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Research Article

Optical Solitons with Differential Group Delay and Inter-Modal Dispersion Singlet

Anwar Ja'afar Mohamad Jawad[1], **Anjan Biswas**[2,3,4,5], **Yakup Yildirim**[6,7,8*], **Ali Saleh Alshomrani**[3]

[1]Department of Computer Technical Engineering, Al-Rafidain University College, Baghdad-10064, Iraq
[2]Department of Mathematics and Physics, Grambling State University, Grambling, LA 71245-2715, USA
[3]Mathematical Modeling and Applied Computation (MMAC) Research Group, Center of Modern Mathematical Sciences and their Applications (CMMSA), Department of Mathematics, King Abdulaziz University, Jeddah-21589, Saudi Arabia
[4]Department of Applied Sciences, Cross-Border Faculty of Humanities, Economics and Engineering, Dunare de Jos University of Galati, 111 Domneasca Street, Galati-800201, Romania
[5]Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Medunsa-0204, Pretoria, South Africa
[6]Department of Computer Engineering, Biruni University, Istanbul-34010, Turkey
[7]Mathematics Research Center, Near East University, 99138 Nicosia, Cyprus
[8]Faculty of Arts and Sciences, University of Kyrenia, 99320 Kyrenia, Cyprus
E-mail: yyildirim@biruni.edu.tr

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Abstract: This paper introduces a state of the art investigation into the interaction between soliton propagation and differential group delay, offering a fresh perspective often neglected in previous studies. Motivated by the imperative to comprehend soliton behavior within inter-modal dispersion environments, it presents three innovative methodologies aimed at uncovering novel optical soliton solutions. Through the utilization of cutting-edge algorithms, these approaches unveil the emergence of solitons in hitherto unexplored contexts. The research makes significant strides through extensive numerical simulations, which not only validate theoretical conjectures but also offer practical insights. Furthermore, it delineates crucial parameter limitations essential for the existence of solitons, thus furnishing valuable guidance for future research endeavors and practical applications.

***Keywords*:** traveling waves, csch-function method, extended simplest equation method, tanh-coth approach

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1. Introduction

One of the inherent hindrances of soliton transmission across trans-continental and trans-oceanic distances is the effect of differential group delay, which, in its cumulative form, leads to the effect of birefringence. Thus, the effect of pulse-splitting ensues, and the solitons are split into two components. The scalar version of the governing nonlinear Schrödinger's equation (NLSE) is now formulated into two components, yielding cross-phase modulation (XPM) in addition to the pre-existing self-phase modulation (SPM). The current paper will address such a model with an additional optoelectronic effect that will be considered. It is the inter-modal dispersion that exists along the first

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component of birefringence only, hence the title of the paper.

This study addresses the coupled NLSE to obtain its optical soliton solutions using three integration algorithms: the csch-function approach, the extended simplest equation method, and the tanh-coth scheme. While innovative, the csch-function method, the extended simplest equation method, and the tanh-coth approach have limitations. They may struggle to capture certain nuances of soliton behavior, particularly in highly complex scenarios. Additionally, their applicability could be restricted in situations where nonlinear effects dominate or when dealing with certain types of dispersion. While they have limitations, the csch-function, the extended simplest equation, and tanh-coth methods offer efficient and systematic approaches for exploring soliton behavior. They provide valuable insights into soliton dynamics across diverse scenarios and parameter regimes, thereby enhancing our understanding of their behavior. A wide range of approaches was implemented in the past to study various nonlinear dynamical structures more proficiently [1-2]. Analytical techniques strive to locate exact mathematical solutions to nonlinear evolution equations (NLEEs), even if they are rendered non-integrable through the Painleve test. Due to the complexity of NLEEs, simpler analytical solutions are frequently required. The traveling wave approach involves employing specific strategies to identify exact solutions for particular NLEEs by focusing on solutions that display distinct traveling wave characteristics. Examples include Kudryashov's approach [3-4], the improved Q-expansion strategy [5], the sine-Gordon expansion method [6], the modified simple equation method [7], the generalized exponential rational function method [8], the Riccati equation method [9], the auxiliary equation method [10], the unified method [11], the improved F-expansion technique [12], the exp(-ζ(ξ)) expansion technique [13], the Khater method [14], and the homotopy analysis transform method (Hatm) [15].

It needs to be noted that such forms of integration architectures are the ones that have been reported during the past couple of decades or so. One must not forget the classic methodologies of integration that were applied to reveal soliton solutions for decades before this variety of integration techniques came into existence. These include the inverse scattering transform (IST) and Hirota's bilinear approach, among others. The advantage of IST is that it can yield N-soliton solutions to any model as long as these NLEEs pass the Painleve test of integrability. This is only for reflectionless potential. However, in alternative circumstances, it is the soliton radiation that is the essential component of soliton solutions. The retrieval of the complete soliton solutions package is absolutely not possible by any of the integration strategies listed above in the previous paragraph. Therefore, those methodologies listed in the previous paragraph are not robust in every sense of the word. It must nevertheless be noted that IST does have its own limitations. This scheme fails to retrieve soliton solutions when SPM is of a non-Kerr type, such as power-law, parabolic-law, polynomial-law, dual or triple-power law, or anti-cubic law, or even logarithmic law, among several others. Moreover, IST does not integrate NLSE in dispersion-flattened fibers or for additional optoelectronic devices such as magneto-optic waveguides or for optical metamaterials. Additionally, it is not applicable to retrieve gap solitons in fiber Bragg gratings. Thus, the extreme need and necessities gave way to the modern methods of integrability that have expanded the horizon in quantum optics.

2. Governing equations

Consider the second order coupled NLSE (C-NLSE) equation [1]:

$$i\psi_t + i\alpha\psi_x - \beta\psi_{xx} + (|\psi|^2 + \sigma|\Phi|^2)\psi = 0, \tag{1}$$

and

$$i\Phi_t + \beta\Phi_{xx} + (|\Phi|^2 + \sigma|\psi|^2)\Phi = 0. \tag{2}$$

In this case, the functions $\psi(x, t)$ and $\Phi(x, t)$ represent the unfamiliar complex functions. Here, α represents the inter-modal dispersion along the first component, while β characterizes the chromatic dispersion along the two components of the optical fiber, and σ denotes the XPM parameter.

This paper introduces new insights into soliton propagation with (1) and (2) by examining its interaction with differential group delay, a previously overlooked factor. Motivated by the desire to understand soliton behavior in the presence of inter-modal dispersion, the study presents three novel approaches to reveal previously undiscovered soliton solutions. These methods employ advanced algorithms to explore soliton emergence in uncharted territory. The findings are validated through numerical simulations, providing both theoretical and practical insights. Additionally, the paper identifies key parameter constraints that are essential for ensuring the existence of these solitons, offering valuable guidance for future research and applications.

3. Travelling wave solution

The solutions of Eq. (2) are supposed as

$$\psi(x, t) = u(\xi)e^{i\theta(x, t)} \tag{3}$$

and

$$\Phi(x, t) = v(\xi)e^{i\theta(x, t)} \tag{4}$$

where $\xi = x - \gamma t$ and the phase component is $\theta(x, t) = -kx + \omega t + \theta_0$. Also, $u(\xi)$ and $v(\xi)$ are the amplitude components of the wave. Moreover, γ is its speed, k is the soliton frequency, ω is its wavenumber and θ_0 is the phase constant.

