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Research Article

Co-Secure Domination in Zero-Divisor Graphs $\Gamma(\mathbb{Z}_n)$

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Abstract: Let $G = (V, E)$ be a zero-divisor graph (ZDG) whose nodes are non-zero zero-divisors of a commutative ring R with unity and there exists an edge $e = (u_i, u_j)$, $i \neq j$ between any two distinct nodes of G if the product of u_i and u_j is zero. Usually this zero-divisor graph is denoted by $\Gamma(R)$. In this paper, the concept of co-secure dominating set of a zero-divisor graph is studied. Also, the co-secure domination number of some zero-divisor graphs (ZDGs) are obtained.

***Keywords*:** zero-divisor graph, co-secure domination, domination number

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1. Introduction

In 1988, the concept of the ZDG was proposed by Beck [1] for the very first time, whose nodes are all the elements of the ring and two nodes are adjacent if their product is equal to 0. After 11 years, it was simplified and redefined by Anderson et al. [2] to the ZDGs. The nodes of the ZDG are all non-zero zero-divisors and two distinct nodes u and v are adjacent if and only if $uv = 0$. On juxtaposed with the Beck's ZDG, Anderson et al. [2] excluded '0' element, thus the properties of the zero-divisors in the ring were briefly discussed. In 1962, Berge [3] introduced the concept of the domination number in graph, he called this as "coefficient of external Stability" and Ore [4] used the name dominating set and domination number for the same notion. In Cockayne et al. [5] made an significant and extensive survey of the results know at that time about dominating sets in graphs. The survey paper of Cockayne et al. [5] has generated lot of interest in the study of domination in graphs. The study of domination parameters and related topics are one of the rich and fast developing field in graph theory. Advanced topics in domination can be referred in [6, 7]. Ring theory is a branch of mathematics, primarily in the discipline of algebra, that has been extensively studied and used in computer science, cryptography and picture segmentation. Mohamed et al. [8] studied the secure domination number of a ZDG. Some recent works on ZDGs can be found in [4, 9–12]. In this paper, co-secure dominating set of a ZDG is defined and also the co-secure domination number of some ZDGs $\Gamma(\mathbb{Z}_n)$ are found.

2. Preliminaries

In this section, some basic definitions of zero-divisor graphs are given.

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Definition 1 [2] Let R be a commutative ring with $1 \neq 0$, and let $\mathbb{Z}(R)$ be its set of zero-divisors. The ZDG of R, denoted by $\Gamma(R) = (V(\Gamma(R)), E(\Gamma(R)))$ is the (undirected) graph with nodes $V(\Gamma(R)) = \mathbb{Z}(R)^* = \mathbb{Z}(R) - \{0\}$, the non-zero zero-divisors of R, and for distinct $u, v \in V(\Gamma(R))$, the nodes u and v are adjacent if and only if $uv = 0$.

Remark 1 $\Gamma(R)$ is the null graph if and only if R is an integral domain.

Note: Here after, we consider a commutative ring R by $\mathbb{Z}_n$ and the ZDG $\Gamma(R)$ by $\Gamma(\mathbb{Z}_n)$.

Definition 2 [9] Let $\Gamma(\mathbb{Z}_n)$ be a ZDG. A subset M of $V(\Gamma(\mathbb{Z}_n))$ is a dominating set if every node in $V(\Gamma(\mathbb{Z}_n)) - M$ is adjacent to atleast one node in M.

The minimum of the cardinality of a dominating set is the domination number of a graph $\Gamma(\mathbb{Z}_n)$ and its denoted by $\gamma(\Gamma(\mathbb{Z}_n))$.

Definition 3 [8] Let $\Gamma(\mathbb{Z}_n)$ be a simple graph with the node set $V(\Gamma(\mathbb{Z}_n))$. The neighborhood of u is the set $N_{\Gamma(\mathbb{Z}_n)}(u) = \{v \in V(\Gamma(\mathbb{Z}_n)) \mid v \odot_n u = 0\}$.

Definition 4 [8] A node $u \in V(\Gamma(\mathbb{Z}_n))$ is said to be a pendant node if and only if it has degree 1.

