


Research Article

Optical Solitons with Dispersive Concatenation Model Having Multiplicative White Noise by the Enhanced Direct Algebraic Method

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Received: 18 December 2023; **Revised:** 8 March 2024; **Accepted:** 18 March 2024

Abstract: This paper investigates the significance of the dispersive concatenation model, incorporating the Kerr law of self-phase modulation in the presence of white noise. Our methodology relies on the enhanced direct algebraic method for integration. We reveal that intermediate solutions are expressed in terms of Jacobi's elliptic functions, leading to soliton solutions as the modulus of ellipticity approaches unity. This discovery culminates in the emergence of a diverse range of optical solitons. Our findings contribute novelty to the existing literature by offering insights into the behavior of optical solitons within the dispersive concatenation model, presenting a significant advancement in understanding this complex phenomenon.

Keywords: solitons, concatenation, dispersion, white noise

AMS: 78A60

1. Introduction

One of the most interesting forms of nonlinear evolution equations that was proposed in nonlinear optics is the concatenation model. This was conceived in 2014 when three of the pre-existing well-known models were conjoined to formulate a concatenated version of the nonlinear evolution equation. This model is the conjunction of the familiar nonlinear Schrödinger's equation (NLSE), the Lakshmanan-Porsezian-Daniel (LPD) model and the Sasa-Satsuma equation. Later during the same year and in the following year another form of the concatenation model was proposed, this time with the dispersive effect being predominant. This is therefore coined as the dispersive concatenation model [1-5]. This version is obtained from the Schrödinger-Hirota equation (SHE), the LPD and the fifth-order NLSE, thus making the model truly dispersive.

A considerable amount of results has been subsequently reported for this model during the past couple of years [6-10]. These include the retrieval of optical soliton solutions and the conservation laws that were recovered by means of multipliers approach. The model was later studied in absence of self-phase modulation (SPM) as well as with the inclusion of spatio-temporal dispersion (STD) in addition to the usual chromatic dispersion (CD). The mitigation of the Internet bottleneck effect was thus proposed. The quiescent optical solitons for the model were also recovered with nonlinear CD and Kerr law of SPM as well as power-law of SPM and finally without SPM. The numerical analysis of the solitons was also carried out for the concatenation model by the aid of Laplace-Adomian decomposition approach.

The current paper addresses the model with the inclusion of the white noise in Itô sense leading to the analysis of the governing stochastic differential equation. The enhanced direct algebraic method is the integration technology that is adopted in the work. A wide spectrum of optical soliton solutions is thus revealed. These are recovered through the intermediate Jacobi's elliptic functions which gave way to soliton solutions when the modulus of ellipticity approached unity. The details are exhibited in the rest of the paper.

This paper's originality lies in its exploration of the dispersive concatenation model with the inclusion of white noise in Itô sense. By utilizing the enhanced direct algebraic method, we unveil a diverse range of optical soliton solutions derived from intermediate Jacobi's elliptic functions. This novel approach sheds light on the behavior of solitons in the presence of stochastic influences, contributing significantly to the existing literature on the dispersive concatenation model.

1.1 Governing model

The dispersive concatenation model with the Kerr law of SPM in the presence of multiplicative white noise in the Itô sense is given for the first time in the current paper:

$$\begin{aligned}
 & iq_t + aq_{xx} + b|q|^2 q - i\delta_1 \left[\sigma_1 q_{xxx} + \sigma_2 |q|^2 q_x \right] + \sigma q W_t(t) \\
 & + \delta_2 \left[\sigma_3 q_{xxxx} + \sigma_4 |q|^2 q_{xx} + \sigma_5 |q|^4 q + \sigma_6 |q_x|^2 q + \sigma_7 q_x^2 q^* + \sigma_8 q_{xx}^* q^2 \right] \\
 & - i\delta_3 \left[\sigma_9 q_{xxxxx} + \sigma_{10} |q|^2 q_{xxx} + \sigma_{11} |q|^4 q_x + \sigma_{12} q q_x q_{xx}^* + \sigma_{13} q^* q_x q_{xx} + \sigma_{14} q q_x^* q_{xx} + \sigma_{15} q_x^2 q^* \right] = 0. \quad (1)
 \end{aligned}$$

The function $q(x, t)$ is a complex-valued expression of the wave profile, where x and t denote the spatial and temporal coordinates, respectively. The coefficients a and b are the parameters associated with the CD and SPM of the NLS equation, respectively. The imaginary unit is commonly represented by the symbol $i = \sqrt{-1}$. The coefficients δ_1 , δ_2 , and δ_3 correspond to the parameters of the SHE, the LPD equation, and the fifth-order NLSE, respectively. The symbol σ represents a non-zero constant that denotes the intensity of white noise. Additionally, the conventional Wiener process, denoted as $W(t)$, is defined as the integral of the function $\Lambda(\eta)$ with respect to the Wiener process $W(\eta)$, where η is a variable that takes values less than t . In the given context, the symbol η is used to represent a stochastic variable, whereas $\Lambda(\eta)$ is used to symbolize conventional Gaussian white noise, which is commonly referred to as multiplicative white noise. The purpose of this noise is to indicate a disruption in the excitation phase of a given process.

The paper follows a structured approach with Section 2 laying the theoretical groundwork by dissecting the dispersive concatenation model. Section 3 revisits the principles of the integration method, ensuring clarity on the methodology. In Section 4, the core findings are unveiled-exploration and analysis of optical soliton solutions within the model. Lastly, Section 5 summarizes key outcomes, underscores their significance, and suggests potential future research directions. Together, these sections provide a cohesive narrative, contributing to the understanding of dispersive concatenation model with Kerr law of SPM and multiplicative white noise.

2. Mathematical analysis

Let us consider the assumed structure of the solution to equation (1) as follows:

$$q(x, t) = U(\xi)e^{i\phi(x, t)}. \quad (2)$$

The wave variable ξ is defined as

$$\xi = k(x - vt), \quad (3)$$

where k and v are non-zero constants. Here, $U(\xi)$ is real valued function, which represents the amplitude component of the soliton solution and v is the speed of the soliton, while the phase component $\phi(x, t)$ is defined as

$$\phi(x, t) = -\kappa x + \omega t + \sigma W(t) - \sigma^2 t + \theta_0. \quad (4)$$