Using Eqs. (3) and (4) and their derivatives, Eqs. (1) and (2) can be decomposing into real and imaginary parts that yield a pair of relations.

The real parts of Eqs. (1) and (2) stick out as

$$-\beta u'' + (\alpha k - \omega + \beta k^2)u + u^3 + \sigma uv^2 = 0, \tag{5}$$

and

$$\beta v'' - (\omega + \beta k^2)v + v^3 + \sigma vu^2 = 0. \tag{6}$$

Also, the imaginary parts appear as

$$-\gamma + \alpha + 2\beta k = 0, \tag{7a}$$

and

$$\gamma + 2\beta k = 0. \tag{7b}$$

From Eq. (7), we get

$$\gamma = -2\beta k, \alpha = -4\beta k. \tag{8}$$

Then Eqs. (5) and (6) become:

$$-\beta u'' - (\omega + 3\beta k^2)u + u^3 + \sigma uv^2 = 0, \tag{9a}$$

and

$$\beta v'' - (\omega + \beta k^2)v + v^3 + \sigma v u^2 = 0. \tag{9b}$$

4. Methodology

In this section, we will apply three different methods to solve Eqs. (5) and (6). These methods are the csch-function method, the tanh-coth method, and the extended simple equation method.

4.1 *Csch function method*

The solutions of many nonlinear equations can be expressed in the form [16]:

$$u(\xi) = A_1 csch^{p_1}(\mu\xi), \tag{10}$$

and

$$v(\xi) = A_2 csch^{p_2}(\mu\xi), \tag{11}$$

and their derivatives stand as

$$u'(\xi) = -A_1 p_1 \mu csch^{p_1}(\mu\xi) coth(\mu\xi), \tag{12}$$

$$u''(\xi) = A_1 p_1 \mu^2((p_1 + 1)csch^{p_1+2}(\mu\xi) + p_1 csch^{p_1}(\mu\xi)), \tag{13}$$

$$v'(\xi) = -A_2 p_2 \mu csch^{p_2}(\mu\xi) coth(\mu\xi), \tag{14}$$

$$v''(\xi) = A_2 p_2 \mu^2((p_2 + 1)csch^{p_2+2}(\mu\xi) + p_2 csch^{p_2}(\mu\xi)), \tag{15}$$

where A_1, A_2, p_1, p_2, and μ are parameters to be determined, and μ is the wave number. We substitute Eqs. (10)-(15) into the reduced equations (5-6), then we get

$$\beta p_1 \mu^2((p_1 + 1)csch^{p_1+2}(\mu\xi) + p_1 csch^{p_1}(\mu\xi)) + (\omega + 3\beta k^2)csch^{p_1}(\mu\xi) -$$

$$A_1^2 csch^{3p_1}(\mu\xi) - \sigma A_2^2 csch^{p_1+2p_2}(\mu\xi) = 0, \tag{16}$$

and

$$\beta p_2 \mu^2((p_2 + 1)csch^{p_2+2}(\mu\xi) + p_2 csch^{p_2}(\mu\xi)) - (\omega + \beta k^2)csch^{p_2}(\mu\xi) +$$

$$A_2^2 csch^{3p_2}(\mu\xi) + \sigma A_1^2 csch^{2p_1+p_2}(\mu\xi) = 0. \tag{17}$$

Equating the exponents and the coefficients of each pair of the Csch functions, we find

$$3p_1 = p_1 + 2p_2,$$

$$3p_2 = p_2 + 2, \tag{18}$$

then

$$p_1 = p_2 = 1. \tag{19}$$

We then collect all terms in Eqs. (16) and (17) with the same power in $csch^k(\mu\xi)$ and set their coefficients to zero to obtain a system of algebraic equations among the unknowns A_1, A_2 and μ, leading to the following system:

$$2\beta\mu^2 - A_1{}^2 - \sigma A_2{}^2 = 0,$$

$$2\beta\mu^2 + A_2{}^2 + \sigma A_1{}^2 = 0,$$

$$\beta\mu^2 + (\omega + 3\beta k^2) = 0,$$

$$\beta\mu^2 - (\omega + \beta k^2) = 0. \tag{20}$$

Solving the system of equations in (20), we get:

$$A_1 = \mp k\sqrt{\frac{2\beta}{(\sigma-1)}},\ A_2 = \mp k\sqrt{\frac{2\beta}{(1-\sigma)}},\ \mu = \mp ik,\ \omega = -2\beta k^2. \tag{21}$$

Thus, singular soliton solutions come out as

$$\psi_1(x, t) = \pm k\sqrt{\frac{2\beta}{(\sigma-1)}}\operatorname{Csc}\left(k(x+2\beta kt)\right)e^{i(-kx-2\beta k^2 t+\theta_0)},\ \sigma < 1, \tag{22}$$

and

$$\Phi_1(x, t) = \pm k\sqrt{\frac{2\beta}{(1-\sigma)}}\operatorname{Csc}\left(k(x+2\beta kt)\right)e^{i(-kx-2\beta k^2 t+\theta_0)},\ \sigma > 1. \tag{23}$$

4.2 *Extended Simple Equation Method (ESEM)*

In this section, the extended form of the simple equation method (ESEM) is introduced to obtain the traveling wave solutions [5-6].

Step 1: Consider the wave forms $\psi(x, t)$ and $\Phi(x, t)$ as in complex form in Eqs. (3) and (4).

Step 2: The forms of the solution for Eqs. (5, 6) appear as

$$u(\xi) = \sum_{j=-1}^{j=N} B_j f^j(\xi), \tag{24}$$

and

$$v(\xi)=\sum_{j=-1}^{j=M} D_j f^j(\xi). \tag{25}$$

Here, B_j and D_j are real constants.

Step 3: Find the positive integers N and M appearing in Eqs. (24) and (25) by employing the balance rule between the non-linear terms of Eqs. (5) and (6) and the highest-order derivative.

Step 4: Suppose that Eqs. (24) and (25) satisfy the following differential equation:

$$f'(\xi)=b_0+b_1 f(\xi)+b_2\left[f(\xi)\right]^2, \tag{26}$$

where b_0, b_1 and b_2 are arbitrary constants.

