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Research Article

Stochastic Perturbation of Optical Solitons for the Concatenation Model with Power-Law of Self-Phase Modulation Having Multiplicative White Noise

Ahmed H. Arnous[1], **Anjan Biswas**[2,3,4,5], **Yakup Yildirim**[6,7*], **Ali Saleh Alshomrani**[3]

[1]Department of Physics and Engineering Mathematics, Higher Institute of Engineering, El-Shorouk Academy, Cairo, Egypt
[2]Department of Mathematics and Physics, Grambling State University, Grambling, LA-71245, USA
[3]Mathematical Modeling and Applied Computation (MMAC) Research Group, Center of Modern Mathematical Sciences and their Applications (CMMSA), Department of Mathematics, King Abdulaziz University, Jeddah-21589, Saudi Arabia
[4]Department of Applied Sciences, Cross-Border Faculty of Humanities, Economics and Engineering, Dunarea de Jos University of Galati, 111 Domneasca Street, Galati-800201, Romania
[5]Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Medunsa-0204, South Africa.
[6]Department of Computer Engineering, Biruni University, Istanbul-34010, Turkey
[7]Department of Mathematics, Near East University, 99138 Nicosia, Cyprus
E-mail: yyildirim@biruni.edu.tr

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Abstract: Addressing the concatenation model, this paper explores the incorporation of power-law self-phase modulation and spatio-temporal dispersion, in addition to the chromatic dispersion. The inclusion of the white noise effect, along with deterministic Hamiltonian perturbation terms of arbitrary intensity, is also examined. Two integration approaches are applied to recover the soliton solutions, leading to the observation that the white noise effect remains confined to the phase component of these solutions.

***Keywords*:** solitons, concatenation, power-law, stochastic, perturbations, Kudryashov

MSC: 78A60

1. Introduction

Modeling the dynamics of optical soliton propagation through monomode fibers involves various forms of nonlinear evolution equations. In 2012, Ankiewicz et al. introduced a new governing model that combines the nonlinear Schrödinger's equation (NLSE), Lakshmanan-Porsezian-Daniel (LPD) model, and the Sasa-Satsuma equation (SSE) [1-2]. Known as the concatenation model, it has yielded significant results in recent years, recovering and reporting optical solitons under Kerr law of self-phase modulation (SPM) and power-law of SPM [3-9]. These results include the retrieval of optical solitons using the method of undetermined coefficients, identification of conservation laws through the multiplier approach, discovery of quiescent solitons with nonlinear chromatic dispersion (CD), numerical analysis through Laplace-Adomian decomposition, and more. The concatenation model extends to birefringent fibers, with recovered soliton solutions. Quiescent optical solitons in birefringent fibers, under nonlinear CD along both components,

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have been observed.

Stepping beyond established concepts, this paper advances the ongoing studies. It focuses on the concatenation model, introducing the power-law of self-phase modulation (SPM) and incorporating perturbation terms. Both deterministic and stochastic perturbation terms are carefully examined, with the former, originating from inter-modal dispersion, self-steepening effect, and self-frequency shift, exhibiting arbitrary intensity. Simultaneously, the stochastic perturbation term arises from the white noise effect. Notably, alongside chromatic dispersion (CD), we include the spatio-temporal dispersion (STD) effect to mitigate the Internet bottleneck [5]. Consequently, the primary emphasis of this paper lies in the extended version of the concatenation model. Two integration schemes-enhanced Kudryashov's scheme and the extended auxiliary equation method-are employed to identify soliton solutions that encapsulate all perturbative effects. The subsequent sections meticulously detail these integration schemes and present the corresponding results.

1.1 *Governing model*

In its dimensionless representation, the perturbed edition of the concatenation model includes both STD and the white noise effect, as detailed in references [3-9]:

$$iq_t + aq_{xx} + bq_{xt} + c|q|^{2n} q + c_1\left[\delta_1 q_{xxxx} + \delta_2 (q_x)^2 q^* + \delta_3 |q_x|^2 q + \delta_4 |q|^{2n} q_{xx} + \delta_5 q^2 q_{xx}^* + \delta_6 |q|^{2n+2} q\right]$$

$$+ic_2\left[\delta_7 q_{xxx} + \delta_8 |q|^{2n} q_x + \delta_9 q^2 q_x^*\right] + \sigma(q - ibq_x)\frac{dW(t)}{dt} = i\left[\alpha q_x + \lambda\left(|q|^{2n} q\right)_x + \beta\left(|q|^{2n}\right)_x q\right]. \tag{1}$$

The expression in equation (1) features the dependent variable $q(x, t)$, signifying the wave amplitude as a complex-valued function with spatial and temporal coordinates x and t respectively. The first four terms characterize the NLSE, with a and b acting as coefficients for CD and STD respectively. The parameter c accounts for SPM with n representing the power-law nonlinearity parameter. Coefficients c_1 and c_2 relate to the LPD model and the SSE respectively. The term $W(t)$ represents the Weiner process arising from stochasticity. In terms of deterministic perturbation, α represents inter-modal dispersion, λ captures the effect of self-steepening, and β signifies soliton self-frequency shift. Finally, $i = \sqrt{-1}$ represents the complex quantity.

2. Overview of the integration algorithms

We could write a nonlinear evolution equation as [10-16]:

$$F(u, u_x, u_t, u_{xt}, u_{xx}, \ldots) = 0. \tag{2}$$

The function $u = u(x, t)$ signifies the wave profile, where t and x serve as descriptors for the time and space variables, respectively, according to references [17-25].

The implementation of the wave transformation [26-35], described by

$$u(x, t) = U(\xi), \; \xi = k(x - \upsilon t), \tag{3}$$

results in the equation (2) being reduced to

$$P(U, -k\upsilon U', kU', k^2U'', \ldots) = 0. \tag{4}$$

Within the given expression, k is used to represent the wave width, ξ is employed as the wave variable, and υ is utilized to denote the wave velocity, according to references [36-45].

