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Research Article

Highly Dispersive Optical Gap Solitons with Kundu-Eckhaus Equation Having Multiplicative White Noise

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Abstract: This paper explores the dynamics of highly dispersive gap solitons within the framework of the Kundu-Eckhaus equation, augmented by the inclusion of multiplicative white noise in the Itô sense, a novel addition to the model. Two integration schemes, the extended simplest equation approach and the generalized Riccati equation mapping scheme, are employed to analyze and integrate the model. Despite yielding bright, singular, and dark-singular straddled solitons, both methods independently fail to capture dark optical solitons. Additionally, our investigation highlights that the influence of white noise predominantly affects the phase aspect of the solitons, with negligible impact on their amplitude. Further details regarding the specific physical system or optical medium under study would provide readers with context, aiding in the comprehension of the significance of our findings.

***Keywords*:** gap solitons, Bragg gratings, Kundu-Eckhaus

MSC: 78A60

1. Introduction

The concept of highly dispersive optical solitons made its debut a couple of years go. This concept was conceived with the absolute need to maintain the delicate balance between chromatic dispersion (CD) and self-phase modulation (SPM) for the soliton propagation to sustain for inter-continental distances. Later, this concept was applied to optical

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fibers with differential group delay. This concept was applied to nonlinear Schrödinger's equation as well as the complex Ginzburg-Landau equation and several of the features have been recovered [1–5]. The retrieval of optical solitons for the models as well as as locating the conservation laws by the method of multipliers have been studied. The numerical analysis of such highly dispersive optical solitons, by the aid of Laplace-Adomian decomposition scheme as well as variational iteration scheme, have also been reported [4, 5].

The current paper addresses such highly dispersive optical solitons in Bragg gratings that would yield gap solitons. The study is made with Kundu-Eckhaus equation as the platform. These gap solitons are being considered with a flavor of stochasticity. The inclusion of multiplicative white noise in Itô sense would yield the necessary gap solitons with the stochastic effect embedded in them. Two integration schemes would make the retrieval of such optical solitons possible. They are the extended simplest equation approach and the generalized Riccati equation mapping scheme. These two algorithms would collectively yield bright and singular optical solitons as well as straddled bright-singular and straddled dark-singular optical gap solitons. An inherent shortcoming with these two integration algorithms is that they fail to recover dark optical gap solitons. The details of the retrieval procedure of such gap solitons are exhibited in the rest of the paper.

1.1 *Governing model*

The dimensionless form of highly dispersive Kundu-Eckhaus equation in fiber Bragg gratings with multiplicative white noise is modeled for the first time as:

$$iq_t + ia_{11}r_x + a_{12}r_{xx} + ia_{13}r_{xxx} + a_{14}r_{xxxx} + ia_{15}r_{xxxxx} + a_{16}r_{xxxxxx} + \left(\xi_1 |q|^4 + \eta_1 |q|^2 |r|^2 + \zeta_1 |r|^4\right) q$$

$$+ b_1 r_{xt} + \left[\lambda_1 \left(|q|^2\right)_x + \theta_1 \left(|r|^2\right)_x\right] q + i\alpha_1 q_x + \beta_1 r + \delta_1 q^* r^2 + \sigma (q - ib_1 r_x) \frac{dW(t)}{dt} = 0, \tag{1}$$

and

$$ir_t + ia_{21}q_x + a_{22}q_{xx} + ia_{23}q_{xxx} + a_{24}q_{xxxx} + ia_{25}q_{xxxxx} + a_{26}q_{xxxxxx} + \left(\xi_2 |r|^4 + \eta_2 |r|^2 |q|^2 + \zeta_2 |q|^4\right) r$$

$$+ b_2 q_{xt} + \left[\lambda_2 \left(|r|^2\right)_x + \theta_2 \left(|q|^2\right)_x\right] r + i\alpha_2 r_x + \beta_2 q + \delta_2 r^* q^2 + \sigma (r - ib_2 q_x) \frac{dW(t)}{dt} = 0, \tag{2}$$

where $q(x,\ t)$ and $r(x,\ t)$ are complex-valued functions that represent the wave profiles and $q^*(x,\ t)$ and $r^*(x,\ t)$ are their complex-conjugate, $i = \sqrt{-1}$. The first term is the linear temporal evolution. The constants $a_{kj}(k = 1,\ 2,\ j = 1-6)$ are the coefficients inter-modal dispersion (IMD), chromatic dispersion (CD), third order dispersion (3OD), fourth order dispersion (4OD), fifth order dispersion (5OD) and sixth order dispersion (6OD), respectively. The constants $b_j(j = 1,\ 2)$ are the coefficients of STD. The constants $\xi_j,\ \lambda_j\,(j = 1,\ 2)$ are the coefficients of self-phase modulation (SPM), while the constants $\eta_j,\ \zeta_j,\ \theta_j\,(j = 1,\ 2)$ are the coefficients of cross-phase modulation (XPM). The constants $\alpha_j,\ \beta_j,\ \delta_j\ (j = 1,\ 2)$ are the coefficients of IMD, detuning parameters and four wave mixing (4WM) terms, respectively. The constant σ is the coefficient of noise strength and $W(t)$ is the standard Wiener process, such that $dW(t)/dt$ is the white noise.

This article's primary goal is to use the extended simplest equation approach, and the generalized Riccati equation mapping strategy to locate the bright, singular, straddled dark-singular soliton solutions of Equations (1) and (2).

This article's structure may be expressed as follows: Section 3 introduces the mathematical preliminaries. We derive the solutions to systems (1) and (2) in Sections 4 and 5. Section 6 concludes the work with a few words on the future plan.