In the given context, κ denotes the frequency of the soliton, ω corresponds to the wave number, σ represents the noise coefficient, and θ_0 signifies the phase constant. By substituting equation (2) into equation (1) and afterward decomposing them into their real and imaginary components, we obtain the following expressions:

$$\begin{aligned} & \kappa^2 (10\sigma_9\kappa^3\delta_3 - 6\sigma_3\kappa^2\delta_2 - 3\sigma_1\kappa\delta_1 + a)U'' + \kappa^4 (\sigma_3\delta_2 - 5\sigma_9\kappa\delta_3)U^{(4)} \\ & + (\kappa^4 (\sigma_3\delta_2 - \sigma_9\kappa\delta_3) + \sigma_1\kappa^3\delta_1 - a\kappa^2 + (\sigma^2 - \omega))U + (\sigma_5\delta_2 - \sigma_{11}\kappa\delta_3)U^5 \\ & + (\kappa^2 ((\sigma_{10} + \sigma_{12} + \sigma_{13} - \sigma_{14} - \sigma_{15})\kappa\delta_3 - (\sigma_4 - \sigma_6 + \sigma_7 + \sigma_8)\delta_2) - \sigma_2\kappa\delta_1 + b)U^3 \\ & + \kappa^2 ((\sigma_4 + \sigma_8)\delta_2 - (3\sigma_{10} + \sigma_{12} + \sigma_{13} - \sigma_{14})\kappa\delta_3)U^2U'' \\ & + \kappa^2 ((2\sigma_{12} - 2(\sigma_{13} + \sigma_{14}) - \sigma_{15})\kappa\delta_3 + (\sigma_6 + \sigma_7)\delta_2)UU'^2 = 0, \end{aligned} \quad (5)$$

$$\begin{aligned} & k(-5\sigma_9\kappa^4\delta_3 + 4\sigma_3\kappa^3\delta_2 + 3\sigma_1\kappa^2\delta_1 - 2a\kappa - v)U' \\ & - \sigma_9\delta_3\kappa^5U^{(5)} - \kappa^3(2\kappa(2\sigma_3\delta_2 - 5\sigma_9\kappa\delta_3) + \sigma_1\delta_1)U^{(3)} - \sigma_{11}\kappa\delta_3U^4U' \\ & - \sigma_{10}\delta_3\kappa^3U^2U^{(3)} - \sigma_{15}\delta_3\kappa^3U'^3 - (\sigma_{12} + \sigma_{13} + \sigma_{14})\delta_3\kappa^3UUU'' \\ & - \kappa(\kappa((-3\sigma_{10} + \sigma_{12} - 3\sigma_{13} + \sigma_{14} + \sigma_{15})\kappa\delta_3 + 2(\sigma_4 + \sigma_7 - \sigma_8)\delta_2) + \sigma_2\delta_1)U^2U' = 0. \end{aligned} \quad (6)$$

The soliton speed is attained from the imaginary component, as shown below

$$v = 4\sigma_3\kappa^3\delta_2 + 3\sigma_1\kappa^2\delta_1 - 2a\kappa. \quad (7)$$

Moreover, the frequency reaches

$$\kappa = \frac{(-2\sigma_2\sigma_3 + \sigma_1(\sigma_4 + \sigma_7 - \sigma_8))\delta_2}{2\sigma_1\sigma_{13}\delta_3}, \quad (8)$$

with the addition of parametric restrictions

$$\sigma_9 = \sigma_{10} = \sigma_{11} = \sigma_{15} = 0, \quad (9)$$

$$\sigma_{12} + \sigma_{13} + \sigma_{14} = 0, \quad (10)$$

and

$$\sigma_{13}\sigma_1^2\delta_1\delta_3 + 2\sigma_3(-2\sigma_2\sigma_3 + \sigma_1(\sigma_4 + \sigma_7 - \sigma_8))\delta_2^2 = 0. \quad (11)$$

Then equation (1) reduces to

$$\begin{aligned} & iq_t + aq_{xx} + b|q|^2 q - i\delta_1 \left[\sigma_1 q_{xxx} + \sigma_2 |q|^2 q_x \right] + \sigma q W_t(t) \\ & + \delta_2 \left[\sigma_3 q_{xxxx} + \sigma_4 |q|^2 q_{xx} + \sigma_5 |q|^4 q + \sigma_6 |q_x|^2 q + \sigma_7 q_x^2 q^* + \sigma_8 q_{xx}^* q^2 \right] \\ & - i\delta_3 \left[\sigma_{12} q q_x q_{xx}^* + \sigma_{13} q^* q_x q_{xx} + \sigma_{14} q q_x^* q_{xx} \right] = 0, \end{aligned} \quad (12)$$

and equation (5) shrinks to

$$\begin{aligned} & k^2(-6\sigma_3\kappa^2\delta_2 - 3\sigma_1\kappa\delta_1 + a)U'' + k^2(2\sigma_{14}\kappa\delta_3 + \sigma_4\delta_2 + \sigma_8\delta_2)U^2U'' \\ & + (\sigma_3\kappa^4\delta_2 + \sigma_1\kappa^3\delta_1 - a\kappa^2 + \sigma^2 - \omega)U + \sigma_3\kappa^4\delta_2U^{(4)} \\ & + (-2\sigma_{14}\kappa^3\delta_3 - \sigma_4\kappa^2\delta_2 + \sigma_6\kappa^2\delta_2 - \sigma_7\kappa^2\delta_2 - \sigma_8\kappa^2\delta_2 - \sigma_2\kappa\delta_1 + b)U^3 \\ & + k^2(-4(\sigma_{13} + \sigma_{14})\kappa\delta_3 + \sigma_6\delta_2 + \sigma_7\delta_2)UU'^2 + \sigma_5\delta_2U^5 = 0. \end{aligned} \quad (13)$$

Equation (13) can be further simplified to

$$k^2U^{(4)}(\xi) + \lambda_6U(\xi)^2U''(\xi) + \lambda_3U''(\xi) + \lambda_4U(\xi)U'(\xi)^2 + \lambda_3U(\xi)^5 + \lambda_2U(\xi)^3 + \lambda_1U(\xi) = 0, \quad (14)$$

with

$$\begin{cases} \lambda_1 = \frac{\sigma_3\kappa^4\delta_2 + \sigma_1\kappa^3\delta_1 - a\kappa^2 + (\sigma^2 - \omega)}{\sigma_3\kappa^2\delta_2}, \\ \lambda_2 = \frac{b - \kappa(\kappa(2\sigma_{14}\kappa\delta_3 + (\sigma_4 - \sigma_6 + \sigma_7 + \sigma_8)\delta_2) + \sigma_2\delta_1)}{\sigma_3\kappa^2\delta_2}, \\ \lambda_3 = \frac{\sigma_5}{\sigma_3\kappa^2}, \\ \lambda_4 = \frac{(\sigma_6 + \sigma_7)\delta_2 - 4(\sigma_{13} + \sigma_{14})\kappa\delta_3}{\sigma_3\delta_2}, \\ \lambda_5 = \frac{-3\kappa(2\sigma_3\kappa\delta_2 + \sigma_1\delta_1) + a}{\sigma_3\delta_2}, \\ \lambda_6 = \frac{2\sigma_{14}\kappa\delta_3 + (\sigma_4 + \sigma_8)\delta_2}{\sigma_3\delta_2}, \end{cases} \quad (15)$$

where $\sigma_3 \neq 0$.