Definition 5 Let $\Gamma(Z_p \times Z_q)$ be a commutative ring and $Z(R)$ be the set of all zero-divisors of $\Gamma(Z_p \times Z_q)$. The adjacent of the graph $\Gamma(Z_p \times Z_q)$ is as follows:

- $(a_i, 0) \sim (0, b_j)$, where $0 < a_i < p$, $0 < b_i < q$,
- $(0, b_j) \sim (a_i, 0)$, where $0 < a_i < p$, $0 < b_i < q$,

where p, q are primes.

3. Co-secure domination number of $\Gamma(\mathbb{Z}_n)$

In this section, the co-secure dominating set of a ZDG is defined and the co-secure domination number of some ZDGs are obtained.

Definition 6 A dominating set M of $V(\Gamma(\mathbb{Z}_n))$ of a ZDG $\Gamma(\mathbb{Z}_n)$ is a co-secure dominating set if for every $u \in M$, there exists $v \in V(\Gamma(\mathbb{Z}_n)) - M$ such that $u \in N_{\Gamma(\mathbb{Z}_n)}(v) \cap M$ and $(M - \{u\}) \cup \{v\}$ is a domination set.

The minimum cardinality taken over all minimal co-secure dominating sets is called a co-secure domination number. The co-secure domination number and its corresponding minimum dominating set are denoted by $\gamma^{csd}(\Gamma(\mathbb{Z}_n))$ and γ^{csd}-set respectively.

Example 1 Consider the ZDG $\Gamma(\mathbb{Z}_{33})$ given in Figure 1.

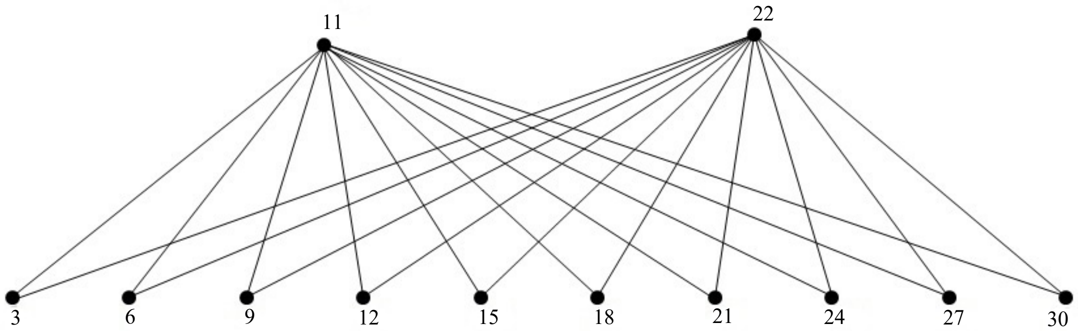

Figure 1. Zero divisor graph $\Gamma(Z_{33})$

The node set $V(\Gamma(\mathbb{Z}_{33})) = \{3, 6, 9, 11, 12, 15, 18, 21, 22, 24, 27, 30\}$ and the edge set $E(\Gamma(\mathbb{Z}_{33})) = \{(3, 11), (6, 11), (9, 11), (12, 11), (15, 11), (18, 11), (21, 11), (24, 11), (27, 11), (30, 11), (3, 22), (6, 22), (9, 22), (12, 22), (15, 22), (18, 22), (21, 22), (24, 22), (27, 22), (30, 22)\}$.

Let $M = \{11, 22\}$ be a dominating set.

Let $11 \in M$, there exist $3 \in V(\Gamma(\mathbb{Z}_{33})) - M$, such that $11 \in N_{\Gamma(\mathbb{Z}_{33})}(3) \cap M$ and $(M - \{11\}) \cup \{3\}$ is a dominating set.

Here, $\Gamma(\mathbb{Z}_{33})$ is a ZDG and the co-secure dominating set $M = \{11, 22\}$.

Hence, the co-secure domination number $\gamma^{csd}(\Gamma(\mathbb{Z}_{33}))$ is 2.

Theorem 1 Let $\Gamma(\mathbb{Z}_{2q})$ be a ZDG and q is a prime number greater than 3, then $\gamma^{csd}(\Gamma(\mathbb{Z}_{2q})) = q-1$.

Proof. Consider $\Gamma(\mathbb{Z}_{2q})$ be a ZDG and $q > 3$ be any prime number.

Then $V(\Gamma(\mathbb{Z}_{2q})) = \{2, 4, 6, .., 2(q-1), q\}$.

Let $u = 2(q-1)$ and $v = q$ be two nodes, then $u \odot_{2q} v = 2(q-1) \odot_{2q} q = 0$, since $2q|uv$.