2.1 *The enhanced Kudryashov's approach*

Within this subsection, an exhaustive overview of the fundamental procedures is presented. The algorithmic steps are listed below:

Step-1: Hereafter is the explicit solution for the reduced model equation (4):

$$U(\xi) = \sigma_0 + \sum_{i=1}^{N} \left\{ \sigma_i R(\xi)^i + \rho_i \left(\frac{R'(\xi)}{R(\xi)} \right)^i \right\}, \tag{5}$$

along with

$$R'(\xi)^2 = R(\xi)^2 (1 - \chi R(\xi)^2). \tag{6}$$

Values for the constants σ_0, χ, σ_i, and ρ_i (for i = 1, ..., N) will be presented, where the value of N is ascertained through the balancing procedure illustrated in equation (4).

Step-2: Soliton wave characteristics are captured by Equation (6) as follows:

$$R(\xi) = \frac{4c}{4c^2 e^{\xi} + \chi e^{-\xi}}, \tag{7}$$

where c is nonzero constant.

Step-3: Through the insertion of equation (5) into equation (4), along with equation (6), we can determine the necessary constants for equations (3) and (5). To incorporate the identified parametric restrictions, they can be substituted into equation (5) along with equation (7). Subsequently, this yields straddled solitons, further categorized as singular, dark, or bright solitons.

2.2 *Extended auxiliary equation method*

Within this subsection, an exhaustive overview of the fundamental procedures is presented.

Step-1: It is hypothesized that the solution to equation (4) can be formulated in the following way:

$$U(\xi) = \alpha_0 + \sum_{i=1}^{N} \left\{ \alpha_i \theta(\xi)^i + \beta_i \theta(\xi)^{-i} \right\}, \tag{8}$$

with

$$\theta'(\xi)^2 = \sum_{l=0}^{4} \tau_l \theta(\xi)^l. \tag{9}$$

Different solution types are yielded by this equation, presented as follows:

Case-1: $\tau_0 = \tau_1 = \tau_3 = 0$.

Respectively, bright and singular soliton solutions are achieved:

$$\theta(\xi)=\sqrt{-\frac{\tau_2}{\tau_4}}\ \operatorname{sech}\left[\sqrt{\tau_2}\,\xi\right],\ \tau_2>0,\ \tau_4<0, \tag{10}$$

and

$$\theta(\xi)=\sqrt{\frac{\tau_2}{\tau_4}}\ \operatorname{csch}\left[\sqrt{\tau_2}\,\xi\right],\ \tau_2>0,\ \tau_4>0. \tag{11}$$

Case-2: $\tau_0=\dfrac{{\tau_2}^2}{4\tau_4},\ \tau_1=\tau_3=0.$

Dark and singular soliton solutions are respectively obtained:

$$\theta(\xi)=\sqrt{-\frac{\tau_2}{2\tau_4}}\ \tanh\left[\sqrt{-\frac{\tau_2}{2}}\,\xi\right],\ \tau_2<0,\ \tau_4>0, \tag{12}$$

and

$$\theta(\xi)=\sqrt{-\frac{\tau_2}{2\tau_4}}\ \coth\left[\sqrt{-\frac{\tau_2}{2}}\,\xi\right],\ \tau_2<0,\ \tau_4>0. \tag{13}$$

Case-3: $\tau_1=\tau_3=0.$

The Weierstrass elliptic function solution is obtained as:

$$\theta(\xi)=\frac{3\wp'(\xi;g_2,g_3)}{\sqrt{\tau_4}\left[6\wp(\xi;g_2,g_3)+\tau_2\right]}, \tag{14}$$

where $g_2=\dfrac{\tau_2^2}{12}+\tau_0\tau_4$ and $g_3=\dfrac{\tau_2\left(36\tau_0\tau_4-\tau_2^2\right)}{216}$ are denoted as invariants of the Weierstrass elliptic function.

Case-4: $\tau_0=\tau_1=0,\ \tau_2,\ \tau_4>0,\ \tau_3\neq\pm2\sqrt{\tau_2\tau_4}.$

Straddled soliton solutions are obtained:

$$\theta(\xi)=\frac{-\tau_2\operatorname{sech}^2\left[\frac{1}{2}\sqrt{\tau_2}\,\xi\right]}{\pm2\sqrt{\tau_2\tau_4}\tanh\left[\frac{1}{2}\sqrt{\tau_2}\,\xi\right]+\tau_3}, \tag{15}$$

and

$$\theta(\xi) = \frac{\tau_2 \operatorname{csch}^2\left[\frac{1}{2}\sqrt{\tau_2}\,\xi\right]}{\pm 2\sqrt{\tau_2\tau_4}\coth\left[\frac{1}{2}\sqrt{\tau_2}\,\xi\right] + \tau_3}. \tag{16}$$

Step-2: The positive integer number N in equation (8) is to be determined by balancing the highest order derivatives and the nonlinear terms in equation (4).

Step-3: Insert (8) into (4) along with (9). Through this substitution, a polynomial in terms of $\theta(\xi)$ is derived. Grouping terms of the same powers in this polynomial and equating them to zero forms an over-determined system of algebraic equations. Solving this system collectively provides the values for the unknown parameters k, v, α_0, α_i, and β_i (i = 1, 2, ...). As a result, the exact solutions for (2) are achieved.

3. Application to the concatenation model

The solution to Eq. (1) is obtained by following the solution structure presented in references [46-53]:

$$\psi(x, t) = U(\xi)e^{i\phi(x,t)}, \tag{17}$$

with

$$\xi = k(x - vt). \tag{18}$$

Within this setting, the wave variable is denoted as ξ, the amplitude component is represented by $U(\xi)$, and v symbolizes the soliton speed. The phase component $\phi(x, t)$ is presented below:

$$\phi(x, t) = -\kappa x + \omega t + \sigma W(t) - \sigma^2 t + \theta_0. \tag{19}$$

