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Research Article

Evaluation of the Komlos Conjecture Using Multi-Objective Optimization

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Abstract: The Komlos conjecture, which explores the existence of a constant upper bound in the realm of n-dimensional vectors, specifically addresses the function $K(n)$. This function, intricately defined as

$$K(n) = \max_{\left\{\vec{V_1}, \ldots, \vec{V_n}\right\} \in \left\{\vec{V} \in \mathbb{R}^n : \left\|\vec{V}\right\|_2 \le 1\right\}^n} \left(\min_{\{\varepsilon_1, \ldots, \varepsilon_n\} \in \{-1, 1\}^n} \left\| \sum_{i=1}^{n} \varepsilon_i \vec{V_i} \right\|_\infty \right),$$

encapsulates the maximal discrepancy within a set of n -dimensional vectors. This paper endeavors to unravel the mysteries of $K(n)$, by meticulously evaluating its behavior for lower dimensions $n \le 5$. Our findings revealed through systematic exploration, showcase intriguing values such as $(2) = \sqrt{2}$, $K(3) = \frac{\sqrt{2}+\sqrt{11}}{3}$, $K(4) = \sqrt{3}$, and $K(5) = \frac{4+\sqrt{142}}{9}$, shedding light on the intricate relationships within n-dimensional spaces. Venturing into higher dimensions, we introduce the function $f(n) = \sqrt{n - \lceil log_2(2^{n-1}/n) \rceil}$ as a potentially robust lower bound for $K(n)$. This innovative approach aims to provide a deeper understanding of the limiting behavior of $K(n)$ as the dimensionality expands. As a culmination of our comprehensive analysis, we arrive at a significant revelation the Komlos conjecture stands refuted. This conclusion stems from the suspected divergence of $K(n)$, as n approaches infinity, as evidenced by $\lim_{n\to\infty} K(n) \ge \lim_{n\to\infty} \sqrt{\log(n) - 1} = +\infty$. This seminal result challenges established notions and added a valuable dimension to the ongoing discourse in optimization and discrepancy theory.

***Keywords*:** Komlos Conjecture, optimization, discrepancy theory

MSC: 65L05, 34K06, 34K28

1. Introduction

The Komlos Conjecture stands at the intersection of convex geometry and norm inequalities, posing the intriguing question of whether a universal constant K exists to govern the configurations of sets of vectors in Euclidean space $\mathbb{R}^n$.

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This mathematical challenge delves into the complexities of high-dimensional spaces, where vectors navigate within convex bodies, and norms dictate their permissible arrangements. Beyond its theoretical implications, the conjecture carries practical significance by influencing algorithm design and computational efficiency. Embedded within the broader context of discrepancy theory, the Komlos Conjecture is not merely a mathematical puzzle but a gateway to understanding fundamental principles governing vectors and their potential applications in computational realms. Researchers engage in this intellectual pursuit, seeking both abstract beauty and practical insights as they unravel the mysteries posed by the conjecture.

J. Komlos has made the following conjecture: For a given dimension n, let $K(n)$ denote the minimum value such that: for any set of n Vectors $\overrightarrow{V_1} \ldots, \overrightarrow{V_n} \in \mathbb{R}^n$ with $\left\|\overrightarrow{V}_i\right\|_2 \leq 1$, there exists weights $\varepsilon_i = +1$ or -1 such that

$$\left\|\sum_{i=1}^{n} \varepsilon_i \overrightarrow{V}_i\right\|_\infty \leq K(n). \tag{1}$$

Kolmos has conjected the existence of a universal constant K such that $K(n) \leq k$ for any dimension n. The l_2 and l_∞ norms in $\mathbb{R}^n$ are denoted by $\|.\|_2$ and $\|.\|_\infty$ respectively.

This conjecture was referred to by Joel Spencer [1] in 1994, where he linked the Komlos Conjecture to Spencer's famous Six Standard Deviation in 1985, see [2].

The main nontrivial result known, which is due to Joel Spencer [2], is that if $k \leq n$ then $\left\|\sum_{i=1}^{k} \varepsilon_i \overrightarrow{V}_i\right\|_\infty = O(\log(n))$. The main result of D. Hajela [3] was very close to disproving the Komlos conjecture, where precisely he has proved the following theorem:

Theorem 1 Let $f(n)$ be a function that goes to infinity when n goes to infinity with $f(n) = O(n)$ and let $0 < \lambda < 1/2$. Then form $n \geq n_0$ (where n_0 depends only on n and λ) and any $A \subseteq \{1, -1\}^n$ with $|A| \leq 2^{n/f(n)}$, there are orthogonal vectors $x_1, \ldots, x_n$ in R^n, $\|x_i\|_2 \leq 1$ for all $1 \leq i \leq n$, and such that

$$\|\varepsilon_1 x_1 + \cdots + \varepsilon_n x_n\|_\infty \geq \exp\left(\frac{\lambda \log\log f(n)}{\log\log\log f(n)}\right) \tag{2}$$

for all $(\varepsilon_1, \ldots, \varepsilon_n) \in A$.

The previous theorem disproves the conjecture of Komlos over the set $A \subseteq \{1, -1\}^n$ where $|A| \leq 2^{n/f(n)}$. The proof of Theorem 1 is based on certain inequalities that arise in the geometry of convex bodies [4–6].

Komlos Conjecture is also related to discrepancy theory, a paper by Becka and Sós [7], where it states that for a global constant K and for any $m \times n$ matrix A, whose columns are inside a unit ball, there exists a vector $X \in \{-1, +1\}^n$ such that $\|AX\|_\infty \leq K$.

The best progress in proving the Komlos conjecture is a result given by Banaszczyk [8], who proved the bound:

$$\min_{x \in \{-1, +1\}^n} \|AX\| \leq K\sqrt{\log(n)} \tag{3}$$

for a global constant.

This is the best-known bound for the Becka-Fiala conjecture as well [9].

Discrepancy is a challenging problem that has applications in geometry, data analysis, and complexity theory. The books Beck and Fiala [9], Matousek [10], and Chazelle [11] provide references for a wide array of applications.

For the lower dimension, the idea is to find a hypercube of a minimum size of $2K$, where all vertices are formed by different combinations of the weights, $\sum_{i=1}^{n} \varepsilon_i \overrightarrow{V}_i$, should be all inside the hypercube. Also, it is not hard to show that $\sqrt{n}$

is an upper bound for the function $K(n)$. The proof can be carried out by funding to find particular weights ε_i^* such that all vectors $\overrightarrow{V_1} \ldots, \ \overrightarrow{V_n} \ \in \mathbb{R}^n$ with $\left\| \overrightarrow{V_i} \ \right\|_2 \leq 1$, so

$$\left\| \sum_{i=1}^{n} \varepsilon_i^* \overrightarrow{V}_i \right\|_\infty \leq K(n) \leq \sqrt{n} \tag{4}$$

The below bullets are the details of the proof:

• We will prove first that $\left\| \sum_{i=1}^{n} \ \varepsilon_i^* \ \overrightarrow{V}_i \right\|_2 \leq \sqrt{n}$, which is a sufficient condition to prove that $\left\| \sum_{i=1}^{n} \varepsilon_i^* \overrightarrow{V}_i \right\|_\infty \leq \sqrt{n}$.

• For dimension 2: the upper bound can be evaluated by using the cosine rule as follows:

$$\begin{aligned} \left\| \varepsilon_1^* \overrightarrow{V}_1 + \varepsilon_2^* \overrightarrow{V}_2 \right\|_2 &= \left\| \overrightarrow{V}_1 + \frac{\varepsilon_2^*}{\varepsilon_1^*} \overrightarrow{V}_2 \right\|_2 \\ &= \sqrt{\left\| \overrightarrow{V}_1 \right\|_2^2 + \left\| \overrightarrow{V}_2 \right\|_2^2 - 2 \left\| \overrightarrow{V}_1 \right\|_2 \left\| \overrightarrow{V}_2 \right\|_2 \cos(\overrightarrow{V}_1, \ \overrightarrow{V}_2)} \\ &\leq \sqrt{2} \end{aligned} \tag{5}$$

where the weight $\frac{\varepsilon_2^*}{\varepsilon_1^*}$ is chosen to have a cos $\left(\overrightarrow{V}_1, \ \overrightarrow{V}_2 \right) \geq 0$.

• If we suppose that $\left\| \sum_{i=1}^{n-1} \ \varepsilon_i^* \ \overrightarrow{V}_i \right\|_2 \leq \sqrt{n-1}$, we need to prove that $\left\| \sum_{i=1}^{n-1} \ \varepsilon_i^* \ \overrightarrow{V}_i + \varepsilon_n^* \ \overrightarrow{V}_n \right\|_2 \leq \sqrt{n}$. Again, by cosine rule, we can write the following:

$$\begin{aligned} \left\| \sum_{i=1}^{n-1} \varepsilon_i^* \overrightarrow{V}_i + \varepsilon_n^* \overrightarrow{V}_n \right\|_2 &\leq \sqrt{\left\| \overrightarrow{V}_n \right\|_2^2 + \left\| \sum_{i=1}^{n-1} \varepsilon_i^* \overrightarrow{V}_i \right\|_2^2} \\ &\leq \sqrt{1 + (n-1)} \\ &\leq \sqrt{n} \end{aligned} \tag{6}$$

where the weight ε_n^* is chosen to have a cos $\left(\sum_{i=1}^{n-1} \ \varepsilon_i^* \ \overrightarrow{V}_i, \ \varepsilon_n^* \ \overrightarrow{V}_n \right) \geq 0$.

• From the principle of induction proof, it is concluded that all vectors $\overrightarrow{V_1}, \ldots, \ \overrightarrow{V_n} \ \in \mathbb{R}^n$ with $\left\| \overrightarrow{V_i} \ \right\|_2 \leq 1$, the weights can find ε_i^* such that:

$$\left\| \sum_{i=1}^{n} \varepsilon_i^* \overrightarrow{V}_i \right\|_2 \leq \sqrt{n} \tag{7}$$

Since $\left\| \sum_{i=1}^{n} \ \varepsilon_i^* \ \overrightarrow{V}_i \right\|_\infty \leq \left\| \sum_{i=1}^{n} \ \varepsilon_i^* \ \overrightarrow{V}_i \right\|_2$, we can conclude that the function $K(n)$ has an upper bound of order $\sqrt{n}$.

We can extend the Komlos conjecture statement to the below lemma, where it summarizes very interesting properties related to special vectors, $\overrightarrow{V_i}^*$, $i = 1, \ldots, n,$ that cannot cancel each other further than $K(n)$.