3. The enhanced direct algebraic method: a quick recapitulation

This section is a quick overview of the basic procedures with the enhanced direct algebraic technique [6-10]. We could include a governing model that represents a nonlinear evolution equation which has the structure:

$$F(u, u_x, u_t, u_{xt}, u_{xx}, \dots) = 0. \quad (16)$$

The $u = u(x, t)$ is a function that represents the wave profile, where t and x denote the temporal and spatial coordinate variables, respectively.

Using the wave transformation

$$u(x, t) = U(\xi), \quad \xi = k(x - vt), \quad (17)$$

causes a reduction of equation (16) to

$$P(U, -kvU', kU', k^2U'', \dots) = 0. \quad (18)$$

In that expression, k represents the wave width, ξ represents the wave variable, and v represents the wave velocity.

Step 1 We assume that the solution of equation (13) can be expressed in the form

$$U(\xi) = \alpha_0 + \sum_{i=1}^N \{ \alpha_i \theta(\xi)^i + \beta_i \theta(\xi)^{-i} \}, \quad (19)$$

where θ satisfies

$$\theta'(\xi)^2 = \sum_{i=0}^4 \tau_i \theta(\xi)^i, \quad (20)$$

where τ_i ($i = 0, 1, 2, 3, 4$) are constants provided that $\tau_4 \neq 0$. Equation (20) provides several kinds of solutions of different types as follows:

Set 1 If we set $\tau_0 = \tau_1 = \tau_3 = 0$, we get bright soliton with $\tau_2 > 0$, $\tau_4 < 0$ and singular soliton with $\tau_2 > 0$, $\tau_4 > 0$:

$$\theta(\xi) = \sqrt{-\frac{\tau_2}{\tau_4}} \operatorname{sech}[\sqrt{\tau_2} \xi], \quad \tau_2 > 0, \quad \tau_4 < 0, \quad (21)$$

$$\theta(\xi) = \sqrt{\frac{\tau_2}{\tau_4}} \operatorname{csch}[\sqrt{\tau_2} \xi], \quad \tau_2 > 0, \quad \tau_4 > 0, \quad (22)$$

respectively.

Set 2 If we set $\tau_0 = \frac{\tau_2^2}{4\tau_4}$, $\tau_1 = \tau_3 = 0$, we have dark and singular solitons for $\tau_2 < 0$, $\tau_4 > 0$:

$$\theta(\xi) = \sqrt{-\frac{\tau_2}{2\tau_4}} \tanh\left[\sqrt{-\frac{\tau_2}{2}} \xi\right], \quad \tau_2 < 0, \quad \tau_4 > 0, \quad (23)$$

$$\theta(\xi) = \sqrt{-\frac{\tau_2}{2\tau_4}} \coth\left[\sqrt{-\frac{\tau_2}{2}} \xi\right], \quad \tau_2 < 0, \quad \tau_4 > 0, \quad (24)$$

respectively.

Set 3 If we set $\tau_1 = \tau_3 = 0$, we get Jacobi elliptic doubly periodic type solution for different choices of τ_0 as follows:

$$\theta(\xi) = \pm \sqrt{\frac{m^2 \tau_2}{(2m^2 - 1)\tau_4}} \operatorname{cn} \left(\sqrt{\frac{\tau_2}{2m^2 - 1}} \xi \middle| m \right), \quad \tau_0 = \frac{m^2(1 - m^2)\tau_2^2}{(2m^2 - 1)^2 \tau_4}, \quad (25)$$

$$\theta(\xi) = \pm \sqrt{\frac{m^2 \tau_2}{(2 - m^2)\tau_4}} \operatorname{dn} \left(\sqrt{\frac{\tau_2}{2 - m^2}} \xi \middle| m \right), \quad \tau_0 = \frac{(1 - m^2)\tau_2^2}{(2 - m^2)^2 \tau_4}, \quad (26)$$

$$\theta(\xi) = \pm \sqrt{\frac{m^2 \tau_2}{(m^2 + 1)\tau_4}} \operatorname{sn} \left(\sqrt{\frac{\tau_2}{m^2 + 1}} \xi \middle| m \right), \quad \tau_0 = \frac{m^2 \tau_2^2}{(m^2 + 1)^2 \tau_4}. \quad (27)$$

Set 4 If we set $\tau_1 = \tau_3 = 0$, we get Weierstrass elliptic doubly periodic type solutions:

$$\theta(\xi) = \frac{3\wp'(\xi; g_2, g_3)}{\sqrt{\tau_4} [6\wp(\xi; g_2, g_3) + \tau_2]}, \quad \tau_4 > 0, \quad (28)$$

$$\theta(\xi) = \frac{\sqrt{\tau_0} [6\wp(\xi; g_2, g_3) + \tau_2]}{3\wp'(\xi; g_2, g_3)}, \quad \tau_0 > 0, \quad (29)$$

where $g_2 = \frac{\tau_2^2}{12} + \tau_0 \tau_4$ and $g_3 = \frac{\tau_2}{216} (36\tau_0 \tau_4 - \tau_2^2)$ are called invariants of the Weierstrass elliptic function.

Set 5 If we set, $\tau_0 = \tau_1 = 0$, we get straddled soliton solutions with $\tau_2 > 0$ as follows:

$$\theta(\xi) = \frac{-\tau_2 \operatorname{sech}^2 \left[\frac{1}{2} \sqrt{\tau_2} \xi \right]}{\pm 2\sqrt{\tau_2 \tau_4} \tanh \left[\frac{1}{2} \sqrt{\tau_2} \xi \right] + \tau_3}, \quad \tau_4 > 0, \quad (30)$$

$$\theta(\xi) = \frac{\tau_2 \operatorname{csch}^2 \left[\frac{1}{2} \sqrt{\tau_2} \xi \right]}{\pm 2\sqrt{\tau_2 \tau_4} \coth \left[\frac{1}{2} \sqrt{\tau_2} \xi \right] + \tau_3}, \quad \tau_4 > 0, \quad (31)$$

$$\theta(\xi) = \frac{2\tau_2 \operatorname{sech} \left[\sqrt{\tau_2} \xi \right]}{\pm \sqrt{\tau_3^2 - 4\tau_2 \tau_4} - \tau_3 \operatorname{sech} \left[\sqrt{\tau_2} \xi \right]}, \quad \tau_3^2 - 4\tau_2 \tau_4 > 0, \quad (32)$$

$$\theta(\xi) = \frac{2\tau_2 \operatorname{csch} \left[\sqrt{\tau_2} \xi \right]}{\pm \sqrt{4\tau_2 \tau_4 - \tau_3^2} - \tau_3 \operatorname{csch} \left[\sqrt{\tau_2} \xi \right]}, \quad \tau_3^2 - 4\tau_2 \tau_4 < 0, \quad (33)$$

$$\theta(\xi) = - \frac{\tau_2 \tau_3 \operatorname{sech}^2 \left[\frac{\sqrt{\tau_2}}{2} \xi \right]}{\tau_3^2 - \tau_2 \tau_4 \left(1 - \tanh \left[\frac{\sqrt{\tau_2}}{2} \xi \right] \right)^2}, \quad \tau_3 \neq 0, \quad (34)$$

$$\theta(\xi) = \frac{\tau_2 \tau_3 \operatorname{csch}^2 \left[\frac{\sqrt{\tau_2}}{2} \xi \right]}{\tau_3^2 - \tau_2 \tau_4 \left(1 - \coth \left[\frac{\sqrt{\tau_2}}{2} \xi \right] \right)^2}, \quad \tau_3 \neq 0. \quad (35)$$

Step 2 Determine the positive integer number N in equation (19) by balancing the highest order derivatives and the nonlinear terms in equation (18).