Step 5: For different values of b_i, the solutions of Eqs. (5) and (6) are given below:

When $b_0 = 0$:

$$f(\xi)=\frac{b_1 e^{b_1(\xi+\xi_0)}}{1-b_2 e^{b_1(\xi+\xi_0)}},\ b_1>0, \tag{27}$$

and

$$f(\xi)=-\frac{b_1 e^{b_1(\xi+\xi_0)}}{1+b_2 e^{b_1(\xi+\xi_0)}},\ b_1<0. \tag{28}$$

When $b_1 = 0$:

$$f(\xi)=\frac{\sqrt{b_0 b_2}\tan\left(\sqrt{b_0 b_2}\,(\xi+\xi_0)\right)}{b_2},\ b_0 b_2>0, \tag{29}$$

and

$$f(\xi)=\frac{\sqrt{-b_0 b_2}\tanh\left(\sqrt{-b_0 b_2}\,(\xi+\xi_0)\right)}{b_2},\ b_0 b_2<0. \tag{30}$$

The general solutions are

$$f(\xi)=\frac{\sqrt{4b_0 b_2-b_1^2}\tan\left(\frac{1}{2}\sqrt{4b_0 b_2-b_1^2}\,(\xi+\xi_0)\right)-b_1}{2b_2},\ 4b_0 b_2>b_1^2 \text{ and } b_2>0, \tag{31}$$

and

$$f(\xi)=\frac{\sqrt{4b_0b_2-b_1^2}\tan\left(\frac{1}{2}\sqrt{4b_0b_2-b_1^2}\,(\xi+\xi_0)\right)+b_1}{2b_2},\ 4b_0b_2>b_1^2 \text{ and } b_2<0. \tag{32}$$

Step 6: By inserting Eqs. (24) and (25) along with Eq. (26) into Eqs. (5) and (6) and equating the coefficients of powers of f^j to zero, a system of equations is obtained. This set of equations is then solved to obtain the values of constant parameters. With these constant values and the $f(\xi)$ values, the solutions of Eqs. (5) and (6) can be achieved.

4.2.1 *Solution by Extended Simple Equation Method (ESEM)*

To find the values of N, apply the homogeneous balance principle to Eq. (9). By balancing u'' and u^3, we get $N+2=3N$, then $N=1$. Similarly, balancing v'' and v^3 yields $M+2=3M$, then $M=1$. Thus, $u(\xi)$ and $v(\xi)$ have the forms given below:

$$u(\xi)=\frac{B_{-1}}{f(\xi)}+B_0+B_1f(\xi),\ B_1\neq 0, \tag{33}$$

and

$$v(\xi)=\frac{D_{-1}}{f(\xi)}+D_0+D_1f(\xi),\ D_1\neq 0, \tag{34}$$

where B_j and D_j ($j=-1, 0, 1$) are constants.

Substitute Eqs. (33) and (34) and their derivatives into Eqs. (5) and (6) to get:

$$\begin{aligned}
&\beta\left(\frac{2b_0{}^2B_{-1}}{f^3}+\frac{3b_0b_1B_{-1}}{f^2}+\left(b_1{}^2+2b_0b_2\right)\frac{B_{-1}}{f}+\left(b_1b_2B_{-1}+b_0b_1B_1\right)+B_1\left(2b_0b_2+b_1{}^2\right)f+\right.\\
&\left.3b_1b_2B_1f^2+2b_2{}^2B_1f^3\right)+\left(\omega+3\beta k^2\right)\left(\frac{B_{-1}}{f(\xi)}+B_0+B_1f(\xi)\right)-\left(\frac{B^3{}_{-1}}{f^3}+3B_0\frac{B^2{}_{-1}}{f^2}+\right.\\
&\left.3B_1\frac{B^2{}_{-1}}{f}+3B_0{}^2\frac{B_{-1}}{f}+6B_0B_1B_{-1}+B_0{}^3+\left(B_{-1}B_1+B_0{}^2\right)3B_1f+3B_0B_1{}^2f^2+B_1{}^3f^3\right)-\\
&\sigma\left(B_{-1}{}^2\left(\frac{B_{-1}}{f(\xi)^3}+\frac{B_0}{f(\xi)^2}+\frac{B_1}{f(\xi)}\right)+2B_{-1}B_0\left(\frac{B_{-1}}{f(\xi)^2}+\frac{B_0}{f(\xi)}+B_1\right)+2B_{-1}B_1\left(\frac{B_{-1}}{f(\xi)^2}+B_0+\right.\right.\\
&\left.B_1f(\xi)\right)+B_0{}^2\left(\frac{B_{-1}}{f(\xi)}+B_0+B_1f(\xi)\right)+2B_0B_1\left(B_{-1}+B_0f(\xi)+B_1f(\xi)^2\right)+B_1{}^2\left(B_{-1}f(\xi)+\right.\\
&\left.\left.B_0f(\xi)^2+B_1f(\xi)^3\right)\right)=0,
\end{aligned} \tag{35}$$

and

$$\beta\left(\frac{2b_0{}^2B_{-1}}{f^3}+\frac{3b_0b_1B_{-1}}{f^2}+\left(b_1{}^2+2b_0b_2\right)\frac{D_{-1}}{f}+\left(b_1b_2D_{-1}+b_0b_1D_1\right)+D_1\left(2b_0b_1+b_1{}^2\right)f+\right.$$

$$\left.3b_1b_2D_1f^2+2b_2{}^2D_1f^3\right)-\left(\omega+\beta k^2\right)\left(\frac{D_{-1}}{f(\xi)}+D_0+D_1f(\xi)\right)+\left(\frac{D^3{}_{-1}}{f^3}+3D_0\frac{D^2{}_{-1}}{f^2}+\right.$$

$$\left.3D_1\frac{D^2{}_{-1}}{f}+3D_0{}^2\frac{D_{-1}}{f}+6D_0D_1D_{-1}+D_0{}^3+\left(D_{-1}D_1+D_0{}^2\right)3D_1f+3D_0D_1{}^2f^2+D_1{}^3f^3\right)+$$

$$\sigma\left(B_{-1}{}^2\left(\frac{D_{-1}}{f(\xi)^3}+\frac{D_0}{f(\xi)^2}+\frac{D_1}{f(\xi)}\right)+2B_{-1}B_0\left(\frac{D_{-1}}{f(\xi)^2}+\frac{D_0}{f(\xi)}+D_1\right)+2B_{-1}B_1\left(\frac{D_{-1}}{f(\xi)}+D_0+\right.\right.$$

$$\left.D_1f(\xi)\right)+B_0{}^2\left(\frac{D_{-1}}{f(\xi)}+D_0+D_1f(\xi)\right)+2B_0B_1\left(D_{-1}+D_0f(\xi)+D_1f(\xi)^2\right)+B_1{}^2\left(D_{-1}f(\xi)+\right.$$

$$\left.\left.D_0f(\xi)^2+D_1f(\xi)^3\right)\right)=0. \tag{36}$$

A set of algebraic system equations is obtained from Eqs. (35) and (36) for different orders of f^j(j = –3, –2, –1, 0, 1, 2, 3), as presented below:

$$2\beta b_0{}^2-B^2{}_{-1}-\sigma D_{-1}{}^2=0,$$

$$3\beta b_0b_1B_{-1}-3B_0B^2{}_{-1}-\sigma\left(D_{-1}{}^2B_0+2D_{-1}D_0B_{-1}\right)=0,$$

$$\beta\left(b_1{}^2+2b_0b_2\right)B_{-1}+\left(\omega+3\beta k^2\right)B_{-1}-\left(3B_1B^2{}_{-1}+3B_0{}^2B_{-1}\right)-\sigma\left(D_{-1}{}^2B_1+\right.$$

$$\left.2D_{-1}D_0B_0+2D_{-1}D_1B_{-1}+D_0{}^2B_{-1}\right)=0,$$

$$\beta\left(b_1b_2B_{-1}+b_0b_1B_1\right)+\left(\omega+5\beta k^2\right)\left(B_0\right)-\left(6B_0B_1B_{-1}+B_0{}^3\right)-\sigma\left(2D_{-1}D_0\left(B_1\right)+\right.$$