Lemma 1 Let C^n be a set of vectors in $\mathbb{R}^n$ have l_2 norm at most 1, and we denote by V^* as a set of vectors in $\mathbb{R}^n$ that satisfies the following equation:

$$V^* = \left\{\overrightarrow{V_1}^*, \ldots, \overrightarrow{V_n}^*\right\} = \underset{\overrightarrow{V}_i \in C^n}{\operatorname{argmax}} \min_{\varepsilon_i} \left\| \sum_{i=1}^{n} \varepsilon_i \overrightarrow{V_i}^* \right\|_\infty . \tag{8}$$

The set V^* satisfies the below properties:

i. For any vector $\overrightarrow{V_i}^*$ in V^* has l_2 norm equal to 1, $\left\|\overrightarrow{V_i}^*\right\|_2 = 1$.

ii. All the vertices have the same distance l_∞, i.e.,

$$\underset{\varepsilon_i \in \{-1, +1\}}{\text{Min}} \left\| \sum_{i=1}^{n} \varepsilon_i \overrightarrow{V_i}^* \right\|_\infty = \underset{\varepsilon_i \in \{-1, +1\}}{\text{Max}} \left\| \sum \varepsilon_i \overrightarrow{V_i}^* \right\|_\infty .$$

iii. $K(n)$ is a strictly increasing sequence, i.e., for all integers $m > n$ implies that $K(n) < K(m)$.

We encourage researchers to explore the proof of the preceding lemma. This proof aims to establish that $K(n)$ exhibits an asymptotic behavior proportional to $\sqrt{log_2(n)}$. This upcoming contribution holds the promise of shedding light on the conjecture, advancing our understanding of the relationship between the universal constant K and the logarithmic dimensionality of the vectors in $\mathbb{R}^n$.

The following sections are consecrated to evaluate the function K for a different dimension; the exact value of K will be calculated for a dimension less or equal to 5, and a lower bound will evaluated for any dimension n.

2. Evaluation of $K(2)$

It is obvious that the constant $K(1)$ for dimension one is equal to one, and it is quite easy to calculate $K(2)$ by using some basic rules in geometry.

To find the value of $K(2)$, it will be useful to analyze the parallelogram formed by four vertices centered at the origin, resulting from the four combinations $\mp\overrightarrow{V}_1 \mp \overrightarrow{V}_2$ (see Figure 1).

By using the cosine rule, we can find the length of the big and the small diagonals, respectively, as follows:

$$\begin{cases} L^2 = \left\|\overrightarrow{V}_1\right\|_2 + \left\|\overrightarrow{V}_2\right\|_2 + 2\left\|\overrightarrow{V}_1\right\|_2\left\|\overrightarrow{V}_2\right\|_2 \cos(\theta) \\ \\ l^2 = \left\|\overrightarrow{V}_1\right\|_2 + \left\|\overrightarrow{V}_2\right\|_2 - 2\left\|\overrightarrow{V}_1\right\|_2\left\|\overrightarrow{V}_2\right\|_2 \cos(\theta) \end{cases} \tag{9}$$

where θ is the acute angle between the two vectors $\overrightarrow{V}_1$ and $\overrightarrow{V}_2$.

We can notice that the small diagonal l has $\sqrt{2}$ as an upper bound, i.e.,

$$\sqrt{\left\|\overrightarrow{V}_1\right\|_2 + \left\|\overrightarrow{V}_2\right\|_2 - 2\left\|\overrightarrow{V}_1\right\|_2\left\|\overrightarrow{V}_2\right\|_2 \cos(\theta)} \leq \sqrt{\left\|\overrightarrow{V}_1\right\|_2 + \left\|\overrightarrow{V}_2\right\|_2} \leq \sqrt{2}. \tag{10}$$

As it was mentioned before, the two weights, ε_1 and ε_2, can be chosen in such a way the length of $\varepsilon_1 \overrightarrow{V}_1 + \varepsilon_2 \overrightarrow{V}_2$ is smaller than the length of diagonal l, which implies that for all vectors $\overrightarrow{V}_i$ inside the circle of center (0,0) and Radius =1, we can find ε_1 and ε_2 such that $\left\| \varepsilon_1 \overrightarrow{V}_1 + \varepsilon_2 \overrightarrow{V}_2 \right\|_\infty \le K(n) \le \sqrt{2}$.

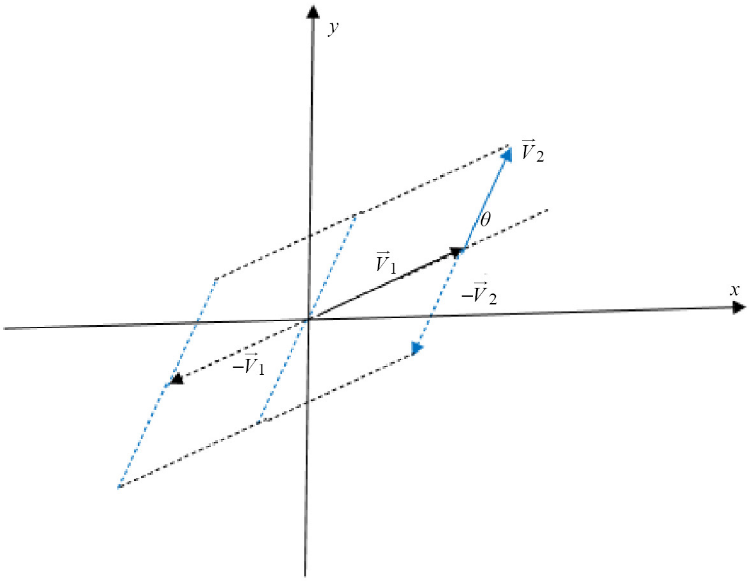

Figure 1. A parallelogram formed by four vertices, $\overrightarrow{V}_1 + \overrightarrow{V}_2$, $\overrightarrow{V}_1 - \overrightarrow{V}_2$, $-\overrightarrow{V}_1 + \overrightarrow{V}_2$, $-\overrightarrow{V}_1 - \overrightarrow{V}_2$

In the different methods, the proof of $K(2) \le \sqrt{2}$ can be carried out by using the technique of proof by contradiction. Let's assume the case where the vertices A, B, C, and D are located outside of the red square of side $2\sqrt{2}$ as shown in Figure 2.

The possibility of having all vertices, $\mp\overrightarrow{V}_1 \mp \overrightarrow{V}_2$, outside the red square in Figure 2 is impossible! Because it contradicts the fact that a small diagonal length is less or equal to $\sqrt{2}$.

From previous proof, we can conclude that $K(2) \le \sqrt{2}$, and it is enough to find a particular case where $\min_{\varepsilon_i} \left\| \varepsilon_1 \overrightarrow{V}_1 + \varepsilon_2 \overrightarrow{V}_2 \right\|_\infty = \sqrt{2}$ to prove that $K(2) = \sqrt{2}$.

Let's consider the case where $\overrightarrow{V}_1 = \frac{1}{\sqrt{2}}\binom{+1}{+1}$ and $\overrightarrow{V}_2 = \frac{1}{\sqrt{2}}\binom{-1}{+1}$, so for any possible values of ε_1 and ε_2, we can calculate the value $\min_{\varepsilon_i} \left\| \varepsilon_1 \overrightarrow{V}_1 + \varepsilon_2 \overrightarrow{V}_2 \right\|_\infty$ as follows:

$$\max\left(\frac{|\varepsilon_1+\varepsilon|}{\sqrt{2}}, \frac{|\varepsilon_1-\varepsilon_2|}{\sqrt{2}}\right) = \max\left(\frac{|1+\varepsilon_2/\varepsilon_1|}{\sqrt{2}}, \frac{|1-\varepsilon_2/\varepsilon_1|}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\max\left(1+|\varepsilon_2/\varepsilon_1|, 1-|\varepsilon_2/\varepsilon_1|\right) = \frac{2}{\sqrt{2}}. \tag{11}$$

Therefore, we can conclude that

$$K(2) = \sqrt{2}. \tag{12}$$

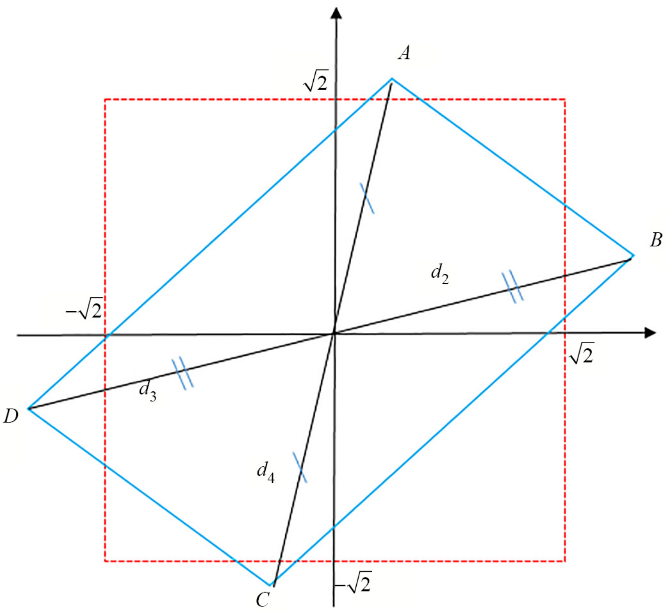

Figure 2. ABCD is a parallelogram with vertices located outside of a square whose side is $\sqrt{2}$ has $min_i(d_i) \geq \sqrt{2}$

3. Evaluation of $K(3)$

Given the vector space $\mathbb{R}^3$, the span of the set S of finite vectors is defined as the set of all linear combinations of the vectors in S, noted as follows:

$$\text{Span}(S) = \left\{ \sum_{i=1}^{k} \alpha_i \vec{V}_i;\ k \in \mathbb{N},\ \vec{V}_i \in S,\ \alpha_i \in \mathbb{R} \right\}. \tag{13}$$

The calculation of $K(3)$ will be split into several cases related to different configurations of the three vectors $\vec{V}_1$, $\vec{V}_2$, and $\vec{V}_3$ in $\mathbb{R}^3$.

Case 1 $\vec{V}_3 \perp Span\left(\vec{V}_2,\ \vec{V}_1\right)$ and $Span\left(\vec{V}_2,\ \vec{V}_1\right) = x - y$ plane.