Within this context, κ is the frequency of the solitons, ω represents the wave number, σ indicates the noise strength, and θ_0 serves as the phase constant. Upon substituting (17) into (1) and then decomposing into real and imaginary parts, one gets the real component

$$\begin{aligned}
&k^2\left(a - bv - 6c_1\delta_1\kappa^2 + 3c_2\delta_7\kappa\right)U'' + \left(-a\kappa^2 - \alpha\kappa - b\kappa\sigma^2 + b\kappa\omega + c_1\delta_1\kappa^4 - c_2\delta_7\kappa^3 + \sigma^2 - \omega\right)U \\
&+c_1\delta_1 k^4 U'''' + c_1\delta_4 k^2 U^{2n}U'' + c_1\delta_5 k^2 U^2 U'' + c_1\left(\delta_2 + \delta_3\right)k^2 UU'^2 + \left(-c_1\delta_4\kappa^2 + c_2\delta_8\kappa + c - \kappa\lambda\right)U^{2n+1} \\
&+c_1\delta_6 U^{2n+3} - \kappa\left(c_1\left(\delta_2 - \delta_3 + \delta_5\right)\kappa + c_2\delta_9\right)U^3 = 0,
\end{aligned} \tag{20}$$

and the imaginary part

$$\begin{aligned}
&k\left(-2a\kappa - \alpha - b\sigma^2 + b\kappa v + b\omega + 4c_1\delta_1\kappa^3 - 3c_2\delta_7\kappa^2 - v\right)U' + k^3\left(c_2\delta_7 - 4c_1\delta_1\kappa\right)U''' \\
&-k\left(2c_1\delta_4\kappa - c_2\delta_8 + \lambda + 2\beta n + 2\lambda n\right)U^{2n}U' + k\left(2c_1\left(\delta_5 - \delta_2\right)\kappa + c_2\delta_9\right)U^2U' = 0.
\end{aligned} \tag{21}$$

Extracting the soliton speed is achieved through examination of the imaginary part

$$v = \frac{2a\kappa + \alpha + b\sigma^2 - b\omega - 4c_1\delta_1\kappa^3 + 3c_2\delta_7\kappa^2}{b\kappa - 1}, \tag{22}$$

and the soliton frequency becomes

$$\kappa = \frac{c_2\delta_9}{2c_1\left(\delta_2 - \delta_5\right)}, \tag{23}$$

with the parametric restrictions

$$\delta_4 = \frac{\left(\delta_2 - \delta_5\right)\left(c_2\delta_8 - \lambda - 2\beta n - 2\lambda n\right)}{c_2\delta_9}, \tag{24}$$

and

$$\delta_1 = \frac{\left(\delta_2 - \delta_5\right)\delta_7}{2\delta_9}. \tag{25}$$

Eq. (20) can be simplified to

$$A_1U^{2n}U'' + A_6U^{2n+1} + A_5U^{2n+3} + A_2U^2U'' + A_3U'' + A_4UU'^2 + A_7U^3 + A_8U + k^2U'''' = 0, \tag{26}$$

with

$$\begin{cases} A_1 = \dfrac{\delta_4}{\delta_1}, \\ A_2 = \dfrac{\delta_5}{\delta_1}, \\ A_3 = \dfrac{a - bv - 6c_1\delta_1\kappa^2 + 3c_2\delta_7\kappa}{c_1\delta_1}, \\ A_4 = \dfrac{\delta_2 + \delta_3}{\delta_1}, \\ A_5 = \dfrac{\delta_6}{\delta_1 k^2}, \\ A_6 = \dfrac{-c_1\delta_4\kappa^2 + c_2\delta_8\kappa + c - \kappa\lambda}{c_1\delta_1 k^2}, \\ A_7 = -\dfrac{\kappa\left(c_1\left(\delta_2 - \delta_3 + \delta_5\right)\kappa + c_2\delta_9\right)}{c_1\delta_1 k^2}, \\ A_8 = -\dfrac{a\kappa^2 + \alpha\kappa + b\kappa\sigma^2 - b\kappa\omega - c_1\delta_1\kappa^4 + c_2\delta_7\kappa^3 - \sigma^2 + \omega}{c_1\delta_1 k^2}, \end{cases} \tag{27}$$

where $c_1\delta_1 \neq 0$.

Using the transformation

$$U = V^{\frac{1}{n}},$$

Eq. (26) collapses to

$$A_6 n^4 V^6 + A_8 n^4 V^4 + A_1 n^3 V^5 V'' + A_3 n^3 V^3 V'' - A_1 (n-1) n^2 V^4 V'^2 - A_3 (n-1) n^2 V^2 V'^2$$

$$+ V^{\frac{2}{n}+2}\left(A_5 n^4 V^4 + A_7 n^4 V^2 + A_2 n^3 V V'' + n^2 \left(A_2 (1-n) + A_4\right) V'^2\right) + k^2 n^3 V^{(4)} V^3 - 3k^2 (n-1) n^2 V^2 V''^2$$

$$-4k^2 (n-1) n^2 V^{(3)} V^2 V' + 6k^2 n \left(2n^2 - 3n + 1\right) V V'^2 V'' + k^2 \left(-6n^3 + 11n^2 - 6n + 1\right) V'^4 = 0. \tag{28}$$

For integrability, we set

$$A_2 = A_4 = A_5 = A_7 = 0. \tag{29}$$

This leads to

$$\delta_5 = \delta_6 = 0,\ \delta_3 = -\delta_2,\ -c_1 \delta_4 \kappa^2 + c_2 \delta_8 \kappa + c - \kappa\lambda = 0, \tag{30}$$

with

$$\kappa = \frac{c_2 \delta_9}{2 c_1 \delta_2}. \tag{31}$$

Then Eq. (1) reaches

$$iq_t + aq_{xx} + bq_{xt} + c\,|q|^{2n}\,q + c_1\left[\delta_1 q_{xxxx} + \delta_2 (q_x)^2 q^* - \delta_2\,|q_x|^2\,q + \delta_4\,|q|^{2n}\,q_{xx}\right]$$

$$+ ic_2\left[\delta_7 q_{xxx} + \delta_8\,|q|^{2n}\,q_x + \delta_9 q^2 q_x^*\right] + \sigma\left(q - ibq_x\right)\frac{dW(t)}{dt}$$

$$= i\left\{\alpha q_x + \lambda\left(|q|^{2n}\,q\right)_x + \beta\left(|q|^{2n}\right)_x q\right\}. \tag{32}$$