2. Mathematical preliminaries

To analyze the model, we postulate assumption:

$$q(x,\ t) = H_1(\xi)\exp i\left[-\kappa x + \Omega t + \theta + \sigma W(t) - \sigma^2 t\right],$$

$$r(x,\ t) = H_2(\xi)\exp i\left[-\kappa x + \Omega t + \theta + \sigma W(t) - \sigma^2 t\right], \tag{3}$$

and

$$\xi = x - Vt, \tag{4}$$

where κ, Ω, θ and V are nonzero real-valued constants which represent frequency of the soliton, wave number, phase constant and soliton velocity, respectively. The functions $H_1(\xi)$ and $H_2(\xi)$ are real functions which represent the amplitude portion of the soliton and the phase component of the soliton, respectively. Inserting (3) and (4) into Eqs. (1) and (2) gives the real parts:

$$\begin{aligned}
&a_{16}H_2^{(6)}(\xi) + \left(a_{14} + 5\kappa a_{15} - 15\kappa^2 a_{16}\right)H_2^{(4)}(\xi)\\
&+ \left(a_{12} + 3a_{13}\kappa - 6a_{14}\kappa^2 - 10a_{15}\kappa^3 + 15a_{16}\kappa^4 - b_1 V\right)H_2''(\xi) + \left(\kappa\alpha_1 - \Omega + \sigma^2\right)H_1(\xi)\\
&+ \left[\beta_1 + a_{11}\kappa - a_{12}\kappa^2 - a_{13}\kappa^3 + a_{14}\kappa^4 + a_{15}\kappa^5 - a_{16}\kappa^6 + b_1\kappa\left(\Omega - \sigma^2\right)\right]H_2(\xi) + \xi_1 H_1^5(\xi) + \eta_1 H_1^3(\xi)H_2^2(\xi)\\
&+ \zeta_1 H_1(\xi)H_2^4(\xi) + 2\lambda_1 H_1^2(\xi)H_1'(\xi) + 2\theta_1 H_1(\xi)H_2(\xi)H_2'(\xi) + \delta_1 H_1(\xi)H_2^2(\xi) = 0,
\end{aligned} \tag{5}$$

$$\begin{aligned}
&a_{26}H_1^{(6)}(\xi) + \left(a_{24} + 5\kappa a_{25} - 15\kappa^2 a_{26}\right)H_1^{(4)}(\xi)\\
&+ \left(a_{22} + 3a_{23}\kappa - 6a_{24}\kappa^2 - 10a_{25}\kappa^3 + 15a_{26}\kappa^4 - b_2 V\right)H_1''(\xi) + \left(\kappa\alpha_2 - \Omega + \sigma^2\right)H_2(\xi)\\
&+ \left[\beta_2 + a_{21}\kappa - a_{22}\kappa^2 - a_{23}\kappa^3 + a_{24}\kappa^4 + a_{25}\kappa^5 - a_{26}\kappa^6 + b_2\kappa\left(\Omega - \sigma^2\right)\right]H_1(\xi) + \xi_2 H_2^5(\xi) + \eta_2 H_2^3(\xi)H_1^2(\xi)\\
&+ \zeta_2 H_2(\xi)H_1^4(\xi) + 2\lambda_2 H_2^2(\xi)H_2'(\xi) + 2\theta_2 H_2(\xi)H_1(\xi)H_1'(\xi) + \delta_2 H_2(\xi)H_1^2(\xi) = 0,
\end{aligned} \tag{6}$$

and the imaginary parts are

$$(\alpha_1 - V) H_1'(\xi) + \left[a_{11} - 2a_{12}\kappa - 3a_{13}\kappa^2 + 4a_{14}\kappa^3 + 5a_{15}\kappa^4 - 6a_{16}\kappa^5 + b_1\left(\Omega - \sigma^2\right) + b_1\kappa V\right] H_2'(\xi)$$

$$+ \left(a_{13} - 4a_{14}\kappa - 10a_{15}\kappa^2 + 20a_{16}\kappa^3\right) H_2'''(\xi) + (a_{15} - 6a_{16}\kappa) H_2^{(5)}(\xi) = 0, \tag{7}$$

$$(\alpha_2 - V) H_2'(\xi) + \left[a_{21} - 2a_{22}\kappa - 3a_{23}\kappa^2 + 4a_{24}\kappa^3 + 5a_{25}\kappa^4 - 6a_{26}\kappa^5 + b_2\left(\Omega - \sigma^2\right) + b_2\kappa V\right] H_1'(\xi)$$

$$+ \left(a_{23} - 4a_{24}\kappa - 10a_{25}\kappa^2 + 20a_{26}\kappa^3\right) H_1'''(\xi) + (a_{25} - 6a_{26}\kappa) H_1^{(5)}(\xi) = 0, \tag{8}$$

where $' = \frac{d}{d\xi}$, $'' = \frac{d^2}{d\xi^2}$, $''' = \frac{d^3}{d\xi^3}$, $^{(4)} = \frac{d^4}{d\xi^4}$, $^{(5)} = \frac{d^5}{d\xi^5}$ and $^{(6)} = \frac{d^6}{d\xi^6}$.

To recover the integer balancing number used in the given integration methods in the current paper, we set

$$H_2(\xi) = \mu\, H_1(\xi), \tag{9}$$

where μ is a non zero constant, such that $\mu \neq 1$. Now, Eqs. (5)-(8) become

$$a_{16}\mu H_1^{(6)}(\xi) + \mu\left(a_{14} + 5\kappa a_{15} - 15\kappa^2 a_{16}\right) H_1^{(4)}(\xi)$$

$$+ \mu\left(a_{12} + 3a_{13}\kappa - 6a_{14}\kappa^2 - 10a_{15}\kappa^3 + 15a_{16}\kappa^4 - b_1 V\right) H_1''(\xi)$$

$$+ \mu\left[\beta_1 + (a_{11} + \alpha_1)\kappa - a_{12}\kappa^2 - a_{13}\kappa^3 + a_{14}\kappa^4 + a_{15}\kappa^5 - a_{16}\kappa^6 - (1 - b_1\kappa)\left(\Omega - \sigma^2\right)\right] H_1(\xi)$$

$$+ \left(\xi_1 + \eta_1\mu^2 + \zeta_1\mu^4\right) H_1^5(\xi) + 2\left(\lambda_1 + \theta_1\mu^2\right) H_1^2(\xi) H_1'(\xi) + \delta_1\mu^2 H_1^3(\xi) = 0, \tag{10}$$

$$a_{26} H_1^{(6)}(\xi) + \left(a_{24} + 5\kappa a_{25} - 15\kappa^2 a_{26}\right) H_1^{(4)}(\xi)$$

$$+ \left(a_{22} + 3a_{23}\kappa - 6a_{24}\kappa^2 - 10a_{25}\kappa^3 + 15a_{26}\kappa^4 - b_2 V\right) H_1''(\xi)$$

$$+ \left[\beta_2 + (a_{21} + \mu\alpha_2)\kappa - a_{22}\kappa^2 - a_{23}\kappa^3 + a_{24}\kappa^4 + a_{25}\kappa^5 - a_{26}\kappa^6 - (\mu - b_2\kappa)\left(\Omega - \sigma^2\right)\right] H_1(\xi)$$