$$\left.2D_{-1}D_1\left(B_0\right)+D_0{}^2\left(B_0\right)+2D_0D_1\left(B_{-1}\right)\right)=0,$$

$$\beta\left(B_1\left(2b_0b_2+b_1{}^2\right)\right)+\left(\omega+5\beta k^2\right)\left(B_1\right)-\left(\left(B_{-1}B_1+B_0{}^2\right)3B_1\right)-\sigma\left(2D_{-1}D_1\left(B_1\right)+\right.$$

$$\left.D_0{}^2\left(B_1\right)+2D_0D_1\left(B_0\right)+D_1{}^2B_{-1}\right)=0,$$

$$\beta\left(3b_1b_2B_1\right)-3B_0B_1^{\,2}-\sigma\left(2D_0D_1\left(B_1\right)+D_1^{\,2}\left(B_0\right)\right)=0,$$

$$2\beta b_2^{\,2}-B_1^{\,2}-\sigma D_1^{\,2}=0,$$

$$2\beta b_0^{\,2}+D^2{}_{-1}+\sigma B_{-1}^{\,2}=0,$$

$$\beta 3b_0b_1D_{-1}+3D_0D^2{}_{-1}+\sigma B_{-1}^{\,2}D_0+2B_{-1}B_0D_{-1}=0,$$

$$\beta\left(b_1^{\,2}+2b_0b_2\right)D_{-1}-\left(\omega+\beta k^2\right)D_{-1}+3D_1D^2{}_{-1}+3D_0^{\,2}D_{-1}+\sigma\left(B_{-1}^{\,2}D_1+\right.$$

$$\left.2B_{-1}B_0D_0+2B_{-1}B_1D_{-1}+B_0^{\,2}D_{-1}\right)=0,$$

$$\beta\left(b_1b_2D_{-1}+b_0b_1D_1\right)-\left(\omega-\beta k^2\right)D_0+\left(6D_0D_1D_{-1}+D_0^{\,3}\right)+\sigma\left(2B_{-1}B_0D_1+\right.$$

$$\left.2B_{-1}B_1D_0+B_0^{\,2}D_0+2B_0B_1D_{-1}\right)=0,$$

$$\beta D_1\left(2b_0b_2+b_1^{\,2}\right)-\left(\omega-\beta k^2\right)D_1+\left(D_{-1}D_1+D_0^{\,2}\right)3D_1+\sigma\left(2B_{-1}B_1\left(D_0+D_1\right)+\right.$$

$$\left.B_0^{\,2}D_1+2B_0B_1D_0+B_1^{\,2}D_{-1}\right)=0,$$

$$\beta 3b_1b_2D_1+3D_0D_1^{\,2}+\sigma\left(2B_0B_1D_1+B_1^{\,2}D_0\right)=0,$$

$$\beta 2b_2^{\,2}-D_1^{\,2}-\sigma B_1^{\,2}=0. \tag{37}$$

Case I

When $b_0 = 0$ we get

$$B_{-1}=0,\; B_0=0,\; B_1=b_2\sqrt{\frac{\beta}{2}},\; D_{-1}=0,\; D_0=\sqrt{-3\beta k^2},\; D_1=b_2\sqrt{\frac{\beta}{2}},$$

$$\omega=-\beta k^2,\; \sigma=-3,\; b_1=-2k. \tag{38}$$

Family I

$$\psi_2\left(x,t\right)=\sqrt{2\beta}b_2k\frac{\exp^{-2k\left(x+2\beta kt\right)}}{-1+b_2\exp^{-2k\left(x+2\beta kt\right)}}\exp^{i\left(-kx-2\beta k^2t+\theta_0\right)}, \tag{39}$$

and

$$\Phi_2(x,t)=\left(\sqrt{-3\beta k^2}+\sqrt{2\beta}b_2k\frac{\exp^{-2k(x+2\beta kt)}}{-1+b_2\exp^{-2k(x+2\beta kt)}}\right)\exp^{i\left(-kx-2\beta k^2t+\theta_0\right)}. \tag{40}$$

Family II

$$\psi_3(x,t)=\sqrt{2\beta}b_2k\frac{\exp^{-2k(x+2\beta kt)}}{1+b_2\exp^{-2k(x+2\beta kt)}}\exp^{i\left(-kx-2\beta k^2t+\theta_0\right)}, \tag{41}$$

and

$$\Phi_3(x,t)=\left(\sqrt{-3\beta k^2}+\sqrt{2\beta}b_2k\frac{\exp^{-2k(x+2\beta kt)}}{1+b_2\exp^{-2k(x+2\beta kt)}}\right)\exp^{i\left(-kx-2\beta k^2t+\theta_0\right)}. \tag{42}$$

4.3 *Tanh-Coth method*

Assume $u = u(\xi)$, by using the ansatz, [17]

$$Y=\tanh(\xi), \tag{43}$$

that leads to the change of variables:

$$\frac{du}{d\xi}=\left(1-Y^2\right)\frac{du}{dY}, \tag{44}$$

and

$$\frac{d^2u}{d\xi^2}=-2Y\left(1-Y^2\right)\frac{du}{dY}+\left(1-Y^2\right)^2\frac{d^2u}{dY^2}. \tag{45}$$

For the next step, assume that the solutions for Eqs. (5) and (6) are expressed in the form

$$u(Y)=\sum_{i=0}^{P_1}a_iY^i+\sum_{i=1}^{P_1}b_iY^{-i}, \tag{46}$$

and

$$v(Y)=\sum_{i=0}^{P_2}c_iY^i+\sum_{i=1}^{P_2}d_iY^{-i}, \tag{47}$$

where the parameters P_1 and P_2 can be found by balancing the highest-order linear term with the nonlinear terms in the reducing equation, as described below:

$$P_1 + 2 = 3P_1,\ P_1 + 2 = 2P_2 + P_1, \text{ then } P_1 = P_2 = 1.$$

Tanh-Coth method admits the use of the finite expansion for:

$$u(Y) = \left(a_0 + a_1 Y + b_1 Y^{-1}\right), \tag{48}$$

and

$$v(Y) = \left(c_0 + c_1 Y + d_1 Y^{-1}\right), \tag{49}$$

where a_0, a_1, b_1, c_0, c_1 and d_1 are constants to be determined. Substituting Eqs. (48) and (49) with their derivatives into Eq. (9), we get