Hence $uv \in E(\Gamma(\mathbb{Z}_{2q}))$.

Similarly, let $u \in V(\Gamma(\mathbb{Z}_{2q})) - \{v\}$ and $v = q$ then $v \odot_{2q} u = 0$

Let $M = V(\Gamma(\mathbb{Z}_{2q})) - \{v\}$ be a subset of $V(\Gamma(\mathbb{Z}_{2q}))$ and it has $q-1$ elements.

For every $u \in M$, there exists $v \in V(\Gamma(\mathbb{Z}_{2q})) - M$, such that $u \in N_{\Gamma(\mathbb{Z}_{2q})}(v) \cap M$ and $(M - \{u\}) \cup \{v\}$ is a dominating set.

Thus M is a $\gamma^{csd} - set$.

Hence $\gamma^{csd}(\Gamma(\mathbb{Z}_{2q})) = q-1$. □

Theorem 2 Let $\Gamma(\mathbb{Z}_{3q})$ be a ZDG with prime number $q > 3$. Then $\gamma^{csd}(\Gamma(\mathbb{Z}_{3q})) = 2$.

Proof.

Let $\Gamma(\mathbb{Z}_{3q})$ be a ZDG and $q > 3$ be a prime number.

The node set $V(\Gamma(\mathbb{Z}_{3q})) = \{3, 6, 9, , , , 3(q-1), q, 2q\}$.

Let $v \in \Gamma(\mathbb{Z}_{3q})$ with $d(v) = \Delta(\Gamma(\mathbb{Z}_{3q}))$ also w be a another node with $d(w) = \Delta$ in $\Gamma(\mathbb{Z}_{3q})$, either $w = q$, $v = 2q$ (or) $w = 2q$, $v = q$

Then $w \odot_{3q} v = 2q \odot_{3q} q \neq 0 \implies wv \notin E(\Gamma(\mathbb{Z}_{3q}))$ (since $3q|wv$)

Let u be any node in $V(\Gamma(\mathbb{Z}_{3q})) - \{w, v\}$ such that $w \odot_{3q} u = v \odot_{3q} u = 0$

Then $wu, vu \in E(\Gamma(\mathbb{Z}_{3q}))$

Let $M = \{w, v\} \subseteq V(\Gamma(\mathbb{Z}_{3q}))$ be a dominating set.

For every $w \in M$, there exists $u \in V(\Gamma(\mathbb{Z}_{3q})) - M$ such that $w \in N_{\Gamma(\mathbb{Z}_{3q})}(u) \cap M$ and $(M - \{w\}) \cup \{u\} = \{v, u\}$ is a dominating set. Thus M is a $\gamma^{csd} - set$.

Hence $\gamma^{csd}(\Gamma(\mathbb{Z}_{3q})) = 2$. □

Theorem 3 Let $\Gamma(\mathbb{Z}_{5q})$ be a ZDG with prime number $q > 3$. Then $\gamma^{csd}(\Gamma(\mathbb{Z}_{5q})) = 4$.

Proof. Let $\Gamma(\mathbb{Z}_{5q})$ be a ZDG with $q > 5$.

The node set of $V(\Gamma(\mathbb{Z}_{5q})) = \{5, 10, 15, ..., 5(q-1), q, 2q, 3q, 4q\}$.

Now, the node set $V(\Gamma(\mathbb{Z}_{5q}))$ can be partition in the two parts V_1 and V_2 with $V_1 \cup V_2 = V(\Gamma(\mathbb{Z}_{5q}))$

Consider $V_1 = \{5, 10, 15, ..., 5(q-1)\}$ and $V_2 = \{q, 2q, 3q, 4q\}$.

Let $u, v \in V_1$ Then $u \odot_{5q} v \neq 0$, (since $u.v$ is not divisible by $5q$).

Similarly,

Let $w, t \in V_2$. Then $w \odot_{5q} t \neq 0$, (since $w.t$ is not divisible by $5q$).

Therefore $uv, wt \notin E(\Gamma(\mathbb{Z}_{5q}))$

Hence no nodes of V_1 is adjacent to any nodes of V_1.

Similarly, no nodes of V_2 is adjacent to any nodes of V_2.

Let $u \in V_1$ and $v \in V_2$.

Then $u \odot_{5q} v = 0$. (since $5q|uv$)

Hence $uv \in E(\Gamma(\mathbb{Z}_{5q}))$.