In this case, Eq. (26) becomes

$$A_1 U^{2n} U'' + A_6 U^{2n+1} + A_3 U'' + A_8 U + k^2 U'''' = 0, \tag{33}$$

with

$$\begin{cases} A_1 = \dfrac{\delta_4}{\delta_1}, \\ A_3 = \dfrac{a - bv - 6c_1\delta_1\kappa^2 + 3c_2\delta_7\kappa}{c_1\delta_1}, \\ A_6 = \dfrac{-c_1\delta_4\kappa^2 + c_2\delta_8\kappa + c - \kappa\lambda}{c_1\delta_1 k^2}, \\ A_8 = -\dfrac{a\kappa^2 + \alpha\kappa + b\kappa\sigma^2 - b\kappa\omega - c_1\delta_1\kappa^4 + c_2\delta_7\kappa^3 - \sigma^2 + \omega}{c_1\delta_1 k^2}. \end{cases} \tag{34}$$

Also, Eq. (28) reads

$$A_6 n^4 V^6 + A_8 n^4 V^4 + A_1 n^3 V^5 V'' + A_3 n^3 V^3 V'' - A_1(n-1)n^2 V^4 V'^2 - A_3(n-1)n^2 V^2 V'^2$$

$$+k^2 n^3 V^{(4)} V^3 - 3k^2(n-1)n^2 V^2 V''^2 - 4k^2(n-1)n^2 V^{(3)} V^2 V' + 6k^2 n\left(2n^2 - 3n + 1\right) V V'^2 V''$$

$$+k^2\left(-6n^3 + 11n^2 - 6n + 1\right) V'^4 = 0. \tag{35}$$

Balancing $V^3 V''''$ with $V^5 V''$ or V^6 in Eq. (29) gives $N = 2$ or $N = 1$.

3.1 *Case-1: N = 1*

3.1.1 *Enhanced Kudryashov's approach*

The solution takes the form outlined below

$$V(\xi) = \sigma_0 + \sigma_1 R(\xi) + \rho_1\left(\frac{R'(\xi)}{R(\xi)}\right). \tag{36}$$

Substituting equation (36) together with equation (6) into equation (35) results in a system of algebraic equations. The solution to these equations is obtained as follows

$$a_0 = 0,\ a_1 = \pm\sqrt{-\frac{A_3\left(6n^3 + 11n^2 + 6n + 1\right)\chi}{A_1\left(4n^3 + 2n^2 + n + 1\right) - A_6 n^2\left(6n^2 + 5n + 1\right)}},$$

$$b_1 = 0,\ k = n\sqrt{\frac{A_1 A_3(n+1)}{A_6 n^2\left(6n^2 + 5n + 1\right) - A_1\left(4n^3 + 2n^2 + n + 1\right)}},$$

$$A_8 = \frac{A_3(2n+1)\left(A_6(3n+1) - 2A_1\right)}{A_1\left(4n^3 + 2n^2 + n + 1\right) - A_6 n^2\left(6n^2 + 5n + 1\right)}. \tag{37}$$

As a result, we achieve the exact solutions of Eq. (1) as outlined below

$$q(x,t)=\left\{\frac{\pm 4c\sqrt{-\frac{A_3\left(6n^3+11n^2+6n+1\right)\chi}{A_1\left(4n^3+2n^2+n+1\right)-A_6n^2\left(6n^2+5n+1\right)}}}{4c^2e^{n\sqrt{\frac{A_1A_3(n+1)}{A_6n^2\left(6n^2+5n+1\right)-A_1\left(4n^3+2n^2+n+1\right)}}(x-tv)}+\chi e^{-n\sqrt{\frac{A_1A_3(n+1)}{A_6n^2\left(6n^2+5n+1\right)-A_1\left(4n^3+2n^2+n+1\right)}}(x-tv)}}\right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\left\{\frac{c_2\delta_9}{2c_1\delta_2}\right\}x+\omega t+\sigma W(t)-\sigma^2t+\theta_0\right)}. \tag{38}$$

Set $\chi=\pm 4c^2$ in the solution given by (38). Consequently, for $A_6n^2(6n^2+5n+1)-A_1(4n^3+2n^2+n+1)$ and $A_1A_3(n+1)>0$, we have bright soliton with $A_3>0$ and singular soliton with $A_3<0$, respectively:

$$q(x,t)=\left\{\pm\sqrt{-\frac{A_3\left(6n^3+11n^2+6n+1\right)}{A_1\left(4n^3+2n^2+n+1\right)-A_6n^2\left(6n^2+5n+1\right)}}\right.$$

$$\left.\times\operatorname{sech}\left[n\sqrt{\frac{A_1A_3(n+1)}{A_6n^2\left(6n^2+5n+1\right)-A_1\left(4n^3+2n^2+n+1\right)}}(x-tv)\right]\right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\left\{\frac{c_2\delta_9}{2c_1\delta_2}\right\}x+\omega t+\sigma W(t)-\sigma^2t+\theta_0\right)}, \tag{39}$$

and

$$q(x,t)=\left\{\pm\sqrt{\frac{A_3\left(6n^3+11n^2+6n+1\right)}{A_1\left(4n^3+2n^2+n+1\right)-A_6n^2\left(6n^2+5n+1\right)}}\right.$$

$$\left.\times\operatorname{csch}\left[n\sqrt{\frac{A_1A_3(n+1)}{A_6n^2\left(6n^2+5n+1\right)-A_1\left(4n^3+2n^2+n+1\right)}}(x-tv)\right]\right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\left\{\frac{c_2\delta_9}{2c_1\delta_2}\right\}x+\omega t+\sigma W(t)-\sigma^2t+\theta_0\right)}. \tag{40}$$

This approach therefore failed to recover dark solitons for the governing model.

3.1.2 *Extended auxiliary equation method*

In line with the technique, the solution takes the form outlined below

$$V(\xi)=\alpha_0+\alpha_1\theta(\xi)+\frac{\beta_1}{\theta(\xi)}. \tag{41}$$

Substituting equation (41) together with equation (9) into equation (35) results in a system of algebraic equations. The solutions to these equations are obtained as follows.