As the vector $\vec{V}_3$ is parallel to y-axe, then without losing generality, we can write the following:

$$\min_{\varepsilon_i} \left\| \vec{\varepsilon}_3 \vec{V}_3 + \vec{\varepsilon}_2 \vec{V}_2 + \vec{\varepsilon}_1 \vec{V}_1 \right\|_\infty = \min_{\varepsilon_i} \left\| \vec{V}_3 + \vec{\varepsilon}_2 \vec{V}_2 + \vec{\varepsilon}_1 \vec{V}_1 \right\|_\infty, \tag{14}$$

by consequence,

$$\min_{\varepsilon_i} \left\| \vec{V}_3 + \vec{\varepsilon}_2 \vec{V}_2 + \vec{\varepsilon}_1 \vec{V}_1 \right\|_\infty = \max \left\{ \left\| \vec{V}_3 \right\|_2,\ \min_{\varepsilon_i} \left\| \vec{\varepsilon}_2 \vec{V}_2 + \vec{\varepsilon}_1 \vec{V}_1 \right\|_\infty \right\}. \tag{15}$$

From the previous section, we know that the constant $K(2) = \sqrt{2}$ and from the fact that $\overrightarrow{\varepsilon}_2\overrightarrow{V}_2 + \overrightarrow{\varepsilon}_1\overrightarrow{V}_1 \in x-y$ plane, we have

$$max\left\{\left\|\overrightarrow{V}_3\right\|_2, \min_{\varepsilon_i}\left\|\overrightarrow{\varepsilon}_2\overrightarrow{V}_2 + \overrightarrow{\varepsilon}_1\overrightarrow{V}_1\right\|_\infty\right\} \leq max\left\{\left\|\overrightarrow{V}_3\right\|_2, \sqrt{2}\right\} \leq \sqrt{2}.$$

By considering $\overrightarrow{V}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} +1 \\ +1 \\ 0 \end{pmatrix}$, $\overrightarrow{V}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ +1 \\ 0 \end{pmatrix}$, and $\overrightarrow{V}_3 = \begin{pmatrix} 0 \\ 0 \\ +1 \end{pmatrix}$, $K(3)$ can be calculated as follows:

$$\left\|\sum_{i=1}^3 \varepsilon_i\overrightarrow{V}_i\right\|_\infty = \max\left(\frac{|\varepsilon_1+\varepsilon_2|}{\sqrt{2}}, \frac{|\varepsilon_1-\varepsilon_2|}{\sqrt{2}}, |\varepsilon_3|\right) = \max\left(\frac{|1+\varepsilon_2|}{\sqrt{2}}, \frac{|1-\varepsilon_2|}{\sqrt{2}}, 1\right) = \sqrt{2}. \tag{16}$$

Therefore, under the case 1, the value $K(3)$ is equal to $\sqrt{2}$.

Case 2 *Span* $\left(\overrightarrow{V}_1, \overrightarrow{V}_2\right) = x-y$ plane.

We split the vector $\overrightarrow{V}_3$ as follows:

$\overrightarrow{V}_3 = \overrightarrow{V}_{31} \bigoplus \overrightarrow{V}_{32}$, where $\overrightarrow{V}_{31} \perp$ XY-plane. Without losing generality, the value of the weight ε_3 can be fixed to 1 in our calculation.

Therefore, for all vectors $\overrightarrow{V_1}$, $\overrightarrow{V_2}$, $\overrightarrow{V_3}$ $\in \mathbb{R}^3$ with $\left\|\overrightarrow{V_i}\right\|_2 \leq 1$

$$\min_{\varepsilon_i}\left\|\sum_{i=1}^3 \varepsilon_i\overrightarrow{V}_i\right\|_\infty = \min_{\varepsilon_i}\left\|\overrightarrow{V}_3 + \sum_{i=1}^2 \varepsilon_i\overrightarrow{V}_i\right\|_\infty = \max\left\{\left\|\overrightarrow{V}_{31}\right\|_2, \min_{\varepsilon_i}\left\|\overrightarrow{V}_{32} + \sum_{i=1}^2 \varepsilon_i\overrightarrow{V}_i\right\|_\infty\right\}. \tag{17}$$

From the previous equation, we can see that the calculation is moved from dimension 3 to dimension 2 by just calculating the following:

For all vectors $\overrightarrow{V_1}$, $\overrightarrow{V_2}$, $\overrightarrow{V_{32}} \in \mathbb{R}^2$ with $\left\|\overrightarrow{V_i}\right\|_2 \leq 1$, the below maximum is needed to be calculated

$$\max_{\overrightarrow{V_i} \in \mathbb{R}^3 \,:\, \left\|\overrightarrow{V_i}\right\|_2 \leq 1} \quad \min_{\varepsilon_i}\left\|\overrightarrow{V}_{32} + \sum_{i=1}^2 \varepsilon_i\overrightarrow{V}_i\right\|_\infty,$$

where $\overrightarrow{V}_{32} = \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix}$, and without losing generality, the two assumptions $\alpha^2+\beta^2 \leq 1$ and $0 \leq \alpha \leq \beta \leq 1$ can be added.

To evaluate the value $K(3)$, the question about the possibility of having all the vertices, $\overrightarrow{V}_{32} \pm \overrightarrow{V}_2 \pm \overrightarrow{V}_1$, outside the square of side $2\sqrt{2}$, as shown in Figure 3, needs to be checked.

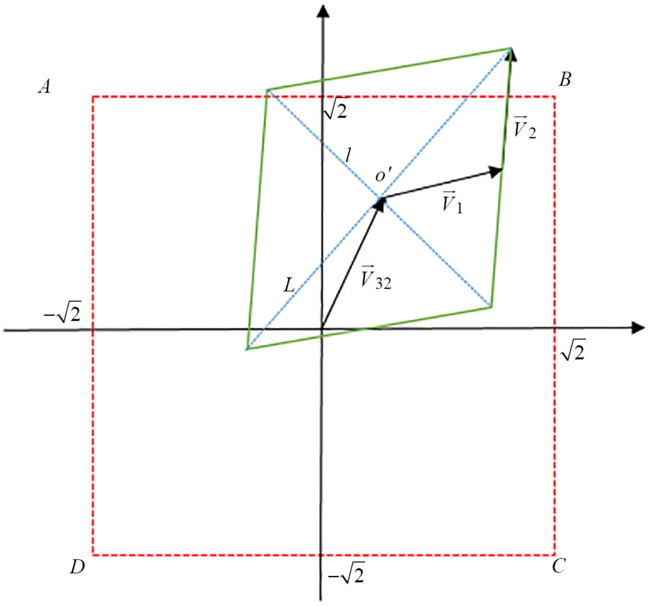

Figure 3. Without losing generality, the vector $\vec{V}_{32}$ can be considered with a slope bigger than one, $m \geq 1$. The two distances l and L are the lengths of the small and the big diagonal, respectively

From Figure 4, the small diagonal, l, of the parallelogram centered at the point O' is at most equal to $\sqrt{2}$; consequently, we must focus only on the green area, highlighted in Figure 5, the possible location of two opposite vertices that form the two small diagonal l.

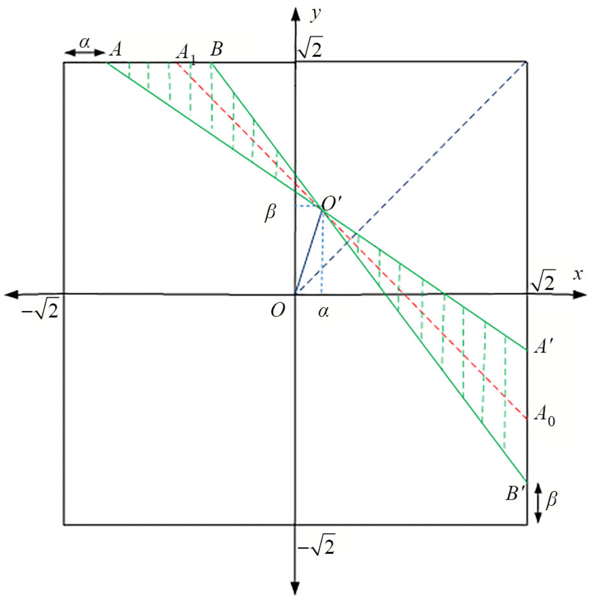

Figure 4. The Green Area is the only possible location of the vertices, $\overrightarrow{V_{32}} \mp \overrightarrow{V_1} \mp \overrightarrow{V_2}$, to have maybe $\left\| \overrightarrow{V_{32}} \mp \overrightarrow{V_1} \mp \overrightarrow{V_2} \right\|_\infty > \sqrt{2}$

The distance between the point O' and the midpoint of any two adjacent vertices is equal to either $\left\| \vec{V}_1 \right\|$ or $\left\| \vec{V}_2 \right\|$, which implies the impossibility of having, on one side of the square, two vertices outside of the square, refer to Figure 4.

This impossibility can be proved by highlighting the fact that the distance between any point inside the area S_1 and any point inside the area S_2 is bigger than or equal to 1, see Figure 5.

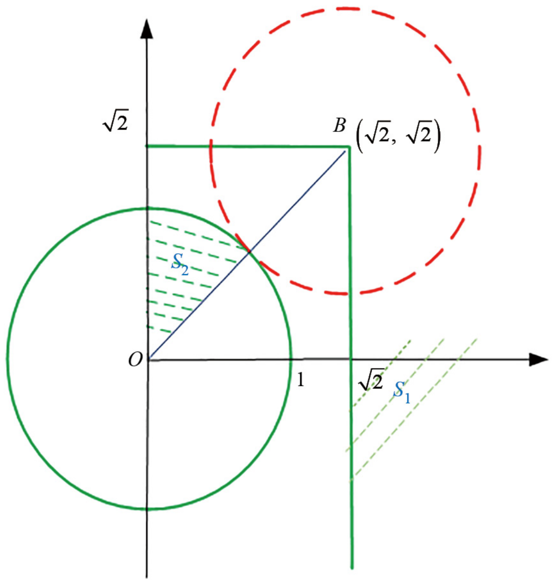

Figure 5. Distance between any point in the area S_1 and any point in the area S_2 has a minimum distance of 1, $S_1 = \left\{x \geq \sqrt{2} \text{ and } y \leq 0\right\}$ and $S_2 = \left\{x^2 + y^2 \leq 1 \text{ and } 0 \leq x \leq y \leq 1\right\}$, where the red and green circles have a radius of 1 and are centered at the point B and O respectively

Therefore, under the case 2, the constant $K(3)$ is upper bounded by $\sqrt{2}$.

To conclude that $K(3) = \sqrt{2}$, it is enough to check the function $K(3)$ for $\overrightarrow{V}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} +1 \\ +1 \\ 0 \end{pmatrix}$, $\overrightarrow{V}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ +1 \\ 0 \end{pmatrix}$, and $\overrightarrow{V}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ +1 \end{pmatrix}$, where

$$\min_{\varepsilon_i}\left\|\sum_{i=1}^{3} \varepsilon_i \overrightarrow{V}_i\right\|_\infty = \sqrt{2}. \tag{18}$$

Case 3 General case.

By symmetry, without losing generality, we can consider the weight $\varepsilon_3 = 1$ in our calculations, as it is proven below:

$$\min_{\varepsilon_i}\left\|\sum_{i=1}^{3} \varepsilon_i \overrightarrow{V}_i\right\|_\infty = \min_{\varepsilon_i}\left\|\varepsilon_3 \sum_{i=1}^{3} \frac{\varepsilon_i}{\varepsilon_3} \overrightarrow{V}_i\right\|_\infty = \min_{\varepsilon_i}\left\|\overrightarrow{V}_3 + \sum_{i=1}^{2} \frac{\varepsilon_i}{\varepsilon_3} \overrightarrow{V}_i\right\|_\infty = \min_{\varepsilon_j}\left\|\overrightarrow{V}_3 + \sum_{j=1}^{2} \varepsilon_j \overrightarrow{V}_i\right\|_\infty. \tag{19}$$

The vector $\sum_{i=1}^{2} \varepsilon_i \overrightarrow{V}_i$ will be evaluated over two perpendicular spaces, x-y plane and z -axe, and a link between the two spaces will be found to maximize the $\left\|\sum_{i=1}^{3} \varepsilon_i \overrightarrow{V}_i\right\|_\infty$.

