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Research Article

Application of Laplace Transform Method to Solve Neutral-Type Delay Differential Equation

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Abstract: This study investigates the use of the Laplace transform method to resolve a particular class of second-order linear initial-value neutral delay differential equations, which are prevalent in fields such as physics and engineering. By employing this analytical approach, we derive exact solutions that facilitate a deeper awareness of the dynamics of the system and its stability characteristics. A thorough illustrative example is provided to illustrate the Laplace transform method's efficacy, showcasing its practical utility in addressing real-world problems. This study not only highlights the advantages of the Laplace transform in providing clear insights into complex delay systems but also emphasizes its relevance in both theoretical and applied contexts.

Keywords: Laplace transform method, initial value problem, neutral delay differential equation

MSC: 34A05, 34A25, 34A30, 34E05, 34K06, 42A10, 44A10

1. Introduction

Neutral delay differential equations (NDDEs) are essential for modeling systems where the current state is influenced not only by past states but also by the rates of change of those states. This characteristic makes NDDEs particularly valuable in various fields, including control engineering, biological modeling, mechanical systems, and fluid dynamics. For instance, in biological contexts, NDDEs effectively capture the intricacies of population dynamics by representing delays that correlate with gestation or maturation periods. In engineering, they are employed in control systems to accommodate delayed feedback inherent in many processes, thus facilitating the analysis of dynamic systems exhibiting memory effects. The growing significance of NDDEs underscores the necessity for methods that accurately reveal detailed system behaviors, a need that continues to drive advancements in NDDE research [1, 2].

Despite their importance, the analytical solution of NDDEs poses significant challenges due to their transcendental properties, particularly in scenarios involving proportional and constant delays. In contrast to simpler delay differential equations, NDDEs introduce complexities that necessitate specialized methods for accurate analysis. Recent attention has shifted toward analytical approaches, such as the Laplace transform method, which can yield exact solutions that elucidate

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system characteristics like stability and oscillations more transparently than numerical methods alone. Although numerical approaches like the finite element and finite difference methods are commonly applied to NDDEs, they may not provide the analytical clarity essential for thorough insights into system properties [3–5].

In recent years, researchers have also explored meshless methods for solving NDDEs, especially those that rely on the singular boundary method (SBM), strong meshless radial point interpolation (SMRPI), and radial basis functions (RBFs). These methods demonstrate promise due to their flexibility in accommodating complex problem domains and boundary conditions. Notable studies have showcased the effectiveness of meshless techniques across various applications; for example, RBF-based methods have been utilized in time-fractional diffusion-wave equations [6], while SBM has been utilised for nonlinear source terms in modified anomalous diffusion equations and telegraph equations on arbitrary domains [7, 8]. Additionally, SMRPI has exhibited stability and convergence in pseudo-parabolic equations [9]. Nonetheless, while RBF, SMRPI, and SBM offer versatility, they encounter stability and scalability challenges, highlighting the necessity for further investigation to enhance their applicability [10–15].

In light of these challenges, the Laplace transform method emerges as a systematic approach for simplifying NDDEs by converting them into algebraic equations, which facilitates the analysis of characteristic roots and stability. Recent studies applying the Laplace transform to second-order NDDEs have successfully derived exact solutions, providing valuable insights into system stability and oscillatory behavior. This research aims to advance the application of the Laplace transform methodology for NDDEs, concentrating on stability analysis and validating the method's effectiveness through specific applications. By integrating insights from recent studies and comparing various solution methods, this work underscores the strengths of the Laplace transform method, making it particularly useful for applications in control engineering and biological modeling [16–21].

2. Literature review

A comprehensive review of the literature reveals a burgeoning interest in both analytical and numerical methods for solving NDDEs. While traditional approaches often focus on numerical solutions, there has been a marked shift towards analytical methods that can provide deeper insights into system dynamics. The Laplace transform method, in particular, has gained traction for its ability to yield exact solutions, thereby elucidating stability characteristics that numerical methods may obscure.