$$+ \mu\left(\mu^4\xi_2 + \mu^2\eta_2 + \zeta_2\right) H_1^5(\xi) + 2\mu\left(\lambda_2\mu^2 + \theta_2\right) H_1^2(\xi) H_1'(\xi) + \delta_2\mu\, H_1^3(\xi) = 0, \tag{11}$$

and

$$\left[\alpha_1 - V + \mu\left(a_{11} - 2a_{12}\kappa - 3a_{13}\kappa^2 + 4a_{14}\kappa^3 + 5a_{15}\kappa^4 - 6a_{16}\kappa^5 + b_1\left(\Omega - \sigma^2\right) + b_1\kappa V\right)\right] H_1'(\xi)$$

$$+\mu\left(a_{13} - 4a_{14}\kappa - 10a_{15}\kappa^2 + 20a_{16}\kappa^3\right) H_1'''(\xi) + \mu\left(a_{15} - 6a_{16}\kappa\right) H_1^{(5)}(\xi) = 0, \tag{12}$$

$$\left[(\alpha_2 - V)\mu + a_{21} - 2a_{22}\kappa - 3a_{23}\kappa^2 + 4a_{24}\kappa^3 + 5a_{25}\kappa^4 - 6a_{26}\kappa^5 + b_2\left(\Omega - \sigma^2\right) + b_2\kappa V\right] H_1'(\xi)$$

$$+\left(a_{23} - 4a_{24}\kappa - 10a_{25}\kappa^2 + 20a_{26}\kappa^3\right) H_1'''(\xi) + \left(a_{25} - 6a_{26}\kappa\right) H_1^{(5)}(\xi) = 0. \tag{13}$$

Integrating Eqs. (12) and (13) with zero-integration constants, we have

$$\left[\alpha_1 - V(1 - b_1\kappa) + \mu\left(a_{11} - 2a_{12}\kappa - 3a_{13}\kappa^2 + 4a_{14}\kappa^3 + 5a_{15}\kappa^4 - 6a_{16}\kappa^5 + b_1\left(\Omega - \sigma^2\right)\right)\right] H_1(\xi)$$

$$+\mu\left(a_{13} - 4a_{14}\kappa - 10a_{15}\kappa^2 + 20a_{16}\kappa^3\right) H_1''(\xi) + \mu\left(a_{15} - 6a_{16}\kappa\right) H_1^{(4)}(\xi) = 0, \tag{14}$$

$$\left[\alpha_2\mu - (\mu - b_2\kappa)V + a_{21} - 2a_{22}\kappa - 3a_{23}\kappa^2 + 4a_{24}\kappa^3 + 5a_{25}\kappa^4 - 6a_{26}\kappa^5 + b_2\left(\Omega - \sigma^2\right)\right] H_1(\xi)$$

$$+\left(a_{23} - 4a_{24}\kappa - 10a_{25}\kappa^2 + 20a_{26}\kappa^3\right) H_1''(\xi) + \left(a_{25} - 6a_{26}\kappa\right) H_1^{(4)}(\xi) = 0. \tag{15}$$

Setting the coefficients of the linearly independent functions of Eqs. (14) and (15) to zero, yields

$$\kappa = \frac{a_{j5}}{6a_{j6}},\ j = 1,\ 2, \tag{16}$$

and

$$V = \frac{\alpha_1 + \mu\left[a_{11} - 2a_{12}\kappa - 3a_{13}\kappa^2 + 4a_{14}\kappa^3 + 5a_{15}\kappa^4 - 6a_{16}\kappa^5 + b_1\left(\Omega - \sigma^2\right)\right]}{1 - b_1\kappa}, \tag{17}$$

$$V = \frac{\alpha_2\mu + a_{21} - 2a_{22}\kappa - 3a_{23}\kappa^2 + 4a_{24}\kappa^3 + 5a_{25}\kappa^4 - 6a_{26}\kappa^5 + b_2\left(\Omega - \sigma^2\right)}{\mu - b_2\kappa}, \tag{18}$$

and the constraints conditions

$$a_{j3} - 4a_{j4}\kappa - 10a_{j5}\kappa^2 + 20a_{j6}\kappa^3 = 0,\ j = 1,\ 2, \tag{19}$$

provided $a_{j5} \neq 0,\ a_{j6} \neq 0,\ (j = 1,\ 2)$, $b_1\kappa \neq 1,\ b_2\kappa \neq \mu$. Eqs. (10) and (11) are equivalent under the constraint conditions:

$$a_{16}\mu = a_{26}, \tag{20}$$

$$\delta_1\mu = \delta_2, \tag{21}$$

$$\lambda_1 + \theta_1\mu^2 = \mu\left(\lambda_2\mu^2 + \theta_2\right), \tag{22}$$

$$\xi_1 + \eta_1\mu^2 + \zeta_1\mu^4 = \mu\left(\mu^4\xi_2 + \mu^2\eta_2 + \zeta_2\right), \tag{23}$$

$$\mu\left(a_{14} + 5\kappa a_{15} - 15\kappa^2 a_{16}\right) = \left(a_{24} + 5\kappa a_{25} - 15\kappa^2 a_{26}\right), \tag{24}$$

$$\begin{aligned}&\mu\left(a_{12} + 3a_{13}\kappa - 6a_{14}\kappa^2 - 10a_{15}\kappa^3 + 15a_{16}\kappa^4 - b_1V\right)\\ &= \left(a_{22} + 3a_{23}\kappa - 6a_{24}\kappa^2 - 10a_{25}\kappa^3 + 15a_{26}\kappa^4 - b_2V\right),\end{aligned} \tag{25}$$

$$\begin{aligned}&\mu\left[\beta_1 + (a_{11} + \alpha_1)\kappa - a_{12}\kappa^2 - a_{13}\kappa^3 + a_{14}\kappa^4 + a_{15}\kappa^5 - a_{16}\kappa^6 - (1 - b_1\kappa)\left(\Omega - \sigma^2\right)\right]\\ &= \beta_2 + (a_{21} + \mu\alpha_2)\kappa - a_{22}\kappa^2 - a_{23}\kappa^3 + a_{24}\kappa^4 + a_{25}\kappa^5 - a_{26}\kappa^6 - (\mu - b_2\kappa)\left(\Omega - \sigma^2\right).\end{aligned} \tag{26}$$