The projection of the vector $\overrightarrow{V}_i$ over the space x-y plane, $Proj_{XY-plan}\left(\overrightarrow{V}_i\right)$, is denoted by vector $\overrightarrow{U}_i$.

From case 2, we have proven that it is not possible to have all vertices, $\overrightarrow{U}_3 \pm \overrightarrow{U}_2 \pm \overrightarrow{U}_1$, outside the square of side $2\sqrt{2}$ centered at the origin. Does an important question arise about the possibility of increasing the l_∞ norm beyond $\sqrt{2}$ for two vertices and compensate for the l_∞ norms of the two other vertices by l_∞ norm over z-axe?

To answer the previous question, we need to find z-coordinates of three vectors $\overrightarrow{V}_i$ that satisfy the following statement:

For each possible weight's vector $(1, \varepsilon_1, \varepsilon_2)$ where $\left\|\overrightarrow{U}_3 + \sum_{j=1}^{2} \varepsilon_j \overrightarrow{U}_i\right\|_\infty < \sqrt{2}$ then

$$\left\|Z_3 + \sum_{j=1}^{2} \varepsilon_j Z_i\right\|_\infty = \left|Z_3 + \sum_{j=1}^{2} \varepsilon_j Z_i\right| > \sqrt{2}. \tag{20}$$

To summarize the above idea, we create an example of vectors $\overrightarrow{V}_i$, where $\left\|\sum_{i=1}^{3} \varepsilon_i \overrightarrow{V}_i\right\|_\infty > \sqrt{2}$, as follows:

$$\overrightarrow{V}_1 = \begin{pmatrix} -x_1 \\ y_1 \\ z_1 \end{pmatrix} \tag{21}$$

$$\overrightarrow{V}_2 = \begin{pmatrix} -x_2 \\ y_2 \\ -z_2 \end{pmatrix} \tag{22}$$

$$\overrightarrow{V}_3 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}, \tag{23}$$

where x_i, y_i, z_i are all non-negative values with $\left\|\overrightarrow{V}_i\right\|_2 = \sqrt{x_i^2 + y_i^2 + z_i^2} \leq 1$,

We assume the following equations:

$$\text{if} \begin{cases} \left\|\overrightarrow{V}_3 - \overrightarrow{V}_2 - \overrightarrow{V}_1\right\|_\infty = x_1 + x_2 + x_3 = K_1 \\ \left\|\overrightarrow{V}_3 + \overrightarrow{V}_2 + \overrightarrow{V}_1\right\|_\infty = y_1 + y_2 + y_3 = K_2 \end{cases} \tag{24}$$

$$\text{then} \begin{cases} \left\|\overrightarrow{V}_3 - \overrightarrow{V}_2 + \overrightarrow{V}_1\right\|_\infty = z_1 + z_2 + z_3 = K_3 \\ \left\|\overrightarrow{V}_3 + \overrightarrow{V}_2 - \overrightarrow{V}_1\right\|_\infty = |z_3 - z_1 - z_2| = K_4 \end{cases} \tag{25}$$

By symmetry, we can consider the constants $K_1 = K_2$ and $K_3 = K_4$.

Then the system that needs to be solved is summarized by the following equations:

$$\begin{cases} x_1 + x_2 + x_3 = K_1 \\ y_1 + y_2 + y_3 = K_1 \\ z_1 + z_2 + z_3 = K_3 \\ z_1 + z_2 - z_3 = K_3 \end{cases} \tag{26}$$

From the last two equations, we conclude that:

$$z_3 = 0. \tag{27}$$

By symmetry, we can conclude that:

$$x_3 = y_3. \tag{28}$$

Since $\left\|\overrightarrow{V}_3\right\| \leq 1$, it is convenient to increase x_3 & y_3 as much as we can to maximize the value of K_1, where it can be found when the coordinates of $\overrightarrow{V}_3$ are:

$$x_3 = y_3 = \tfrac{1}{\sqrt{2}}. \tag{29}$$

Therefore, the system will be simplified again as follows:

$$\begin{cases} x_1 + x_2 = K_1 - \frac{\sqrt{2}}{2} \\ y_1 + y_2 = K_1 - \frac{\sqrt{2}}{2} \\ z_1 + z_2 = K_3 \end{cases} \tag{30}$$

Again, by symmetry, we can consider the following equations:

$$x_1 = y_1 = \alpha. \tag{31}$$

$$x_2 = y_2 = \beta. \tag{32}$$

$$z_1 = z_2 = \gamma. \tag{33}$$

To maximize K and by symmetry, we need to impose that $K_1 = K_3$, then the final system that needs to be solved is as follows:

$$\begin{cases} \alpha+\beta = K - \frac{\sqrt{2}}{2} \\ \gamma = \frac{K}{2} \\ \alpha^2+\beta^2+\gamma^2 \leq 1 \end{cases} \tag{34}$$

The last inequality comes from the constraint that $\left\|\overrightarrow{V}_i\right\|_2 \leq 1,$ for $i = 1, 2$.
Again, without losing generality, we can assume that $\alpha = \beta$,
The maximum value of K can be calculated by

$$\begin{cases} \alpha = \frac{K}{2} - \frac{\sqrt{2}}{4} \\ \gamma = \frac{K}{2} \\ 2\alpha^2+\gamma^2 = 1 \end{cases} \tag{35}$$

So, we end up solving the below quadratic equation:

$$2\left(\frac{K}{2} - \frac{\sqrt{2}}{4}\right)^2 + \left(\frac{K}{2}\right)^2 = 1. \tag{36}$$

After simplification, we find:

$$3K^2 - 2\sqrt{2}K - 3 = 0. \tag{37}$$

The solution of the previous quadratic equation is when the value K is equal to $\frac{\sqrt{2}+\sqrt{11}}{3}$.
Hence

$$K(3) \geq \frac{\sqrt{2}+\sqrt{11}}{3}. \tag{38}$$

A simulation is used to answer the question if the value $K(3)$ is equal to $\frac{\sqrt{2}+\sqrt{11}}{3}$ or not. A cylindrical Coordinate has been used in our simulation to check most of the cases; the possible coordinate values of the vector $\overrightarrow{V}_i$ are summarized as follows:

$$x = r\cos\theta\sin\alpha, \tag{39}$$

$$y = r\cos\theta\sin\alpha, \tag{40}$$

$$z = r\cos\alpha, \tag{41}$$

where θ = [start value : step : end value] = [0 : 0.001 : 2π], α = [start value : step : end value] = [0 : 0.001 : 2π], and r = [start value : step : end value] = [0 : 0.01 : 1].

The simulation has shown that the value $K(3)$ is equal to $\frac{\sqrt{2}+\sqrt{11}}{3}$ i.e., $K(3) = \frac{\sqrt{2}+\sqrt{11}}{3}$.

4. Evaluation of $K(4)$

Before giving the approach for dimension 4, we will review the evaluation $K(3)$ for dimensions 2 and 3 in different ways to be generalized later.

For dimension 2, we denote by $\overrightarrow{V}_1 = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$ and $\overrightarrow{V}_2 = \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$ the two particular vectors that verified:

$$\min_{\varepsilon_1} \left\| \overrightarrow{V}_1 + \varepsilon_2 \overrightarrow{V}_2 \right\|_\infty = K(2). \tag{42}$$

By symmetry, we can assume that:

$$\left\| \overrightarrow{V}_1 + \overrightarrow{V}_2 \right\|_\infty = \alpha_1 + \alpha_2 = K(2). \tag{43}$$

$$\left\| \overrightarrow{V}_1 - \overrightarrow{V}_2 \right\|_\infty = \beta_1 - \beta_2 = K(2). \tag{44}$$

From the definition of $K(2)$, to get the maximum value, the coordinates of the two vectors should be non-negative values except the coordinate β_2 should be a negative value.

By symmetry, we denote $\alpha_1 = \alpha_2 = \alpha$ & $\beta_1 = -\beta_2 = \beta$.

To find $K(2)$, it is enough to solve the following system:

$$\begin{cases} 2\alpha = K(2) \\ 2\beta = K(2) \end{cases} \tag{45}$$

under the constraint $\alpha^2 + \beta^2 \leq 1$.

The maximum $K(2)$ can be found by considering $\alpha^2 + \beta^2 = 1$, so the previous system is equivalent to the following quadratic equation:

$$\left(\frac{K(2)}{2}\right)^2+\left(\frac{K(2)}{2}\right)^2=1. \tag{46}$$

Therefore,

$$K(2)=\sqrt{2}. \tag{47}$$

For dimension 3, we would like to find $\overrightarrow{V}_1=\begin{pmatrix}\alpha_1\\ \beta_1\\ \gamma_1\end{pmatrix}$, $\overrightarrow{V}_2=\begin{pmatrix}\alpha_2\\ \beta_2\\ \gamma_2\end{pmatrix}$ and $\overrightarrow{V}_3=\begin{pmatrix}\alpha_3\\ \beta_3\\ \gamma_3\end{pmatrix}$ that verify

$$K(3)=\min_{\varepsilon_1,\,\varepsilon_2}\left\|\overrightarrow{V}_3+\varepsilon_2\overrightarrow{V}_2+\varepsilon_1\overrightarrow{V}_1\right\|_\infty. \tag{48}$$

All possible cases of the vector $\overrightarrow{\varepsilon}=\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ 1\end{pmatrix}$ will be gathered under a matrix named A_3, where its rows r_i form all cases of the vector $\overrightarrow{\varepsilon}$.

The matric A_3 is defined as follows:

$$A_3=\begin{bmatrix}+1 & +1 & +1\\ +1 & -1 & +1\\ +1 & +1 & -1\\ +1 & -1 & -1\end{bmatrix}. \tag{49}$$

The four rows are not independent vectors because it is noted that $r_4=r_2+r_3-r_1$.

