Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

Palindromic Antimagic Labeling of Products of Paw and Banner Graph

P. Reka[*], **S. P. Soundariya**

Department of Mathematics, A.V.V.M Sri Pushpam College (Affiliated to Bharathidasan University, Tiruchirapalli), Poondi, Thanjavur 613 503, Tamilnadu, India
E-mail: prekabharathi@gmail.com

Received: 29 December 2023; **Revised:** 22 January 2024; **Accepted:** 31 January 2024

Abstract: This article presents a novel classification of Antimagic labeling referred to as Palindromic antimagic. Palindromic Antimagic labeling pertains to the assignment of palindromic numbers $\{\wp_1, \wp_2, \wp_3, \ldots, \wp_q\}$ to the set of edges of a graph $G = (V, E)$, where G consists of p vertices and q edges. This labeling is characterized by being invertible, meaning that each edge is uniquely associated with a palindromic number. Additionally, it involves an injective mapping of vertex labeling, where the sum of incident edges for any given vertex is distinct from one another. In this study, we examine the palindromic antimagic labeling of Cartesian and Tensor product of certain Tadpole graphs, such as the paw and banner graph.

Keywords: palindromic antimagic labeling, cartesian product, tensor product, paw graph, banner graph

MSC: 05C78, 05C76

1. Introduction

A Palindromic antimagic labeling (New category of Antimagic labeling) of a graph $G = (V, E)$ with p vertices and q edges whose edge set E assure an invertible mapping $\phi : E(G) \to \{\wp_1, \wp_2, \wp_3, \ldots, \wp_q\}$ where $\wp_1, \wp_2, \wp_3, \ldots, \wp_q$ is a palindromic number (Numbers remain carbon copy when its terms are overturned) that launch an injective mapping $\theta : V(G) \to Z^+$ defined

$$\theta(V(G)) = \Sigma_{e\in\psi_G(Z^+)}\phi(e)$$

$\psi_G(Z^+)$ is an incidence function associated with corresponding Z^+ and those vertex labelings are incompatible each other.

The Cartesian product of G and H is indicated by $G\square H$ has vertex set, $V(G\square H) = \{(y_i, z_j) : y_i \in V(G), z_j \in V(H)\}$ and the edge set $E(G\square H)$ has two prospect

(i) $y_i = y_k$ and $N(z_j)$ is z_s in H;

(ii) $z_j = z_s$ and $N(y_i)$ is y_k in G.

The Tensor product of G and H is indicated by $G \times H$ has vertex set, $V(G \times H) = \{(y_i, z_j) : y_i \in V(G), z_j \in V(H)\}$ and the edge set $E(G \times H)$ has two prospect

DOI: https://doi.org/10.37256/cm.5220244180

(i) $N(y_i)$ is y_k in G, and

(ii) $N(z_j)$ is z_s in H.

A Paw graph is a tadpole graph obtained by joining a cycle C_3 and path P_1 with a bridge and it is denoted by $T_{3,\ 1}$ or (3, 1) Tadpole graph. A Banner graph is a tadpole graph obtained by joining a cycle C_4 and path P_1 with a bridge and it is denoted by $T_{4,\ 1}$ or (4, 1) Tadpole graph. In this article, paw and banner graph is indicated by $T_{d,\ w}$ and $T_{m,\ w}$.

The concept of antimagic was first developed by Hartsfield and Ringel [1] in 1990. The conclusion was reached that all connected graphs, with the exception of K_2, and all trees, with the exception of K_2, are considered to be antimagic. Furthermore, a demonstration was presented to show that all paths $P_n(n \geqslant 3)$, cycles $C_n(n \geqslant 3)$, complete graph $K_n(n \geqslant 3)$ had antimagic properties. Gallian [2] did a comprehensive examination on the subject of Graph Labeling in a work extending back to 1996. Liang and zhu [3] demonstrated that the cartesian product of graph G and H permits antimagic labeling if G is a k regular graph and H is any arbitrary graph with $1 \leqslant \mid V(H) \mid -1 \leqslant \mid E(H) \mid$. Cheng [4], takes into account a regular graph G and H that is bounded with some inequality on the degree, and in the ensuing situation that the cartesian product of G and H admits antimagic labeling and he demonstrated that the cartesian product of two or more regular graphs with positive degrees is antimagic. He also illustrated in his article [5] that the cartesian product of two path (namely lattice grids) for some integer $m,\ n \geqslant 1$, cycles and path (specially, prism graph) exhibit antimagic properties. Additionally, he established that if G is a j regular graph and H has a maximum degree of atmost k and a minimum degree of atleast one (where G and H may not connected), then the cartesian product of $G \times H$ is also antimagic. This holds true when j is odd and $j^2 - j \geqslant 2k$ or j is even and $j^2 \geqslant 2k$. In a study of Wang [6], it was demonstrated that the toroidal grid $C_{n_1} \times C_{n_2} \times \ldots \times C_{n_k}$ posses antimagic properties. Furthermore, he (see [7]) found that the graphs of the type $G \times C_n$ exhibit antimagic properties if G is an r regular antimagic graph with r being greater than 1. Richard et al. provides a comprehensive explanation of the concept of graph products in his book [8]. Now, let us examine the primary outcome of the research paper.