Step 4 Substitute (19) into (18) along with (20). As a result of this substitution, we get a polynomial of ϕ . In this polynomial, we gather all terms of the same powers and equate them to zero. We get an overdetermined system of algebraic equations. Mathematica can solve to get the unknown parameters in (17) and (19). Consequently, we obtain the exact solutions of (16).

4. Optical soliton solutions

In equation (14), balancing $U^{(4)}(\xi)$ with $U(\xi)^5$ yields $N = 1$. The solution is expressed in the following structure:

$$U(\xi) = \alpha_0 + \alpha_1 \theta(\xi) + \frac{\beta_1}{\theta(\xi)}. \quad (36)$$

Inserting equation (36) together with equation (20) into equation (14), we get a system of algebraic equations. Solving these equations together yields the following results:

Case 1 Choosing $\tau_0 = \tau_1 = \tau_3 = 0$, yields

$$\alpha_0 = 0, \quad \alpha_1 = \sqrt{\frac{2\tau_4(9\lambda_5\tau_2 + 10\lambda_1)}{\tau_2((\lambda_4 + \lambda_6)\tau_2 + \lambda_2)}}, \quad \beta_1 = 0, \quad k = \frac{\sqrt{-(\lambda_5\tau_2 + \lambda_1)}}{\tau_2}$$

$$\lambda_3 = \frac{((\lambda_4 + \lambda_6)\tau_2 + \lambda_2)(3\lambda_5\tau_2((\lambda_4 - 2\lambda_6)\tau_2 + 4\lambda_2) + 2\lambda_1((\lambda_4 - 4\lambda_6)\tau_2 + 6\lambda_2))}{2(9\lambda_5\tau_2 + 10\lambda_1)^2}. \quad (37)$$

As a result, the solutions of equation (1) reach

$$q(x, t) = \pm \sqrt{-\frac{2(9\lambda_5\tau_2 + 10\lambda_1)}{(\lambda_4 + \lambda_6)\tau_2 + \lambda_2}} \operatorname{sech} \left[\sqrt{-\frac{\lambda_5\tau_2 + \lambda_1}{\tau_2}} (x - vt) \right] \times e^{i \left\{ -\left\{ \frac{(-2\sigma_2\sigma_3 + \sigma_1(\sigma_4 + \sigma_7 - \sigma_8))\delta_2}{2\sigma_1\sigma_{13}\delta_3} \right\} x + \omega t + \sigma W(t) - \sigma^2 t + \theta_0 \right\}}, \quad (38)$$

and

$$q(x, t) = \pm \sqrt{\frac{2(9\lambda_5\tau_2 + 10\lambda_1)}{(\lambda_4 + \lambda_6)\tau_2 + \lambda_2}} \operatorname{csch} \left[\sqrt{-\frac{\lambda_5\tau_2 + \lambda_1}{\tau_2}} (x - vt) \right] \times e^{i \left\{ -\left\{ \frac{(-2\sigma_2\sigma_3 + \sigma_1(\sigma_4 + \sigma_7 - \sigma_8))\delta_2}{2\sigma_1\sigma_{13}\delta_3} \right\} x + \omega t + \sigma W(t) - \sigma^2 t + \theta_0 \right\}}. \quad (39)$$

Equation (38) is a bright soliton with $(\lambda_5\tau_2 + \lambda_1)\tau_2 < 0$, and $(9\lambda_5\tau_2 + 10\lambda_1)((\lambda_4 + \lambda_6)\tau_2 + \lambda_2) < 0$, while equation (39) is a singular soliton with $(\lambda_5\tau_2 + \lambda_1)\tau_2 < 0$, and $(9\lambda_5\tau_2 + 10\lambda_1)((\lambda_4 + \lambda_6)\tau_2 + \lambda_2) > 0$.

Case 2 Choosing $\tau_0 = \frac{\tau_2^2}{4\tau_4}$, $\tau_1 = \tau_3 = 0$, yields

$$\alpha_0 = 0, \alpha_1 = \sqrt{\frac{2\tau_4(3\lambda_5\tau_2 + 5\lambda_1)}{\tau_2(4\lambda_2 - (\lambda_4 - 4\lambda_6)\tau_2)}}, \beta_1 = 0,$$

$$\lambda_3 = \frac{(4\lambda_2 - (\lambda_4 - 4\lambda_6)\tau_2)(6\lambda_2\lambda_5\tau_2 + \lambda_1((\lambda_4 - 4\lambda_6)\tau_2 + 6\lambda_2))}{4(3\lambda_5\tau_2 + 5\lambda_1)^2},$$

$$k = \sqrt{\frac{-2\lambda_1((\lambda_4 + \lambda_6)\tau_2 + \lambda_2) - \lambda_5\tau_2((\lambda_4 + 2\lambda_6)\tau_2 + 2\lambda_2)}{2\tau_2^2(4\lambda_2 - (\lambda_4 - 4\lambda_6)\tau_2)}}. \quad (40)$$

As a result, the solutions of equation (1) are dark and singular solitons with $(3\lambda_5\tau_2 + 5\lambda_1)((\lambda_4 - 4\lambda_6)\tau_2 - 4\lambda_2) > 0$, and $\tau_2(4\lambda_2 - (\lambda_4 - 4\lambda_6)\tau_2)(2\lambda_1((\lambda_4 + \lambda_6)\tau_2 + \lambda_2) + \lambda_5\tau_2((\lambda_4 + 2\lambda_6)\tau_2 + 2\lambda_2)) > 0$:

$$q(x, t) = \pm \sqrt{\frac{2(3\lambda_5\tau_2 + 5\lambda_1)}{(\lambda_4 - 4\lambda_6)\tau_2 - 4\lambda_2}} \tanh \left[\frac{1}{2} \sqrt{\frac{2\lambda_1((\lambda_4 + \lambda_6)\tau_2 + \lambda_2) + \lambda_5\tau_2((\lambda_4 + 2\lambda_6)\tau_2 + 2\lambda_2)}{\tau_2(4\lambda_2 - (\lambda_4 - 4\lambda_6)\tau_2)}} (x - vt) \right]$$

$$\times e^{i \left(\left\{ \frac{(-2\sigma_2\sigma_3 + \sigma_1(\sigma_4 + \sigma_7 - \sigma_8))\delta_2}{2\sigma_1\sigma_{13}\delta_3} \right\} x + \omega t + \sigma W(t) - \sigma^2 t + \theta_0 \right)}, \quad (41)$$