Case 1: $\tau_0=\tau_1=\tau_3=0$.

$$\alpha_0=\beta_1=0,\ \tau_2=\frac{n^2\left(A_6k^2\left(6n^2+5n+1\right)-A_1A_3(n+1)\right)}{A_1k^2\left(4n^3+2n^2+n+1\right)},\ \tau_4=-\frac{\alpha_1^2A_1n^2}{k^2\left(6n^2+5n+1\right)},$$

$$A_8=-\frac{(2n+1)\left(A_6k^2(3n+1)+2A_1A_3n^2\right)\left(A_6k^2(n(6n+5)+1)-A_1A_3(n+1)\right)}{A_1^2k^2\left(4n^3+2n^2+n+1\right)^2}. \tag{42}$$

For this case, the solution of (1) reads

$$q(x,t)=\left\{\pm\sqrt{\frac{(2n+1)(3n+1)\left(A_6k^2(n(6n+5)+1)-A_1A_3(n+1)\right)}{A_1^2\left(4n^3+2n^2+n+1\right)}}\right.$$

$$\left.\times\operatorname{sech}\left[n\sqrt{\frac{A_6k^2\left(6n^2+5n+1\right)-A_1A_3(n+1)}{A_1\left(4n^3+2n^2+n+1\right)}}(x-vt)\right]\right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\left\{\frac{c_2\delta_9}{2c_1\delta_2}\right\}x+\omega t+\sigma W(t)-\sigma^2t+\theta_0\right)}, \tag{43}$$

and

$$q(x,t)=\left\{\pm\sqrt{-\frac{(2n+1)(3n+1)\left(A_6k^2(n(6n+5)+1)-A_1A_3(n+1)\right)}{A_1^2\left(4n^3+2n^2+n+1\right)}}\right.$$

$$\left.\times\operatorname{csch}\left[n\sqrt{\frac{A_6k^2\left(6n^2+5n+1\right)-A_1A_3(n+1)}{A_1\left(4n^3+2n^2+n+1\right)}}(x-vt)\right]\right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\left\{\frac{c_2\delta_9}{2c_1\delta_2}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}. \tag{44}$$

These solitons with $\tau_2 > 0$ are bright for $\tau_4 < 0$ and singular for $\tau_4 > 0$, respectively.

Case 2: $\tau_1 = 0,\ \tau_3 = 0,\ \tau_0 = \frac{\tau_2^2}{4\tau_4}$.

$$\alpha_0 = 0,\ \alpha_1 = \frac{\sqrt{-k^2A_1^3\left(6n^2+5n+1\right)\left(4n^3+2n^2+n+1\right)^2\tau_4}}{A_1^2 n\left(4n^3+2n^2+n+1\right)},$$

$$\beta_1 = \frac{n\sqrt{6n^2+5n+1}\left(A_6k^2\left(6n^2+5n+1\right)-A_1A_3(n+1)\right)}{4\sqrt{A_1^3\left(-k^2\right)\left(4n^3+2n^2+n+1\right)^2\tau_4}},$$

$$\tau_2 = -\frac{n^2\left(A_6k^2\left(6n^2+5n+1\right)-A_1A_3(n+1)\right)}{2A_1k^2\left(4n^3+2n^2+n+1\right)},$$

$$A_8 = -\frac{(2n+1)\left(A_6k^2(3n+1)+2A_1A_3n^2\right)\left(A_6k^2(n(6n+5)+1)-A_1A_3(n+1)\right)}{A_1^2k^2\left(4n^3+2n^2+n+1\right)^2}. \tag{45}$$

For this case, the solution of (1) reads

$$q(x,t)=\left\{\pm\sqrt{\frac{(2n+1)(3n+1)\left(A_1A_3(n+1)-A_6k^2(n(6n+5)+1)\right)}{A_1^2\left(4n^3+2n^2+n+1\right)}}\right.$$

$$\times \operatorname{csch}\left[n\sqrt{\frac{A_6k^2(n(6n+5)+1)-A_1A_3(n+1)}{A_1\left(4n^3+2n^2+n+1\right)}}(x-vt)\right]\Bigg\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\left\{\frac{c_2\delta_9}{2c_1\delta_2}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}. \tag{46}$$

The obtained soliton is singular with $\tau_2 < 0$ and $\tau_4 > 0$.

Case 3: $\tau_1 = 0$, $\tau_3 = 0$.

$$\alpha_0=0,\ \tau_0=0,\ \beta_1=0,\ \tau_2=\frac{n^2\left(A_6k^2\left(6n^2+5n+1\right)-A_1A_3(n+1)\right)}{A_1k^2\left(4n^3+2n^2+n+1\right)},\ \tau_4=-\frac{\alpha_1^2A_1n^2}{k^2\left(6n^2+5n+1\right)},$$

$$A_8=-\frac{(2n+1)\left(A_6k^2(3n+1)+2A_1A_3n^2\right)\left(A_6k^2(n(6n+5)+1)-A_1A_3(n+1)\right)}{A_1^2k^2\left(4n^3+2n^2+n+1\right)^2}. \tag{47}$$

For this case, the solution of (1) reads

$$q(x,t)=\left\{\frac{3\wp'(\xi;g_2,g_3)}{\sqrt{-\frac{A_1n^2}{k^2\left(6n^2+5n+1\right)}}\left[6\wp(\xi;g_2,g_3)+\frac{n^2\left(A_6k^2\left(6n^2+5n+1\right)-A_1A_3(n+1)\right)}{A_1k^2\left(4n^3+2n^2+n+1\right)}\right]}\right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\left\{\frac{c_2\delta_9}{2c_1\delta_2}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}, \tag{48}$$

where

$$g_2=\frac{\tau_2^2}{12},\ g_3=-\frac{\tau_2^3}{216}. \tag{49}$$

The Weierstrass' elliptic function solution (48) with its restricted invariants (49) can be converted to a singular soliton

$$q(x,t)=\left\{\pm\sqrt{\frac{(2n+1)(3n+1)\left(A_1A_3(n+1)-A_6k^2(n(6n+5)+1)\right)}{A_1^2\left(4n^3+2n^2+n+1\right)}}\right.$$

$$\left.\times\operatorname{csch}\left[n\sqrt{\frac{A_6k^2(n(6n+5)+1)-A_1A_3(n+1)}{A_1\left(4n^3+2n^2+n+1\right)}}(x-vt)\right]\right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\left\{\frac{c_2\delta_9}{2c_1\delta_2}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}. \tag{50}$$