Therefore, every node in V_1 is adjacent to any nodes in V_2.

Consider $M = \{q, 2q, 3q, 4q\} \subseteq V(\Gamma(\mathbb{Z}_{5q}))$ is a dominating set.

For every $u \in M$ there exists $v \in V(\Gamma(\mathbb{Z}_{5q})) - M$, such that $u \in N_{\mathbb{Z}_{5q}}(v) \cap M$ and $(M - \{u\}) \cup \{v\}$ is a dominating set. That is for every $q \in M$ there exists $5 \in V(\Gamma(\mathbb{Z}_{5q})) - M$, such that $q \in N_{\mathbb{Z}_{5q}}(5) \cap M$ and $(M - \{q\}) \cup \{5\} = \{5, 2q, 3q, 4q\}$ is a dominating set. Thus M is a $\gamma^{csd} - set$.

Hence $\gamma^{csd}(\Gamma(\mathbb{Z}_{5q})) = 4$. □

Theorem 4 Let $\Gamma(\mathbb{Z}_{7q})$ be a ZDG with prime number $q > 7$. Then $\gamma^{csd}(\Gamma(\mathbb{Z}_{7q})) = 6$.

Proof. Let $\gamma^{sd}(\Gamma(\mathbb{Z}_{7q}))$ be a ZDG and $q > 7$ be a prime number.

The node set of $\gamma^{sd}(\Gamma(\mathbb{Z}_{7q})) = \{7, 14, 21, .., 7(q-1), q, 2q, 3q, 4q, 5q, 6q\}$.

Clearly , the node V can be partition in the two parts V_1 and V_2.

$V_1 = \{7, 14, 21, ..., 7(q-1)\}$ and $V_2 = \{q, 2q, 3q, 4q, 5q, 6q\}$.

Let $u \in V_1$, $w \in V_2$

Then $u \odot_{7q} w = 0$ (since $7q|uw$)

Hence $uw \in E(\Gamma(\mathbb{Z}_{7q}))$.

Let $u, v \in V_1$

Then $u \odot_{7q} v \neq 0$ (since $u.v$ is not divisible by $7q$)

Similarly,

Let $w, t \in V_2$

Then $w \odot_{7q} t \neq 0$ (since $w.t$ is not divisible by $7q$) $uv, wt \notin E((\Gamma(\mathbb{Z}_{7q})))$

Therefore, every node in V_1 is adjacent to any node in V_2 and vice versa.

Let $M = \{q, 2q, 3q, 4q, 5q, 6q\} \subseteq V(\Gamma(\mathbb{Z}_{7q}))$ be a dominating set.

For every $u \in M$, there exists $v \in V(\Gamma(\mathbb{Z}_{7q})) - M$, such that $u \in N_{\mathbb{Z}_{7q}}(v) \cap M$ and $(M - \{u\}) \cup \{v\}$ is a dominating set.

Thus M is a $\gamma^{csd} - set$.

Hence $\gamma^{csd}(\Gamma(\mathbb{Z}_{7q})) = 6$. □

Theorem 5 Let $\Gamma(\mathbb{Z}_{q^2})$ be a ZDG with prime number q. Then $\gamma^{csd}(\Gamma(\mathbb{Z}_{q^2}) = 1$.

Proof. Let $\Gamma(\mathbb{Z}_{q^2})$ be a ZDG with prime number q.

The node set of $V(\Gamma(\mathbb{Z}_{q^2})) = \{q, 2q, 3q, .., (q-1)q\}$.

Clearly q is adjacent to all the nodes in $\Gamma(\mathbb{Z}_{q^2})$ also each nodes adjacent to remaining all the nodes in $\Gamma(\mathbb{Z}_{q^2})$.

Let $M = \{q\} \subseteq V(\Gamma(\mathbb{Z}_{q^2}))$ be a dominating set.

For every $q \in S$, there exists $2q \in V(\Gamma(\mathbb{Z}_{q^2})) - M$ such that $q \in N_{\mathbb{Z}_{q^2}}(2q) \cap M$ and $(M - \{q\} \cup 2q) = \{2q\}$ is a dominating set.

Thus M is a γ^{csd}-set.

Hence $\gamma^{csd}(\Gamma(\mathbb{Z}_{q^2}) = 1$. □

Theorem 6 Let $\Gamma(\mathbb{Z}_{3^n})$ be a ZDG with a positive integer $n > 2$. Then $\gamma^{csd}(\Gamma(\mathbb{Z}_{3^n})) = 1$.

