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A Note on Wiener and Hyper-Wiener Indices of Abid-Waheed Graph

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Abstract: A Hosoya polynomial is a polynomial connected to a molecular graph, which is a graph representation of a chemical compound with atoms as vertices and chemical bonds as edges. A graph invariant is the Hosoya polynomial; it is a graph attribute that does not change under graph isomorphism. It provides information about the number of unique non-empty subgraphs in a given graph. A molecular graph's size and branching complexity are determined by a topological metric known as the Wiener index. The Wiener index of each pair of vertices in a molecular network is the sum of those distances. The topological index, one of the various classes of graph invariants, is a real number related to a connected graph's structure .The goal of this article is to compute the Hosoya polynomial of some class of Abid-Waheed graph. Further, this research focused on a C++ algorithm to calculate the wiener index of AW_m^9 and AW_m^{11}. The Wiener index (W^*I) *a*nd Hyper-Wiener index (H^*W^*I) are calculated using Hosoya polynomial (H^*-polynomial) of some family of Abid-Waheed graphs AW_m^9 and AW_m^{11}. Illustrations and applications are given to enhance the research work.

***Keywords*:** abid waheed graph, hosoya polynomial, shortest path, topological index, wiener index

MSC: 05C35, 05C07, 05C40

1. Introduction

Let B be a simple and connected graph. The points and lines of the graph B are each represented by a collection of symbols $V(B)$ and $E(B)$ respectively. Hosoya [1] introduced some counting polynomials. The authors [2–4] discussed to demonstrate that in an application of generalised hierarchical structures constructed from graphs, the reformulated first Zagreb index of a few chemical graphs is elegant labelling, and that every binomial and the power of three acyclic graphs is cube sum labelling for every non-negative integer. The authors [5–7] studied several techniques, including the interpolation approach, the cut method, and the Hosoya polynomial method, that can be used to calculate the hyper-wiener index. Recently Shahid et al. [8] established the Mathematical modeling and topological graph description of dominating David derived networks based on edge partitions also calculated the topological indices. Asad et al. [9]

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studied and computed the different types of indices and also applied the numerical result and graphical representation to illustrate their findings. Examined to calculate some useful topological indices for three kinds of dendrimer networks and derived a closed mathematical formula, investigating QSPRs /QSARs of such molecules in many fields of science, such as chemistry, physics, biochemistry, and the structure from the first to the third generation, the dendrimer's by Shahid et al. [10]. Compared the numerical results and physio-chemical properties studied by [11] Correlating the π-electronic energies of the underlying chemical structures of these nanotubes and nanocones,were assisted in [12]. Asad et al. [13] Calculated the different types of Zagreb connection indices for triangular chain structures and derived closed formulas and the results of the physicochemical properties. Determined the topology of the nanosheet [14]. Exploring random walks of networks, particularly biassed random walks, practical problems such as search and routing on networks were solved in [15]. Yu et al. [16] discussed through extensive matrix analysis , obtained an explicit closed-form formula of the global mean-first-passage time for hexagonal model,and spectrums provides an easy calculation in terms of large networks. Abdul et al. [17] determined a graph theory-based edge partitioning technique is used to model the molecular topology of γ-Graphyne and Zigzag graphene nanoribbons, and mathematical closed-form expressions for degree-based molecular descriptors were derived. The authors [18] discovered the various types of topological indices and also performed the numerical and graphical comparative analysis of molecular. The supramolecular sheet of fuchsine acid was modelled topologically based on the edge partition, and closed formulae were derived for some of its important irregular molecular descriptors [19]. The Wiener index has been the subject of numerous articles because of its practical applications to the tree graph found in chemistry and mathematics [20–22]. Unambiguously determined from the structure graph of a molecule, the topological index is a numerical value. Alkane, alcohol, amine, and their comparable chemical physicochemical properties were correlated using the index, which Wiener applied to trees and investigated [23, 24]. A few of their characteristics were used to determine their efficacy, and they talked about the practical use of the symmetric difference network using a minimal spanning tree algorithm, which is produced to reach the minimum effective productivity to finish the tasks in a social network. They obtained a few degree-based indices of networks formed from honeycomb networks [25, 26]. Meenakshi et al. [27] investigated of the neutrosophic optimum network use in graph operations. It is used to assess the effectiveness and efficiency of the network because this type of graph structure has applications in computer networking. The generalisation of the Abid-Waheed graph for Hosoya polynomial and Wiener index are derived, and the hyper Wiener index can be found using a variety of techniques [28, 29]. The generalisations of the Abid-Waheed graph and the Jahangir graph are identical.

