)| \leq \frac{1}{40}\|x-y\|,$$

$$|f(t,x)-f(t,y)| \leq \frac{1}{30}||x-y||.$$

Obviously

$$\|\lambda_1\| = \frac{49}{360}, \|\lambda_2\| = \frac{31}{30}, \lambda = \max\{\lambda_1,\lambda_2\} = \frac{31}{30}, L = \max\{L_1,L_2\} = \frac{1}{30},\ L\Lambda_1 \simeq 0.3058091177 < 1.$$

Obviously, the hypotheses of Theorem 7 all hold. Therefore, we can infer that the boundary value problem (27) *has at least one solution on* $[0,1]$ *with* $f(t,x(t))$ *and* $h(t,x(t))$ *given by* (29).

Example 3. *Using the data for Example 2 , we find that*

$$L\Lambda \simeq 0.4105393890 < 1.$$

Finally, one can notice that boundary value problem (27) with $f(t,x(t))$ and $h(t,x(t))$ given by (29) has a unique solution on $[0,1]$ as the hypothesis of Theorem 8 holds true.

6. Conclusions

In this article, we have obtained some results for a Caputo fractional proportional type nonlinear boundary value problem coupled with Caputo fractional proportional type slit-strips and Riemann-Stieltjes integral boundary conditions. Our first results deal with single-valued maps and are obtained by applying Leray-Schauder nonlinear alternative fixed point theorem, Karasnoselekii's fixed point theorem, and Banach contraction mapping principle. The second result is concerned with multivalued maps and is established by means of another version Karasnoselekii's fixed point theorem. Our results correspond to several special classes of the problem at hand mentioned in Remark 1.

Confilict of interest

The authors declare that they have no conflict of interest.

References

[1] Kilbas AA, Srivastava HM, Trujillo JJ. *Theory and Applications of Fractional Differential Equations*. London, UK: Elsevier; 2006.

[2] Podlubny I. *Fractional Differential Equations*. San Diego, CA, USA: Academic Press; 1999.

[3] Katugampola U. A new approach to a generalized fractional integral. *Appl. Math. Comput*. 2011; 218(3): 860-865.

[4] Katugampola UN. New approach to generalized fractional derivative. *Bull. Math. Anal. Appl*. 2014; 6(4): 1-15.

[5] Almeida R. A Caputo fractional derivative of a function with respect to another function. *Commun. Nonlinear Sci. Numer Simul*. 2017; 44: 460-481. Available from: https://doi.org/10.1016/j.cnsns.2016.09.006.

[6] Kucche KD, Mali AD. On the Nonlinear (k,ψ)-Hilfer Fractional Differential Equations. *Chaos Solitons Fractals*. 2021; 152: 1-24. Available from: https://doi.org/10.22541/au.162264976.62662516/v1.

[7] Jarad F, Abdeljawad T. Generalized fractional derivatives and Laplace transforms. *Discrete Contin. Dyn. Syst. Ser. S*. 2020; 13(3): 709-722. Available from: https://doi.org/10.3934/dcdss.2020039.

[8] Jarad F, Abdeljawad T, Alzabut J. Generalized fractional derivatives generated by a class of local proportional derivatives. *Eur. Phys. J. Spec. Top*. 2017; 226: 3457-3471. Available from: https://doi.org/10.1140/epjst/e2018-00021-7.
[9] Jarad F, Alqudah MA, Abdeljawad T. On more generalized form of proportional fractional operators. *Open Math*. 2020; 18: 167-176.
[10] Jarad F, Abdeljawad T, Rashid S, Hammouch Z. More properties of the proportional fractional integrals and derivatives of a function with respect to another function. *Advances in Difference Equations*. 2020; 1: 1-16.
[11] Ahmad B, Ntouyas SK. Nonlinear fractional differential inclusions with anti-periodic type integral boundary conditions. *Discuss Math. Differ. Incl. Control Optim*. 2012; 32: 45-62.
[12] Shammakh W, Alzumi HZ, Alqahtani BA. On More General Fractional Differential Equations Involving Mixed Generalized Derivatives with Nonlocal Multipoint and Generalized Fractional Integral Boundary Conditions. *J. Funct Spaces*. 2020; 4: 1-19.
[13] Shammakh W, Alzumi HZ, Albarqi Z. On multi-term proportional fractional differential equations and inclusions. *Adv. Differ Equ*. 2020; 2020(1): 1-28. Available from: https://doi.org/10.1186/s13662-020-03104-y.
[14] Benchohra M, Bouriah S, Nieto JJ. Boundary value problem for differential equations with generalized Hilfer-type fractional derivative. *Fixed Point Theory*. 2021; 22(2): 527-542.
[15] Abbas MI, Snezhana H. On the Initial Value Problems for Caputo-Type Generalized Proportional Vector-Order Fractional Differential Equations. *Mathematics*. 2021; 9(21): 1-10. Available from: https://doi.org/10.3390/math9212720.
[16] Agarwal RP, Hristova S. Boundary Value Problem for Multi-Term Nonlinear Delay Generalized Proportional Caputo Fractional Differential Equations. *Fractal Fract*. 2022; 6(12): 1-17.
[17] Ahmad B, Alsaedi A, Salem S, Ntouyas SK. Fractional differential equation involving mixed nonlinearities with nonlocal multi-point and Riemann-Stieltjes integral-multi-strip conditions. *Fractal Fract*. 2019; 3(34): 1-16.
[18] Laadjal Z, Jarad F. On a Langevin equation involving Caputo fractional proportional derivatives with respect to another function. *AIMS Mathematics*. 2022; 7: 1273-1292.
[19] Hurd RA, Hayashi Y. Low-frequency scattering by a slit in a conducting plane. *Radio Sci*. 1980; 15: 1171-1178.
[20] Otsuki T. Diffraction by multiple slits. *J. Opt. Soc. Am*. 1990; 7: 646-652.
[21] Asghar S, Hayat T, Ahmad B. Acoustic diffraction from slit in an infinite absorbing sheet. *Jpn. J. Ind. Appl. Math*. 1996; 13: 519-532.
[22] Lundqvist M. *Silicon Strip detectors for Scanned Multi-Slit X-Ray Imaging*. KTH Royal Institute of Technology, Stockholm, Doctoral thesis, 2003.
[23] Ahmad B, Alsaedi A, Alghamdi BS. Analytic approximation of solutions of the forced Duffing equation with integral boundary conditions. *Nonlinear Anal. Real World Appl*. 2008; 9: 1727-1740.
[24] Ciegis R, Bugajev A. Numerical approximation of one model of the bacterial self-organization. *Nonlinear Anal. Model Control*. 2012; 17: 253-270.
[25] Granas A, Dugundji J. *Fixed Point Theory*. New York, NY, USA: Springer Science Business Media; 2003.
[26] Dhage BC. Multi-valued mappings and fixed points II. *Tamkang J. Math*. 2006; 37(1): 27-46.
[27] Krasnoselskii MA. Two remarks on the method of successive approximations. *Uspekhi Mat. Nauk*. 1955; 10: 123-127.
[28] Petrusel A. Fixed points and selections for multivalued operators. *Seminar on Fixed Point Theory, Cluj-Napoca*. 2001; 2: 3-22.
[29] Deimling K. *Multivalued Differential Equations*. Berlin, Germany: Walter de Gruyter; 1992.
[30] Lasota A, Opial Z. An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations. *Bull. Acad. Pol. Sci*. 1965; 13: 781-786.