2. Main result

Theorem 1 The Cartesian product of Paw and Banner (G and H) graphs is palindromic antimagic.

Proof. The Cartesian product of G and H is denoted by $G\Box H$, the vertex set $V(G\Box H) = \{x_{i,\ j} : 1 \leq i \leq m,\ 1 \leq j \leq (m+w)\}$ and the edge set $E(G\Box H) = \{e_{hk} : 1 \leq k \leq [d(m+w)+w] \cup e_{vk} : 1 \leq k \leq [d(m+w)] \cup e_{hck} : 1 \leq k \leq m \cup e_{vck} : 1 \leq k \leq (m+w)\}$ where e_{hk} is an horizontal edges, e_{vk} is a vertical edges, e_{hck} is an horizontal curve edges, e_{vck} is a vertical curve edges. In this graph $G\Box H$ has $\mid V \mid = [m(m+w)]$ and $\mid E \mid = \{(d-w)[m(m+w)]\}$ and it has m rows and $(m+w)$ columns (horizontal manner) respectively. In the graph $G\Box H$, there are w vertex has $(d-w)$ degree, $(m+w)$ vertices has d degree, $[m(d-w)]$ vertices has m degree, $(m+w)$ vertices has $(m+w)$ degree, w vertex has $[d(d-w)]$ degree.

Consider a edge labeling of $G\Box H$ an invertible mapping $\phi : E(G\Box H) \rightarrow \{\wp_1,\ \wp_2,\ \wp_3, \ldots,\ \wp_q\}$ where $\wp_1,\ \wp_2,\ \wp_3,\ \ldots,$ $\wp_q$ is a palindromic number. The number of edges (q) in the graph $G\Box H$ equal to $\{(d-w)[m(m+w)]\}$ so the assignment of distinct palindromic number from $\wp_1$ to $\wp_{\{(d-w)[m(m+w)]\}}$. The process of assigning palindromic numbers to the edges may be categorized into m phases. Initially, the horizontal edges (e_{hk}) are labeled, followed by the labeling of the vertical edges (e_{vk}). Thirdly, it is important to allocate the labeling on the edge of the horizontal curve (e_{hck}). Subsequently, it is essential to proceed with labeling the edge of the vertical curve (e_{vck}). The functionality of this method may be succinctly demonstrated as follows,

$\phi[e_{hw}$ to $e_{h[d(m+w)+w]}] = \wp_w$ to $\wp_{[d(m+w)+w]}$ (Left to Right in rowwise);

$\phi[e_{vw}$ to $e_{v[d(m+w)]}] = \wp_{[(d-w)(m+d)+d]}$ to $\wp_{\{[d(d-w)(m+w)+w]\}}$ (Up to Down in columnwise);

$\phi[e_{hcw}$ to $e_{hcm}] = \wp_{[d(d-w)(m+w)+2w]}$ to $\wp_{[(m+w)(m+d)]}$ (Up to Down);

$\phi[e_{vcw}$ to $e_{vc(m+w)}] = \wp_{[(m+w)(m+d)+w]}$ to $\wp_{(d-w)[m(m+w)]}$ (Left to Right).

Consider a vertex labeling of $G\Box H$ is an injective mapping $\theta : V(G\Box H) \rightarrow Z^+$ defined

$$\theta(x_{i,\ j}) = \Sigma[\psi_{G\Box H}(x_{i,\ j})]$$

where $\psi_{G\square H}(x_{i,\ j})$ is an incidence function of vertex $x_{i,\ j}$ of graph $G\square H$. Labeling of edges and vertices of $G\square H$ followed by the above pattern is shown in Figure 1.