$$\begin{aligned}
&2\beta\left[-a_1 Y + a_1 Y^3 + b_1 Y^{-3} - b_1 Y^{-1}\right] + \left(\omega + 3\beta k^2\right)\left(a_0 + a_1 Y + b_1 Y^{-1}\right) - \left[a_0^3 + \right.\\
&6a_0 a_1 b_1 + 3a_1\left(b_1 a_1 + a_0^2\right)Y + 3a_0 a_1^2 Y^2 + a_1^3 Y^3 + 3b_1\left(a_1 b_1 + a_0^2\right)Y^{-1} + \\
&\left.3a_0 b_1^2 Y^{-2} + b_1^3 Y^{-3}\right] - \sigma\left[a_0^2 c_0 + 2(a_0 c_1 d_1 + c_0 a_1 d_1) + 2c_1 b_1 (c_0 + d_1) + \left[a_0^2 c_1 + \right.\right.\\
&\left.2a_0 c_1\left(c_0 + d_1\right) + 2a_1 c_1 d_1 + c_0^2 a_1 + c_1^2 b_1\right]Y + \left(2a_0 d_1 c_0 + c_0^2 b_1 + 2b_1 c_1 d_1 + \right.\\
&\left.d_1^2 a_1\right)Y^{-1} + \left[2a_1 c_1 (c_0 + d_1) + c_1^2 a_0\right]Y^2 + \left(d_1^2 a_0 + 2c_0 b_1 d_1\right)Y^{-2} + c_1^2 a_1 Y^3 + \\
&\left.d_1^2 b_1 Y^{-3}\right] = 0,
\end{aligned} \tag{50}$$

and

$$\begin{aligned}
&2\beta\left[-c_1 Y + c_1 Y^3 + d_1 Y^{-3} - d_1 Y^{-1}\right] - \left(\omega + \beta k^2\right)\left(c_0 + c_1 Y + d_1 Y^{-1}\right) + \left[c_0^3 + 6c_0 c_1 d_1 + \right.\\
&3c_1\left(d_1 c_1 + c_0^2\right)Y + 3c_0 c_1^2 Y^2 + c_1^3 Y^3 + 3d_1\left(c_1 d_1 + c_0^2\right)Y^{-1} + 3c_0 d_1^2 Y^{-2} + \\
&\left.d_1^3 Y^{-3}\right] + \sigma\left[c_0^2 a_0 + 2(c_0 a_1 b_1 + a_0 c_1 b_1) + 2a_1 d_1 (a_0 + b_1) + \left[c_0^2 a_1 + 2c_0 a_1\left(a_0 + \right.\right.\right.\\
&\left.\left.b_1\right) + 2c_1 a_1 b_1 + a_0^2 c_1 + a_1^2 d_1\right]Y + \left(2c_0 b_1 a_0 + a_0^2 d_1 + 2b_1 a_1 d_1 + b_1^2 c_1\right)Y^{-1} + \\
&\left.\left[2c_1 a_1 (a_0 + b_1) + a_1^2 c_0\right]Y^2 + \left(b_1^2 c_0 + 2a_0 b_1 d_1\right)Y^{-2} + a_1^2 c_1 Y^3 + b_1^2 d_1 Y^{-3}\right] = 0.
\end{aligned} \tag{51}$$

Equating expressions at Y^i, ($i = -3, -2, -1, 0, 1, 2, 3$), to zero, we have the following algebraic system equations:

$$2\beta b_1 - b_1^3 - \sigma d_1^2 b_1 = 0,$$

$$3a_0 b_1^2 + \sigma\left(d_1^2 a_0 + 2c_0 b_1 d_1\right) = 0,$$

$$-2\beta b_1 + \left(\omega + 3\beta k^2\right)b_1 - 3b_1\left(a_1 b_1 + a_0^2\right) - \sigma\left(2a_0 d_1 c_0 + c_0^2 b_1 + 2b_1 c_1 d_1 + d_1^2 a_1\right) = 0,$$

$$\left(\omega + 3\beta k^2\right)a_0 - a_0^3 + 6a_0 a_1 b_1 - \sigma\left[a_0^2 c_0 + 2\left(a_0 c_1 d_1 + c_0 a_1 d_1\right) + 2c_1 b_1\left(c_0 + d_1\right)\right] = 0,$$

$$-2\beta a_1 + \left(\omega + 3\beta k^2\right)a_1 - 3a_1\left(b_1 a_1 + a_0^2\right) - \sigma\left[a_0^2 c_1 + 2a_0 c_1\left(c_0 + d_1\right) + 2a_1 c_1 d_1 + c_0^2 a_1 + c_1^2 b_1\right] = 0,$$

$$3a_0 a_1^2 + \sigma\left[2a_1 c_1 (c_0 + d_1) + c_1^2 a_0\right] = 0,$$

$$a_1^3 + \sigma c_1^2 a_1 = 0,$$

$$2\beta d_1 + d_1^3 + \sigma b_1^2 d_1 = 0,$$

$$3c_0 d_1^2 + \sigma\left(b_1^2 c_0 + 2a_0 b_1 d_1\right) = 0,$$

$$-2\beta d_1 - \left(\omega - \beta k^2\right)d_1 - 3d_1\left(c_1 d_1 + c_0^2\right) + \sigma\left(2c_0 b_1 a_0 + a_0^2 d_1 + 2b_1 a_1 d_1 + b_1^2 c_1\right) = 0,$$

$$-\left(\omega - \beta k^2\right)c_0 - c_0^3 + 6c_0 c_1 d_1 + \sigma\left[c_0^2 a_0 + 2\left(c_0 a_1 b_1 + a_0 c_1 b_1\right) + 2a_1 d_1\left(a_0 + b_1\right)\right] = 0,$$

$$-2c_1\beta - \left(\omega - \beta k^2\right)c_1 + 3c_1\left(d_1 c_1 + c_0^2\right) + \sigma\left[c_0^2 a_1 + 2c_0 a_1\left(a_0 + b_1\right) + 2c_1 a_1 b_1 + a_0^2 c_1 + a_1^2 d_1\right] = 0,$$

$$3c_0 c_1^2 + \sigma\left[2c_1 a_1 (a_0 + b_1) + a_1^2 c_0\right] = 0,$$

$$c_1^3 + \sigma a_1^2 c_1 = 0. \tag{52}$$

Solving the algebraic system equations (52), one gets the following cases:
Case I

$$a_0 = c_0 = a_1 = c_1 = 0,\ \omega = -2\beta k^2,\ b_1 = \sqrt{\frac{2\beta}{(1-\sigma)}},\ d_1 = \sqrt{\frac{2\beta}{(\sigma - 1)}}. \tag{53}$$

Thus, singular soliton solutions shape up as

$$\Psi_4(x, t) = \sqrt{\frac{2\beta}{(1-\sigma)}} \coth(x + 2\beta kt) \exp^{i\left(-kx - 2\beta k^2 t + \theta_0\right)}, \ \sigma < 1, \tag{54}$$

and

$$\Phi_4(x, t) = \sqrt{\frac{2\beta}{(\sigma - 1)}} \coth(x + 2\beta kt) \exp^{i\left(-kx - 2\beta k^2 t + \theta_0\right)}, \ \sigma > 1. \tag{55}$$

Case II

$$a_0 = c_0 = b_1 = d_1 = 0, \ \sigma = -1, \ c_1 = a_1, \ \omega = -2\beta k^2. \tag{56}$$

Thus, dark soliton solutions turn out to be

$$\Psi_5(x, t) = a_1 \tanh(x + 2\beta kt) \exp^{i\left(-kx - 2\beta k^2 t + \theta_0\right)}, \tag{57}$$

and

$$\Phi_5(x, t) = a_1 \tanh(x + 2\beta kt) \exp^{i\left(-kx - 2\beta k^2 t + \theta_0\right)}. \tag{58}$$