From (26), we have the wave number of the soliton:

$$\Omega = \frac{\begin{array}{c}[a_{21} - \mu(a_{11} + \alpha_1 - \alpha_2)]\kappa + (\mu a_{16} - a_{26})\kappa^6 + (a_{25} - \mu a_{15})\kappa^5 + (a_{24} - \mu a_{14})\kappa^4\\ + (a_{13}\mu - a_{23})\kappa^3 + (\mu a_{12} - a_{22})\kappa^2 + \beta_2 - \mu\beta_1 + \sigma^2\kappa(\mu b_1 - b_2)\end{array}}{\kappa(\mu b_1 - b_2)}, \tag{27}$$

provided $\mu b_1 \neq b_2$. From (17) and (18), we deduce that

$$\sigma^2 = \frac{A(\mu - b_2\kappa) - B(1 - b_1\kappa) + \Omega[\mu b_1(\mu - b_2\kappa) - b_2(1 - b_1\kappa)]}{\mu b_1(\mu - b_2\kappa) - b_2(1 - b_1\kappa)}, \tag{28}$$

where

$$\begin{aligned}A &= \alpha_1 + \mu\left[a_{11} - 2a_{12}\kappa - 3a_{13}\kappa^2 + 4a_{14}\kappa^3 + 5a_{15}\kappa^4 - 6a_{16}\kappa^5\right],\\ B &= \alpha_2\mu + a_{21} - 2a_{22}\kappa - 3a_{23}\kappa^2 + 4a_{24}\kappa^3 + 5a_{25}\kappa^4 - 6a_{26}\kappa^5.\end{aligned} \tag{29}$$

Now, Eq. (10) can be rewritten in the form

$$H_1^{(6)}(\xi)+L_1H_1^{(4)}(\xi)+L_2H_1''(\xi)+2L_3H_1^2(\xi)H_1'(\xi)+L_4H_1(\xi)+L_5H_1^3(\xi)+L_6H_1^5(\xi)=0, \tag{30}$$

where

$$L_1=\frac{a_{14}+5\kappa a_{15}-15\kappa^2a_{16}}{a_{16}},$$

$$L_2=\frac{a_{12}+3a_{13}\kappa-6a_{14}\kappa^2-10a_{15}\kappa^3+15a_{16}\kappa^4-b_1V}{a_{16}},$$

$$L_3=\frac{\lambda_1+\theta_1\mu^2}{a_{16}\mu},$$

$$L_4=\frac{\beta_1+(a_{11}+\alpha_1)\kappa-a_{12}\kappa^2-a_{13}\kappa^3+a_{14}\kappa^4+a_{15}\kappa^5-a_{16}\kappa^6-(1-b_1\kappa)\left(\Omega-\sigma^2\right)}{a_{16}},$$

$$L_5=\frac{\delta_1\mu}{a_{16}},$$

$$L_6=\frac{\xi_1+\eta_1\mu^2+\zeta_1\mu^4}{a_{16}\mu}, \tag{31}$$

Now, balancing the terms $H_1^{(6)}(\xi)$ and $H_1^5(\xi)$ in Eq. (30) yields the balance number $N=\frac{3}{2}$. Thus, we take the transformation:

$$H_1(\xi)=Z^{\frac{3}{2}}(\xi), \tag{32}$$

where $Z(\xi)$ is a new positive function of ξ. Next, Eq. (30) changes to:

$$Z^5(\xi)Z^{(6)}(\xi)+3Z^4(\xi)Z'(\xi)Z^{(5)}(\xi)-\frac{15}{4}\left[Z'^2(\xi)-2Z(\xi)Z''(\xi)\right]\left[Z^3(\xi)Z^{(4)}(\xi)-2Z^2(\xi)Z'(\xi)Z'''(\xi)\right]$$
$$+5Z^4(\xi)Z'''^2(\xi)-\frac{15}{4}Z^3(\xi)Z''^3(\xi)+\frac{135}{8}Z^2(\xi)Z'^2(\xi)Z''^2(\xi)+\frac{105}{32}Z'^6(\xi)-\frac{225}{16}Z(\xi)Z'^4(\xi)Z''(\xi)$$
$$+\frac{L_1}{8}\left[8Z^3(\xi)Z^{(4)}(\xi)+16Z^2(\xi)Z'(\xi)Z'''(\xi)+3Z'^4(\xi)+12\left(Z(\xi)Z''(\xi)-Z'^2(\xi)\right)Z(\xi)Z''(\xi)\right]Z^2(\xi)$$
$$+\frac{L_2}{2}\left[Z'^2(\xi)+2Z(\xi)Z''(\xi)\right]Z^4(\xi)+2L_3Z^8(\xi)Z'(\xi)+\frac{2}{3}L_4Z^6(\xi)+\frac{2}{3}L_5Z^9(\xi)+\frac{2}{3}L_6Z^{12}(\xi)=0. \tag{33}$$

In Eq. (33), we balance $Z^5(\xi)Z^{(6)}(\xi)$ and $Z^{12}(\xi)$ produces the equilibrium number $N = 1$. Eq. (33) will be solved using the following two integration techniques in the next sections.

3. Extended simplest equation approach

The formal solution to Equation (33) is [6–8]:

$$Z(\xi) = \chi_0 + \chi_1 \left[\frac{\Phi'(\xi)}{\Phi(\xi)}\right] + \rho_0 \left[\frac{1}{\Phi(\xi)}\right], \tag{34}$$

and $\Phi(\xi)$ is the solution of the equation

$$\Phi''(\xi) + \tau\Phi(\xi) = \upsilon, \tag{35}$$

where $\tau,\ \upsilon,\ \chi_0,\ \chi_1$ and ρ_0 are variables, such $\chi_1^2 + \rho_0^2 \neq 0$.