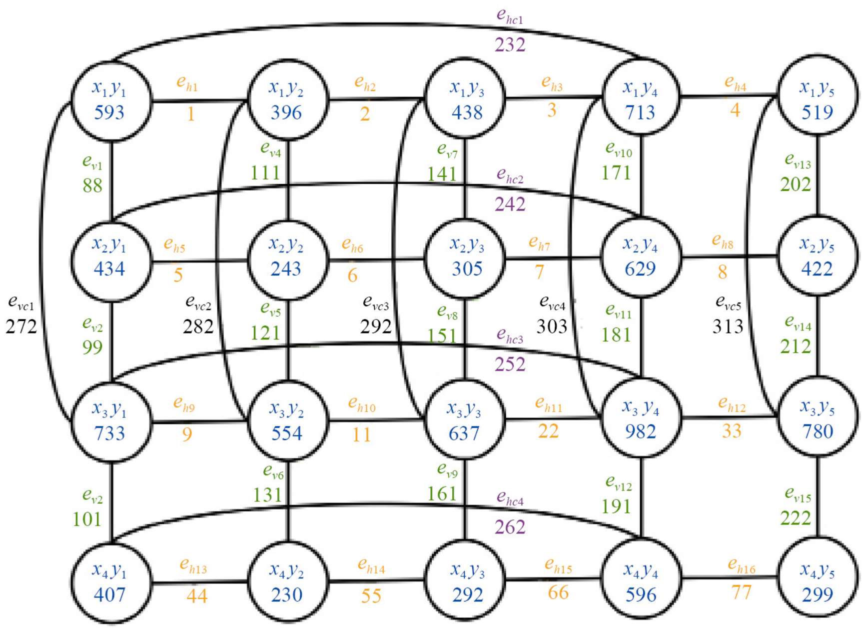

Figure 1. Cartesian product of Paw and Banner graph

The Palindromic antimagicness of graph $G\square H$ is establish through distinct vertex labeling. In order to verify the uniqueness of vertex labeling, it is necessary to employ numbers from the field of number theory.

In our approach, we employ a technique in which each vertex is assigned a distinct number, while ensuring that the remaining vertices do not satisfy the criterion associated with that particular number.

Let us commence the implementation of the technique, $\theta(x_{w,\ w})$ is a Leyland number, a number k is a leyland number if it can be written as a format $a^b + b^a$, $\forall a,\ b > 1$. Here $\theta(x_{w,\ w})$ written as $[m(d-w)+w]^{d-w} + (d-w)^{[m(d-w)+w]}$. No further vertex labeling is able to be written in the format $a^b + b^a$, so $\theta(x_{w,\ w})$ is distinctive from each other.

Next, $\theta(x_{w,\ (d-w)})$ is a gapful number, a number k is considered to be gapful if it is divisible by the number generated by its first and last digits. $\theta(x_{w,\ (d-w)}) = \{(d-w)^{d-w}d^{d-w}[(d-w)(m+w)+w]\}$ is divisible by $[(d-w)^{d-w}d^{d-w}]$ which is the first and last digits of $\theta(x_{w,\ (d-w)})$. Here, it is not possible to express any further vertex labeling $G\square H$ divisible by the number formed by first and last digit of those vertex labeling.

Therefore, $\theta(x_{n,\ l})$ is a smith number, which is a composite number k, for which the total of its digits is equal to the sum of the digits of its prime factors. $\theta(x_{w,\ d}) = \{d(d-w)[(d-w)^d d^{d-w} + w]\}$ has $d(m+w)$ as a sum of its digits and sum of the digits of prime factors. All the other vertex labeling of $G\square H$ has unequal sum of digits and sum of the digits of prime factors.

Next, $\theta(x_{w,\ m})$ is a junction number, where junction number can be expressed as the $k + sod(k)$, where $sod(k)$ is a sum of the digits. $\theta(x_{w,\ m}) = \{(d-w)[[(d-w)^{(d-w)}d(((d-w)d(m+w))-w)]-w]\} + [d(d-w)] + [m(d-w)+w] + m$. $\theta(x_{(d-w),\ d})$ is also a junction number also expressed as $k + sod(k)$. Thus, $\theta(x_{(d-w),\ d}) = \{(d+m)[[(d-w)d(d+m)] - w]\} + d + w$. It is necessary that we tackle the conflict between the vertex labelings $\theta(x_{w,\ m})$ and $\theta(x_{(d-w),\ d})$ because both posses a junction number. Here $\theta(x_{w,\ m})$ has a k as $\{(d-w)[[(d-w)^{(d-w)}d(((d-w)d(m+w))-w)]-w]\}$ and $\theta(x_{(d-w),\ d})$ has a k as $\{(d+m)[[(d-w)d(d+m)]-w]\}$. Since both entities posses a unique k value so $\theta(x_{w,\ m})$ and $\theta(x_{(d-w),\ d})$ is distinctive from each other. No other vertex labeling of $G\square H$ can be expressed as a form $k + sod(k)$.