Several studies have explored the application of the Laplace transform to NDDEs, particularly in contexts requiring detailed stability analysis. For example, recent work has demonstrated the method's efficacy in deriving exact solutions for second-order NDDEs, which not only clarify the stability criteria but also reveal oscillatory behaviors critical to understanding system dynamics. These analytical approaches have been shown to complement numerical methods, offering a more comprehensive understanding of the underlying systems. However, despite the promising results, challenges remain regarding the stability and scalability of these meshless methods, which warrant further investigation. The literature suggests that while analytical methods such as the Laplace transform provide significant insights, the integration of meshless techniques could enhance solution methodologies, particularly in complex real-world applications.

Torelli [22] addresses the stability of numerical methods for delay differential equations (DDEs), focusing on the backward Euler method. This work is critical in ensuring that numerical simulations of DDEs yield reliable results. Gopalsamy and Zhang [23] study the stability of impulsively perturbed DDEs, providing sufficient conditions for asymptotic stability and oscillatory behavior. This research extends the understanding of nonlinear dynamics in systems with both delays and impulses.

Wolfrum [24] explore DDEs with large delays, introducing novel approaches to stability analysis. Their work on strong and weak instabilities offers insights into the complex dynamics of delay systems, including control systems and semiconductor lasers. Lin and Wang [25] and Li [26] investigate systems with multiple discrete delays. These studies contribute to understanding how multiple delays interact to influence the stability and bifurcation of solutions, with applications ranging from motor control to biological systems. Yan and Zhao [27] discuss the oscillation and stability of linear impulsive DDEs, finding that these properties are equivalent to those of corresponding non-impulsive DDEs. Their results can enhance the stability analysis of systems with sudden changes.

Enright and Hayashi [28] develop a solver for neutral DDEs based on a continuous Runge-Kutta method with defect control. Their DDVERK algorithm ensures accurate error and step size control, offering a robust tool for solving DDEs numerically. Keane [29] review the use of DDEs in climate models, focusing on delayed feedback loops in global energy balance and the El Niño Southern Oscillation (ENSO) system. Their study demonstrates that DDEs can capture complex climate dynamics with relatively simple models, offering insights into the predictability of climate phenomena.

The Laplace transform method is a versatile and effective tool for solving delay differential equations (DDEs). It has been widely used in various fields, including biology, medicine, and economics, to analyze and solve complex systems with delays. Research has shown that choosing the right software is crucial for accurately and efficiently solving DDEs. The field is constantly evolving, with a focus on improving analytical methods like the Laplace transform and developing new numerical approaches. This research aims to build on existing studies, enhance the Laplace transform method, and explore its applications.

3. Neutral delay differential equation

A neutral delay differential equation (NDDE) is a type of differential equation that includes both derivative terms and delayed terms. It can be written in the general form

$$F(t,\, x(t),\, x'(t),\, x(t-\tau_1),\, x'(t-\tau_2),\, \ldots,\, x'(t-\tau_k)) = 0,$$

where $x(t)$ is the unknown function to be solved for, F is a given function, τ is the independent variable (time), $x'(t)$ represents the derivative of x with respect to τ, and $x(t-\tau_i)$ represents the delayed term with delay τ_i.

The presence of delayed terms introduces memory effects into the equation, making the behavior of solutions more complex compared to ordinary differential equations. The delays τ_i can be constant or time-varying, and the function F can be nonlinear.

In this section the authors discussed with new concept of Neutral delay differential equation. It is represented by

$$z''(t) - Cz''(t-\tau) + p_1 z'(t) + p_2 z'(t-\tau) + q_1 z(t) + q_2(t-\tau) = f(t),\ z \in R^n, \tag{1}$$

$$z(t) = \phi(t),\ \tau \le t \le 0;\ z'(0) = \gamma, \tag{2}$$

where $C,\ p_1,\ p_2,\ q_1,\ q_2$ are real constants, $f(t)$ and $\phi(t)$ are given real valued function, τ is a positive delay constant and γ is real number.