5. Results and discussion

By employing the three techniques: csch, ESEM, and tanh-coth, we have successfully derived exact analytical soliton solutions for the second-order C-NLSE as described in Eqs. (1) and (2). The soliton solutions obtained have broad physical applications. They can enhance optical communication systems for high-speed data transmission, aid in nonlinear optics for compact signal processing devices, and improve fiber optic sensing for precise environmental measurements like temperature and strain. The exact solutions for the system are represented as $\psi(x, t)$ and $\Phi(x, t)$. After considering specific parameter values, we transform the C-NLSE into a system of real and imaginary equations described in Eqs. (5)-(7). Subsequently, we explore the analytical solutions of this system using the three methods. We examine dark soliton solutions in Figures 1 and 2. In the realm of optical communication systems, dark solitons, characterized by localized intensity minima, offer valuable insights. These robust solitons maintain their shape during propagation, resisting dispersion effects. Understanding these dark soliton solutions is crucial for optimizing optical communication systems, where preserving waveform integrity is essential for reliable data transmission. Moreover, in nonlinear optics, dark solitons play a pivotal role in phenomena such as soliton collisions and interactions, opening avenues for exploring innovative optical functionalities and device applications.

In Figures 1 and 2, surface and 2D plots depict dark soliton solutions defined by Eqs. (57) and (58), with specific parameter values: $\beta = 1$, $k = 1$, and $a_1 = 1$. Meanwhile, Figure 2 illustrates the behavior of these dark soliton solutions as we vary the parameter a_1, while keeping $\beta = 1$, and $k = 1$ constant. Consequently, it becomes apparent that modifying the parameter a_1 produces distinct outcomes for $\psi_5(x, t)$ and $\Phi_5(x, t)$. This approach yields dependable and robust results.

Unlike previous studies that primarily focused on mathematical techniques for solving the coupled nonlinear Schrödinger equation [1], our research explores the practical implications of soliton propagation interacting with differential group delay. While prior work relied on the improved, modified, and extended tanh-function method to derive accurate solutions for quantum systems, our study adopts three innovative approaches to uncover new soliton solutions amidst inter-modal dispersion. Through the utilization of advanced algorithms, we unveil soliton emergence in previously unexplored scenarios, providing valuable insights for optical communication and nonlinear optics applications. Furthermore, whereas previous research predominantly emphasized analytical and numerical solutions employing implicit finite difference methods, our study offers a fresh perspective on soliton behavior dynamics and presents practical guidance for future research and applications in the field.

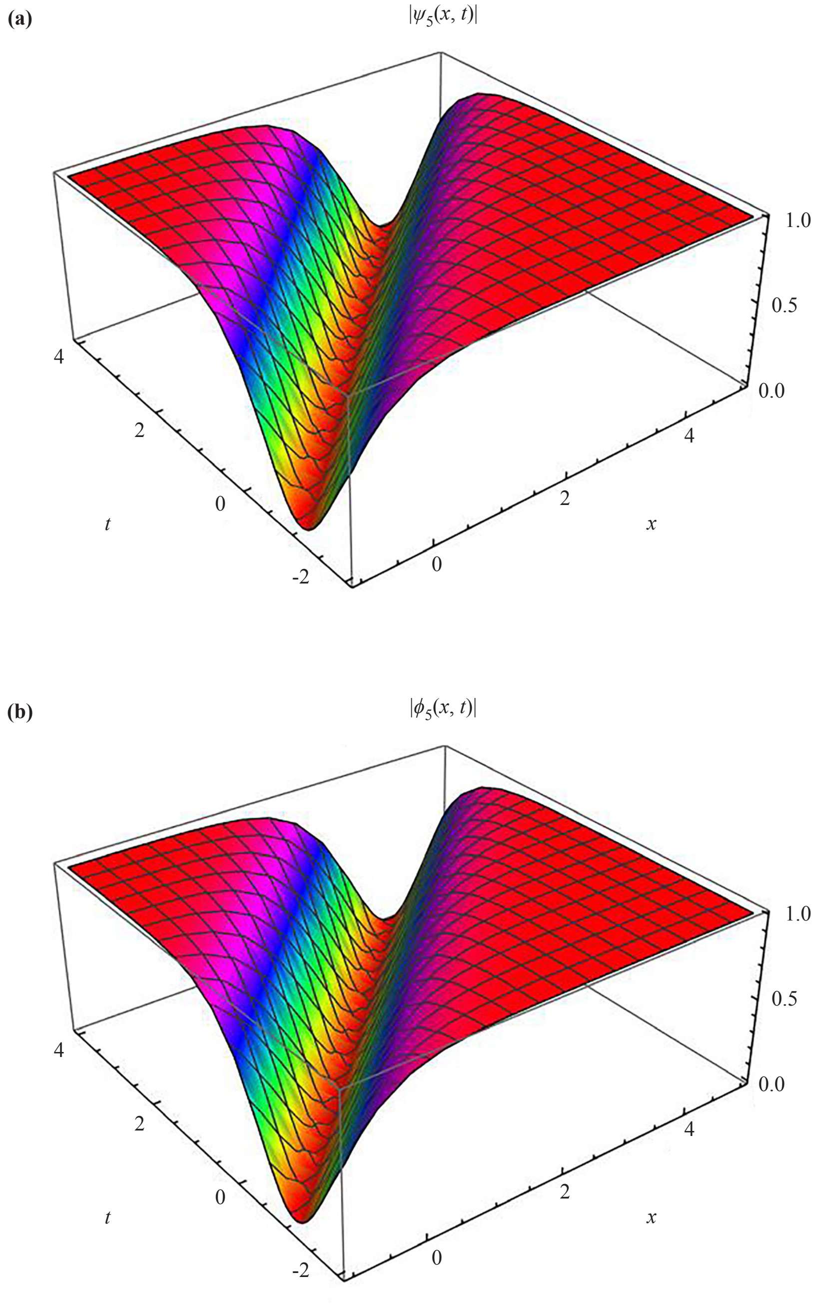

Figure 1. Surface plots of dark soliton solutions (57) and (58)