Now, $\theta(x_{w,\ (m+w)})$ is a lucky number, defined as the set of numbers that successfully pass through a sieving procedure related to the eratosthenes sieve, commonly employed for the computation of prime number. Let's proclaim that 1 is lucky and begin with a sieve that only contains odd numbers. Since k is the first integer bigger than 1, we declare it to be lucky and remove all the numbers in multiple positions from the sieve, continuing this ideal method we may locate all the lucky number. $\theta(x_{w,\ (m+w)}) = \{[(d-w)^d(m+w)(3m+w)] - w\}$ is a lucky number cannot be eliminated by sieve process while, all the other vertex labeling of $G\Box H$ can be eliminated by sieve process. so $\theta(x_{w,\ (m+w)})$ is distinct from each other.

At the moment, $\theta(x_{(d-w),\ w})$ and $\theta(x_{m,\ d})$ are palindromic numbers. $\theta(x_{(d-w),\ w})$ is a $\wp_{[(d-w)^{d-w}(3m-w)]^{th}}$ palindromic number and $\theta(x_{m,\ d})$ is a $\wp_{[(d-w)^{d-w}(d)^{d-w}]^{th}}$ palindromic number so there is no clash between $\theta(x_{(d-w),\ w})$ and $\theta(x_{m,\ d})$. All the other vertex labeling of $G\Box H$ are non palindromic number so these two vertex labelings are distinct from each other.

Next, $\theta(x_{(d-w),\ (d-w)})$ is a powerful number, when every prime p divides k and p^2 also divide k, then an integer k is said to be powerful. $\theta(x_{(d-w),\ (d-w)}) = d^{m+w}$ which is divided by d and d^2. None of the another vertex labeling of $G\Box H$ cannot divided by both prime p and p^2.

Therefore, $\theta(x_{(d-w),\ m})$ is a proth number, if k is an odd number, $a > 0$ and $2^a > k$, then $(k)2^a + 1$ is a proth number. $\theta(x_{(d-w),\ m})$ can be written as $\{[m(m+w) - w]2^{(m+w)} + 1\}$. No further vertex labeling is able to written in the format $(k)2^a + 1$, so $\theta(x_{(d-w),\ m})$ is distinct from each other.

Now, $\theta(x_{(d-w),\ (m+w)}), \theta(x_{d,\ w})$ and $\theta(x_{m,\ (m+w)})$ are modest number, a number k is considered modest if its digits can be partitioned into two distinct integers, a and b in such a way that dividing k by b yields a as the remainder. Diving $\theta(x_{(d-w),\ (m+w)})$ by $(d-w)[2(m+w)+w]$ gives m as a remainder where $\theta(x_{(d-w),\ (m+w)}) = (d-w)\{[(d-w)d(m+w)(m+d)]+w\}$. Dividing $\theta(x_{d,\ w})$ by $d[2(m+w)+w]$ gives $d+m$ as a remainder where $\theta(x_{d,\ w}) = [((d-w)^{d-w}d(((d-w)^{(d-w)}d(m+w))+w))]$. Finally, dividing $\theta(x_{m,\ (m+w)})$ by $(2(m+w)+w)d^{d-w}$ gives $(d-w)$ as a remainder then $\theta(x_{m,(m+w)}) = [d^{(d-w)}(2(m+w)+w)]$. Hence $\theta(x_{(d-w),\ (m+w)}), \theta(x_{d,\ w})$ and $\theta(x_{m,\ (m+w)})$ are distinct from each other because each has distinct b which gives distinct a as a remainder. All the other vertex labelings of $G\Box H$ can be partitioned into a and b but divide by b of those vertex labeling not gives a as a remainder.

At the moment, $\theta(x_{d,\ (d-w)})$ is a magnanimous number, a number k, $k \geq 2$ whose sum is prime when a " + " sign is added to any of its digits. $\theta(x_{d,\ (d-w)}) = (d-w)[((d-w)^{d-w}d[m(m+w)+d])+w]$ inserting " + " to any of its digits gives $\{[(d-w)^{d-w}d(m+w)] - w\}$ which is prime. After the process of magnanimous number procedure to the another vertex labeling of $G\Box H$ gives a composite number so $\theta(x_{d,\ (d-w)})$ is distinctive from each other.