Case 4: $\tau_0=0$, $\tau_1=0$.

$$\alpha_0=\beta_1=0,\ \tau_4=-\frac{\alpha_1^2A_1n^2}{k^2\left(6n^2+5n+1\right)},\ \tau_2=\frac{n^2\left(A_6k^2(n(6n+5)+1)-A_1A_3(n+1)\right)}{A_1k^2\left(4n^3+2n^2+n+1\right)},\ \tau_3=0,$$

$$A_8=-\frac{(2n+1)\left(A_6k^2(3n+1)+2A_1A_3n^2\right)\left(A_6k^2(n(6n+5)+1)-A_1A_3(n+1)\right)}{A_1^2k^2\left(4n^3+2n^2+n+1\right)^2}. \tag{51}$$

For this case, the obtained solitons together with $\tau_2>0$ are bright for $\tau_4<0$ and singular for $\tau_4>0$, respectively:

$$q(x,t)=\left\{\pm\sqrt{\frac{(2n+1)(3n+1)\left(A_6k^2(n(6n+5)+1)-A_1A_3(n+1)\right)}{A_1^2\left(4n^3+2n^2+n+1\right)}}\right.$$

$$\left.\times\operatorname{sech}\left[n\sqrt{\frac{A_6k^2\left(6n^2+5n+1\right)-A_1A_3(n+1)}{A_1\left(4n^3+2n^2+n+1\right)}}(x-vt)\right]\right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\left\{\frac{c_2\delta_9}{2c_1\delta_2}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}, \tag{52}$$

and

$$q(x,t)=\left\{\pm\sqrt{-\frac{(2n+1)(3n+1)\left(A_6k^2(n(6n+5)+1)-A_1A_3(n+1)\right)}{A_1^2\left(4n^3+2n^2+n+1\right)}}\right.$$

$$\left.\times\operatorname{csch}\left[n\sqrt{\frac{A_6k^2\left(6n^2+5n+1\right)-A_1A_3(n+1)}{A_1\left(4n^3+2n^2+n+1\right)}}(x-vt)\right]\right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\left\{\frac{c_2\delta_9}{2c_1\delta_2}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}. \tag{53}$$

3.2 *Case-2: N = 2*

3.2.1 *Enhanced Kudryashov's apprpoach*

Following the technique, the solution takes the form outlined below

$$V(\xi)=a_0+a_1R(\xi)+b_1\left(\frac{R'(\xi)}{R(\xi)}\right)+a_2R(\xi)^2+b_2\left(\frac{R'(\xi)}{R(\xi)}\right)^2. \tag{54}$$

By substituting (54) together with (6) into Eq. (35), a system of algebraic equations is formed. Solving these equations collectively results in the following outcome

$$a_0=-b_2,\ a_2=b_2\chi\pm\sqrt{\frac{(n+1)(n+2)(3n+2)A_3\chi^2}{n^2(n(n+2)+2)A_6}},\ k=\frac{1}{2}\sqrt{-\frac{n^2A_3}{n(n+2)+2}},$$

$$A_8=-\frac{4(n+1)^2A_3}{n^2\left(n^2+2n+2\right)},\ A_1=0. \tag{55}$$

This leads to the derivation of the solution for Eq. (1):

$$q(x,t)=\left\{\chi\left(b_2\pm\sqrt{\frac{(n+1)(n+2)(3n+2)A_3}{n^2(n(n+2)+2)A_6}}\right)\left(\frac{4c}{4c^2e^{\frac{1}{2}\sqrt{-\frac{n^2A_3}{n(n+2)+2}}(x-vt)}+\chi e^{-\frac{1}{2}\sqrt{-\frac{n^2A_3}{n(n+2)+2}}(x-vt)}}\right)^2\right.$$

$$\left.+b_2\left(\frac{\chi-4c^2e^{\sqrt{-\frac{n^2A_3}{n(n+2)+2}}(x-vt)}}{\chi+4c^2e^{\sqrt{-\frac{n^2A_3}{n(n+2)+2}}(x-vt)}}\right)^2-b_2\right\}^{\frac{1}{n}}e^{i\left(-\left\{\frac{c_2\delta_9}{2c_1\delta_2}\right\}x+\omega t+\theta_0\right)}. \tag{56}$$

Selecting $\chi = \pm 4c^2$, we recover bright and singular solitons for $A_6 < 0$ and $A_3 < 0$, respectively:

$$q(x, t) = \left\{ \pm \sqrt{\frac{(n+1)(n+2)(3n+2)A_3}{n^2(n(n+2)+2)A_6}} \operatorname{sech}^2 \left[\frac{1}{2} \sqrt{-\frac{n^2 A_3}{n(n+2)+2}} (x - vt) \right] \right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\left\{ \frac{c_2 \delta_9}{2c_1 \delta_2} \right\} x + \omega t + \theta_0 \right)}, \tag{57}$$

and

$$q(x, t) = \left\{ \mp \sqrt{\frac{(n+1)(n+2)(3n+2)A_3}{n^2(n(n+2)+2)A_6}} \operatorname{csch}^2 \left[\frac{1}{2} \sqrt{-\frac{n^2 A_3}{n(n+2)+2}} (x - vt) \right] \right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\left\{ \frac{c_2 \delta_9}{2c_1 \delta_2} \right\} x + \omega t + \theta_0 \right)}. \tag{58}$$

3.2.2 *Extended auxiliary equation method*

In line with the technique, the solution takes the form outlined below

$$V(\xi) = \alpha_0 + \alpha_1 \theta(\xi) + \frac{\beta_1}{\theta(\xi)} + \alpha_2 \theta(\xi)^2 + \frac{\beta_2}{\theta(\xi)^2}. \tag{59}$$