and

$$q(x, t) = \pm \sqrt{\frac{2(3\lambda_5\tau_2 + 5\lambda_1)}{(\lambda_4 - 4\lambda_6)\tau_2 - 4\lambda_2}} \coth \left[\frac{1}{2} \sqrt{\frac{2\lambda_1((\lambda_4 + \lambda_6)\tau_2 + \lambda_2) + \lambda_5\tau_2((\lambda_4 + 2\lambda_6)\tau_2 + 2\lambda_2)}{\tau_2(4\lambda_2 - (\lambda_4 - 4\lambda_6)\tau_2)}} (x - vt) \right]$$

$$\times e^{i \left(\left\{ \frac{(-2\sigma_2\sigma_3 + \sigma_1(\sigma_4 + \sigma_7 - \sigma_8))\delta_2}{2\sigma_1\sigma_{13}\delta_3} \right\} x + \omega t + \sigma W(t) - \sigma^2 t + \theta_0 \right)}, \quad (42)$$

respectively.

Case 3 In this case, we choose $\tau_1 = \tau_3 = 0$ and provide three possible choices for τ_0 as follows:

$$(i): \tau_0 = \frac{m^2(1 - m^2)\tau_2^2}{(2m^2 - 1)^2 \tau_4},$$

$$\alpha_0 = 0, \beta_1 = 0, \alpha_1 = \sqrt{\frac{\tau_4(18\lambda_5\tau_2 + 20\lambda_1)}{\tau_2((\lambda_4 + \lambda_6)\tau_2 + \lambda_2)}},$$

$$\lambda_3 = \frac{(\tau_2(\lambda_4 m_2 + \lambda_6 m_1) + \lambda_2 m_1)(2\lambda_1 m_4^2((\lambda_4 - 4\lambda_6)\tau_2 + 6\lambda_2) + 3\lambda_5\tau_2((\lambda_4 - 2\lambda_6)m_1\tau_2 + 4\lambda_2 m_4^2))}{2(3\lambda_5(5 - 2m_1)\tau_2 + 10\lambda_1 m_4^2)^2},$$

$$k = \sqrt{\frac{\lambda_1 m_4^2((\lambda_4 + \lambda_6)\tau_2 + \lambda_2) + \lambda_5\tau_2(\tau_2(\lambda_6 m_4^2 + \lambda_4 m_3) + \lambda_2 m_4^2)}{\tau_2^2(\lambda_4\tau_2 + \tau_2(\lambda_6 + 4(3\lambda_4 - 2\lambda_6)(m^2 - 1)m^2) + \lambda_2 m_1)}}. \quad (43)$$

where $m_1 = -8m^4 + 8m^2 + 1$, $m_2 = 12m^4 - 12m^2 + 1$, $m_3 = 6m^4 - 6m^2 + 1$, $m_4 = 1 - 2m^2$, As a consequence, the solutions of equation (1) reach

$$\begin{aligned}
 q(x, t) = & \pm \sqrt{-\frac{2m^2(9\lambda_5\tau_2 + 10\lambda_1)}{(2m^2 - 1)((\lambda_4 + \lambda_6)\tau_2 + \lambda_2)}} \\
 & \times \operatorname{cn} \left(\sqrt{\frac{\lambda_1 m_4^2 ((\lambda_4 + \lambda_6)\tau_2 + \lambda_2) + \lambda_5 \tau_2 (\tau_2 (\lambda_6 m_4^2 + \lambda_4 m_3) + \lambda_2 m_4^2)}{m_4 \tau_2 (\lambda_4 \tau_2 + \tau_2 (\lambda_6 + 4(3\lambda_4 - 2\lambda_6)(m^2 - 1)m^2) + \lambda_2 m_1)}}} (x - vt) \middle| m \right) \\
 & \times e^{\left(i \left[\frac{(-2\sigma_2\sigma_3 + \sigma_1(\sigma_4 + \sigma_7 - \sigma_8))\delta_2}{2\sigma_1\sigma_{13}\delta_3} \right] x + \omega t + \sigma W(t) - \sigma^2 t + \theta_0 \right)}.
 \end{aligned} \tag{44}$$

When the modulus of ellipticity approaches unity ($m \rightarrow 1^-$), we obtain a bright soliton solution

$$\begin{aligned}
 q(x, t) = & \pm \sqrt{-\frac{2(9\lambda_5\tau_2 + 10\lambda_1)}{(\lambda_4 + \lambda_6)\tau_2 + \lambda_2}} \operatorname{sech} \left[\sqrt{-\frac{\lambda_1 ((\lambda_4 + \lambda_6)\tau_2 + \lambda_2) + \lambda_5 \tau_2 ((\lambda_4 + \lambda_6)\tau_2 + \lambda_2)}{\tau_2 (\lambda_4 \tau_2 + \lambda_6 \tau_2 + \lambda_2)}} (x - vt) \right] \\
 & \times e^{\left(i \left[\frac{(-2\sigma_2\sigma_3 + \sigma_1(\sigma_4 + \sigma_7 - \sigma_8))\delta_2}{2\sigma_1\sigma_{13}\delta_3} \right] x + \omega t + \sigma W(t) - \sigma^2 t + \theta_0 \right)},
 \end{aligned} \tag{45}$$

provided that $(9\lambda_5\tau_2 + 10\lambda_1)((\lambda_4 + \lambda_6)\tau_2 + \lambda_2) < 0$, and

$$\tau_2(\lambda_4\tau_2 + \lambda_6\tau_2 + \lambda_2)(\lambda_1((\lambda_4 + \lambda_6)\tau_2 + \lambda_2) + \lambda_5\tau_2((\lambda_4 + \lambda_6)\tau_2 + \lambda_2)) < 0.$$

$$\text{(ii): } \tau_0 = \frac{m^2(1 - m^2)\tau_2^2}{(2m^2 - 1)^2 \tau_4},$$

$$\alpha_0 = 0, \beta_1 = 0, \alpha_1 = \sqrt{\frac{\tau_4(6\lambda_5(5m^4 - 2m_7)\tau_2 + 20\lambda_1 m_6^2)}{\tau_2(\tau_2(\lambda_4 m_7 + \lambda_6 m_5) + \lambda_2 m_5)}},$$

$$\lambda_3 = \frac{(\tau_2(\lambda_4 m_7 + \lambda_6 m_5) + \lambda_2 m_5)(3\lambda_5\tau_2((\lambda_4 - 2\lambda_6)m^4\tau_2 + 4\lambda_2 m_6^2) + 2\lambda_1 m_6^2((\lambda_4 - 4\lambda_6)\tau_2 + 6\lambda_2))}{2(3\lambda_5(5m^4 - 2m_7)\tau_2 + 10\lambda_1 m_6^2)^2},$$