Next, $\theta(x_{d,\ d})$ and $\theta(x_{m,\ (d-w)})$ are happy number, the concept of happy number k is defined according to the following procedure. The iterative procedure involves replacing k, with the sum of the squares of its individual digits. This process is repeated until k reaches the value of 1. Happy number are defined as number for which the iterative process terminates at the value of 1. Hence $\theta(x_{d,\ d}) = [(d+m)^{d-w}(3m+w)]$ which is ends up with 1 in $(m+w)$ steps, and $\theta(x_{m,\ (d-w)}) = [(d-w)(m+w)[((d-w)^d d) - w]]$ which is end up with 1 in d steps so there is no clash between $\theta(x_{d,\ d})$ and $\theta(x_{m,\ (d-w)})$. No further vertex labelings of $G\Box H$ cannot ends up with 1.

At the moment, $\theta(x_{d,\ m})$ is a untouchable number, which are not the sum of the proper divisors of any number. $\theta(x_{d,\ m}) = \{(d-w)[((d-w)(m+w)(d+m)^{d-w}) + w]\}$ which are not the sum of proper divisors of any number. All the other vertex labeling of $G\Box H$ be a sum of proper divisors of any natural number.

Now, $\theta(x_{d,\ (m+w)})$ is triangular number, which is a concept of figurate number T_n can be mathematically represented as a triangular grid of points, whereby the first row consists of just one component and each consecutive row has one more element than the proceeding row. $\theta(x_{d,\ (m+w)}) = [(d-w)^{d-w}d(m+w)(3m+w)]$ is a $[d(dm+w)]$ triangular number. None of the other vertex labeling of $G\Box H$ are triangular number.

Next, $\theta(x_{m,\ w})$ and $\theta(x_{m,\ m})$ are pancake number, which is denoted by p_k refers to the maximum number of pieces that can be obtained by dividing a pancake into k linear cut (i.e) $p_k = \dfrac{k^2+k+2}{2}$. Here $\theta(x_{m,\ w}) = (2m+d)\{[(d-w)^{d-w}(d)^d] + w\}$ which is obtained by $[m(m+d)]$ straight cuts. $\theta(x_{m,\ m}) = (d-w)^{d-w}[[(d-w)d(m+w)^{d-w}] - w]$ is obtained by $(d-w)\{[d(d-w)(m+w)] + m\}$. So there is no clash in between $\theta(x_{m,\ w})$ and $\theta(x_{m,\ m})$. None of the other vertex labeling of $G\Box H$ satisfy pancake number.

Therefore, it can be concluded that any vertex labeling of graph $G\Box H$ is distinctive from each other, indicating that $G\Box H$ possesses the property of being palindromic antimagic.

Theorem 2 The Tensor product of Paw and Banner (G and H) graphs is palindromic antimagic.

Proof. The Tensor product of G and H is denoted by $G\times H$, the vertex set $V(G\times H)=\{x_{i,\ j}:1\leq i\leq m, 1\leq j\leq (m+w)\}$ and the edge set $E(G\times H)=\{e_{sk}:1\leq k\leq \{(d-w)[m(m+w)]\}$ where e_{sk} is a slope edges. In this graph $G\times H$ has $|V|=[m(m+w)]$ and $|E|=\{(d-w)[m(m+w)]\}$ and it has m rows and $(m+w)$ columns (horizontal manner) respectively. In the graph $G\times H$, there are w vertex has w degree, $(m+w)$ vertices has $(d-w)$ degree, $(d-w)$ vertices has d degree, $[d(d-w)]$ vertices has m degree, $(m+w)$ vertex has $[d(d-w)]$ degree, w vertex has $[m(d-w)+w]$ degree.