Proof. Let $\Gamma(\mathbb{Z}_{3^n})$ be a ZDG with $n \geq 3$ is a positive integer.

The node set $V(\Gamma(\mathbb{Z}_{3^n})) = \{3, 6, 9, ..., 3^{n-1}, ..., 2(3^{n-1}), ..., 3(3^{n-1} - 1)\}$.

Since, $\Gamma(\mathbb{Z}_{3^n})$ has no pendent node.

Therefore, there exists $u, v \in \Gamma(\mathbb{Z}_{3^n})$ is adjacent to all the nodes in $\Gamma(\mathbb{Z}_{3^n})$.

Let $w \in \Gamma(\mathbb{Z}_{3^n})$, then $w \odot_{3^n} u = w \odot_{3^n} v = 0$.

And let $u = 3^{n-1}$ and $v = 2(3^{n-1})$ Then $u \odot_{3^n} v = 3^{n-1} \odot_{3^n} 2(3^{n-1}) = 0$ (since $3^n|uv$)

Hence $uv \in E(\Gamma(\mathbb{Z}_{3^n}))$.

Let $M = \{3^{n-1}\} \subseteq V(\Gamma(\mathbb{Z}_{3^n}))$ be a dominating set.

For every $3^{n-1} \in M$, there exists $2(3^{n-1}) \in V(\Gamma(\mathbb{Z}_{3^n})) - S$ such that $3^{n-1} \in N_{\mathbb{Z}_{3^n}}(2(3^{n-1})) \cap M$ and $(M - \{3^{n-1}\}) \cup \{2(3^{n-1})\} = \{2(3^{n-1})\}$ is a dominating set.

Thus M is a γ^{csd}-set.

Hence $\gamma^{csd}(\Gamma(\mathbb{Z}_{3^n})) = 1$. □

Theorem 7 Let $\Gamma(\mathbb{Z}_{q^n})$ be a ZDG with prime $q > 2$ and $n > 2$. Then $\gamma^{csd}(\Gamma(\mathbb{Z}_{q^n})) = 1$.

Proof. Theorem 5 and 6 guarantees that, there exist $u, v \in \Gamma(\mathbb{Z}_{q^n})$ is adjacent to all the nodes in $\Gamma(\mathbb{Z}_{q^n})$.

The node set $V(\Gamma(\mathbb{Z}_{q^n}))$ is $\{q, 2q, 3q, ..., q^{n-1}, ..., 2q^{n-1}, ...3q^{n-1}, ..., q(q^{n-1} - 1)\}$

Clearly $\Gamma(\mathbb{Z}_{q^n})$ has no pendent node.

Let $u = q^{n-1}$, $v = 2q^{n-1}$ then $u \odot_{q^n} v = 0$ (since $u.v$ is a multiple of q^n)

Hence $uv \in E(\Gamma(\mathbb{Z}_{q^n}))$

Let $M = \{q^{n-1}\} \subseteq V(\Gamma(\mathbb{Z}_{q^n}))$ be a dominating set.

For every $q^{n-1} \in M$, there exists $2(q^{n-1}) \in V(\Gamma(\mathbb{Z}_{q^n})) - M$ such that $q^{n-1} \in N_{\mathbb{Z}_{q^n}}(2q^{n-1}) \cap M$ and $(M - \{q^{n-1}\}) \cup \{2q^{n-1}\} = \{2q^{n-1}\}$ is a dominating set.

Thus M is a γ^{csd}-set.

Hence $\gamma^{csd}(\Gamma(\mathbb{Z}_{q^n})) = 1$. □

Theorem 8 Let $\Gamma(Z_2 \times Z_q)$ be a ZDG with prime number q. Then $\gamma^{csd}(\Gamma(Z_2 \times Z_q)) = q-1$.

Proof. Let $V(\Gamma(Z_2 \times Z_q)) = \{(0, 1), (0, 2), ..., (0, q-1), (1, 0)\}$.

Let $u = (0, 1)$ and $w = (1, 0)$ be two nodes, then $u \times w = (0, 1) \times (1, 0) = (0, 0)$ that is an edge between u and w, hence $uw \in E(\Gamma(Z_2 \times Z_q))$.