$$k = \sqrt{\frac{\lambda_1(-m^2)((\lambda_4 + \lambda_6)\tau_2 + \lambda_2) - \lambda_5\tau_2(\tau_2(\lambda_4(m^4 - 2m^2 + 2) + \lambda_6 m_6^2) + \lambda_2 m_6^2)}{\tau_2^2(\tau_2(\lambda_4 m_7 + \lambda_6 m_5) + \lambda_2 m_5)}}, \tag{46}$$

where $m_5 = m^4 - 16m^5 + 16$, $m_6 = m^2 - 2$, $m_7 = m^4 + 4m^2 - 4$. As a consequence, the solutions of equation (1) reach

$$\begin{aligned}
q(x, t) = & \pm \sqrt{\frac{2m^2(3\lambda_5(5m^4 - 2m_7)\tau_2 + 10\lambda_1 m_6^2)}{(m^2 - 2)(\lambda_4 m_7 \tau_2 + m_5(\lambda_6 \tau_2 + \lambda_2))}} \\
& \times \operatorname{dn} \left[\sqrt{\frac{\lambda_4 \lambda_5 (m^4 - 2m^2 + 2)\tau_2^2 + m_6^2(\lambda_5 \tau_2(\lambda_6 \tau_2 + \lambda_2) + \lambda_1((\lambda_4 + \lambda_6)\tau_2 + \lambda_2))}{m_6 \tau_2(\lambda_4 m_7 \tau_2 + m_5(\lambda_6 \tau_2 + \lambda_2))}}(x - vt) \middle| m \right] \\
& \times e^{i \left\{ \frac{(-2\sigma_2 \sigma_3 + \sigma_1(\sigma_4 + \sigma_7 - \sigma_8))\delta_2}{2\sigma_1 \sigma_{13} \delta_3} \right\} x + \omega t + \sigma W(t) - \sigma^2 t + \theta_0} .
\end{aligned} \tag{47}$$

When the modulus of ellipticity approaches unity ($m \rightarrow 1^-$), we obtain a bright soliton solution

$$\begin{aligned}
q(x, t) = & \pm \sqrt{\frac{2(9\lambda_5 \tau_2 + 10\lambda_1)}{(\lambda_4 + \lambda_6)\tau_2 + \lambda_2}} \operatorname{sech} \left[\sqrt{\frac{\lambda_5 \tau_2 + \lambda_1}{\tau_2}}(x - vt) \right] \\
& \times e^{i \left\{ \frac{(-2\sigma_2 \sigma_3 + \sigma_1(\sigma_4 + \sigma_7 - \sigma_8))\delta_2}{2\sigma_1 \sigma_{13} \delta_3} \right\} x + \omega t + \sigma W(t) - \sigma^2 t + \theta_0} ,
\end{aligned} \tag{48}$$

provided that $(9\lambda_5 \tau_2 + 10\lambda_1)((\lambda_4 + \lambda_6)\tau_2 + \lambda_2) < 0$, and $\tau_2(\lambda_5 \tau_2 + \lambda_1) < 0$.

$$(iii): \tau_0 = \frac{m^2 \tau_2^2}{(m^2 + 1)^2 \tau_4},$$

$$\alpha_0 = 0, \beta_1 = 0, \alpha_1 = \sqrt{\frac{\tau_4(6\lambda_5 m_{11} \tau_2 + 20\lambda_1 m_9^2)}{\tau_2(\tau_2(\lambda_4 m_{10} + \lambda_6 m_8) + \lambda_2 m_8)}},$$

$$\lambda_3 = \frac{(\lambda_4 m_{10} \tau_2 + m_8(\lambda_6 \tau_2 + \lambda_2))(3\lambda_5(\lambda_4 - 2\lambda_6)(m^2 - 1)^2 \tau_2^2 + 2m_9^2(6\lambda_2 \lambda_5 \tau_2 + \lambda_1((\lambda_4 - 4\lambda_6)\tau_2 + 6\lambda_2)))}{2(3\lambda_5 m_{11} \tau_2 + 10\lambda_1 m_9^2)^2},$$

$$k = \sqrt{\frac{-m_9^2 \lambda_1((\lambda_4 + \lambda_6)\tau_2 + \lambda_2) - \lambda_5 \tau_2(\tau_2(\lambda_4(m^4 + 1) + \lambda_6 m_9^2) + \lambda_2 m_9^2)}{\tau_2^2(\tau_2(\lambda_4 m_{10} + \lambda_6 m_8) + \lambda_2 m_8)}}, \tag{49}$$

where $m_8 = m^4 + 14m^2 + 1$, $m_9 = m^2 + 1$, $m_{10} = m^4 - 6m^2 + 1$, $m_{11} = 3m^4 + 2m^2 + 3$. As a consequence, the solutions of equation (1) reach

$$\begin{aligned}
q(x, t) = & \pm \sqrt{\frac{2m^2(3\lambda_5(3m^4 + 2m^2 + 3)\tau_2 + 10\lambda_1 m_9^2)}{m_9(\lambda_4 m_{10} \tau_2 + m_8(\lambda_6 \tau_2 + \lambda_2))}} \\
& \times \operatorname{sn} \left[\sqrt{\frac{\lambda_4 \lambda_5 (m^4 + 1)\tau_2^2 + m_9^2(\lambda_5 \tau_2(\lambda_6 \tau_2 + \lambda_2) + \lambda_1((\lambda_4 + \lambda_6)\tau_2 + \lambda_2))}{m_9 \tau_2(\lambda_4 m_{10} \tau_2 + m_8(\lambda_6 \tau_2 + \lambda_2))}}(x - vt) \middle| m \right] \\
& \times e^{i \left\{ \frac{(-2\sigma_2 \sigma_3 + \sigma_1(\sigma_4 + \sigma_7 - \sigma_8))\delta_2}{2\sigma_1 \sigma_{13} \delta_3} \right\} x + \omega t + \sigma W(t) - \sigma^2 t + \theta_0} .
\end{aligned} \tag{50}$$

When the modulus of ellipticity approaches unity ($m \rightarrow 1^-$), we obtain a dark soliton solution as:

$$q(x, t) = \pm \sqrt{\frac{24\lambda_5\tau_2 + 40\lambda_1}{16(\lambda_6\tau_2 + \lambda_2) - 4\lambda_4\tau_2}} \times \tanh \left[\sqrt{\frac{2\lambda_4\lambda_5\tau_2^2 + 4(\lambda_5\tau_2(\lambda_6\tau_2 + \lambda_2) + \lambda_1((\lambda_4 + \lambda_6)\tau_2 + \lambda_2))}{2\tau_2(16(\lambda_6\tau_2 + \lambda_2) - 4\lambda_4\tau_2)}}(x - vt) \right] \times e^{\left\{ \left[\frac{(-2\sigma_2\sigma_3 + \sigma_1(\sigma_4 + \sigma_7 - \sigma_8))\delta_2}{2\sigma_1\sigma_{13}\delta_3} \right]_{x+\omega t + \sigma W(t) - \sigma^2 t + \theta_0} \right\}} \quad (51)$$

provided that $(24\lambda_5\tau_2 + 40\lambda_1)(16(\lambda_6\tau_2 + \lambda_2) - 4\lambda_4\tau_2) < 0$, and

$$\tau_2(16(\lambda_6\tau_2 + \lambda_2) - 4\lambda_4\tau_2) \left(2\lambda_4\lambda_5\tau_2^2 + 4(\lambda_5\tau_2(\lambda_6\tau_2 + \lambda_2) + \lambda_1((\lambda_4 + \lambda_6)\tau_2 + \lambda_2)) \right) > 0.$$