Consider an edge labeling of $G\times H$ is an invertible mapping $\phi: E(G\times H)\rightarrow\{\wp_1,\ \wp_2,\ \wp_3,\ \ldots,\ \wp_q\}$ where $\wp_1,\ \wp_2,\ \wp_3,\ \ldots,\ \wp_q$ is a palindromic number. The number of edges (q) in the graph $G\times H$ equal to $\{(d-w)[m(m+w)]\}$ so the assignment of palindromic number from $\wp_1$ to $\wp_{\{(d-w)[m(m+w)]\}}$. The procedure of assignment of palindromic number to the edges of $G\times H$ in rowwise. In row n, the assignment of palindromic number from an incident edges of vertex $\{x_{w,\ w}\}$ to an incident edges of vertex $\{x_{w,\ (m+w)}\}$. In vertex $\{x_{w,\ w}\}$ the assignment of palindromic number to the incident edges in clockwise and the same procedure to all the another vertices in row w. In row $(d-w)$, the assignment of palindromic number from an incident edges of vertex $\{x_{(d-w),\ w}\}$ to an incident edges of vertex $\{x_{(d-w),\ (m+w)}\}$. In vertex $\{x_{(d-w),\ w}\}$ the assignment of palindromic number to the incident edges in clockwise except the edges incident with row w and the same procedure to all the another vertices in row $(d-w)$. In row d, the assignment of palindromic number from an incident edges of vertex $\{x_{d,\ w}\}$ to an incident edges of vertex $\{x_{d,\ (m+w)}\}$. In vertex $\{x_{d,\ w}\}$ the assignment of palindromic number to the incident edges in clockwise except the edges incident with row w and row $(d-w)$ and the same procedure to all the another vertices in row d. Finally, In row m there is no edges for labeling because all the incident edges are already labeled in row d.

Consider a vertex labeling of $G\times H$ is an injective mapping $\theta:\ V(G\times H)\rightarrow Z^+$ defined

$$\theta(x_{i,\ j})=\Sigma[\psi_{G\times H}(x_{i,\ j})] \tag{1}$$

where $\psi_{G\ \times\ H}(x_{i,\ j})$ is an incidence function of vertex $x_{i,\ j}$ of graph $G\times H$. Labeling of edges and vertices of $G\times H$ followed by the above pattern is shown in Figure 2.

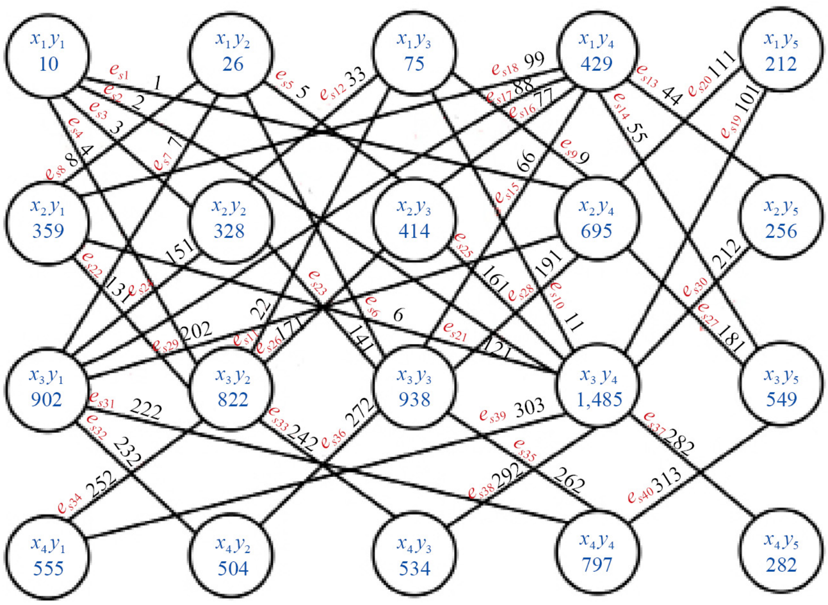

Figure 2. Tensor product of Paw and Banner graph

The palindromic antimagicness of a graph $G \times H$ is established by the utilization of distinct vertex labeling. To determine the distinctiveness of vertex labeling of $G \times H$, we employ a methodology akin to that employed in $G\square H$.

Now, $\theta(x_{w,\ w}), \theta(x_{w,\ (d-w)})$ and $\theta(x_{(d-w),\ m})$ are semi prime number. The product of two prime is consider as semi prime number. To prove there is no clash in between the above 3 vertex labelings. $\theta(x_{w,\ w}) = (d-w)(m+w)$ which is a brilliant number (i.e) A Semiprime when both primes have the same number of digits. $\theta(x_{w,\ (d-w)}) = (d-w)(3m+w)$ is emirpimes (i.e) A number is a semiprime and its reverse is another semiprime. $\theta(x_{(d-w),\ m}) = (m+w)[((d-w)^{d-w}(m+w)(m+d)) - w]$ which is neither brilliant nor emirpimes. So there is no conflict observed among the three aforementioned vertex labelings. None of the other vertex labeling of $G \times H$ can be expressed as the product of two primes.