4. Preliminaries on the Laplace transform

A popular and effective mathematical method for analysing linear time-invariant systems is the Laplace transform, especially for addressing differential equations that come up in a variety of disciplines, including control theory, physics, and engineering. This segment gives a summary of the essential concepts, properties, and results related to the Laplace transform, forming the foundation for its application in the subsequent sections of this research (see [30–34]).

Definition 1 A time-domain function $f(t)$ is transformed using the Laplace transform,defined for $t \ge 0$, into a complex frequency-domain function $F(s)$, where s is a complex variable. The transform is defined as:

$$L(F(t)) = \int_0^\infty e^{-st} F(t) dt.$$

Here, $s = \sigma + i\omega$, where σ and ω are real numbers. Differential equations can be easily converted into algebraic equations using the Laplace transform in the s-domain, greatly simplifying their analysis and solution.

4.1 *Conditions for existence*

A function $f(t)$ must be piecewise continuous on any finite interval $[0, T]$, with $T > 0$, for the Laplace transform to exist, of exponential order, which indicates that for $t \geq 0$, there are constants M and α such that

$$|f(t)| \leq Me^{at}. \tag{3}$$

These prerequisites guarantee the convergence of the integral in the Laplace transform definition.

4.2 *Basic properties of the Laplace transform*

A number of the Laplace transform's essential characteristics are in its application to differential equations. Some key properties are presented below.

4.2.1 *Linearity*

For functions $f(t)$ and $g(t)$, and constants a and b

$$L\{af(t) + bg(t)\} = aF(s) + bG(s).$$

4.2.2 *Derivative property*

The Laplace transform of the first derivative of $f(t)$, denoted $f'(t)$, is

$$L\{f'(t)\} = sF(s) - f(0).$$

For the second derivative $f''(t)$, the Laplace transform is:

$$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0).$$

These properties allow differential equations to be transformed into algebraic equations, simplifying their manipulation and solution in the s-domain.

4.2.3 *Time shifting*

If $f(t-a)$ represents a delayed function, where $a > 0$

$$L\{f(t-a)u(t-a)\}=e^{-as}F(s),$$

where the Heaviside function is represented by $u(t-a)$ shifting the function by a units in time.

4.2.4 *Frequency shifting*

For a function $e^{at}f(t)$, the Laplace transform is

$$L\{e^{at}f(t)\}=F(s-a).$$

This property enables manipulation of the Laplace-transformed function in the s-domain by shifting its frequency component.

4.3 *Inverse Laplace transform*

The inverse Laplace transform is applied to retrieve the original time-domain function from its Laplace-transformed counterpart. Denoted as L^{-1}, which is defined as:

$$f(t)=L^{-1}\{F(s)\}.$$

5. Existence of the solution to NDDE using Laplace transform

Theorem 1 [17] Suppose the following characteristics of the F function:
1. F is continuously piecewise in all finite interval $0<t<t_1 \ \ (t_1>0)$.
2. For an exponential order F, there is $(\alpha,\ M>0$, and $t_0>0)$ with

$$e^{-\alpha t}|F(t)|<M,\ \text{for } t>t_0. \tag{4}$$

Then the Laplace transform and function F exists for values of s greater than α.

Theorem 2 [Gronwall's inequality [17]]. Consider functions $h(t)\geq 0,\ K(t),\ g(t)\geq 0$ which are real and continuous on $(0,\ \infty)$. If

$$v(t)\leq g(t)+K(t)\int_0^t h(\tau)v(\tau)d\tau,$$

then

$$v(t)\leq g(t)+K(t)\int_0^t h(\tau)h(\tau)e^{\int_0^t h(\xi)K(\xi)d\xi}d\tau. \tag{5}$$

Theorem 3 If the conditions specified in (4) are met by the function $F(t)$ in equation (1), then the validity of the Laplace transformation of $y(t)$ extends to all possible values of s, when s is greater than α.