For $\tau < 0$, we switch (34) with (33) and apply Eq. (35) with the following relation

$$\left(\frac{\Phi'(\xi)}{\Phi(\xi)}\right)^2 = T_1\left(\frac{1}{\Phi(\xi)}\right)^2 - \tau + \frac{2\upsilon}{\Phi(\xi)}, \tag{36}$$

where $T_1 = \tau\left(W_1^2 - W_2^2\right) - \frac{\upsilon^2}{\tau}$, while W_1 and W_2 are parameters, allows for results

$$\chi_0 = 0,\ \chi_1 = 0,\ \rho_0 = \frac{1}{2}\left[-\frac{117855\tau^3\left(W_1^2 - W_2^2\right)^3}{L_6}\right]^{\frac{1}{6}},\ \upsilon = 0, \tag{37}$$

and

$$L_1 = \frac{3247}{84}\tau,\ L_2 = \frac{135679}{336}\tau^2,\ L_3 = 0,\ L_4 = \frac{45873}{64}\tau^3,\ L_5 = 0, \tag{38}$$

provided $\left(W_1^2 - W_2^2\right)L_6 > 0$. Consequently, we obtain bright-singular straddled optical solitons as:

$$q(x,\ t) = \left\{\frac{1}{2}\left[-\frac{117855\tau^3\left(W_1^2 - W_2^2\right)^3}{L_6}\right]^{\frac{1}{6}}\left[\frac{1}{W_1\cosh\left(\sqrt{-\tau}\xi\right) + W_2\sinh\left(\sqrt{-\tau}\xi\right)}\right]\right\}^{\frac{3}{2}}$$
$$e^{i\left[-\kappa x + \Omega t + \theta + \sigma W(t) - \sigma^2 t\right]}, \tag{39}$$

$$r(x,\ t) = \mu \left\{ \frac{1}{2} \left[-\frac{117855\tau^3 \left(W_1^2 - W_2^2\right)^3}{L_6} \right]^{\frac{1}{6}} \left[\frac{1}{W_1 \cosh\left(\sqrt{-\tau}\xi\right) + W_2 \sinh\left(\sqrt{-\tau}\xi\right)} \right] \right\}^{\frac{3}{2}}$$

$$e^{i\left[-\kappa x+\Omega t+\theta+\sigma W(t)-\sigma^2 t\right]}. \tag{40}$$

The bright soliton solution is obtained when $W_1 \neq 0, W_2 = 0$, we have:

$$q(x,\ t) = \left\{ \frac{1}{2} \left[-\frac{117855\tau^3}{L_6} \right]^{\frac{1}{6}} \operatorname{sech}\left(\sqrt{-\tau}\xi\right) \right\}^{\frac{3}{2}} e^{i\left[-\kappa x+\Omega t+\theta+\sigma W(t)-\sigma^2 t\right]}, \tag{41}$$

$$r(x,\ t) = \mu \left\{ \frac{1}{2} \left[-\frac{117855\tau^3}{L_6} \right]^{\frac{1}{6}} \operatorname{sech}\left(\sqrt{-\tau}\xi\right) \right\}^{\frac{3}{2}} e^{i\left[-\kappa x+\Omega t+\theta+\sigma W(t)-\sigma^2 t\right]}, \tag{42}$$

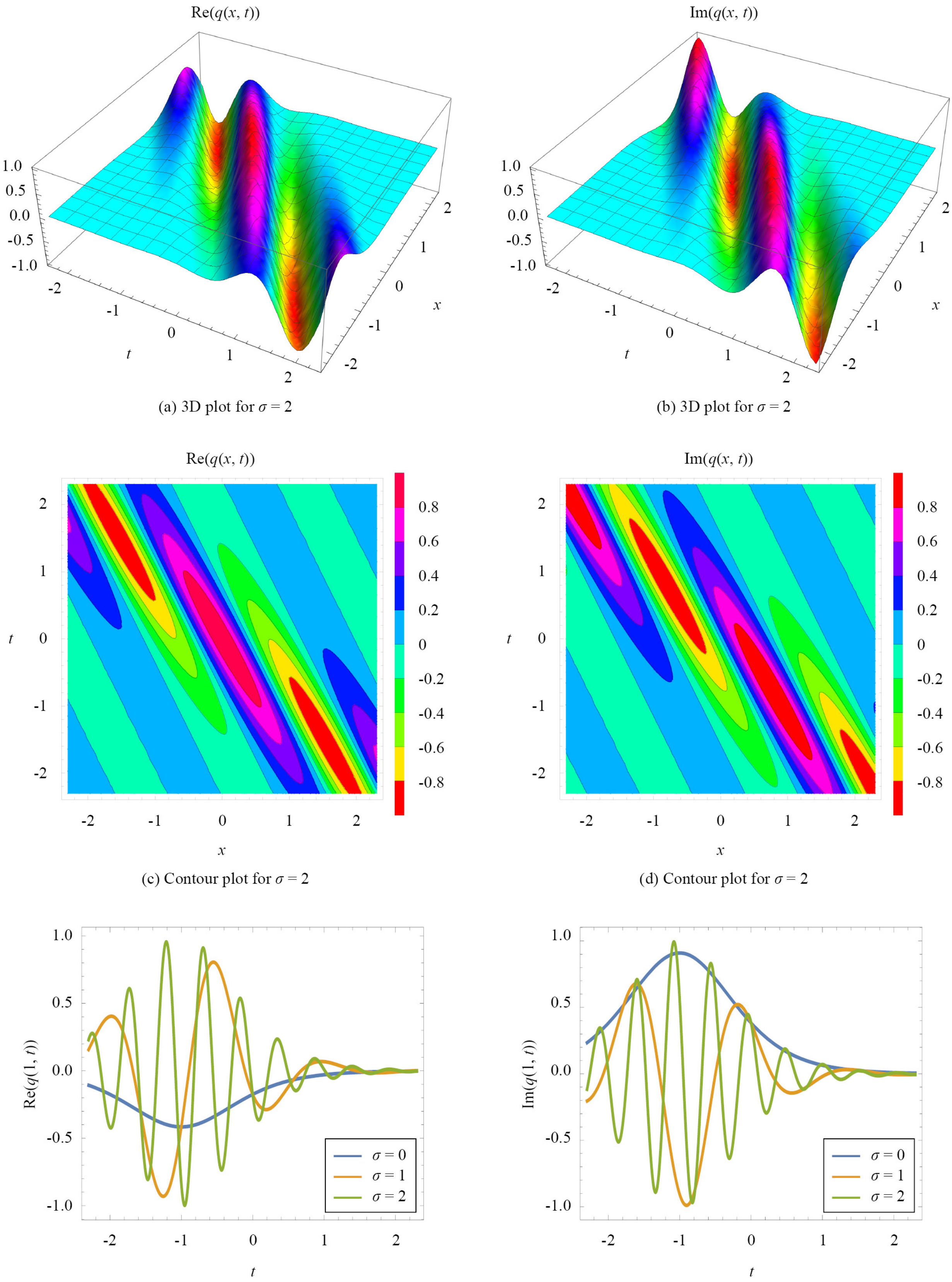
provided $L_6 > 0$. The singular soliton solution is obtained when $W_1 = 0,\ W_2 \neq 0$, we have:

$$q(x,\ t) = \left\{ \frac{1}{2} \left[\frac{117855\tau^3}{L_6} \right]^{\frac{1}{6}} \operatorname{csch}\left(\sqrt{-\tau}\xi\right) \right\}^{\frac{3}{2}} e^{i\left[-\kappa x+\Omega t+\theta+\sigma W(t)-\sigma^2 t\right]}, \tag{43}$$

$$r(x,\ t) = \mu \left\{ \frac{1}{2} \left[\frac{117855\tau^3}{L_6} \right]^{\frac{1}{6}} \operatorname{csch}\left(\sqrt{-\tau}\xi\right) \right\}^{\frac{3}{2}} e^{i\left[-\kappa x+\Omega t+\theta+\sigma W(t)-\sigma^2 t\right]}. \tag{44}$$

for $L_6 < 0$.