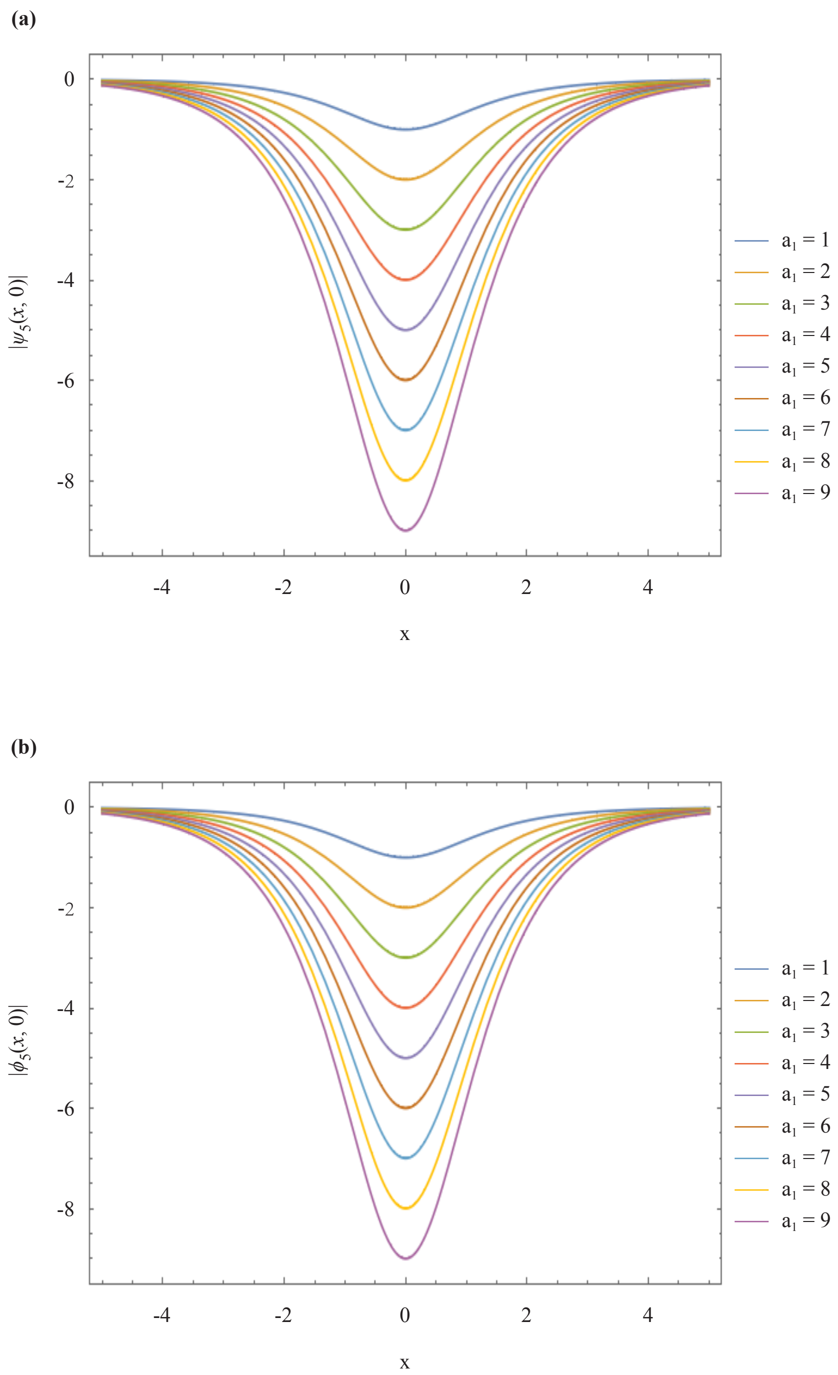

Figure 2. 2D plots of dark soliton solutions (57) and (58)

6. Conclusion

In this study, we have obtained exact soliton solutions for propagating waves in the corresponding C-NLSE (1) using three approaches. Additionally, we have incorporated 2D plots to demonstrate the behavior of dark soliton solutions as the parameter varies while other variables remain constant. The techniques employed in this research may have potential applications in other nonlinear partial differential equations within the field of natural sciences [17-36].

Naturally, as pointed out, the three integration architectures never revealed multiple-soliton solutions. Only the

1-soliton solutions were recovered. Moreover, the soliton radiation solutions are completely out of the question, as only IST can provide such solutions. Nevertheless, these integration schemes are indeed helpful in quickly revealing 1-soliton solutions to the model, which can be applied to carry out additional studies such as the recovery of conservation laws or the retrieval of 1-soliton solutions when the model is considered with perturbation terms, among others.

In conclusion, this paper pioneers the study of soliton propagation interacting with differential group delay, unveiling new solutions. Through innovative approaches and advanced algorithms, it uncovers soliton emergence in unexplored scenarios, validated by numerical simulations. Future research can build upon these findings to further explore soliton dynamics and develop novel methodologies.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Alharbi TG, Alharbi A. Constructions of the soliton solutions to coupled nonlinear Schrödinger equation with advanced mathematical techniques. *AIP Advances*. 2023; 13(12): 125006. Available from: https://doi.org/10.1063/5.0177321.

[2] Jawad A, Biswas A. Solutions of resonant nonlinear schrödinger's equation with exotic non-kerr law nonlinearities. *Al-Rafidain Journal of Engineering Sciences*. 2024; 2(1): 43-50. Available from: https://doi.org/10.61268/2bz73q95.

[3] Malik S, Hashemi MS, Kumar S, Rezazadeh H, Mahmoud W, Osman MS. Application of new Kudryashov method to various nonlinear partial differential equations. *Optical and Quantum Electronics*. 2022; 55(1): 8. Available from: https://doi.org/10.1007/s11082-022-04261-y.

[4] Kumar S, Nisar KS, Niwas M. On the dynamics of exact solutions to a (3+1)-dimensional YTSF equation emerging in shallow sea waves: Lie symmetry analysis and generalized Kudryashov method. *Results in Physics*. 2023; 48: 106432-106432. Available from: https://doi.org/10.1016/j.rinp.2023.106432.

[5] Almatrafi MB, Alharbi A. New soliton wave solutions to a nonlinear equation arising in plasma physics. *Computer Modeling in Engineering & Sciences*. 2023; 137(1): 827-841. Available from: https://doi.org/10.32604/cmes.2023.027344.

[6] Kundu PR, Fahim RA, Islam E, Akbar MA. The sine-Gordon expansion method for higher-dimensional NLEEs and parametric analysis. *Heliyon*. 2021; 7(3): e06459-e06459. Available from: https://doi.org/10.1016/j.heliyon.2021.e06459.

[7] Biswas A, Yildirim Y, Yasar E, Zhou Q, Moshokoa SP, Belic M. Optical solitons for Lakshmanan-Porsezian-Daniel model by modified simple equation method. *Optik*. 2018; 160: 24-32. Available from: https://doi.org/10.1016/j.ijleo.2018.01.100.

[8] Ghanbari B, Gómez JF. The generalized exponential rational function method for Radhakrishnan-Kundu-Lakshmanan equation with β-conformable time derivative. *Revista Mexicana de Física*. 2019; 65: 503-518. Available from: https://doi.org/10.31349/revmexfis.65.503.

[9] Akram G, Sadaf M, Arshed S, Sameen F. Bright, dark, kink, singular and periodic soliton solutions of Lakshmanan-Porsezian-Daniel model by generalized projective Riccati equations method. *Optik*. 2021; 241: 167051. Available from: https://doi.org/10.1016/j.ijleo.2021.167051.

[10] Rezazadeh H, Korkmaz A, Eslami M, Mirhosseini-Alizamini SM. A large family of optical solutions to Kundu-Eckhaus model by a new auxiliary equation method. *Optical and Quantum Electronics*. 2019; 51(3): 84. Available from: https://doi.org/10.1007/s11082-019-1801-4.

[11] Fokas AS, Lenells J. The unified method: I. Nonlinearizable problems on the half-line. *Journal of Physics A*. 2012; 45(19): 195201-195201. Available from: https://doi.org/10.1088/1751-8113/45/19/195201.