Proof. The initial integration of Equation (1) over the interval $(0,\ t)$, as derived from

$$z'(t)-\gamma-C[z'(t-\tau)-z'(-\tau)]+p_1[z(t)-z(0)]+p_2[z(t-\tau)-z(-\tau)]$$

$$+q_1\int_0^{\tau} z(x)dx+q_2\int_0^{\tau} z(x-\tau)dx=\int_0^{\tau} f(x)dx. \tag{6}$$

If we integrate this equation again over $(0,\ t)$, we have

$$z(t)-Cz(t-\tau)-\phi(0)-[\gamma-Cz'(-\tau)-p_1z(0)-p_2z(-\tau)]\tau+\int_0^t [p_{1+}q_1(t-x)]z(x)dx$$

$$+\int_{-\tau}^{0}[p_2+q_2(t-x-\tau)]z(\tau)d\tau+\int_0^{t-\tau}[p_2+q_2(t-x-\tau)]z(x)dx=\int_0^{\tau}(t-x)f(x)dx. \tag{7}$$

Substituting integral variable by $x=\tau+\tau_1$ can be obtained as

$$\int_0^t z(x-\tau)dx=\int_{-\tau}^{t-\tau} z(\tau_1)d\tau_1=\int_{-\tau_1}^{0}\phi(\tau_1)d\tau_1+\int_0^{t-\tau} z(\tau_1)d\tau_1$$

and

$$\int_0^t (t-x)z(x-\tau)dx=\int_{-\tau}^{t-\tau}(t-\tau_1-\tau)z(\tau_1)d\tau_1=\int_{-\tau_1}^{0}(\tau-\tau_1-\tau)z(\tau_1)d\tau_1+\int_0^{t-\tau}(\tau-\tau_1-\tau)z(\tau_1)d\tau_1.$$

The equation (4), is transformed into

$$z(t)-Cz(t-\tau)+h_1(t)+\int_0^t [p_{1+}q_1(t-x)]z(x)dx+\int_0^{\tau-\tau}[p_2+q_2(t-x-\tau)]z(x)dx=\int_0^{\tau}(t-x)f(x)dx$$

$$h_1(t)=-\phi(0)-[\gamma-Cz'(-\tau_1)-p_1z(0)-p_2z(-\tau_1)]t+\int_{-\tau}^{0}[p_2+q_2(t-\tau_1-\tau)]\phi(\tau_1)d\tau_1$$

$$\left|\int_0^{t-\tau}[p_2+q_2(t-x-\tau)]z(x)dx\right|\leq\int_0^t |[p_2+q_2(t-x-\tau)]z(x)|\,dx.$$

The equation (7) is obtained using the above inequality as follows

$$|z(t)| - |Cz(t-\tau)| \leq |h_1(t)| + \int_0^t [|p_{1+}q_1(t-x) + [|p_2 + q_2(t-x-\tau)|]||z(x)|]dx$$

$$+ \int_0^t (\tau - x)|f(x)|dx$$

$$\leq |h_1(t)| + h_2(t) \int_0^\tau [|z(x)|]dx + \int_0^\tau (t-x)|f(x)|dx, \tag{8}$$

where $h_2(t) = |p_1 + q_1(t)| + |[p_2 + q_2(t+\tau)]|$.