The degree-based connectivity descriptors, the sum-connectivity, the first and second Zagreb indices, and its co-indices have a strong correlation with the total π-electron energy and other properties that are significant for the synthesis of coronoid polycyclic aromatic compounds [30–32].

The first and second Zagreb indices of a graph are among the most established, well-known and in-depth studies among the various vertex degree-based topological indices. In an application of generalised hierarchies constructed from graphs, the reformulated first Zagreb index of a few chemical graphs was estimated [33, 34]. Deo [35] investigated the use of graph single-valued numbers in networking issues and issues involving group decision-making. Munir et al. [36] discussed finding closed forms of M-polynomials in polyhex nanotubes. Also compute the closed forms of various degree-based topological indices of these tubes. Distinct structure and set of capabilities, dendrimers find use in a wide range of fields, such as materials science, imaging, catalysis, drug delivery, and sensing. Dendrimers are useful for creating innovative materials and solutions for a variety of scientific and technical problems because of their regulated and customisable character. Cash [37] derived the second derivative of the Hyper-wiener index using the Hosoya polynomial. The Wiener index offers details on the location of the graph because it is a distance-based index. Topological indices are used to measure the structural features of chemical compounds and identify connections between molecular structure and properties. Topological indices are essential. This connection is essential to the domains of medicinal chemistry, drug design, and computational chemistry, among others. Many topological indices have been developed, and much time and effort has gone into computing the indices of different molecular graphs and networks [38–40]. Numerous writers have expanded this field of study by researching different kinds of domination [41–43]. Proposes the max product of three graphs and forming a single-valued neutrosophic graph to find efficient time management in the flow of information on a single-source time-dependent network of single-valued neutrosophic network [44]. Investigated the use of fuzzy graph edge coloring for various fuzzy graph operations, and it focused on the efficacy and efficiency of the fuzzy network product using the minimal spanning tree, the chromatic index of the fuzzy network product, and novel combinatorial method for encrypting and decrypting confidential numbers by leveraging an efficient dominant notion and labelled graph [45, 46]. The Bismuth Tri-Iodide chain and sheet's degree-based indices are studied by Shao et al. [47]. The literature contains several algebraic polynomials that can be used to circumvent the tedious procedure of computing a specific type of indices for a given category of graphs. For example, the Hosoya polynomial is important in the field of distance-based topological indices, whose differentiation at one yields the Wiener and hyper-Wiener indices. Further details of the Wiener index and Hyper-Wiener index readers may read [48]. A few degree-based indices of networks derived from honeycomb networks were obtained by Wei et al. [49]. It was introduced by Deutsch et al. [50], where its role in computing degree-based indices was shown to be parallel to the role of the Hosoya polynomial for distance-based indices. Describe how the topological analysis of the molecule's electrostatic potential might be used to investigate a wide range of physicochemical phenomena. How one scalar field can be used to explore so many different aspects of the molecular world is remarkable [51].

The Wiener index of a connected graph was the first topological index to be utilised in chemistry, and it is one of the most well-known distance-based topological indices discussed by Hassan et al. [52].

In medicinal chemistry, chemical compounds' molecular topology can be described using the Wiener and Hyper-Wiener indices. By identifying structural characteristics that affect a molecule's biological action, this information is useful in the creation and optimisation of new pharmaceuticals. Applications for it go beyond chemistry; examples include social network analysis and network analysis for transportation and communication. The Wiener and Hyper-Wiener indices advance the field of graph theory and shed light on the mathematical characteristics of molecular graphs. The links between graph parameters and interesting properties are explored with their help. In this research, inspired by earlier studies, the main objective of this section is to determine the W^*-index and H^*W^*index for the Abid-Waheed graphs AW_m^9 and AW_m^{11} using Hosoya polynomials. Some illustrations have been given for better understanding.