[12] Islam N, Khan K, Islam MH. Travelling wave solution of Dodd-Bullough-Mikhailov equation: A comparative study between Generalized Kudryashov and improved F-expansion methods. *Journal of Physics Communications*. 2019; 3(5): 055004. Available from: https://doi.org/10.1088/2399-6528/ab1a47.

[13] Lakestani M, Manafian J. Analytical treatment of nonlinear conformable time-fractional Boussinesq equations

by three integration methods. *Optical and Quantum Electronics*. 2017; 50(1): 4. Available from: https://doi.org/10.1007/s11082-017-1268-0.
[14] Qin H, Raghda AMA, Mostafa MAK, Lu D. Ample soliton waves for the crystal lattice formation of the conformable time-fractional (N + 1) Sinh-Gordon equation by the modified Khater method and the Painlevé property. *Journal of Intelligent and Fuzzy Systems*. 2020; 38(3): 2745-2752. Available from: https://doi.org/10.3233/jifs-179560.
[15] Korpinar Z. On comparison of approximate solutions for linear and nonlinear schrodinger equations. *Acta Scientiarum. Technology*. 2019; 41(1): 36596. Available from: https://doi.org/10.4025/actascitechnol.v41i1.36596.
[16] Jawad A, Abu-AlShaeer M. Highly dispersive optical solitons with cubic law and cubic-quintic-septic law nonlinearities by two methods. *Al Rafidain Journal of Engineering Sciences*. 2023; 1(1): 1-8. Available from: https://doi.org/10.61268/sapgh524.
[17] Wang S. Novel soliton solutions of CNLSEs with Hirota bilinear method. *Journal of Optics*. 2023; 52(3): 1602-1607. Available from: https://doi.org/10.1007/s12596-022-01065-x.
[18] Kopçasız B, Yaşar E. The investigation of unique optical soliton solutions for dual-mode nonlinear Schrödinger's equation with new mechanisms. *Journal of Optics*. 2022; 52(3): 1513-1527. Available from: https://doi.org/10.1007/s12596-022-00998-7.
[19] Tang L. Bifurcations and optical solitons for the coupled nonlinear Schrödinger equation in optical fiber Bragg gratings. *Journal of Optics*. 2022; 52(3): 1388-1398. Available from: https://doi.org/10.1007/s12596-022-00963-4.
[20] Thị TN, Van LC. Supercontinuum generation based on suspended core fiber infiltrated with butanol. *Journal of Optics*. 2023; 52(4): 2296-2305. Available from: https://doi.org/10.1007/s12596-023-01323-6.
[21] Li Z, Zhu EY. Optical soliton solutions of stochastic Schrödinger-Hirota equation in birefringent fibers with spatiotemporal dispersion and parabolic law nonlinearity. *Journal of Optics*. 2023; 55: 407. Available from: https://doi.org/10.1007/s12596-023-01287-7.
[22] Han T, Zhao L, Li C, Zhao L. Bifurcations, stationary optical solitons and exact solutions for complex Ginzburg-Landau equation with nonlinear chromatic dispersion in non-Kerr law media. *Journal of Optics*. 2022; 52(2): 831-844. Available from: https://doi.org/10.1007/s12596-022-01041-5.
[23] Tang L. Phase portraits and multiple optical solitons perturbation in optical fibers with the nonlinear Fokas-Lenells equation. *Journal of Optics*. 2023; 52: 2214-2223. Available from: https://doi.org/10.1007/s12596-023-01097-x.
[24] Nandy S, Lakshminarayanan V. Adomian decomposition of scalar and coupled nonlinear Schrödinger equations and dark and bright solitary wave solutions. *Journal of Optics.* 2015; 44: 397-404. Available from: https://doi.org/10.1007/s12596-015-0270-9.
[25] Chen W, Shen M, Kong Q, Wang Q. The interaction of dark solitons with competing nonlocal cubic nonlinearities. *Journal of Optics*. 2015; 44: 271-280. Available from: https://doi.org/10.1007/s12596-015-0255-8.
[26] Xu SL, Petrović N, Belić MR. Two-dimensional dark solitons in diffusive nonlocal nonlinear media. *Journal of Optics*. 2015; 44(2): 172-177. Available from: https://doi.org/10.1007/s12596-015-0243-z.
[27] Dowluru RK, Bhima PR. Influences of third-order dispersion on linear birefringent optical soliton transmission systems. *Journal of Optics*. 2011; 40(3): 132-142. Available from: https://doi.org/10.1007/s12596-011-0045-x.
[28] Singh M, Sharma AK, Kaler RS. Investigations on optical timing jitter in dispersion managed higher order soliton system. *Journal of Optics*. 2011; 40: 1-7. Available from: https://doi.org/10.1007/s12596-010-0021-x.
[29] Janyani V. Formation and propagation-dynamics of primary and secondary soliton-like pulses in bulk nonlinear media. *Journal of Optics*. 2008; 37(1): 1-8. Available from: https://doi.org/10.1007/bf03354831.
[30] Hasegawa A. Application of optical solitons for information transfer in fibers-a tutorial review. *Journal of Optics*. 2004; 33(3): 145-156. Available from: https://doi.org/10.1007/bf03354760.
[31] Mahalingam A, Uthayakumar A, Anandhi P. Dispersion and nonlinearity managed multisoliton propagation in an erbium doped inhomogeneous fiber with gain/loss. *Journal of Optics*. 2013; 42(3): 182-188. Available from: https://doi.org/10.1007/s12596-012-0105-x.
[32] Seadawy AR, Arnous AH, Biswas A, Belic MR. Optical solitons with Sasa-Satsuma equation by F-expansion scheme. *Optoelectronics and Advanced Materials-Rapid Communications*. 2019; 13: 31-36. Available from: https://oam-rc.inoe.ro/articles/optical-solitons-with-sasa-satsuma-equation-by-f-expansion-scheme/
[33] Chou D, Rehman HU, Amer A, Amer A. New solitary wave solutions of generalized fractional Tzitzéica-type evolution equations using Sardar sub-equation method. *Optical and Quantum Electronics*. 2023; 55(13): 1148. Available from: https://doi.org/10.1007/s11082-023-05425-0.
[34] Shahzad MU, Rehman HU, Awan AU, Zafar Z, Hassan AM, Iqbal I. Analysis of the exact solutions of nonlinear coupled Drinfeld-Sokolov-Wilson equation through ϕ^6-model expansion method. *Results in Physics*. 2023; 52(52):

106771-106771. Available from: https://doi.org/10.1016/j.rinp.2023.106771.

[35] Razzaq W, Zafar A, Alsharidi AK, Alomair MA. New three wave and periodic solutions for the nonlinear (2+1)-dimensional burgers equations. *Symmetry*. 2023; 15(8): 1573. Available from: https://doi.org/10.3390/sym15081573.

[36] Razzaq W, Zafar A, Ahmed HM, Rabie WB. Construction of solitons and other wave solutions for generalized kudryashov's equation with truncated M-fractional derivative using two analytical approaches. *International Journal of Applied and Computational Mathematics*. 2023; 10(1): 21. Available from: https://doi.org/10.1007/s40819-023-01660-x.