By symmetry, we can assume the following equations:

$$\left\|\overrightarrow{V}_3+\overrightarrow{V}_2+\overrightarrow{V}_1\right\|_\infty=|\alpha_3+\alpha_2+\alpha_1|=K(3), \tag{50}$$

$$\left\|\overrightarrow{V}_3+\overrightarrow{V}_2-\overrightarrow{V}_1\right\|_\infty=|\beta_3+\beta_2-\beta_1|=K(3), \tag{51}$$

$$\left\|\overrightarrow{V}_3-\overrightarrow{V}_2+\overrightarrow{V}_1\right\|_\infty=|\gamma_3-\gamma_2+\gamma_1|=K(3), \tag{52}$$

$$\left\|\overrightarrow{V}_3-\overrightarrow{V}_2-\overrightarrow{V}_1\right\|_\infty=|\gamma_3-\gamma_2-\gamma_1|=K(3). \tag{53}$$

To maximize the value of $K(3)$, it is suitable to consider the coordinates β_1, γ_1, and γ_2 as negative values, so the coordinate of the three vectors $\overrightarrow{V}_i$ will be summarized as follows:

$$\overrightarrow{V}_1 = \begin{pmatrix} \alpha_1 \\ -\beta_1 \\ -\gamma_1 \end{pmatrix}, \tag{54}$$

$$\overrightarrow{V}_2 = \begin{pmatrix} \alpha_2 \\ \beta_2 \\ -\gamma_2 \end{pmatrix}, \tag{55}$$

$$\overrightarrow{V}_3 = \begin{pmatrix} \alpha_3 \\ \beta_3 \\ \gamma_3 \end{pmatrix}. \tag{56}$$

where all parameters, $(\alpha_i\ ,\ \beta_i\ ,\ \gamma_i)$ are non-negative values.

To calculate $K(3)$, it is enough to solve the below system:

$$\begin{cases} \alpha_3 + \alpha_2 + \alpha_1 = K(3) \\ \beta_3 + \beta_2 + \beta_1 = K(3) \\ \gamma_3 + \gamma_2 + \gamma_1 = K(3) \\ -\gamma_3 + \gamma_2 + \gamma_1 = K(3) \end{cases}. \tag{57}$$

Under the constraints

$$\alpha_1^2 + \beta_1^2 + \gamma_1^2 \leq 1, \tag{58}$$

$$\alpha_2^2 + \beta_2^2 + \gamma_2^2 \leq 1, \tag{59}$$

$$\alpha_3^2 + \beta_3^2 + \gamma_3^2 \leq 1. \tag{60}$$

From the last two equations of the system, we can conclude that $\gamma_3 = 0$.

By symmetry also, we can assume that:

$$\alpha_1 = \alpha_2 = \beta_2 = \beta_1 = \beta, \tag{61}$$

$$\gamma_2 = \gamma_1 = \gamma, \tag{62}$$

$$\alpha_3 = \beta_3 = \alpha. \tag{63}$$

Therefore,

$$\begin{cases} 2\beta + \alpha = K(3) \\ 2\gamma = K(3) \end{cases}, \tag{64}$$

under the constraints

$$2\beta^2 + \alpha^2 \leq 1, \tag{65}$$

$$2\alpha^2 \leq 1. \tag{66}$$

To maximize the value of $K(3)$, the two constraints can be considered as

$$2\beta^2 + \alpha^2 = 1, \tag{67}$$

$$2\alpha = 1. \tag{68}$$

Then, the system will be simplified as follows:

$$\begin{cases} 2\beta = K(3) - \frac{\sqrt{2}}{2} \\ 2\gamma = K(3) \end{cases}. \tag{69}$$

The below quadratic equation needs to be solved to calculate the value $K(3)$,

$$2\left(\frac{K(3)}{2} - \frac{\sqrt{2}}{4}\right)^2 + \left(\frac{K(3)}{2}\right)^2 = 1. \tag{70}$$

After simplification, the quadratic equation can be as follows:

$$3K(3)^2 - 2\sqrt{2}K(3) - 3 = 0. \tag{71}$$

As a consequence, it concludes that:

$$K(3) = \frac{\sqrt{2}+\sqrt{11}}{3}. \tag{72}$$

The particular vectors that cannot cancel each other further than $K(3)$ are defined as follows:

$$\overrightarrow{V}_1 = \begin{pmatrix} \frac{K(3)}{4} - \frac{\sqrt{2}}{4} \\ \frac{-K(3)}{4} + \frac{\sqrt{2}}{4} \\ -\frac{K(3)}{2} \end{pmatrix}, \tag{73}$$

$$\overrightarrow{V}_2 = \begin{pmatrix} \frac{K(3)}{4} - \frac{\sqrt{2}}{4} \\ \frac{K(3)}{4} - \frac{\sqrt{2}}{4} \\ -\frac{K(3)}{2} \end{pmatrix}, \tag{74}$$

$$\overrightarrow{V}_3 = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}. \tag{75}$$

Note that these particular three vectors are not unique solutions of $\arg_{\overrightarrow{V}_i} \min_{\varepsilon_i} \left\| \sum \varepsilon_i \overrightarrow{V}_i \right\|_\infty$.

The idea is to generalize the previous approach in evaluating the function $K(n)$, for that let's denote by $V = \{\overrightarrow{V}_4, \overrightarrow{V}_3, \overrightarrow{V}_2, \overrightarrow{V}_1\}$ the set of particular vectors that satisfy the below equation:

$$K(4) = \min_{\varepsilon_1, \varepsilon_2, \varepsilon_3} \left\| \overrightarrow{V}_4 + \sum_{i=1}^{3} \varepsilon_i \overrightarrow{V}_i \right\|_\infty. \tag{76}$$

The previous matrix A_3 can be extended to the matrix A_4 to fit all possible 4-dimension vectors $(1, \varepsilon_1, \varepsilon_2, \varepsilon_3)$ as follows:

$$A_4 = \begin{bmatrix} +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & +1 \\ +1 & +1 & -1 & +1 \\ +1 & -1 & -1 & +1 \\ +1 & +1 & +1 & -1 \\ +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 \\ +1 & -1 & -1 & -1 \end{bmatrix}, \tag{77}$$

where it is noted that $r_{i+2} = r_{i+1} + r_i - r_{i-1}$, for $i > 1$, and $\dim(span(r_1, r_2, r_3)) = 3$.

The idea is to well assign each row r_i to one of the fourth dimensions to avoid zero coordinates in $\overrightarrow{V}_i$, which is a consequence of maximizing the value of $K(4)$, i.e., the axes where l_∞ norm of $\overrightarrow{V}_4 + \sum_{i=1}^{3} \varepsilon_i \overrightarrow{V}_i$ is located will be distributed over all possible combinations of $(1, \varepsilon_1, \varepsilon_2, \varepsilon_3)$ in a way to maximize the value $K(4)$.

The below diagram, in Figure 6, identifies which coordinate will be eliminated, being zero, when we associate two rows to the same axes.

From the previous diagram, the rows are gathered as follows:

$$\bullet (r_1, r_2): \left\| \overrightarrow{V}_4 \pm \overrightarrow{V}_3 + \overrightarrow{V}_2 + \overrightarrow{V}_1 \right\|_\infty = \left| V_4^1 \pm V_3^1 + V_2^1 + V_1^1 \right| = K(4), \tag{78}$$

$$\bullet (r_5, r_7): \left\| \overrightarrow{V}_4 + \overrightarrow{V}_3 \pm \overrightarrow{V}_2 - \overrightarrow{V}_1 \right\|_\infty = \left| V_4^2 + V_3^2 \pm V_2^2 - V_1^2 \right| = K(4), \tag{79}$$

$$\bullet (r_4, r_8): \left\| \overrightarrow{V}_4 - \overrightarrow{V}_3 - \overrightarrow{V}_2 \pm \overrightarrow{V}_1 \right\|_\infty = \left| V_4^3 - V_3^3 - V_2^3 \pm V_1^3 \right| = K(4), \tag{80}$$

$$\bullet (r_3, r_6): \left\| \pm \overrightarrow{V}_4 + \overrightarrow{V}_3 - \overrightarrow{V}_2 + \overrightarrow{V}_1 \right\|_\infty = \left| \pm V_4^4 + V_3^4 - V_2^4 + V_1^4 \right| = K(4), \tag{81}$$

where V_i^j is the jth coordinate of the vector $\overrightarrow{V}_i$.

To maximize the value $K(4)$, the coordinate's sign of $\overrightarrow{V}_i$ can be found as follows:

$$\overrightarrow{V}_1 = \begin{pmatrix} \alpha_1 \\ -\beta_2 \\ \gamma_2 \\ w_2 \end{pmatrix}, \tag{82}$$

$$\overrightarrow{V}_2 = \begin{pmatrix} \alpha_2 \\ \beta_2 \\ -\gamma_2 \\ -w_2 \end{pmatrix}, \tag{83}$$

$$\overrightarrow{V}_3 = \begin{pmatrix} \alpha_3 \\ \beta_3 \\ -\gamma_3 \\ w_4 \end{pmatrix}, \tag{84}$$

$$\overrightarrow{V}_4 = \begin{pmatrix} \alpha_4 \\ \beta_4 \\ \gamma_4 \\ w_4 \end{pmatrix}, \tag{85}$$

where α_i , β_i , γ_i and w_i are non-negative values.

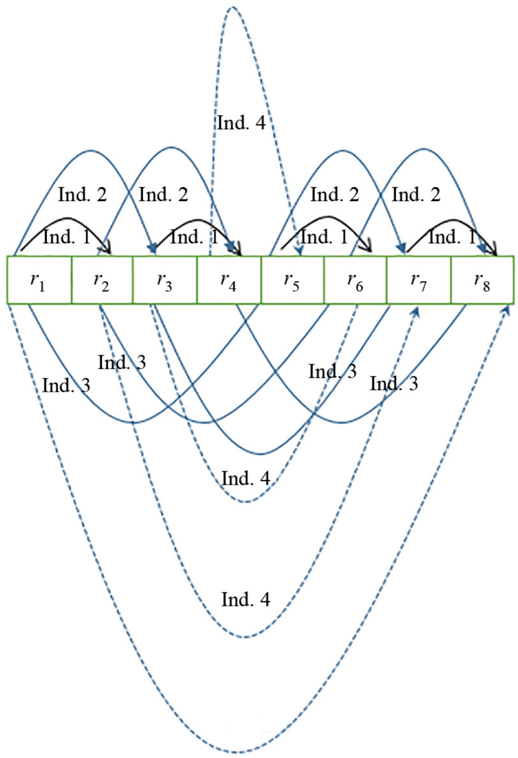

Figure 6. A link between gathering two rows as a system of equations and the index variable that will be eliminated