In Figures 1 and 2, we can see 3D plots, contour plots, 2D plots of a bright soliton solution defined by Eq. (41). The parameters have specific values: $V = 1.1$, $\tau = -1$, $\mu = 1.2$, $\xi_1 = 1.9$, $\eta_1 = 1.3$, $\zeta_1 = 1.4$, $a_{16} = 1.7$, $\kappa = 1.6$, $\Omega = 2.1$, $\theta = 2.4$, and $W(t) = t$.



(a) 3D plot for $\sigma = 2$

(b) 3D plot for $\sigma = 2$

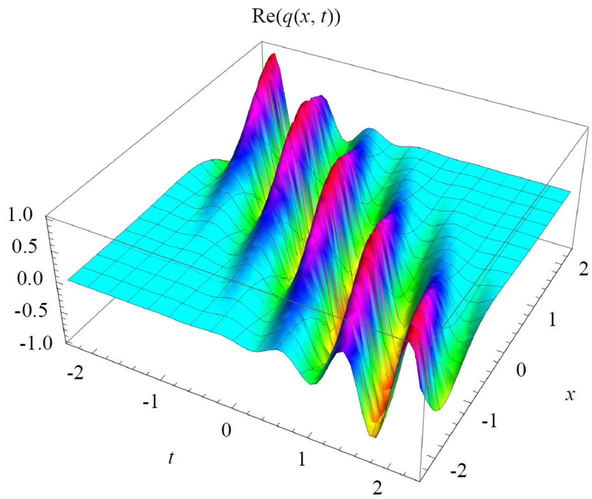
(c) Contour plot for $\sigma = 2$

(d) Contour plot for $\sigma = 2$

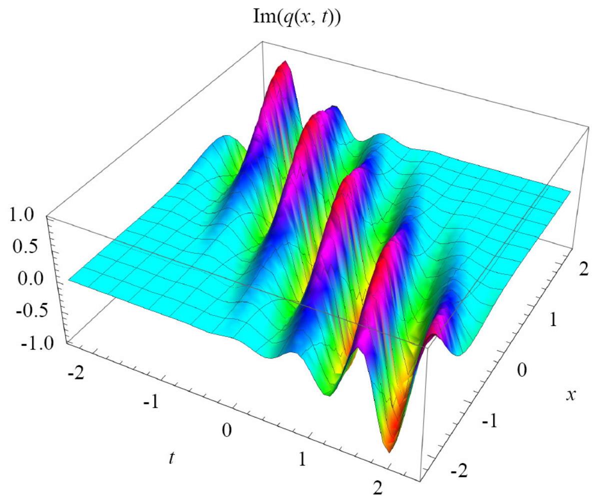
(e) 2D plot for $\sigma = 2$

(f) 2D plot for $\sigma = 2$

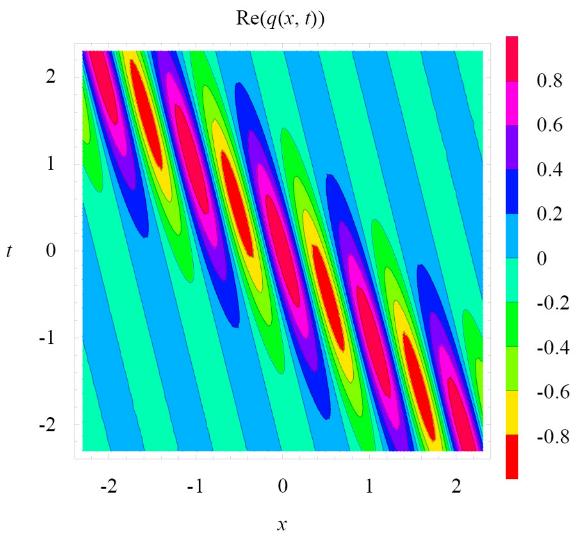
Figure 1. Profile of a bright soliton solution