Next, if we consider $e^{-at}|f(t)| < M_1,\ (\tau < 0)$, we have

$$\int_0^\zeta (\tau - x)|f(x)|dx \leq \int_0^\zeta M_1(t-x)e^{ax}dx = \frac{M_1}{a^2}[e^{at} - 1 - at] \leq \frac{M_1 e^{at}}{a^2}.$$

Thus, we can write

$$|z(t)| - |Cz(t-\tau)| \leq +|h_1(t)| + h_2(t) \int_0^t [|z(x)|]dx + \frac{M_1 e^{at}}{a^2}$$

Whereas

$$h_1(t) = -[\gamma - Cz'(-\tau) - p_2 z(-\tau)]t + \int_{-r}^0 [p_2 + q_2(t-x-\tau)]z(\tau_1)d\tau_1$$

$$|h_1(t)| \leq (1 + |p_1|)\phi(0) + t[\gamma - Cz'(-\tau_1) - p_2 z(-\tau) + q_2 \int_{-\tau}^0 |\phi(\tau_1)|d\tau_1]$$

$$+ \int_{-\tau}^0 [|p_2| + |q_2|(|\tau_1| + \tau)]\phi(\tau_1)d\tau_1$$

$$h_2(t) \leq |p_1| + [|q_1| + |q_2|]t + |p_2 + q_2\tau|.$$

Applying Theorem 2 in equation (8), the inequality is obtained as follows

$$|z(t)| - |Cz(t-\tau)| \leq |h_1(t)| + h_2(t) \int_0^t [|z(x)|]dx + \frac{M_1 e^{at}}{a^2}$$

$$|z(t)| - |Cz(t-\tau)| \leq |h_1(t)| + \frac{M_1 e^{at}}{a^2} + h_2(t) \int_0^\tau \left[|h_1(t)| + \frac{M_1 e^{a\tau}}{a^2} \right] e^{\int_\tau^t h_2(\xi)d\xi} d\tau$$

$$\leq A_1 + B_1 t + \frac{M_1 e^{at}}{a^2} + (A_1 + B_1 t) e^{A_1 t + \frac{B_1 t^2}{2}} \left[\left(A_1 t + \frac{B_1 t^2}{2} \right) + \frac{M_1 (e^{a\tau} - 1)}{a^3} \right],$$

where $A_1,\ B_1,\ A_2,\ B_2$ are constants given as

$$A_1 = (1 + |p_1|)\phi(0) + \int_{-r}^0 [|p_2| + |q_2|(|\tau_1| + \tau)]\phi(\tau_1)d\tau_1$$

$$B_1 = |\gamma| + |C|\ |z'(-\tau_1)| + |p_2|\ |z(-\tau)| + |q_2| \int_{-r}^0 |\phi(\tau_1)|d\tau_1$$

$$A_2 = |a| + + |p_2 + q_2 \tau|$$

$$B_2 = |q_1| + |q_2|.$$

This extension can be taken a step further to include $z'(t)$ and $z''(t)$. □

Theorem 4 Consider $F(s)$ to denote the Laplace transform of the function $f(t)$ in equation (1). Assume $\phi(t),\ \phi'(t)$ be continuous on the interval $[-\tau,\ 0]$ and then equation (1)-(2) has its exact solution as

$$z(t) = L^{-1}\left(\frac{F(S) + T(s)}{K(s)}\right),$$

where

$$K(s) = s^2 + p_1 s + q_1 - Cs^2 e^{-s\tau} + p_2 s e^{-s\tau} + q_2 e^{-s\tau}$$

$$T(s) = \gamma + \left[s + p_1 - C\ s\ e^{-s\tau} + p_2 e^{-s\tau}\right] \phi(0) + \bar{\bar{\phi}}(s) - p_2 \bar{\bar{\phi}}(s) - q_2 \bar{\phi}(s)$$

$$\bar{\phi}(s) = \int_{-r}^0 e^{-s(t+\tau)} \phi(t)dt$$

$$\bar{\bar{\phi}}(s) = \int_{-r}^0 e^{-s(t+\tau)} \phi'(t)dt.$$

Proof. In order to apply the Laplace transform approach to find the solution to the problems (1)-(2), it is important to keep in mind that the derivatives of $z(t)$ have the following Laplace transforms:

$$L(z'(t)) = sL(z(t)) - \phi(0)$$

$$L(z''(t)) = s^2L(z(t)) - s\phi(0) - \phi'(0).$$

By applying the definition of Laplace transform for $u(t-\rho)$, we obtain as

$$L(z(t-\tau)) = \int_0^{\infty} e^{-st}z(t-\tau)dt.$$

By substituting t with $x+\tau$ as the new integral variable, we obtain

$$L(z(t-\tau)) = \int_0^{\infty} e^{-s(x+\tau)}z(x)dx = \int_0^{\infty} e^{-s(x+\tau)}z(x)dx + e^{-s\tau}\int_0^{\infty} e^{-sx}z(x)dx.$$