The negative sign is highlighted at the coordinate of $\overrightarrow{V_i}$ comes from rows r_1, r_5, r_4, *and* r_3. For instance, if we assume that $\left\|\overrightarrow{V}_4 + \sum_{i=1}^{3} \varepsilon_i \overrightarrow{V}_i\right\|_\infty = \left|\sum_{i=1}^{3} \varepsilon_i \alpha_i + \alpha_4\right|$, where $r_4 = (1, -1, -1, 1)$, in order to get maximum value of $K(4)$, it is preferable to consider the two coordinates α_2 and α_3 as negative values such that the equation $1(\alpha_1) - 1(\alpha_2) - 1(\alpha_3) + 1(\alpha_4) = K(4)$ will be equivalent to the equation

$$|\alpha_1| + |\alpha_2| + |\alpha_3| + |\alpha_4| = K(4). \tag{86}$$

The row distribution can be formulated by the following systems of equations:

$$(r_1, r_2): \begin{cases} \alpha_4 + \alpha_3 + \alpha_2 + \alpha_1 = K \\ \\ \alpha_4 - \alpha_3 + \alpha_2 + \alpha_1 = K \end{cases} \Longrightarrow \alpha_3 = 0, \tag{87}$$

$$(r_5, r_7): \begin{cases} \beta_4 + \beta_3 + \beta_2 + \beta_1 = K \\ \\ \beta_4 + \beta_3 - \beta_2 + \beta_1 = K \end{cases} \Longrightarrow \beta_2 = 0, \tag{88}$$

$$(r_4, r_8): \begin{cases} \gamma_4 + \gamma_3 + \gamma_2 + \gamma_1 = K \\ \\ \gamma_4 + \gamma_3 + \gamma_2 - \gamma_1 = K \end{cases} \Longrightarrow \gamma_1 = 0, \tag{89}$$

$$(r_3, r_6): \begin{cases} w_4 + w_3 + w_2 + w_1 = K \\ \\ -w_4 + w_3 + w_2 + w_1 = K \end{cases} \Longrightarrow w_4 = 0. \tag{90}$$

The system can be simplified further by

$$\begin{cases} \alpha_4 + \alpha_2 + \alpha_1 = K \\ \\ \beta_4 + \beta_3 + \beta_1 = K \\ \\ \gamma_4 + \gamma_3 + \gamma_2 = K \\ \\ w_3 + w_2 + w_1 = K \end{cases} \tag{91}$$

Under the below constraints

$$\begin{cases} \alpha_4^2 + \beta_4^2 + \gamma_4^2 = 1 \\ \\ w_4^2 + \beta_3^2 + \gamma_3^2 = 1 \\ \\ \alpha_2^2 + w_2^2 + \gamma_2^2 = 1 \\ \\ \alpha_1^2 + \beta_1^2 + w_1^2 = 1 \end{cases} \tag{92}$$

As before, the previous system of equations needs to be matched with the coordinates of the four vectors to maximize the value of $K(4)$, and then:

$$\left[\overrightarrow{V}_1, \overrightarrow{V}_2, \overrightarrow{V}_3, \overrightarrow{V}_3\right] = \begin{bmatrix} \alpha_1 & \alpha_2 & 0 & \alpha_4 \\ -\beta_1 & 0 & \beta_3 & \beta_4 \\ 0 & -\gamma_2 & -\gamma_3 & \gamma_4 \\ w_1 & -w_2 & w_3 & 0 \end{bmatrix}. \tag{93}$$

By symmetry, we can assume that:

$$\alpha_4 = \alpha_2 = \alpha_1 = \alpha, \tag{94}$$

$$\beta_4 = \beta_3 = \beta_1 = \beta, \tag{95}$$

$$\gamma_4 = \gamma_3 = \gamma_2 = \gamma, \tag{96}$$

$$w_3 = w_2 = w_1 = w. \tag{97}$$

Therefore

$$\begin{cases} \alpha = \frac{K}{3} \\ \beta = \frac{K}{3} \\ \gamma = \frac{K}{3} \\ w = \frac{K}{3} \end{cases} \tag{98}$$

To maximize K, the constraints can be assumed to be as follows:

$$\begin{aligned} 1 =& \alpha^2 + \beta^2 + \gamma^2 \\ =& \alpha^2 + \beta^2 + w^2 \\ =& \alpha^2 + w^2 + \gamma^2 \\ =& w^2 + \beta^2 + \gamma^2 \end{aligned} \tag{99}$$

To find the value of K, it is enough to solve the below quadratic equation:

$$\alpha^2 + \beta^2 + \gamma^2 = 3\left(\frac{K}{3}\right)^2 = 1. \tag{100}$$

It implies that

$$K(4) \geq \sqrt{3}. \tag{101}$$

and the coordinates of the particular set of vectors $\overrightarrow{V}_i$ are summarized under the below matrix:

$$\left[\overrightarrow{V}_1, \overrightarrow{V}_2, \overrightarrow{V}_3, \overrightarrow{V}_4\right] = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}. \tag{102}$$

Note Other distributions can be formulated by the following configuration:
The matrix can be formulated differently as follows:

$$A_4 = \begin{bmatrix} +1 & +1 & +1 & +1 \\ -1 & +1 & +1 & +1 \\ +1 & -1 & +1 & +1 \\ -1 & -1 & +1 & +1 \\ +1 & +1 & -1 & +1 \\ -1 & +1 & -1 & +1 \\ +1 & -1 & -1 & +1 \\ -1 & -1 & -1 & +1 \end{bmatrix}, \tag{103}$$

From Figure 7, the row distribution can be configured by the following systems of equations:

$$(r_1, r_2): \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = K \\ -\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = K \end{cases} \Longrightarrow \alpha_1 = 0, \tag{104}$$

$$(r_6, r_8): \begin{cases} \beta_1 + \beta_2 + \beta_3 + \beta_4 = K \\ \beta_1 - \beta_2 + \beta_3 + \beta_4 = K \end{cases} \Longrightarrow \beta_2 = 0, \tag{105}$$

$$(r_3, r_7): \begin{cases} \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 = K \\ \gamma_1 + \gamma_2 - \gamma_3 + \gamma_4 = K \end{cases} \Longrightarrow \gamma_3 = 0, \tag{106}$$

$$(r_4, r_5): \begin{cases} w_1 + w_2 + w_3 + w_4 = K \\ \\ w_1 + w_2 + w_3 - w_4 = K \end{cases} \Longrightarrow w_4 = 0. \tag{107}$$

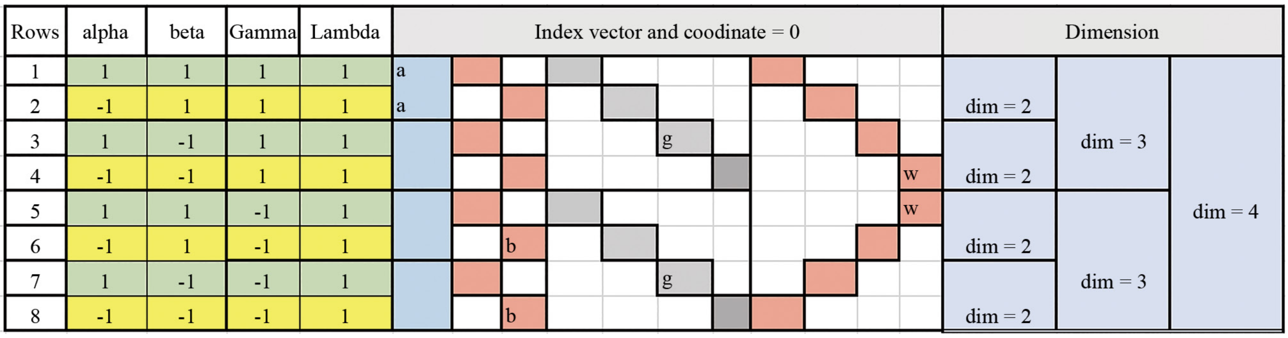

Figure 7. How to gather two rows of the matrix to eliminate a given index coordinate.

By finishing the calculation, we find that:

$$K(4) \geq \sqrt{3}. \tag{108}$$

The Cauchy-Schwarz inequality, also known as the Cauchy-Bunyakovsky-Schwarz inequality, can be used to optimize the following system:

Maximizing the variable K, under the fourth objective functions:

$$\begin{cases} \alpha_2 + \alpha_3 + \alpha_4 = K \\ \beta_1 + \beta_3 + \beta_4 = K \\ \gamma_1 + \gamma_2 + \gamma_4 = K \\ w_1 + w_2 + w_3 = K \end{cases} \tag{109}$$

Under the below constraints:

$$\begin{cases} \alpha_4^2 + \beta_4^2 + \gamma_4^2 = 1 \\ w_3^2 + \beta_3^2 + \gamma_3^2 = 1 \\ \alpha_2^2 + w_2^2 + \gamma_2^2 = 1 \\ \gamma_1^2 + \beta_1^2 + w_1^2 = 1 \end{cases} \tag{110}$$

The system can be modified by Cauchy as follows:

$$\begin{cases} K^2 = (\alpha_2+\alpha_3+\alpha_4)^2 \le (\alpha_2^2+\alpha_3^2+\alpha_4^2)(1^2+1^2+1^2) \\ K^2 = (\beta_1+\beta_3+\beta_4)^2 \le (\beta_1^2+\beta_3^2+\beta_4^2)(1^2+1^2+1^2) \\ K^2 = (\gamma_1+\gamma_2+\gamma_4)^2 \le (\gamma_1^2+\gamma_2^2+\gamma_4^2)(1^2+1^2+1^2) \\ K^2 = (w_1+w_2+w_3)^2 \le (w_1^2+w_2^2+w_3^2)(1^2+1^2+1^2) \end{cases} \tag{111}$$

By adding all four equations, you will get

$$4K^2 \le 3(\alpha_4^2+\alpha_2^2+\alpha_3^2+\beta_4^2+\beta_3^2+\beta_1^2+\gamma_4^2+\gamma_2^2+\gamma_1^2+w_3^2+w_2^2+w_1^2),$$

from constraints, the maximum of the value K can be calculated as follows:

$$4K^2 = 12. \tag{112}$$

Then, the constant of the optimization is found to be as follows

$$K = \sqrt{3}. \tag{113}$$

Then, the Komlos constant has a lower bound as follows

$$K(4) \ge \sqrt{3}. \tag{114}$$

The coordinate of the four vectors can be calculated from the equality of the Cauchy-Schwarz inequality property that states that:

$$\begin{cases} (\alpha_4+\alpha_2+\alpha_3)^2 = (\alpha_4^2+\alpha_2^2+\alpha_3^2)\left(1^2+1^2+1^2\right) \Longrightarrow \frac{\alpha_3}{1} = \frac{\alpha_2}{1} = \frac{\alpha_4}{1} \\ (\beta_4+\beta_3+\beta_1)^2 = \left(\beta_4^2+\beta_3^2+\beta_1^2\right)\left(1^2+1^2+1^2\right) \Longrightarrow \frac{\beta_1}{1} = \frac{\beta_3}{1} = \frac{\beta_4}{1} \\ (\gamma_4+\gamma_2+\gamma_1)^2 = \left(\gamma_4^2+\gamma_1^2+\gamma_2^2\right)\left(1^2+1^2+1^2\right) \Longrightarrow \frac{\gamma_1}{1} = \frac{\gamma_2}{1} = \frac{\gamma_4}{1} \\ (w_3+w_2+w_1)^2 = \left(w_3^2+w_2^2+w_1^2\right)\left(1^2+1^2+1^2\right) \Longrightarrow \frac{w_1}{1} = \frac{w_2}{1} = \frac{w_3}{1} \end{cases} . \tag{115}$$

Therefore

$$\alpha_i = \beta_i = \gamma_i = w_i = \frac{\sqrt{3}}{3}. \tag{116}$$

The coordinate of the particular vectors $\overrightarrow{V_i}$ are summarized under the below matrix:

$$\begin{bmatrix} \overrightarrow{V_1} & \overrightarrow{V_2} & \overrightarrow{V_3} & \overrightarrow{V_4} \end{bmatrix} = \frac{\sqrt{3}}{3} \begin{bmatrix} 0 & +1 & +1 & +1 \\ +1 & 0 & -1 & +1 \\ +1 & -1 & 0 & +1 \\ -1 & -1 & +1 & 0 \end{bmatrix}. \tag{117}$$

In the case where the dimension is under the form of 2^m, for a certain integer m, the optimization is perfect, but for other cases of dimension, an upper bound can be found for the constant K if the Cauchy-Schwarz inequality is applied as above.