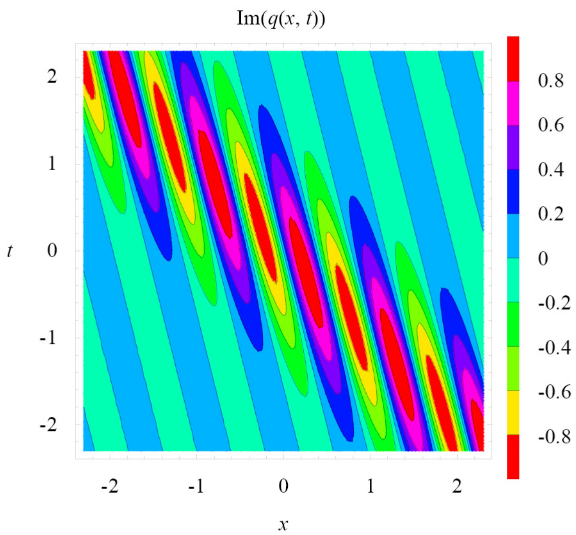
(a) 3D plot for $\sigma = 4$



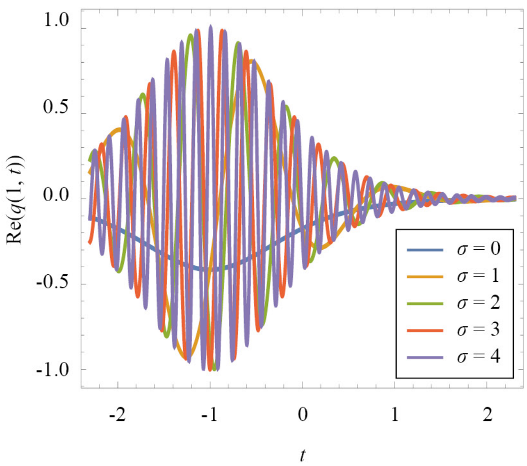
(b) 3D plot for $\sigma = 4$



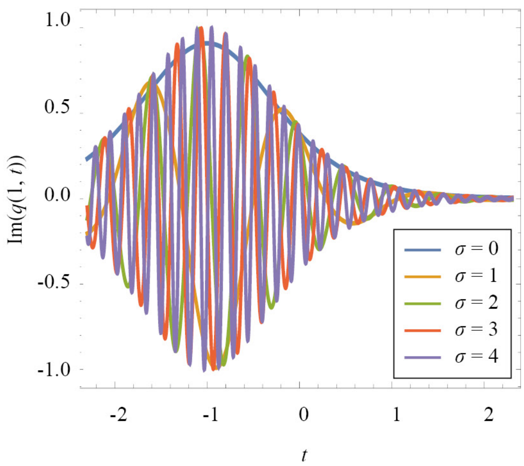
(c) Contour plot for $\sigma = 4$



(d) Contour plot for $\sigma = 4$



(e) 2D plot for $\sigma = 4$



(f) 2D plot for $\sigma = 4$

Figure 2. Profile of a bright soliton solution

4. Generalized Riccati equation mapping scheme

Eq. (33) obtains the formal solution [6–8]:

$$Z(\xi) = \Delta_0 + \Delta_1 Q(\xi) + \frac{\Delta_{-1}}{Q(\xi)}, \tag{45}$$

and $Q(\xi)$ satisfies the generalized Riccati equation:

$$Q'(\xi) = h_0 + h_1\, Q(\xi) + h_2\, Q^2(\xi), \tag{46}$$

where $\Delta_0,\ \Delta_1,\ \Delta_{-1},\ h_0,\ h_1$ and h_2 are arbitrary constants to be determined provided $\Delta_1 \neq 0$ or $\Delta_{-1} \neq 0$ and $h_2 \neq 0$. Plugging (45) together with (46) into Eq. (33), obtains the following results:

$$\Delta_1 = \frac{h_2}{2}\left(-\frac{117855}{L_6}\right)^{\frac{1}{6}},\ \Delta_0 = 0,$$

$$\Delta_{-1} = \frac{21\, L_1}{6494\, h_2}\left(-\frac{117855}{L_6}\right)^{\frac{1}{6}}, h_0 = \frac{21\, L_1}{3247\, h_2},\ h_1 = 0, \tag{47}$$

and

$$L_2 = \frac{2849259\, L_1^2}{10543009},\ L_3 = 0,\ L_4 = \frac{424829853\, L_1^3}{34233150223},\ L_5 = 0, \tag{48}$$

provided $L_6 < 0$. From (45) and (47), then we have the solutions:

$$H_1(\xi) = \left\{\frac{1}{2}\left(-\frac{117855}{L_6}\right)^{\frac{1}{6}}\left[h_2 Q(\xi) + \frac{21\, L_1}{3247\, h_2}\left(\frac{1}{Q(\xi)}\right)\right]\right\}^{\frac{3}{2}}. \tag{49}$$

The following solutions are thus yielded.

If $\Upsilon = h_1^2 - 4h_0h_2 > 0$ and $h_1h_2 \neq 0$ or $h_0h_2 \neq 0$, then, we have the straddled dark-singular soliton solutions:

$$q(x,\ t) = \left\{\left(-\frac{117855}{L_6}\right)^{\frac{1}{6}}\sqrt{-\frac{21\, L_1}{12988}}\left[\tanh\left(\sqrt{-\frac{21\, L_1}{3247}}\,\xi\right) + \coth\left(\sqrt{-\frac{21\, L_1}{3247}}\,\xi\right)\right]\right\}^{\frac{3}{2}}$$

$$e^{i\left[-\kappa x + \Omega t + \theta + \sigma W(t) - \sigma^2 t\right]}, \tag{50}$$

$$r(x,\ t)=\mu\left\{\left(-\frac{117855}{L_6}\right)^{\frac{1}{6}}\sqrt{-\frac{21\,L_1}{12988}}\left[\tanh\left(\sqrt{-\frac{21\,L_1}{3247}}\xi\right)+\coth\left(\sqrt{-\frac{21\,L_1}{3247}}\xi\right)\right]\right\}^{\frac{3}{2}}$$

$$e^{i\left[-\kappa x+\Omega t+\theta+\sigma W(t)-\sigma^2 t\right]}, \tag{51}$$

$$q(x,\ t)=e^{i\left[-\kappa x+\Omega t+\theta+\sigma W(t)-\sigma^2 t\right]}\times$$

$$\left\{\left(-\frac{117855}{L_6}\right)^{\frac{1}{6}}\sqrt{-\frac{21\,L_1}{51952}}\left[\tanh\left(\sqrt{-\frac{21\,L_1}{12988}}\xi\right)+\coth\left(\sqrt{-\frac{21\,L_1}{12988}}\xi\right)\right.\right.$$

$$\left.\left.+\frac{4}{\tanh\left(\sqrt{-\frac{21\,L_1}{12988}}\xi\right)+\coth\left(\sqrt{-\frac{21\,L_1}{12988}}\xi\right)}\right]\right\}^{\frac{3}{2}}, \tag{52}$$

$$r(x,\ t)=\mu\times e^{i\left[-\kappa x+\Omega t+\theta+\sigma W(t)-\sigma^2 t\right]}\times$$

$$\left\{\left(-\frac{117855}{L_6}\right)^{\frac{1}{6}}\sqrt{-\frac{21\,L_1}{51952}}\left[\tanh\left(\sqrt{-\frac{21\,L_1}{12988}}\xi\right)+\coth\left(\sqrt{-\frac{21\,L_1}{12988}}\xi\right)\right.\right.$$

$$\left.\left.+\frac{4}{\tanh\left(\sqrt{-\frac{21\,L_1}{12988}}\xi\right)+\coth\left(\sqrt{-\frac{21\,L_1}{12988}}\xi\right)}\right]\right\}^{\frac{3}{2}}. \tag{53}$$