This research gap is a novel feature selection strategy to be adopted to make the decision to raise the Hosoya polynomial, Wiener, and Hyper Wiener indices values. This research has focused on various types of graphs, which help us study the physical and chemical properties of the chemical structures.

2. Methodology

The Abid Waheed graph AW_m^n is a graph produced by m-cycles (with each of order n), meeting at an external vertex of degree m, and it has $mn+1$ vertices and $m(n+1)$ edges for all $n \geq 3$ and $m \geq 1$, shown in Figure 1.

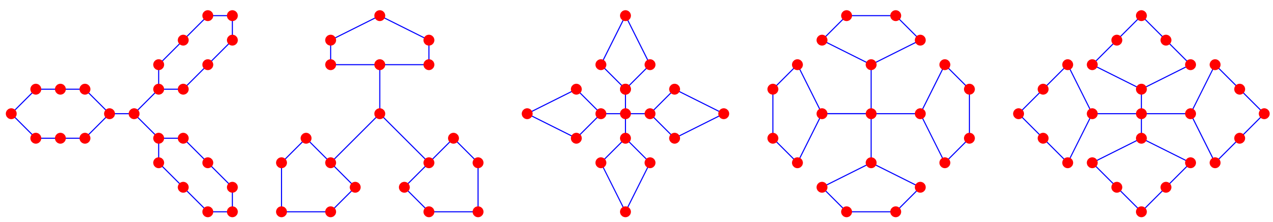

Figure 1. Abid-Waheed graph AW_m^n

With the aid of TIs like the Wiener index (W^*I) and Hyper-Wiener index (H^*W^*I), we attempt to explore the properties of the Abid Waheed graph in this section. With the use of the Hosoya polynomial, we obtain the Wiener index (W^*I). The sum of all shortest pathways between all pairs of vertices in the chemical graph that stand in for the non-hydrogen atoms in the molecule. The (W^*I) of a graph B is defined as

$$W^*(B, \gamma) = \frac{1}{2} \sum_{\tau,\ \psi \in M(B)} \lambda(\tau, \psi)$$

In 1988, Hosoya developed the Hosoya polynomial of a graph, a counting polynomial that counts the number of paths in the graph with different lengths that are separated by different distances. The polynomial was introduced by Hosoya

$$H^*(B, \gamma) = \sum_{\tau,\ \psi \in M(B)} \lambda(\tau, \psi) \gamma^{\lambda(\tau, \psi)}$$

where $\lambda(\tau, \psi)$ displays the separation between τ and ψ. Many metric features of a graph, such as the W^*-index (average distance) and H^*W^*-index, are captured by Hosoya polynomials. We obtain the W^*-index from the first-time differentiation of the H^*-polynomial at $\gamma = 1$. The Wiener index (W^*-index) of the Hosoya polynomial (H^*-polynomial) is defined as

$$W^*(B) = \left\{ \frac{\partial H^*(B, \gamma)}{\partial \gamma} \right\} \bigg|_{\gamma=1}$$

First derivative of the H^*-polynomial is defined as

$$W^*(B) = (H^*)'(B, \gamma) = \left\{ \frac{\partial H^*(B, \gamma)}{\partial \gamma} \right\} \bigg|_{\gamma=1}$$

Second derivative of the H^*-polynomial is defined as

$$W^*(B) = (H^*)''(B, \gamma) = \left\{ \frac{\partial^2 H^*(B, \gamma)}{\partial \gamma} \right\} \bigg|_{\gamma=1}$$

Hyper-Wiener index (H^*W^*-index) is defined as

$$H^*W^*(B) = (H^*)'(B, \gamma) + \frac{1}{2}(H^*)''(B, \gamma)$$

Abid-Waheed introduced the Abid-Waheed graph AW_m^n where $m \geq 1$ and $n \geq 3$, and used the Hosoya polynomial to find the Wiener index of *(AW)$_r$*[7]. The aim of this paper is to continue the studies of the classes of AW_m^9 and AW_m^{11}, and find the Wiener index and Hyper-Wiener index of this class of graphs using the Hosoya polynomial.