5. Evaluation of $K(5)$

By using the same idea of the previous section, in dimension 4, we denote by $\overrightarrow{V_1}, \ldots, \overrightarrow{V_5}$ as a special vector satisfying:

$$K(5) = \min_{\varepsilon_i} \left\| \overrightarrow{V}_5 + \sum_{i=1}^{4} \varepsilon_i \overrightarrow{V}_i \right\|_\infty. \tag{118}$$

All the different combinations of $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, 1)$ are summarized at the rows of the matrix A_5 defined as follows:

$$A_5 = \begin{bmatrix} +1 & +1 & +1 & +1 & +1 \\ -1 & +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & +1 & +1 \\ -1 & -1 & +1 & +1 & +1 \\ +1 & +1 & -1 & +1 & +1 \\ -1 & +1 & -1 & +1 & +1 \\ +1 & -1 & -1 & +1 & +1 \\ -1 & -1 & -1 & +1 & +1 \\ +1 & +1 & +1 & -1 & +1 \\ -1 & +1 & +1 & -1 & +1 \\ +1 & -1 & +1 & -1 & +1 \\ -1 & -1 & +1 & -1 & +1 \\ +1 & +1 & -1 & -1 & +1 \\ -1 & +1 & -1 & -1 & +1 \\ +1 & -1 & -1 & -1 & +1 \\ -1 & -1 & -1 & -1 & +1 \end{bmatrix}. \tag{119}$$

where it is noted that $r_{i+2} = r_{i+1} + r_i - r_{i-1}$, for $i > 1$ and $dim\{r_i, i = 1, \ldots, 16\} = 5$.

The target is to distribute the 16 rows among five dimensions, named $\{\propto, \beta, \gamma, \lambda, w\}$ in such a way to minimize the number of zeros in the 5 vectors, $\overrightarrow{V}_i = \begin{pmatrix} \alpha_i \\ \beta_i \\ \gamma_i \\ \lambda_i \\ w_i \end{pmatrix}$, $i = 1, \ldots, 5$.

The row distributions are summarized as follows:

• Four rows will be assigned to each axe except axe α, where one row is a linear combination of the others $r_{i+2} = r_{i+1} + r_i - r_{i-1}$, it looks like each three independent rows will be assigned to one axe,

• Two rows will be assigned to axe α.

Formulating the previous distribution of the 16 rows to the below 16 equations as follows:

For α-Axe: $\left\|\overrightarrow{\boldsymbol{V}}_{\mathbf{5}} + \sum_{i=1}^{4} \boldsymbol{\varepsilon_i} \overrightarrow{\boldsymbol{V}}_{\boldsymbol{i}}\right\|_{\infty} = \sum_{i=1}^{4} |\boldsymbol{\varepsilon_i \alpha_i}| + |\boldsymbol{\alpha_5}|$

$$\begin{cases} r_{15}\colon \alpha_5 + \alpha_4 + \alpha_3 + \alpha_2 + \alpha_1 = K(5) \\ r_{16}\colon \alpha_5 + \alpha_4 + \alpha_3 + \alpha_2 - \alpha_1 = K(5) \end{cases} \Longrightarrow \alpha_1 = 0. \tag{120}$$

For β-Axe: $\left\|\overrightarrow{\boldsymbol{V}}_{\mathbf{5}} + \sum_{i=1}^{4} \boldsymbol{\varepsilon_i} \overrightarrow{\boldsymbol{V}}_{\boldsymbol{i}}\right\|_{\infty} = \sum_{i=1}^{4} |\boldsymbol{\varepsilon_i \beta_i}| + |\boldsymbol{\beta_5}|$

$$\begin{cases} r_1\colon \beta_5 + \beta_4 + \beta_3 + \beta_2 + \beta_1 = K(5) \\ r_3\colon \beta_5 + \beta_4 + \beta_3 - \beta_2 + \beta_1 = K(5) \\ r_5\colon \beta_5 + \beta_4 - \beta_3 + \beta_2 + \beta_1 = K(5) \\ r_7\colon \beta_5 + \beta_4 - \beta_3 - \beta_2 + \beta_1 = K(5) \end{cases} \Longrightarrow \beta_2 = \beta_3 = 0. \tag{121}$$

Note that the last equations depend on the 3 first equations.

For γ-Axe: $\left\|\overrightarrow{\boldsymbol{V}}_{\mathbf{5}} + \sum_{i=1}^{4} \boldsymbol{\varepsilon_i} \overrightarrow{\boldsymbol{V}}_{\boldsymbol{i}}\right\|_{\infty} = \sum_{i=1}^{4} |\boldsymbol{\varepsilon_i}\ \boldsymbol{\gamma_i}| + |\ \boldsymbol{\gamma_5}|$

$$\begin{cases} r_2\colon \gamma_5 + \gamma_4 + \gamma_3 + \gamma_2 + \gamma_1 = K(5) \\ r_6\colon \gamma_5 + \gamma_4 - \gamma_3 + \gamma_2 + \gamma_1 = K(5) \\ r_{10}\colon \gamma_5 - \gamma_4 + \gamma_3 + \gamma_2 + \gamma_1 = K(5) \\ r_{14}\colon \gamma_5 - \gamma_4 - \gamma_3 + \gamma_2 + \gamma_1 = K(5) \end{cases} \Longrightarrow \gamma_4 = \gamma_3 = 0. \tag{122}$$

Note that the last equations depend on the 3 first equations.

For λ-Axe: $\left\|\overrightarrow{\boldsymbol{V}}_{\mathbf{5}} + \sum_{i=1}^{4} \boldsymbol{\varepsilon_i} \overrightarrow{\boldsymbol{V}}_{\boldsymbol{i}}\right\|_{\infty} = \sum_{i=1}^{4} |\boldsymbol{\varepsilon_i \lambda_i}| + |\ \boldsymbol{\lambda_5}|$

$$\begin{cases} r_4\colon \lambda_5+\lambda_4+\lambda_3+\lambda_2+\lambda_1=K(5) \\ r_5\colon \lambda_5+\lambda_4-\lambda_3-\lambda_2-\lambda_1=-K(5) \\ r_{12}\colon \lambda_5-\lambda_4+\lambda_3+\lambda_2+\lambda_1=K(5) \\ r_{13}\colon \lambda_5-\lambda_4-\lambda_3-\lambda_2-\lambda_1=K(5) \end{cases} \Longrightarrow \lambda_4=\lambda_5=0. \tag{123}$$

Note that the last equations depend on the 3 first equations.
For w-Axe: $\left\|\overrightarrow{\boldsymbol{V}}_5+\sum_{i=1}^{4}\boldsymbol{\varepsilon}_i\overrightarrow{\boldsymbol{V}}_i\right\|_\infty=\sum_{i=1}^{4}|\boldsymbol{\varepsilon}_i\boldsymbol{w}_i|+|\ \boldsymbol{w}_5|$

$$\begin{cases} r_6\colon w_5+w_4+w_3+w_2+w_1=K(5) \\ r_8\colon w_5+w_4+w_3-w_2+w_1=-K(5) \\ r_9\colon w_5-w_4-w_3+w_2+-w_1=K(5) \\ r_{11}\colon w_5-w_4-w_3-w_2-w_1=K(5) \end{cases} \Longrightarrow w_5=w_2=0. \tag{124}$$

Note that the last equations depend on the 3 first equations.
From the previous systems of equations, we can shape our five vectors $\overrightarrow{V}_i$ in order to maximize $K(5)$ as follows:

$$\begin{bmatrix} \overrightarrow{V_1} & \overrightarrow{V_2} & \overrightarrow{V_3} & \overrightarrow{V_4} & \overrightarrow{V_5} \end{bmatrix} = \begin{bmatrix} 0 & -\alpha_2 & -\alpha_3 & -\alpha_4 & \alpha_5 \\ \beta_1 & 0 & 0 & \beta_4 & \beta_5 \\ -\gamma_1 & \gamma_2 & 0 & 0 & \gamma_5 \\ -\lambda_1 & -\lambda_2 & \lambda_3 & 0 & 0 \\ -w_1 & 0 & -w_3 & w_4 & 0 \end{bmatrix}. \tag{125}$$

where α_i, β_i , γ_i , λ_i , and w_i are non-negative values.

The negative sign is highlighted at the coordinate of $\overrightarrow{V_i}$ comes from rows r_{15}, r_1, r_2, r_4 *and* r_6 , for instance, if we assume that $\left\|\overrightarrow{V}_5+\sum_{i=1}^{4}\varepsilon_i\overrightarrow{V}_i\right\|_\infty=\left|\sum_{i=1}^{4}\varepsilon_i\alpha_i+\alpha_5\right|$ where $r_5=(1,\ -1,\ -1,\ -1,\ 1)$ and our target is to maximize the value of $K(5)$, then it is preferable to considerα_2, α_3, and α_4 are negative values such that the equation $1(\alpha_1)-1(\alpha_2)-1(\alpha_3)-1(\alpha_4)+1(\alpha_5)=K(5)$ will be equivalent to the below equation,

$$|\alpha_1|+|\alpha_2|+|\alpha_3|+|\alpha_4|+|\alpha_5|=K(5), \tag{126}$$

for notation simplification notation, we write any negative parameter as $-\alpha_i$.