Thus we have $L(z(t-\tau)) = \bar{\phi}(s) + e^{-s\tau}L(z(t))$.
Similarly, Laplace transformation for $z'(t-\tau)$, we get

$$L(z'(t-\tau)) = \int_0^{\infty} e^{-st}z'(t-\tau)dt$$

$$= \int_{-\tau}^{\infty} e^{-s(x+\tau)}z'(x)dx$$

$$= \int_{-\tau}^{0} e^{-s(x+\tau)}\phi'(x)dx + e^{-s\tau}\int_0^{\infty} e^{-sx}z'(x)dx.$$

Hence $L(z'(t-\tau)) = \bar{\bar{\phi}}(s) - e^{-s\tau}L(z'(t))$.
The resultant obtained by applying Laplace transform to the equation (1) is

$$s^2L(z(t)) - s\phi(0) - \phi'(0) - C[\bar{\bar{\bar{\phi}}}(s) - e^{-s\tau}L(z''(t))] + p_1[sL(z(t)) - \phi(0)] + p_2[\bar{\bar{\phi}}(s) - e^{-s\tau}L(z'(t))]$$

$$+ q_1[L(z(t)) + q_2\bar{\phi}(s) + q_2e^{-s\tau}L((z)(t))] = F(s)$$

$$s^2L(z(t)) - s\phi(0) - \phi'(0) - C\bar{\bar{\bar{\phi}}}(s) + Ce^{-s\tau}[s^2L(z(t)) - s\phi(0) - \phi'(0)] + p_1sL(z(t)) - p_1\phi(0) + q_2e^{-s\tau}L(z)(t)$$

$$+ p_2\bar{\bar{\phi}}(s) - p_2e^{-s\tau}[sL(z(t)) - \phi(0)] + q_1L(z(t)) + q_2\bar{\phi}(s) = F(s)$$

$$L(z(t))[s^2+Ce^{-s\tau}s^2+p_1s-p_2e^{-s\tau}s+q_1+q_2e^{-s\tau}]-s\phi(0)-\phi'(0)-C\bar{\bar{\bar{\phi}}}(s)-Ce^{-s\tau}s\phi(0)$$

$$-Ce^{-s\tau}\phi'(0)-p_1\phi(0)+p_2\bar{\bar{\phi}}(s)+p_2e^{-s\tau}\phi(0)+q_2\bar{\phi}(s)=F(s)$$

$$L(z(t))[s^2+Ce^{-s\tau}s^2+p_1s-p_2e^{-s\tau}s+q_1+q_2e^{-s\tau}]$$

$$=s\phi(0)+\phi'(0)+C\bar{\bar{\bar{\phi}}}(s)+Ce^{-s\tau}s\phi(0)+Ce^{-s\tau}\phi'(0)+p_1\phi(0)-p_2\bar{\bar{\phi}}(s)-p_2e^{-s\tau}\phi(0)-q_2\bar{\phi}(s)$$

$$=F(s),$$

where

$$K(S)0=s^2+p_1s+q_1+Cs^2e^{-s\tau}-p_2se^{-s\tau}+q_2e^{-s\tau}$$

$$T(S)=\gamma+Ce^{-s\tau}\gamma+[s+Ce^{-s\tau}s+p_1-p_2e^{-s\tau}]\phi(0)+C\bar{\bar{\bar{\phi}}}(s)-p_2\bar{\bar{\phi}}(s)-q_2\bar{\phi}(s).$$