Substituting equation (59) together with equation (9) into equation (35) results in a system of algebraic equations. The solutions to these equations are obtained as follows:

Case 1: $\tau_0 = \tau_1 = \tau_3 = 0$.

$$\alpha_0 = \alpha_1 = \beta_1 = \beta_2 = 0,\ \alpha_2 = \pm \frac{2k\sqrt{3n^3 + 11n^2 + 12n + 4}\tau_4}{\sqrt{-A_6} n^2},\ \tau_2 = -\frac{A_3 n^2}{4k^2\left(n^2 + 2n + 2\right)},$$

$$A_1 = 0,\ A_8 = \frac{A_3^2 (n+1)^2}{k^2 \left(n^2 + 2n + 2\right)^2}. \tag{60}$$

For this case, the solution of (1) reads

$$q(x,t)=\left\{\pm\frac{A_3\sqrt{(n+1)(n+2)(3n+2)}}{2\sqrt{-A_6}k(n(n+2)+2)}\operatorname{sech}^2\left[\frac{n}{2}\sqrt{-\frac{A_3}{n^2+2n+2}}(x-vt)\right]\right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\left\{\frac{c_2\delta_9}{2c_1\delta_2}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}, \tag{61}$$

and

$$q(x,t)=\left\{\mp\frac{A_3\sqrt{(n+1)(n+2)(3n+2)}}{2\sqrt{-A_6}k(n(n+2)+2)}\operatorname{csch}^2\left[\frac{n}{2}\sqrt{-\frac{A_3}{n^2+2n+2}}(x-vt)\right]\right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\left\{\frac{c_2\delta_9}{2c_1\delta_2}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}. \tag{62}$$

Singular and bright solitons are the outcomes, signified by $A_6 < 0$ and $A_3 < 0$.

Case 2: $\tau_0=\frac{\tau_2^2}{4\tau_4}$, $\tau_1=\tau_3=0$.

$$\alpha_1=\beta_1=\beta_2=0,\ \alpha_0=\pm\frac{A_3}{2}\sqrt{-\frac{(n+1)(n+2)(3n+2)}{A_6k^2(n(n+2)+2)^2}},\ \tau_2=\frac{A_3n^2}{2k^2(n(n+2)+2)},$$

$$\tau_4=\mp\frac{\alpha_2\sqrt{-A_6}n^2}{2k\sqrt{3n^3+11n^2+12n+4}},\ A_1=0,\ A_8=\frac{A_3^2(n+1)^2}{k^2\left(n^2+2n+2\right)^2}. \tag{63}$$

For this specific situation, the solution of (1) takes the form:

$$q(x,t)=\left\{\pm\frac{A_3\sqrt{(n+1)(n+2)(3n+2)}}{2\sqrt{-A_6}k(n(n+2)+2)}\operatorname{sech}^2\left[\frac{n}{2}\sqrt{-\frac{A_3}{n^2+2n+2}}(x-vt)\right]\right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\left\{\frac{c_2\delta_9}{2c_1\delta_2}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}, \tag{64}$$

and

$$q(x,t)=\left\{\mp\frac{A_3\sqrt{(n+1)(n+2)(3n+2)}}{2\sqrt{-A_6}k(n(n+2)+2)}\operatorname{csch}^2\left[\frac{n}{2}\sqrt{-\frac{A_3}{n^2+2n+2}}(x-vt)\right]\right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\left\{\frac{c_2\delta_9}{2c_1\delta_2}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}. \quad (65)$$

The obtained solutions are singular and bright solitons, with $A_6 < 0$ and $A_3 < 0$.

Case 3: $\tau_1 = \tau_3 = 0$.

$$\alpha_0=0,\ \alpha_1=\beta_1=\beta_2=0,\ \tau_0=0,\ \tau_2=-\frac{A_3n^2}{4k^2\left(n^2+2n+2\right)},$$

$$\tau_4=\pm\frac{\alpha_2\sqrt{A_6}n^2}{2\sqrt{-k^2\left(3n^3+11n^2+12n+4\right)}},\ A_1=0,\ A_8=\frac{A_3^2(n+1)^2}{k^2\left(n^2+2n+2\right)^2}. \quad (66)$$

For this case, the solution of (1) reads

$$q(x,t)=\left\{\mp\frac{92\sqrt{-k^2\left(3n^3+11n^2+12n+4\right)}\left(\wp'(\xi;g_2,g_3)\right)^2}{n^2\sqrt{A_6}\left[6\wp(\xi;g_2,g_3)-\frac{A_3n^2}{4k^2\left(n^2+2n+2\right)}\right]^2}\right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\left\{\frac{c_2\delta_9}{2c_1\delta_2}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}, \quad (67)$$

where

$$g_2=\frac{\tau_2^2}{12},\ g_3=-\frac{\tau_2^3}{216}. \quad (68)$$

The Weierstrass' elliptic function solutions (67) with its restricted invariants (68) can be converted to a singular soliton solution

$$q(x,t)=\left\{\mp\frac{23A_3\sqrt{-k^2(n+1)(n+2)(3n+2)}}{9\sqrt{A_6}k^2(n(n+2)+2)}\text{csch}^2\left[\frac{n}{2}\sqrt{-\frac{A_3}{n^2+2n+2}}(x-vt)\right]\right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\left\{\frac{c_2\delta_9}{2c_1\delta_2}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}. \tag{69}$$

Case 4: $\tau_0 = \tau_1 = 0$.

$$\alpha_0=0,\ \alpha_1=\beta_1=\beta_2=0,\ \tau_3=0,\ \tau_2=-\frac{A_3n^2}{4k^2\left(n^2+2n+2\right)},$$

$$\tau_4=\pm\frac{\alpha_2\sqrt{A_6}n^2}{2\sqrt{-k^2\left(3n^3+11n^2+12n+4\right)}},\ A_1=0,\ A_8=\frac{A_3^2(n+1)^2}{k^2\left(n^2+2n+2\right)^2}. \tag{70}$$