The main objective of this part is to compute the values of the Hosoya polynomial, Wiener index, and Hyper-Wiener index for the AW_m^9 and AW_m^{11} Abid-Waheed graphs. The Wiener index and Hyper-Wiener index for AW_m^9 and AW_m^{11} can be obtained by initially computing the Hosoya polynomial by figuring out the different distances and their cardinality in a graph. The symbol $\varphi(x, y)$ represents the number of unordered pairs of nodes, and the distance $\varphi(x, y)$ is represented by $\lambda(B, \alpha)$. The topological diameter of a graph B is referred to as D(B). α may have a maximum value equal to D(B), but it lies between 1 to D(B).

2.1 *Examples for abid-waheed graph of AW_3^9 , AW_4^9, AW_3^{11} and AW_4^{11}*

Example (i) Let $B = AW_3^9$ be a simple connected graph. Then the H^*-polynomial, W^*-index and H^*W^*-index are discussed below

H^*-polynomial is defined as

$$H^*(B, \gamma) = \sum_{\alpha=1}^{8} \lambda(B, \alpha)\gamma^{\alpha}$$

$$H^*(B, \gamma) = \begin{cases} \lambda(B, 1)\gamma + \lambda(B, 2)\gamma^2 + \lambda(B, 3)\gamma^3 + \lambda(B, 4)\gamma^4 \\ \\ +(B, 5)\gamma^5 + \lambda(B, 6)\gamma^6 + \lambda(B, 7)\gamma^7 + \lambda(B, 8)\gamma^8 \end{cases}$$

The longest distance between the edges AW_3^9 is 8. The values of AW_3^9, where α lies between 1 to 8. We can easily compute the H^*-polynomial using the values from the Table 1. It displays the relevant frequencies and distances.

Table 1. Cardinality of $\lambda(B, \alpha)$ for each pair of points of AW_3^9 is 8, where $1 \leq \alpha \leq 8$

$\lambda(B, \alpha)$	Cardinality	$\lambda(B, \alpha)$	Cardinality
$\lambda(B, 1)$	30	$\lambda(B, 5)$	42
$\lambda(B, 2)$	36	$\lambda(B, 6)$	48
$\lambda(B, 3)$	45	$\lambda(B, 7)$	48
$\lambda(B, 4)$	57	$\lambda(B, 8)$	36

Using the distance values from Table 1, we obtain:

$$H^*\left(AW_3^9, \gamma\right) = 30\gamma + 36\gamma^2 + 45\gamma^3 + 57\gamma^4 + 42\gamma^5 + 48\gamma^6 + 48\gamma^7 + 36\gamma^8$$

First derivative of the H^*-polynomial is defined as

$$W^*(AW_3^9) = (H^*)'(B, \gamma) = \left\{ \frac{\partial H^*(AW_3^9, \gamma)}{\partial \gamma} \right\} \Big|_{\gamma=1}$$

$$W^*(AW_3^9) = \frac{\partial}{\partial \gamma} \left\{ 30\gamma + 36\gamma^2 + 45\gamma^3 + 57\gamma^4 + 42\gamma^5 + 48\gamma^6 + 48\gamma^7 + 36\gamma^8 \right\} \Big|_{\gamma=1}$$

The W^*-index is obtained by first differentiation at $\gamma = 1$. W^*-index is as follows:

$$W^*\left(AW_3^9\right) = 30 + 72\gamma + 135\gamma^2 + 228\gamma^3 + 210\gamma^4 + 288\gamma^5 + 336\gamma^6 + 288\gamma^7$$

$$W^*\left(AW_3^9\right) = 1587$$

First derivative of the H^*-polynomial is

$$(H^*)'\left(AW_3^9, \gamma\right)_{\gamma=1} = 1587$$

Second derivative of the H^*-polynomial is

$$W^*(AW_3^9) = (H^*)''(B, \gamma) = \left\{\frac{\partial^2 H^*(AW_3^9, \gamma)}{\partial\gamma}\right\}\Big|_{\gamma=1}$$