Additionally, it can be refined as

$$L(z(t))=\left(\frac{F(S)+T(s)}{K(s)}\right).$$

□

Example 1 we consider the following problem:

$$z''(x)-2z''(x-\tau)+z'(x)+z'(x-\tau)-2z(x)+z(x-\tau)=0,\ x>0$$

subject to the initial function, $z(x)=e^x,\ -1\leq x\leq 0,\ z'(0)=1.$

If we take into consideration

$$F(0)=0,\ T(\lambda)=\lambda+2+(-2\lambda-1)e^{-\lambda},\ K(\lambda)=\lambda^2+\lambda-2+(-2\lambda^2+\lambda+1)e^{-\lambda}.$$

Applying Laplace transform we get

$$L(z(x))=\frac{\lambda+2+(-2\lambda-1)e^{-\lambda}}{\lambda^2+\lambda-2+(-2\lambda^2+\lambda+1)e^{-\lambda}}=\frac{1}{\lambda-1}$$

and $z(x) = L^{-1}\left[\frac{1}{\lambda - 1}\right] = e^x$.

When analytical solutions of Laplace transforms for neutral delay differential equations (NDDEs) are infeasible, numerical methods become essential, and MATLAB is a particularly powerful tool for this purpose.

Example 2 We consider the following problem:

$$x''(t) + x''(t-\tau) - x'(t) - x'(t-\tau) + x(t) + x(t-\tau) = 0,\ t > 0$$

subject to the initial function, $x(t) = 1,\ -1 \leq t \leq 0,\ x'(0) = 0$.

If we take into consideration $F(0) = 0,\ T(s) = (s-1)(1-e^{-s}),\ K(s) = s^2 - s^2e^{-s} - s + se^{-s} + 1 - 1e^{-s}$.

Applying Laplace transform, we get

$$L(x(t)) = \frac{(s-1)(1-e^{-s}))}{s^2 - s + 1 - (s^2 - s + 1)e^{-s}}$$

and

$$x(t) = L^{-1}\left[\frac{(s-1)}{s^2 - s + 1}\right] = \exp\left(\frac{t}{2}\right)\left(\cos\left(\frac{\sqrt{3}t}{2}\right) - \frac{\sqrt{3}\sin\left(\frac{\sqrt{3}t}{2}\right)}{3}\right).$$

The characteristic roots of neutral delay differential equations (NDDEs), such as $0.5000 - 0.8660i$, $0.0000 - 3.1416i$, $0.0000 - 9.4248i$, $-0.0000 - 15.7080i$, $0.0000 - 21.9911i$, $0.0000 - 28.2743i$, $-0.0000 - 34.5575i$, $-0.0000 - 40.8407i$ and others, obtained using the Quasi-Polynomial mapping-based Rootfinder (QPmR) in MATLAB, show consistency with those derived from the Laplace transform method. While the Laplace transform is effective for finding analytical solutions, it can be limited when handling variable or complex delays. In such cases, numerical methods become essential for accurate solutions. This allows researchers to explore system dynamics, validate analytical results, and analyze complex scenarios that are impractical to solve analytically. Through MATLAB's capabilities, insights into stability, response, and oscillatory behavior are gained, ensuring effective study of NDDEs even when exact solutions are not feasible.

6. Conclusion

This study examines to solve second-order Neutral Delay Differential Equation using Laplace transformation method which is and effective technique to find the analytical solution, which is inferred by an example. NDDEs with variable delays can be very challenging to solve analytically using Laplace transform, especially if the delays are not explicitly given as functions of time. Further research on Laplace transforms for NDDEs can lead to advances in both theoretical understanding and practical applications of these equations in various fields. It can also contribute to the development of new analytical and numerical tools for solving complex delay differential equations.

Availability of data and materials

All data used in this manuscript have been presented within the article.

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Authors contributions

All authors equally contributed to the research work.

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Conflict of interest

The authors declare that they have no competing interests.

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