Next, $\theta(x_{w,\ d})$ and $\theta(x_{d,\ (m+w)})$ are duffinian number, refers composite number k that do not share any prime factors with the sum of their divisors $\sigma(k)$. Here, $\theta(x_{w,\ d}) = [d(m+w)^{d-w}]$ is relatively prime to sum of their divisors $(d-w)^{d-w}[((d-w)d(m+w))+w]$. $\theta(x_{d,\ (m+w)}) = d^{d-w}[[(d-w)^{d-w}d(m+w)]+w]$ is relatively prime to sum of their $\{(d-w)(3m+w)[((d-w)d(m+w))+w]\}$. Since $\theta(x_{w,\ d})$ and $\theta(x_{d,\ (m+w)})$ have different sum of divisors, it is obvious that they are distinct. All the other vertex labelings are cannot relatively prime with the sum of their divisors.

At the moment, $\theta(x_{w,\ (m+w)})$, $\theta(x_{m,\ (m+w)})$, $\theta(x_{(d-w),\ d})$, $\theta(x_{m,\ w})$ and $\theta(x_{m,\ m})$ are palindromic number. $\theta(x_{w,\ (m+w)})$ is a ${\wp_{(d-w)d(m+w)}}^{th}$ palindromic number. $\theta(x_{m,\ (m+w)})$ is a ${\wp_{\{[(d-w)^{d-w}d^{d-w}]+w\}}}^{th}$ palindromic number. $\theta(x_{(d-w),\ d})$ is a ${\wp_{\{[(d-w)(m+w)^{d-w}]\}}}^{th}$ palindromic number. $\theta(x_{m,\ w})$ is a ${\wp_{[(d-w)^{m+2w}]}}^{th}$ palindromic number. $\theta(x_{m,\ m})$ is a ${\wp_{[(d-w)^{d}(3m-w)]}}^{th}$ palindromic number. So there is no clash between the above 5 vertex labeling. All the other vertex labeling of $G \times H$ are non palindromic number.

Next, $\theta(x_{w,\ m})$ is an astonishing number if its representation can be divided into two parts a and b, such that k equals the sum of the integers from a and b. $\theta(x_{w,\ m}) = [d.(2m+d)(3m+w)]$ which can be written as $m+(m+w)+\ldots[((d-w)d(m+w))-w]$. No further vertex labeling in $G \times H$ cannot be written as sum of the integer from a and b.

Now, $\theta(x_{(d-w),\ w})$ is a emirps number, that is a number is prime and its reverse is also a prime. Here $\theta(x_{(d-w),\ w}) = \{[(d-w)d^{d-w}(((d-w)d^{d}) - w)] - w\}$ its reverse is also a prime. The other vertex labeling of $G \times H$ has its reverse is a composite number. so $\theta(x_{(d-w),\ w})$ is distinct from each other.

At the moment, $\theta(x_{(d-w),\ (d-w)})$ and $\theta(x_{m,\ d})$ are unprimeable number, a composite number, denoted by k is referred to as unprimeable if it is incapable of being transformed into a prime number through the alteration of a single digit. Since k digits numbers has $9k$ possible of numbers by changing single digit. $\theta(x_{(d-w),\ (d-w)}) = \{(d-w)^{d-w}[(d-w)d(m+d)-w]\}$ changing any single digits of $\{(d-w)^{d-w}[(d-w)d(m+d)-w]\}$ cannot be form a prime number. $\theta(x_{m,d}) = \{(d-w)d[((d-w)d^{d-w}(m+w))-w]\}$ is also unprimeable. It is necessary to prove there is no conflict in between $\theta(x_{(d-w),\ (d-w)})$ and $\theta(x_{m,\ d})$ because both posses unprimeable. $\theta(x_{(d-w),\ (d-w)})$ lies between $\{(d-w)^{m}(m+w)^{d-w}\}$ and $\{(d-w)^{d}d(m+w)^{d-w}\}$. $\theta(x_{m,\ d})$ lies between $\{(d-w)^{d-w}d(m+w)^{d-w}\}$ and $\{(d-w)^{m}(m+w)^{d-w}\}$. Hence $\theta(x_{(d-w),\ (d-w)}), \theta(x_{m,\ d})$ are distinctive from each other. Any other vertex labeling of $G \times H$ has capable of being transformed into a prime number through the alteration of a single digit.