The W^*-index is obtained by second differentiation at $\gamma = 1$. W^*-index is as follows

$$W^*\left(AW_3^9\right) = 72 + 270\gamma + 684\gamma^2 + 840\gamma^3 + 1440\gamma^4 + 2016\gamma^5 + 2016\gamma^6$$

$$W^*\left(AW_3^9\right) = 7338$$

Second derivative of the H^*-polynomial is

$$(H^*)''\left(AW_3^9, \gamma\right) = 7338$$

Hyper-Wiener index is defined as

$$H^*W^*(B) = (H^*)'\left(AW_3^9, \gamma\right) + \frac{1}{2}(H^*)''\left(AW_3^9, \gamma\right)$$

$$H^*W^*\left(AW_3^9\right) = 1587 + \frac{7338}{2}$$

$$H^*W^*\left(AW_3^9\right) = 5256$$

Example (ii) Let $B = AW_4^9$ be a simple connected graph. Then the H^*-polynomial, W^*-index and H^*W^*-index are discussed below

H^*-polynomial is defined as

$$H^*(B, \gamma) = \sum_{\alpha=1}^{8} \lambda(B, \alpha)\gamma^\alpha$$

$$H^*(B, \gamma) = \begin{cases} \lambda(B, 1)\gamma + \lambda(B, 2)\gamma^2 + \lambda(B, 3)\gamma^3 + \lambda(B, 4)\gamma^4 \\ +(B, 5)\gamma^5 + \lambda(B, 6)\gamma^6 + \lambda(B, 7)\gamma^7 + \lambda(B, 8)\gamma^8 \end{cases}$$

The longest distance between the edges AW_4^9 is 8. The values of AW_4^9, where α lies between 1 to 8. We can easily compute the H^*-polynomial using the values from the Table 2. It displays the relevant frequencies and distances.

Table 2. Cardinality of $\lambda\,(B,\,\alpha)$ for each pair of points of AW_4^9 is 8, where $1 \leq \alpha \leq 8$

$\lambda\,(B,\,\alpha)$	Cardinality	$\lambda\,(B,\,\alpha)$	Cardinality
$\lambda\,(B,\,1)$	40	$\lambda\,(B,\,5)$	80
$\lambda\,(B,\,2)$	50	$\lambda\,(B,\,6)$	96
$\lambda\,(B,\,3)$	68	$\lambda\,(B,\,7)$	96
$\lambda\,(B,\,4)$	92	$\lambda\,(B,\,8)$	72

Using the distance values from Table 2, we obtain:

$$H^*\left(AW_4^9,\,\gamma\right) = 40\gamma + 50\gamma^2 + 68\gamma^3 + 92\gamma^4 + 80\gamma^5 + 96\gamma^6 + 96\gamma^7 + 72\gamma^8$$

First derivative of the H^*-polynomial is defined as

$$W^*(AW_4^9) = (H^*)'\,(B,\,\gamma) = \left\{\frac{\partial H^*(AW_4^9,\,\gamma)}{\partial\gamma}\right\}\bigg|_{\gamma=1}$$

$$W^*(AW_3^9) = \frac{\partial}{\partial\gamma}\left\{40\gamma + 50\gamma^2 + 68\gamma^3 + 92\gamma^4 + 80\gamma^5 + 96\gamma^6 + 96\gamma^7 + 72\gamma^8\right\}\bigg|_{\gamma=1}$$

The W^*-index is obtained by differentiating at $\gamma = 1$. W^*-index is as follows:

$$W^*\left(AW_4^9\right) = 40 + 100\gamma + 204\gamma^2 + 368\gamma^3 + 400\gamma^4 + 576\gamma^5 + 672\gamma^6 + 576\gamma^7$$

$$W^*\left(AW_4^9\right) = 2936$$

First derivative of the H^*-polynomial is

$$(H^*)'\left(AW_4^9,\,\gamma\right)_{\gamma=1} = 2936$$

Second derivative of the H^*-polynomial is

$$W^*(AW_4^9) = (H^*)''\,(B,\,\gamma) = \left\{\frac{\partial^2 H^*(AW_4^9,\,\gamma)}{\partial\gamma}\right\}\bigg|_{\gamma=1}$$