To calculate the constant $K(5)$, we need to solve the below system:

$$
\begin{cases}
\alpha_2+\alpha_3+\alpha_4+\alpha_5=K(5)\\
\beta_1+\beta_4+\beta_5=K(5)\\
\gamma_1+\gamma_2+\gamma_5=K(5)\\
\lambda_1+\lambda_2+\lambda_3=K(5)\\
w_1+w_3+w_4=K(5)
\end{cases} \tag{127}
$$

Under the constraint $\left\|\overrightarrow{V}_i\right\| \leq 1,\ i=1,\ldots,5$.
By symmetry, we can assume the following:

$$\alpha_5=\alpha_4=\alpha_3=\alpha_2=\alpha, \tag{128}$$

$$\gamma_5=\gamma_2=\gamma, \tag{129}$$

$$\lambda_3=\lambda_2=\lambda, \tag{130}$$

$$w_4=w_3=w, \tag{131}$$

$$\beta_1=\gamma_1=\lambda_1=w_1. \tag{132}$$

As $\left\|\overrightarrow{V}_1\right\|_2 \leq 1$, we can put:

$$\beta_1=\gamma_1=\lambda_1=w_1=\frac{1}{2}, \tag{133}$$

The system will be summarized as follows:

$$
\begin{cases}
4\alpha=K(5)\\
2\beta=K(5)-\frac{1}{2}\\
2\gamma=K(5)-\frac{1}{2}\\
2\lambda=K(5)-\frac{1}{2}\\
2w=K(5)-\frac{1}{2}
\end{cases} \tag{134}
$$

Under the constraints

$$
\begin{aligned}
1 =& \alpha^2+\beta^2+\gamma^2 \\
=& \alpha^2+\beta^2+w^2 \\
=& \alpha^2+\lambda^2+w^2 \\
=& \alpha^2+\lambda^2+\gamma^2.
\end{aligned} \tag{135}
$$

The system is equivalent to the quadratic equation:

$$
\left(\frac{K(5)}{4}\right)^2+2\left(\frac{K(5)}{2}-\frac{1}{4}\right)^2=1. \tag{136}
$$

So, we can conclude that the value of $K(5)$ is lower bounded by $\frac{4+\sqrt{142}}{9}$ i.e.

$$
K(5) \geq \frac{4+\sqrt{142}}{9}. \tag{137}
$$

To see the importance of the way of distributing the rows among the axes is very important, we try to make, as an example, another configuration as follows:

$$
\begin{cases}
(r_7, r_8, r_9, r_{10}) & \text{for } \alpha-Axe \\
(r_2, r_3, r_4) & \text{for } \beta-Axe \\
(r_{12}, r_{14}, r_{16}) & \text{for } \gamma-Axe \\
(r_5, r_{13}, r_1) & \text{for } \lambda-Axe \\
(r_6, r_{11}, r_{15}) & \text{for } w-Axe
\end{cases}
$$

The five vector coordinates will be summarized under the below matrix:

$$
\left[\begin{array}{ccccc} \overrightarrow{V_1}, & \overrightarrow{V_2}, & \overrightarrow{V_3}, & \overrightarrow{V}_4, & \overrightarrow{V}_5 \end{array}\right] = \begin{bmatrix} 0 & -\alpha_2 & -\alpha_3 & \alpha_4 & 0 \\ 0 & 0 & \beta_3 & \beta_4 & \beta_5 \\ -\gamma_1 & 0 & 0 & -\gamma_4 & \gamma_5 \\ \lambda_1 & \lambda_2 & 0 & 0 & \lambda_5 \\ -w_1 & w_2 & 0 & w_4 & 0 \end{bmatrix}. \tag{138}
$$

The system that needs to be solved is formulated as follows:

$$\alpha_2 + \alpha_3 + \alpha_4 = K(5),$$

$$\beta_3 + \beta_4 + \beta_5 = K(5),$$

$$\gamma_1 + \gamma_4 + \gamma_5 = K(5),$$

$$\lambda_1 + \lambda_2 + \lambda_5 = K(5),$$

$$w_1 + w_2 + w_4 = K(5). \tag{139}$$

Under the constraints: $\left\|\overrightarrow{V}_i\right\|_2 \leq 1$.

By using this type of distribution, the symmetry of the matrix $[\overrightarrow{V}_1, \ldots, \overrightarrow{V}_5]$ is not preserved, which makes the system hard to solve analytically, and the number of zero coordinates in the set of vectors $\overrightarrow{V_i}$ has been increased from 9 times to 10 times.

Therefore, the system that needs to be optimized is as follows:

$$\begin{aligned}\text{Max } K(5) = &\alpha_2 + \alpha_3 + \alpha_4\\ = &\beta_3 + \beta_4 + \beta_5\\ = &\gamma_1 + \gamma_4 + \gamma_5\\ = &\lambda_1 + \lambda_2 + \lambda_5\\ = &w_1 + w_2 + w_4.\end{aligned} \tag{140}$$

Under the constraints: $\left\|\overrightarrow{V}_i\right\|_2 \leq 1$.

The value of $K(5)$ is very sensitive to the distribution choices, please refer the below Figure 8 for different choices.

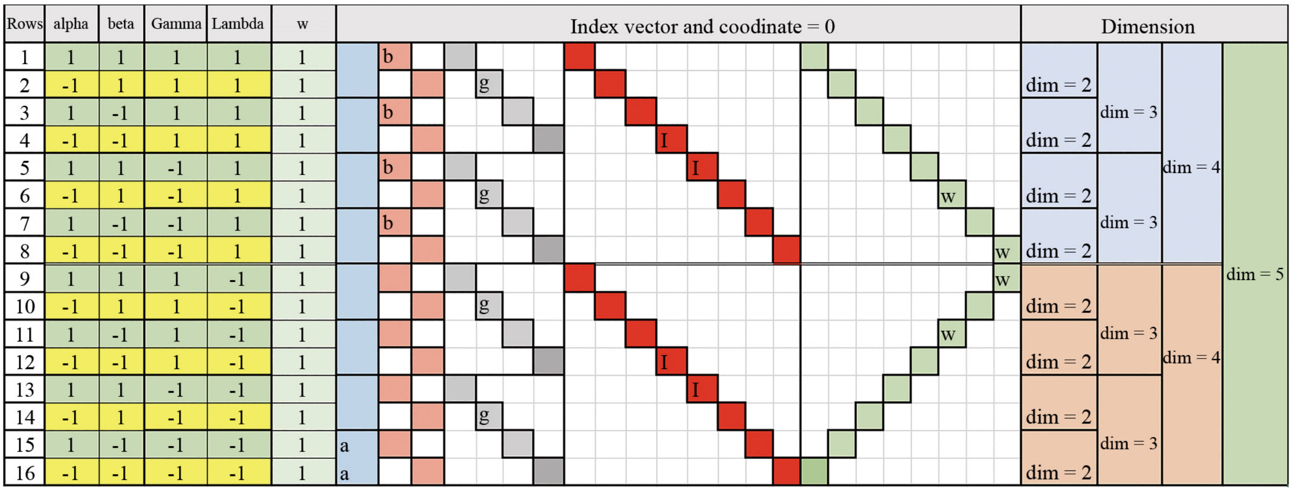

Figure 8. How to gather rows of the matrix to eliminate certain axes-coordinates

6. Intuitively assessing the lower bound $K(n)$

In dimension n, it is very crucial to find the best way to distribute all possible combinations of the vectors $\overrightarrow{\varepsilon} = (1, \varepsilon_1, \ldots, \varepsilon_{n-1})$ among the n axes. We assume that we have $\left\lceil \frac{2^{n-1}}{n} \right\rceil$ different combinations of the vector of $\overrightarrow{\varepsilon}$ for which we have:

$$\left\| \overrightarrow{V}_n + \sum_{i=1}^{n-1} \varepsilon_i \overrightarrow{V}_i \right\|_\infty = \left| \sum \varepsilon_i x_i + x_n \right| = K(n), \tag{141}$$

where x_i is the coordinate of the vector $\overrightarrow{V}_i$ corresponding to x-Axe.

The $\left\lceil \frac{2^{n-1}}{n} \right\rceil$ vectors that have been assigned to one axes has a dimension of order $\left\lceil log_2\left(\frac{2^{n-1}}{n}\right) \right\rceil$ and as a consequence, it implies that each vector $\overrightarrow{V}_i$ has at most $\left\lceil log_2\left(\frac{2^{n-1}}{n}\right) \right\rceil$ null coordinates.

To evaluate the constant $K(n)$, it is enough to solve the below optimization equation:

$$K(n) = \max_{x_i} \left(\sum_{i=1}^{n-\left\lceil log_2\left(\frac{2^{n-1}}{n}\right) \right\rceil} x_i \right). \tag{142}$$

By imposing the symmetry conditions by choosing a good way of distribution, the non-null coordinate in each axe has constant values, i.e., $x_i = x$.

Let B a subset $\{1, \ldots, n\}$ of cardinality around $n - \left\lceil log_2\left(\frac{2^{n-1}}{n}\right) \right\rceil$ and from the condition that $\left\| \overrightarrow{V}_n \right\|_2 \leq 1$, the upper bound of x can be found as follows:

$$\sum_{j \in B} \left(x_i^j\right)^2 = \sum_{j \in B} (x)^2 \leq 1 \implies x \leq \frac{\sqrt{n - \left\lceil log_2\left(\frac{2^{n-1}}{n}\right) \right\rceil}}{n - \left\lceil log_2\left(\frac{2^{n-1}}{n}\right) \right\rceil}. \tag{143}$$

The lower bound of the Komlos conjecture can be calculated as follows:

$$K(n) \geq \sqrt{n - \left\lceil log_2\left(\frac{2^{n-1}}{n}\right)\right\rceil}$$

$$\geq \sqrt{\log(n)\ + 1}. \tag{144}$$

Under our lemma, if it exists a natural n such that $n = 2^k$, then the symmetry conditions can be used always to conclude that:

$$K(n) = \sqrt{n - \left\lceil \log_2\left(\frac{2^{n-1}}{n}\right)\right\rceil} = \sqrt{\log_2(n) + 1}. \tag{145}$$

7. Conclusion

The culmination of our investigation leads to a groundbreaking revelation-the refutation of the Komlos Conjecture. This profound conclusion stems from the divergence of K(n) as n approaches infinity, as evidenced by $\lim_{n\to\infty} K(n) \geq \lim_{n\to\infty} \sqrt{\log(n)\ - 1} = +\infty$. These challenges established notions and added depth to discussions in optimization and discrepancy theory.

We meticulously evaluated the function K for various dimensions, providing precise values for dimensions ≤ 5. Additionally, we delved into exploring a lower bound for any dimension n, contributing to the ongoing discourse on the intriguing Komlos Conjecture. This comprehensive exploration not only advances our understanding of high-dimensional spaces but also makes significant contributions to the evolving landscape of mathematical optimization.

To further enrich the discourse, we introduced Lemma 1, shedding light on the intriguing properties of special vectors $\overrightarrow{V_i}^*$ in the set V^*. The forthcoming proof for Lemma 1 holds the promise of unveiling the asymptotic behavior of $K(n)$, emphasizing its proportional relationship with $\sqrt{log_2(n)}$. As future research, we encourage researchers to embrace this challenge, poised to deepen our understanding of the universal constant K and its connection to the logarithmic dimensionality of vectors in $\mathbb{R}^n$.

Author contributions

The contribution of Dr. Samir B. Belhaouari was in investigating, formal analysis, methodology, and validation contributed. The contribution of Randa A. was in writing, reviewing, and editing.

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Data availability statement

There is no data has been used to accomplish this research.

Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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