For this case, the solution of (1) reads

$$q(x,t)=\left\{\pm\frac{A_3\sqrt{(n+1)(n+2)(3n+2)}}{2\sqrt{-A_6}k(n(n+2)+2)}\text{sech}^2\left[\frac{n}{2}\sqrt{-\frac{A_3}{n^2+2n+2}}(x-vt)\right]\right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\left\{\frac{c_2\delta_9}{2c_1\delta_2}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}, \tag{71}$$

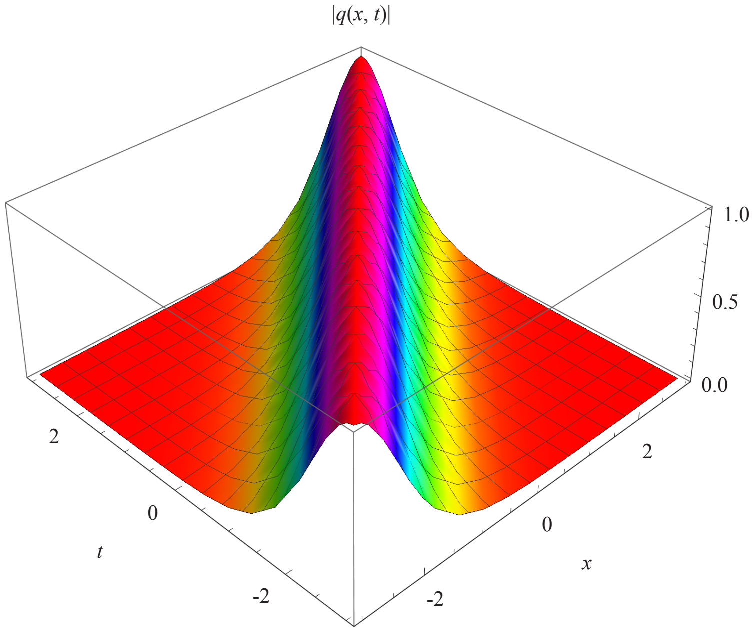
and

$$q(x,t)=\left\{\mp\frac{A_3\sqrt{(n+1)(n+2)(3n+2)}}{2\sqrt{-A_6}k(n(n+2)+2)}\text{csch}^2\left[\frac{n}{2}\sqrt{-\frac{A_3}{n^2+2n+2}}(x-vt)\right]\right\}^{\frac{1}{n}}$$

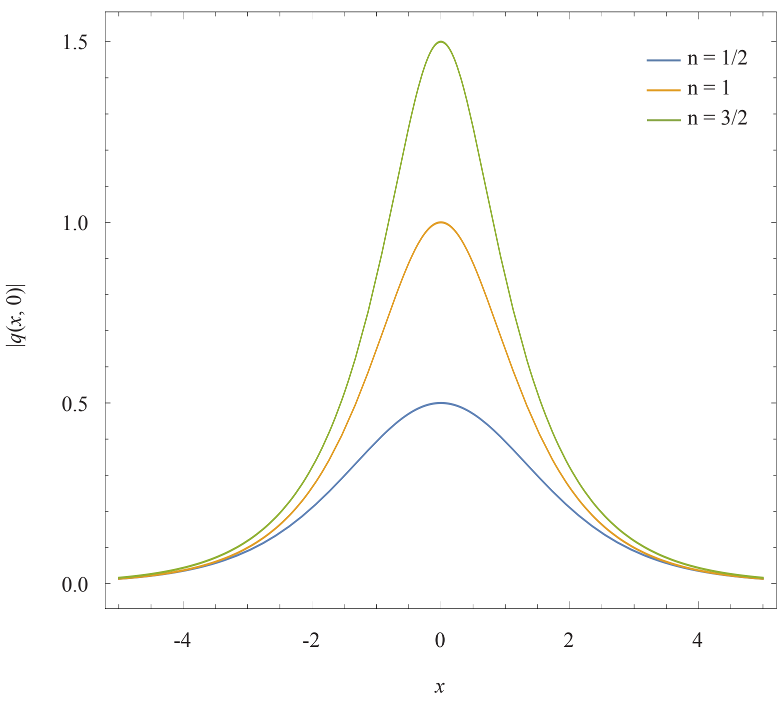
$$\times e^{i\left(-\left\{\frac{c_2\delta_9}{2c_1\delta_2}\right\}x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right)}. \tag{72}$$

These solutions are singular and bright solitons with $A_6 < 0$ and $A_3 < 0$.

The 2D and surface plots of a bright optical soliton (71) are displayed in Figure 1, with parameter values set as $a = 1$, $b = 2$, $c = 1$, $k = 1$, $c_1 = 1$, $c_2 = 1$, $\delta_1 = 1$, $\delta_2 = 1$, $\delta_4 = 1$, $\delta_7 = 1$, $\delta_8 = 1$, $\delta_9 = 1$, $\alpha = 1$, $\omega = 1$, $\sigma = 1$ and $\lambda = 1$.



(a) Surface plot



(b) 2D plot

Figure 1. Examination of the unique characteristics displayed by a bright optical soliton

4. Conclusion

This study addressed the concatenation model, considering both deterministic and stochastic perturbation terms introduced by white noise. Singular and bright solitons were recovered using two integration approaches: enhanced Kudryashov's method and the extended auxiliary equation method. The solutions, presented in terms of Weierstrass' elliptic functions, revealed two key observations. Firstly, only bright and singular soliton solutions were successfully obtained through both integration approaches. Dark optical solitons remained unrecoverable, primarily due to the requirement that the power-law nonlinearity must collapse to Kerr law of SPM for the concatenation model to support dark solitons [7]. Secondly, the influence of white noise was found to be confined to the phase components of both soliton types.

While the manuscript's results are presented, they set the stage for additional studies with the model. Future investigations will include a study of the model with differential group delay. Following this, the model will be explored in the context of dispersion-flattened fibers, incorporating the analysis of white noise effects using the integration schemes adopted in this study, as well as additional integration methodologies. Additionally, the model is yet to be examined for phenomena such as gap solitons, optical metamaterials, optical couplers, PCFs, and various other optoelectronic devices. The diverse outcomes from these upcoming studies will be reported comprehensively, aligning them with pre-existing findings [10-20].

Conflict of interest

The authors declare no competing financial interest.

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