The W^*-index is obtained by second differentiation at $\gamma = 1$. W^*-index is as follows

$$W^*\left(AW_4^9\right) = 100 + 408\gamma + 1104\gamma^2 + 1600\gamma^3 + 2880\gamma^4 + 4032\gamma^5 + 4032\gamma^6 W^*\left(AW_4^9\right) = 14156$$

Second derivative of the H^*-polynomial is

$$(H^*)''\left(AW_4^9,\, \gamma\right)_{\gamma=1} = 14156$$

Hyper-Wiener index is defined as

$$H^*W^*(B) = (H^*)'\left(AW_4^9,\, \gamma\right) + \frac{1}{2}(H^*)''\left(AW_4^9,\, \gamma\right)$$

$$H^*W^*\left(AW_4^9\right) = 2936 + \frac{14156}{2}$$

$$H^*W^*\left(AW_4^9\right) = 10014.$$

Example (iii) Let $B = AW_3^{11}$ be a simple connected graph. Then the H^*-polynomial, W^*-index and H^*W^*-index are discussed below

H^*-polynomial is defined as

$$H^*(B,\, \gamma) = \sum_{\alpha=1}^{10} \lambda(B,\, \alpha)\gamma^{\alpha}$$

$$H^*(B,\, \gamma) = \begin{cases} \lambda(B,\, 1)\gamma + \lambda(B,\, 2)\gamma^2 + \lambda(B,\, 3)\gamma^3 + \lambda(B,\, 4)\gamma^4 + (B,\, 5)\gamma^5 \\ \\ +\lambda(B,\, 6)\gamma^6 + \lambda(B,\, 7)\gamma^7 + \lambda(B,\, 8)\gamma^8 + (B,\, 5)\gamma^9 + (B,\, 5)\gamma^{10} \end{cases}$$

The longest distance between the edges AW_3^{11} is 10. The values of AW_3^{11}, where α lies between 1 to 10. We can easily compute the H^*-polynomial using the values from the Table 3. It displays the relevant frequencies and distances.

Table 3. Cardinality of $\lambda(B,\, \alpha)$for each pair of points of AW_3^{11} is 10, where $1 \leq \alpha \leq 10$

$\lambda(B,\, \alpha)$	Cardinality	$\lambda(B,\, \alpha)$	Cardinality
$\lambda(B,\, 1)$	36	$\lambda(B,\, 6)$	54
$\lambda(B,\, 2)$	42	$\lambda(B,\, 7)$	60
$\lambda(B,\, 3)$	51	$\lambda(B,\, 8)$	60
$\lambda(B,\, 4)$	63	$\lambda(B,\, 9)$	48
$\lambda(B,\, 5)$	75	$\lambda(B,\, 10)$	36

Using the distance values from Table 3, we obtain:

$$H^*\left(AW_3^{11},\gamma\right)=\begin{cases}36\gamma+42\gamma^2+51\gamma^3+63\gamma^4+75\gamma^5\\ \\ +54\gamma^6+60\gamma^7+60\gamma^8+48\gamma^9+36\gamma^{10}\end{cases}$$

First derivative of the H^*-polynomial is defined as

$$W^*(AW_3^{11})=(H^*)'(B,\gamma)=\left\{\frac{\partial H^*(AW_3^{11},\gamma)}{\partial\gamma}\right\}\Bigg|_{\gamma=1}$$

$$W^*(AW_3^{11})=\frac{\partial}{\partial\gamma}\left\{\begin{matrix}l36\gamma+42\gamma^2+51\gamma^3+63\gamma^4+75\gamma^5\\ \\ +54\gamma^6+60\gamma^7+60\gamma^8+48\gamma^9+36\gamma^{10}\end{matrix}\right\}\Bigg|_{\gamma=1}$$