Also, let $v \in V(\Gamma(Z_2 \times Z_q)) - \{w\}$ and $w = (1, 0)$ then $v \times w = (0, 0)$.

Clearly, $w = (1, 0)$ is adjacent to all other nodes.

Let $M = V(\Gamma(Z_2 \times Z_q)) - \{w\} \subseteq V(\Gamma(Z_2 \times Z_q))$ is a dominating set having $q-1$ elements.

For every $u \in M,$ there exists $w \in V(\Gamma(Z_2 \times Z_q)) - M$ such that $w \in N_{\Gamma(Z_2 \times Z_q)}(u) \cap M$ and $(M - \{u\}) \cup \{w\}$ is a dominating set.

Thus M is a $\gamma^{csd} - set$.

Hence $\gamma^{csd}(\Gamma(Z_2 \times Z_q)) = q-1$. □

Theorem 9 Let $\Gamma(Z_2 \times Z_{q^2})$ be a ZDG with prime number q. Then $\gamma^{csd}(\Gamma(Z_2 \times Z_{q^2})) = q^2 - 1$.

Proof. Let $V(\Gamma(Z_2 \times Z_{q^2})) = \{(0, 1), (0, 2), ..., (0, q^2-1), (1, 0)\}$.

Let $u = (0, 1)$ and $w = (1, 0)$ be two nodes, then $u \times w = (0, 1) \times (1, 0) = (0, 0)$ that is the edge between u and w, $uw \in E(\Gamma(Z_2 \times Z_{q^2}))$. also, let $v \in V(\Gamma(Z_2 \times Z_{q^2})) - \{w\}$ and $w = (1, 0)$ then $v \times w = (0, 0)$. Clearly $w = (1, 0)$ is adjacent to all other nodes.

Let $M = V(\Gamma(Z_2 \times Z_{q^2})) - \{w\} \subseteq V(\Gamma(Z_2 \times Z_{q^2}))$ is a dominating set having $q^2 - 1$ elements.

For every $u \in M,$ there exists $w \in V(\Gamma(Z_2 \times Z_{q^2})) - M$ such that $w \in N_{\Gamma(Z_2 \times Z_{q^2})}(u) \cap M$ and $(M - \{u\}) \cup \{w\}$ is a dominating set. Thus M is a $\gamma^{csd} - set$.

Hence $\gamma^{csd}(\Gamma(Z_2 \times Z_{q^2})) = q^2 - 1$. □

Theorem 10 Let $\Gamma(Z_2 \times Z_{nq})$ be a ZDG with prime number q and $n \geq 2$. Then $\gamma^{csd}(\Gamma(Z_2 \times Z_{nq})) = nq-1$.

Proof. Let $V(\Gamma(Z_2 \times Z_{nq})) = \{(0, 1), (0, 2), ..., (0, nq-1), (1, 0)\}$.

Let $u = (0, 1)$ and $w = (1, 0)$ be two nodes, then $u \times w = (0, 1) \times (1, 0) = (0, 0)$ that is the edge between u and w, $uw \in E(\Gamma(Z_2 \times Z_{nq}))$.

Also, let $v \in V(\Gamma(Z_2 \times Z_{nq})) - \{w\}$ and $w = (1, 0)$ then $v \times w = (0, 0)$. Clearly $w = (1, 0)$ is adjacent to all other nodes.

Let $M = V(\Gamma(Z_2 \times Z_{nq})) - \{w\} \subseteq V(\Gamma(Z_2 \times Z_{nq}))$ is a dominating set having $nq-1$ elements.

For every $u \in M,$ there exists $w \in V(\Gamma(Z_2 \times Z_{nq})) - M$ such that $w \in N_{\Gamma(Z_2 \times Z_{nq})}(u) \cap M$ and $(M - \{u\}) \cup \{w\}$ is a dominating set. Thus M is a $\gamma^{csd} - set$.

Hence $\gamma^{csd}(\Gamma(Z_2 \times Z_{nq})) = nq-1$ □

Theorem 11 Let $\Gamma(Z_3 \times Z_q)$ be a ZDG with prime number q. Then $\gamma^{csd}(\Gamma(Z_3 \times Z_q)) = 2$.

Proof. Let $V(\Gamma(Z_3 \times Z_q)) = \{(0, 1), (0, 2), ..., (0, q-1), (1, 0), (2, 0)\}$.