Case 4 Choosing $\tau_1 = \tau_3 = 0$, yields

$$\alpha_0 = 0, \alpha_1 = 0, \beta_1 = \sqrt{\frac{2\tau_0(9\lambda_5\tau_2 + 10\lambda_1)}{\tau_2((\lambda_4 + \lambda_6)\tau_2 + \lambda_2)}}, k = \sqrt{-\frac{\lambda_5\tau_2 + \lambda_1}{\tau_2^2}},$$

$$\lambda_3 = \frac{((\lambda_4 + \lambda_6)\tau_2 + \lambda_2)(3\lambda_5\tau_2((\lambda_4 - 2\lambda_6)\tau_2 + 4\lambda_2) + 2\lambda_1((\lambda_4 - 4\lambda_6)\tau_2 + 6\lambda_2))}{2(9\lambda_5\tau_2 + 10\lambda_1)^2}. \quad (52)$$

As a consequence, the solutions of equation (1) reach

$$q(x, t) = \frac{\sqrt{2\tau_0\tau_4(9\lambda_5\tau_2 + 10\lambda_1)} \left[6\wp(k(x - vt); g_2, g_3) + \tau_2 \right]}{\sqrt{\tau_2((\lambda_4 + \lambda_6)\tau_2 + \lambda_2)} \left[3\wp'(k(x - vt); g_2, g_3) \right]} \times e^{\left\{ \left[\frac{(-2\sigma_2\sigma_3 + \sigma_1(\sigma_4 + \sigma_7 - \sigma_8))\delta_2}{2\sigma_1\sigma_{13}\delta_3} \right]_{x+\omega t + \sigma W(t) - \sigma^2 t + \theta_0} \right\}}, \quad (53)$$

and

$$q(x, t) = \frac{\sqrt{2(9\lambda_5\tau_2 + 10\lambda_1)} \left[3\wp'(k(x - vt); g_2, g_3) \right]}{\sqrt{\tau_2((\lambda_4 + \lambda_6)\tau_2 + \lambda_2)} \left[6\wp(k(x - vt); g_2, g_3) + \tau_2 \right]} \times e^{\left\{ \left[\frac{(-2\sigma_2\sigma_3 + \sigma_1(\sigma_4 + \sigma_7 - \sigma_8))\delta_2}{2\sigma_1\sigma_{13}\delta_3} \right]_{x+\omega t + \sigma W(t) - \sigma^2 t + \theta_0} \right\}}. \quad (54)$$

Setting $\tau_0 = 0$ in solution (54), we achieve a singular soliton with $(9\lambda_5\tau_2 + 10\lambda_1)((\lambda_4 + \lambda_6)\tau_2 + \lambda_2) > 0$, and $\tau_2(\lambda_5\tau_2 + \lambda_1) < 0$:

$$q(x, t) = \sqrt{\frac{2(9\lambda_5\tau_2 + 10\lambda_1)}{(\lambda_4 + \lambda_6)\tau_2 + \lambda_2}} \operatorname{csch} \left[\sqrt{-\frac{\lambda_5\tau_2 + \lambda_1}{\tau_2}}(x - vt) \right] \times e^{\left\{ \left[\frac{(-2\sigma_2\sigma_3 + \sigma_1(\sigma_4 + \sigma_7 - \sigma_8))\delta_2}{2\sigma_1\sigma_{13}\delta_3} \right]_{x+\omega t + \sigma W(t) - \sigma^2 t + \theta_0} \right\}}. \quad (55)$$

Case 5 Choosing $\tau_0 = \tau_1 = 0$, yields

$$\alpha_0 = 0, \alpha_1 = 2\sqrt{\frac{3\lambda_5\tau_4}{(2\lambda_4 + 3\lambda_6)\tau_2}}, \beta_1 = 0, \lambda_1 = -\frac{4}{5}\lambda_5\tau_2, k = \sqrt{-\frac{\lambda_5}{5\tau_2}},$$

$$\lambda_2 = \frac{3\lambda_6(3\tau_3^2 - 4\tau_2\tau_4) + 2\lambda_4(3\tau_3^2 - 8\tau_2\tau_4)}{24\tau_4}, \lambda_3 = -\frac{(2\lambda_4^2 + 11\lambda_6\lambda_4 + 12\lambda_6^2)\tau_2}{60\lambda_5}. \quad (56)$$

As a consequence, we get straddled soliton solutions for (1) as follows:

$$q(x, t) = \frac{2\sqrt{\frac{3\lambda_5\tau_4\tau_2}{2\lambda_4+3\lambda_6}} \operatorname{sech}^2\left[\frac{1}{2}\sqrt{-\frac{\lambda_5}{5}}(x-vt)\right]}{\pm 2\sqrt{\tau_2\tau_4} \tanh\left[\frac{1}{2}\sqrt{-\frac{\lambda_5}{5}}(x-vt)\right] + \tau_3} \times e^{i\left\{-\left[\frac{(-2\sigma_2\sigma_3+\sigma_1(\sigma_4+\sigma_7-\sigma_8))\delta_2}{2\sigma_1\sigma_{13}\delta_3}\right]x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right\}}, \quad (57)$$

and

$$q(x, t) = \frac{2\sqrt{\frac{3\lambda_5\tau_4\tau_2}{2\lambda_4+3\lambda_6}} \operatorname{csch}^2\left[\frac{1}{2}\sqrt{-\frac{\lambda_5}{5}}(x-vt)\right]}{\pm 2\sqrt{\tau_2\tau_4} \coth\left[\frac{1}{2}\sqrt{-\frac{\lambda_5}{5}}(x-vt)\right] + \tau_3} \times e^{i\left\{-\left[\frac{(-2\sigma_2\sigma_3+\sigma_1(\sigma_4+\sigma_7-\sigma_8))\delta_2}{2\sigma_1\sigma_{13}\delta_3}\right]x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right\}}. \quad (58)$$