Also we obtain the straddled singular solitons

$$q(x,\ t) = e^{i\left[-\kappa x+\Omega t+\theta+\sigma W(t)-\sigma^2 t\right]}\times$$

$$\left\{-\sqrt{-\frac{21L_1}{12988}}\left(-\frac{117855}{L_6}\right)^{\frac{1}{6}}\left[\coth\left(\sqrt{-\frac{84\,L_1}{3247}}\xi\right)\pm\operatorname{csch}\left(\sqrt{-\frac{84\,L_1}{3247}}\xi\right)\right.\right.$$

$$\left.\left.+\frac{1}{\coth\left(\sqrt{-\frac{84\,L_1}{3247}}\xi\right)\pm\operatorname{csch}\left(\sqrt{-\frac{84\,L_1}{3247}}\xi\right)}\right]\right\}^{\frac{3}{2}}, \tag{54}$$

$$r(x,\ t) = e^{i\left[-\kappa x+\Omega t+\theta+\sigma W(t)-\sigma^2 t\right]}\times$$

$$\mu\left\{-\sqrt{-\frac{21L_1}{12988}}\left(-\frac{117855}{L_6}\right)^{\frac{1}{6}}\left[\coth\left(\sqrt{-\frac{84\,L_1}{3247}}\xi\right)\pm\operatorname{csch}\left(\sqrt{-\frac{84\,L_1}{3247}}\xi\right)\right.\right.$$

$$\left.\left.+\frac{1}{\coth\left(\sqrt{-\frac{84\,L_1}{3247}}\xi\right)\pm\operatorname{csch}\left(\sqrt{-\frac{84\,L_1}{3247}}\xi\right)}\right]\right\}^{\frac{3}{2}}, \tag{55}$$

A few additional soliton solutions are structured as:

$$q(x,\ t) = \left\{\left(-\frac{117855}{L_6}\right)^{\frac{1}{6}}\sqrt{-\frac{21\,L_1}{12988}}\left[\frac{\sqrt{A^2+B^2}-A\cosh\left(\sqrt{-\frac{84\,L_1}{3247}}\xi\right)}{A\sinh\left(\sqrt{-\frac{84\,L_1}{3247}}\xi\right)+B}\right.\right.$$

$$\left.\left.+\frac{A\sinh\left(\sqrt{-\frac{84\,L_1}{3247}}\xi\right)+B}{\sqrt{A^2+B^2}-A\cosh\left(\sqrt{-\frac{84\,L_1}{3247}}\xi\right)}\right]\right\}^{\frac{3}{2}} e^{i\left[-\kappa x+\Omega t+\theta+\sigma W(t)-\sigma^2 t\right]}, \tag{56}$$

$$r(x,\,t) = \mu\left\{\left(-\frac{117855}{L_6}\right)^{\frac{1}{6}}\sqrt{-\frac{21\,L_1}{12988}}\left[\frac{\sqrt{A^2+B^2}-A\cosh\left(\sqrt{-\frac{84\,L_1}{3247}}\,\xi\right)}{A\sinh\left(\sqrt{-\frac{84\,L_1}{3247}}\,\xi\right)+B}\right.\right.$$

$$\left.\left.+\frac{A\sinh\left(\sqrt{-\frac{84\,L_1}{3247}}\,\xi\right)+B}{\sqrt{A^2+B^2}-A\cosh\left(\sqrt{-\frac{84\,L_1}{3247}}\,\xi\right)}\right]\right\}^{\frac{3}{2}} e^{i\left[-\kappa x+\Omega t+\theta+\sigma W(t)-\sigma^2 t\right]}, \tag{57}$$

$$q(x,\,t) = \left\{-\left(-\frac{117855}{L_6}\right)^{\frac{1}{6}}\sqrt{-\frac{21\,L_1}{12988}}\left[\frac{\sqrt{B^2-A^2}+A\sinh\left(\sqrt{-\frac{84\,L_1}{3247}}\,\xi\right)}{A\cosh\left(\sqrt{-\frac{84\,L_1}{3247}}\,\xi\right)+B}+\right.\right.$$

$$\left.\left.\frac{A\cosh\left(\sqrt{-\frac{84\,L_1}{3247}}\,\xi\right)+B}{\sqrt{B^2-A^2}+A\sinh\left(\sqrt{-\frac{84\,L_1}{3247}}\,\xi\right)}\right]\right\}^{\frac{3}{2}} e^{i\left[-\kappa x+\Omega t+\theta+\sigma W(t)-\sigma^2 t\right]}, \tag{58}$$

$$r(x,\,t) = \mu\left\{-\left(-\frac{117855}{L_6}\right)^{\frac{1}{6}}\sqrt{-\frac{21\,L_1}{12988}}\left[\frac{\sqrt{B^2-A^2}+A\sinh\left(\sqrt{-\frac{84\,L_1}{3247}}\,\xi\right)}{A\cosh\left(\sqrt{-\frac{84\,L_1}{3247}}\,\xi\right)+B}+\right.\right.$$

$$\left.\left.\frac{A\cosh\left(\sqrt{-\frac{84\,L_1}{3247}}\,\xi\right)+B}{\sqrt{B^2-A^2}+A\sinh\left(\sqrt{-\frac{84\,L_1}{3247}}\,\xi\right)}\right]\right\}^{\frac{3}{2}} e^{i\left[-\kappa x+\Omega t+\theta+\sigma W(t)-\sigma^2 t\right]}, \tag{59}$$

provided $L_1 < 0$, A and B are two non-zero real constants satisfying $B^2 - A^2 > 0$.

5. Conclusions

The current paper conducts a detailed analysis and constitutes the retrieval of highly dispersive gap optical solitons that emerge from the Kundu-Eckhaus equation. To give the model a flavor of stochasticity, the effect of multiplicative white noise is included in the Itô sense for the first time. Two integration algorithms shed light on the model. They are the extended simplest equation approach and the generalized Riccati equation mapping scheme. These methods together yield bright and singular 1-soliton solutions as well as dark-singular and bright-singular straddled optical solitons. Both of the schemes have a severe shortcoming. They fail to recover dark 1-soliton solution to the model. Another feature that is observed in this paper is that the effect of white noise stays confined to the phase component of the solitons and thus does not affect the amplitude component of any of the solitons.

The results are thus indeed promising to traverse along additional avenues to proceed with this model. It is imperative to consider more integration schemes that would reveal dark 1-soliton solutions to the model. The present paper thus stands incomplete in the sense that the two adopted algorithms fail to present a full spectrum of optical gap solitons to the model. This is one of the very many avenues to venture into in the upcoming days and the recovered results would be reported after aligning them with the various pre-existing ones [9–24].

Confilict of interest

The authors declare no competing financial interest.

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