Now, $\theta(x_{m,\ (d-w)})$ is a tribonacci number, so that the sequence is characterized by the recursive formula $T_1 = T_2 = T_3$ = 2 and $T_k = T_{k-1} + T_{k-2} + T_{k-3}$ for $k > 3$. This sequence has resemblance to the fibonacci sequence, with the exception that each term is obtained by summing the preceeding three terms. $\theta(x_{m,\ (d-w)}) = \{(d-w)^{d}(d)^{d-w}(m+d)\}$. No further vertex labeling of $G \times H$ are tribonacci.

Next, $\theta(x_{(d-w),\ (m+w)})$ is a enlightened number, start with the concatenation of its unique prime factors. $\theta(x_{(d-w),\ (m+w)}) = (d-w)^{2m}$ is concatenation of $(d-w)$. None of the other vertex labeling of $G \times H$ concatenation of its unique prime factors.

At the moment, $\theta(x_{d,\ w})$ is untouchable number, $\theta(x_{d,\ w}) = \{(d-w)(3m-w)[(d-w)^{d}(m+w)+w]\}$ which is not the sum of the proper divisors of any number. No further vertex labelings of $G \times H$ are untouchable because they are sum of the proper divisors of any number.

Now, $\theta(x_{d,\ d})$ is a magnanimous number, $\theta(x_{d,\ d}) = \{(d-w)(d+m)[(d-w)d(2m+d)+w]\}$, insert " + " among the digits of $\theta(x_{d,\ d})$ gives $(2m+w) + \{(d-w)[m(m+w)-w]\}$ is a prime or $\{d[(d-w)d(m+w)+w]\} + 2m$ is also a prime. All the other vertex labelings of $G \times H$ does not resulted a prime number after insert "+"among the digits.

Next, $\theta(x_{d,\ m})$ and $\theta(x_{d,\ (d-w)})$ are apocalyptic number, where a number of the form 2^k if its digits include the substring 666. $\theta(x_{d,\ m}) = \{d^{d}(m+w)(2m+d)\}$ and $\theta(x_{d,\ (d-w)}) = (d-w)d[((d-w)^{d}(m^{d-w}+w))+w]$. Here $\theta(x_{d,\ m})$

has m digits, otherwise $\theta(x_{d,\,(d-w)})$ has d digits, so $\theta(x_{d,\,m})$ and $\theta(x_{d,\,(d-w)})$ are distinctive from each other. None of the other vertex labelings of $G \times H$ has written in form 2^k its digits contains 666.

Hence it may be inferred that each vertex labeling of the graph $G \times H$ is distinct from one another, so suggesting that $G \times H$ exhibits the characteristic of being palindromic antimagic.

3. Conclusion

This article demonstrates that the cartesian product of the paw and banner graph, as well as the tensor product of the paw and banner graph, exhibit palindromic antimagic properties. This is achieved by establishing the uniqueness of vertex labelings. We propose a novel notion termed "palindromic antimagic labeling" that involves combining it with certain types of graph products. We expect to find numerous unresolved issues and have confidence that these findings may be further developed.

Conflict of interest

Authors declare there is no conflict of interest at any point with reference to research findings.

References

[1] Hartsfield N, Ringel G. *Pearls in Graph Theory. A Comprehensive Introduction*. NY, USA: Dover Publications; 2003.

[2] Gallian JA. A dynamic survey of graph labeling. *The Electronic Journal of Combinatorics*. 2023; DS(6). Available from: doi:10.37236/27.

[3] Liang YC, Zhu X. Anti-magic labeling of Cartesian product of graphs. *Theoretical Computer Sciences*. 2013; 477: 1-5. Available from: doi:10.1016/j.tcs.2012.12.023.

[4] Cheng Y. Lattice grids and prisms are antimagic. *Theoretical Computer Sciences*. 2007; 374(1-3): 66-73.

[5] Cheng Y. A new class of antimagic Cartesian product graphs. *Discrete Mathematics*. 2008; 308(24): 6441-6448.

[6] Wang T, Hsiao CC. On antimagic labeling for graph products. *Discrete Mathematics*. 2008; 308(16): 3624-3633.

[7] Richard H, Imrich W, Sandiklavzar. *Handbook of Product Graphs*. 2nd Edition. Boca Raton, FL, USA: CRC Press; 2011.

[8] Wang T. Toroidal grids are antimagic. *Lecture Notes in Computer Science*. 2005; 3595: 671-679. Available from: doi:10.1007/11533719_68.