Clearly, the vertx set $V(\Gamma(Z_3 \times Z_q))$ can be partition in the two parts $V_1(\Gamma(Z_3 \times Z_q))$ and $V_2(\Gamma(Z_3 \times Z_q))$

Let $V_1(\Gamma(Z_3 \times Z_q)) = \{(0, 1), (0, 2), ..., (0, q-1)\}$ and $V_2(\Gamma(Z_3 \times Z_q)) = \{(1, 0), (2, 0)$

Let $u = (0, 1) \in V_1(\Gamma(Z_3 \times Z_q))$ and $v = (1, 0) \in V_2(\Gamma(Z_3 \times Z_q))$ Then $u \times v = (0, 0)$, that is the edge between u and v, $uv \in E(\Gamma(Z_3 \times Z_q))$.

Let $u, w \in V_1(\Gamma(Z_3 \times Z_q))$, then $u \times w \neq (0, 0)$, u is not adjacent to w.

Also $v, w \in V_2(\Gamma(Z_3 \times Z_q))$, then $v \times w \neq (0, 0)$, v is not adjacent to w.

Therefore, every node in $V_1(\Gamma(Z_3 \times Z_q))$ is adjacent to any node in $V_2(\Gamma(Z_3 \times Z_q))$ and vice versa.

Let $M = \{u, v\}$ is a dominating set, where $u, v \in V_1(\Gamma(Z_3 \times Z_q))$.

For every $u \in M$, there exists $w \in V(\Gamma(Z_3 \times Z_q)) - M$, such that $u \in N_{\Gamma(Z_3 \times Z_q)}(w) \cap M$ and $(M - \{u\}) \cup \{w\} = \{v, w\}$ is a dominating set. Thus M is a $\gamma^{csd} - set$.

Hence $\gamma^{csd}(\Gamma(Z_3 \times Z_q)) = 2$. □

Theorem 12 Let $\Gamma(Z_p \times Z_q)$ be a ZDG with prime numbers p and q where $p > 4$ and $p < q$. Then $\gamma^{csd}(\Gamma(Z_p \times Z_q)) = 4$.

Proof. Let $V(\Gamma(Z_p \times Z_q)) = \{(0, 1), (0, 2), ..., (0, q-1), (1, 0), (2, 0), ..., (p-1, 0)\}$. Clearly, the node set $V(\Gamma(Z_p \times Z_q))$ can be partition in the two parts
$V_1(\Gamma(Z_p \times Z_q))$ and $V_2(\Gamma(Z_p \times Z_q))$

Let $V_1(\Gamma(Z_p \times Z_q)) = \{(0, 1), (0, 2), ..., (0, q-1)\}$ and $V_2(\Gamma(Z_p \times Z_q)) = \{(1, 0), (2, 0), ..., (p-1, 0)$

Let $u = (0, 1) \in V_1(\Gamma(Z_p \times Z_q))$ and $v = (1, 0) \in V_2(\Gamma(Z_p \times Z_q))$ Then $u \times v = (0, 0)$ that is the edge between u and v, $uv \in E(\Gamma(Z_p \times Z_q))$.

Let $u, w \in V_1(\Gamma(Z_p \times Z_q))$, then $u \times w \neq (0, 0)$, u is not adjacent to w. also $v, w \in V_2(\Gamma(Z_p \times Z_q))$, then $v \times w \neq (0, 0)$, v is not adjacent to w.

Therefore, every node in $V_1(\Gamma(Z_p \times Z_q))$ is adjacent to any node in $V_2(\Gamma(Z_p \times Z_q))$ and vice versa.

Let $M = V_2(\Gamma(Z_p \times Z_q))$ be a dominating set.

For every $u \in M$, there exists $x \in V(\Gamma(Z_p \times Z_q)) - M$, such that $u \in N_{\Gamma(Z_p \times Z_q)}(x) \cap M$ and $(M - \{u\}) \cup \{x\}$ is a dominating set. Thus M is a $\gamma^{csd} - set$.

Hence $\gamma^{csd}(\Gamma(Z_p \times Z_q)) = 4$. □

4. Conclusions

The ZDGs of commutative rings have been applied to construct connections between ring theory and graph theory. In this research article, the co-secure dominating set of a zero-divisor graph is defined and the co-secure domination number of some zero-divisor graphs are obtained. In the future, we plan to extend this study to ideal co-secure dominating sets.

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Conflict of interest

The authors declare no competing financial interest.

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