Setting $\tau_3 = \pm 2\sqrt{\tau_2\tau_4}$ into solutions (57) and (58) yields

$$q(x, t) = \frac{\mp \sqrt{\frac{3\lambda_5}{2\lambda_4+3\lambda_6}} \operatorname{sech}^2\left[\frac{1}{2}\sqrt{-\frac{\lambda_5}{5}}(x-vt)\right]}{\tanh\left[\frac{1}{2}\sqrt{-\frac{\lambda_5}{5}}(x-vt)\right] + 1} \times e^{i\left\{-\left[\frac{(-2\sigma_2\sigma_3+\sigma_1(\sigma_4+\sigma_7-\sigma_8))\delta_2}{2\sigma_1\sigma_{13}\delta_3}\right]x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right\}}, \quad (59)$$

and

$$q(x, t) = \frac{\pm \sqrt{\frac{3\lambda_5}{2\lambda_4+3\lambda_6}} \operatorname{csch}^2\left[\frac{1}{2}\sqrt{-\frac{\lambda_5}{5}}(x-vt)\right]}{\coth\left[\frac{1}{2}\sqrt{-\frac{\lambda_5}{5}}(x-vt)\right] + 1} \times e^{i\left\{-\left[\frac{(-2\sigma_2\sigma_3+\sigma_1(\sigma_4+\sigma_7-\sigma_8))\delta_2}{2\sigma_1\sigma_{13}\delta_3}\right]x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right\}}, \quad (60)$$

provided that $\lambda_5 < 0$, and $2\lambda_4 + 3\lambda_6 < 0$.

Setting $\tau_3 = 0$ into solutions (57) and (58) yields singular soliton solution

$$q(x, t) = \mp 2\sqrt{\frac{3\lambda_5}{2\lambda_4+3\lambda_6}} \operatorname{csch}\left[\sqrt{-\frac{\lambda_5}{5}}(x-vt)\right] \times e^{i\left\{-\left[\frac{(-2\sigma_2\sigma_3+\sigma_1(\sigma_4+\sigma_7-\sigma_8))\delta_2}{2\sigma_1\sigma_{13}\delta_3}\right]x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right\}}. \quad (61)$$

Another possible straddled soliton solutions are obtained as presented below

$$q(x, t) = \frac{4\sqrt{\frac{3\lambda_5\tau_4\tau_2}{2\lambda_4+3\lambda_6}} \operatorname{sech}\left[\sqrt{-\frac{\lambda_5}{5}}(x-vt)\right]}{\pm\sqrt{\tau_3^2-4\tau_2\tau_4}-\tau_3 \operatorname{sech}\left[\sqrt{-\frac{\lambda_5}{5}}(x-vt)\right]} \times e^{i\left\{-\left[\frac{(-2\sigma_2\sigma_3+\sigma_1(\sigma_4+\sigma_7-\sigma_8))\delta_2}{2\sigma_1\sigma_{13}\delta_3}\right]x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0\right\}}, \quad (62)$$

and

$$q(x, t) = \frac{4\sqrt{\frac{3\lambda_5\tau_4\tau_2}{2\lambda_4+3\lambda_6}} \operatorname{csch}\left[\sqrt{-\frac{\lambda_5}{5}}(x-vt)\right]}{\pm\sqrt{\tau_3^2-4\tau_2\tau_4}-\tau_3\operatorname{csch}\left[\sqrt{-\frac{\lambda_5}{5}}(x-vt)\right]} \times e^{i\left\{-\left[\frac{(-2\sigma_2\sigma_3+\sigma_1(\sigma_4+\sigma_7-\sigma_8))\delta_2}{2\sigma_1\sigma_{13}\delta_3}\right]_{x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0}\right\}}. \quad (63)$$

Setting $\tau_3 = 0$ into solutions (62) and (63) yields bright and singular soliton solutions

$$q(x, t) = \mp 2\sqrt{-\frac{3\lambda_5}{2\lambda_4+3\lambda_6}} \operatorname{sech}\left[\sqrt{-\frac{\lambda_5}{5}}(x-vt)\right] \times e^{i\left\{-\left[\frac{(-2\sigma_2\sigma_3+\sigma_1(\sigma_4+\sigma_7-\sigma_8))\delta_2}{2\sigma_1\sigma_{13}\delta_3}\right]_{x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0}\right\}}, \quad (64)$$

and

$$q(x, t) = \pm 2\sqrt{\frac{3\lambda_5}{2\lambda_4+3\lambda_6}} \operatorname{csch}\left[\sqrt{-\frac{\lambda_5}{5}}(x-vt)\right] \times e^{i\left\{-\left[\frac{(-2\sigma_2\sigma_3+\sigma_1(\sigma_4+\sigma_7-\sigma_8))\delta_2}{2\sigma_1\sigma_{13}\delta_3}\right]_{x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0}\right\}}, \quad (65)$$

respectively.

Finally, straddled soliton solutions are obtained as described below

$$q(x, t) = \frac{\tau_2\tau_3\operatorname{sech}^2\left[\frac{1}{2}\sqrt{-\frac{\lambda_5}{5}}(x-vt)\right]}{\tau_3^2-\tau_2\tau_4\left(1-\tanh\left[\frac{1}{2}\sqrt{-\frac{\lambda_5}{5}}(x-vt)\right]\right)^2} \times e^{i\left\{-\left[\frac{(-2\sigma_2\sigma_3+\sigma_1(\sigma_4+\sigma_7-\sigma_8))\delta_2}{2\sigma_1\sigma_{13}\delta_3}\right]_{x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0}\right\}}, \quad (66)$$

and

$$q(x, t) = \frac{\tau_2\tau_3\operatorname{csch}^2\left[\frac{1}{2}\sqrt{-\frac{\lambda_5}{5}}(x-vt)\right]}{\tau_3^2-\tau_2\tau_4\left(1-\coth\left[\frac{1}{2}\sqrt{-\frac{\lambda_5}{5}}(x-vt)\right]\right)^2} \times e^{i\left\{-\left[\frac{(-2\sigma_2\sigma_3+\sigma_1(\sigma_4+\sigma_7-\sigma_8))\delta_2}{2\sigma_1\sigma_{13}\delta_3}\right]_{x+\omega t+\sigma W(t)-\sigma^2 t+\theta_0}\right\}}. \quad (67)$$

5. Conclusions

The current paper retrieved optical solitons for the dispersive concatenation model with Kerr law of nonlinear refractive index change, that was considered with white noise in Itô sense. The enhanced direct algebraic approach recovered the soliton solutions to the model. A wide variety of soliton solutions is presented. The intermediate solutions are in terms of Jacobi's elliptic functions. When the modulus of ellipticity approached unity, these solutions yielded soliton solutions and they have been presented. The results are thus interesting and are meaningful. It was observed that the white noise effect stays confined to the phase component of all forms of soliton solutions that are recovered from the model.

The results are thus indeed promising for further future research with this model and its extensions and generalizations. Later, the stochasticity for this model will be studied with power-law of SPM. Subsequently the model will be studied with differential group delay and later with dispersion-flattened fibers. Such results are currently awaited and they will be disseminated later, once they are recovered and aligned with the pre-existing ones [11-26]. Moreover the model will be taken up in additional optoelectronic devices. A few of them are PCF, optical metamaterials, and optical couplers. These are just a tip of the iceberg.

Conflict of interest

The authors declare no competing financial interest.

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