The W^*-index is obtained by first differentiation at $\gamma=1$. W^*-index is as follows:

$$W^*\left(AW_3^{11}\right)=\begin{cases}36+84\gamma+153\gamma^2+252\gamma^3+375\gamma^4\\ \\ +324\gamma^5+420\gamma^6+480\gamma^7+432\gamma^8+360\gamma^9\end{cases}$$

$$W^*\left(AW_3^{11}\right)=2916$$

First derivative of the H^*-polynomial is

$$(H^*)'\left(AW_3^{11},\gamma\right)_{\gamma=1}=2916$$

Second derivative of the H^*-polynomial is

$$W^*(AW_3^{11})=(H^*)''(B,\gamma)=\left\{\frac{\partial^2 H^*(AW_3^{11},\gamma)}{\partial\gamma}\right\}\Bigg|_{\gamma=1}$$

The W^*-index is obtained by second differentiation at $\gamma=1$. W^*-index is as follows

$$W^*\left(AW_3^{11}\right)=\begin{cases}84+306\gamma+756\gamma^2+1500\gamma^3+1620\gamma^4\\ \\ +2520\gamma^5+3360\gamma^6+3456\gamma^7+3240\gamma^8\end{cases}$$

$$W^*\left(AW_3^{11}\right)=16842$$

Second derivative of the H^*-polynomial is

$$(H^*)''\left(AW_3^9,\, \gamma\right) = 16842$$

Hyper-Wiener index is defined as

$$H^*W^*(B) = (H^*)'\left(AW_3^{11},\, \gamma\right) + \frac{1}{2}(H^*)''\left(AW_3^{11},\, \gamma\right)$$

$$H^*W^*\left(AW_3^{11}\right) = 2916 + \frac{16842}{2}$$

$$H^*W^*\left(AW_3^{11}\right) = 11337$$

Example (iv) Let $B = AW_4^{11}$ be a simple connected graph. Then the H^*-polynomial, W^*-index and H^*W^*-index are discussed below

H^*-polynomial is defined as

$$H^*(B,\, \gamma) = \sum_{\alpha=1}^{10} \lambda(B,\, \alpha)\gamma^{\alpha}$$

$$H^*(B,\, \gamma) = \begin{cases} \lambda(B,\, 1)\gamma + \lambda(B,\, 2)\gamma^2 + \lambda(B,\, 3)\gamma^3 + \lambda(B,\, 4)\gamma^4 + (B,\, 5)\gamma^5 \\ \\ +\lambda(B,\, 6)\gamma^6 + \lambda(B,\, 7)\gamma^7 + \lambda(B,\, 8)\gamma^8 + (B,\, 5)\gamma^9 + (B,\, 5)\gamma^{10} \end{cases}$$

The longest distance between the edges AW_4^{11} is 10. The values of AW_4^{11}, where α lies between 1 to 10. We can easily compute the H^*-polynomial using the values from the Table 4. It displays the relevant frequencies and distances.

Table 4. Cardinality of $\lambda\,(B,\, \alpha)$ for each pair of points of AW_4^{11} is 10 where $1 \le \alpha \le 10$

$\lambda\,(B,\, \alpha)$	Cardinality	$\lambda\,(B,\, \alpha)$	Cardinality
$\lambda\,(B,\, 1)$	48	$\lambda\,(B,\, 6)$	104
$\lambda\,(B,\, 2)$	58	$\lambda\,(B,\, 7)$	120
$\lambda\,(B,\, 3)$	76	$\lambda\,(B,\, 8)$	120
$\lambda\,(B,\, 4)$	100	$\lambda\,(B,\, 9)$	96
$\lambda\,(B,\, 5)$	124	$\lambda\,(B,\, 10)$	72

Using the distance values from Table 4, we obtain:

$$H^*\left(AW_4^{11}, \gamma\right) = \begin{cases} 48\gamma + 58\gamma^2 + 76\gamma^3 + 100\gamma^4 + 124\gamma^5 \\ \\ +104\gamma^6 + 120\gamma^7 + 120\gamma^8 + 96\gamma^9 + 72\gamma^{10} \end{cases}$$

First derivative of the H^*-polynomial is defined as

$$W^*(AW_4^{11}) = (H^*)'(B, \gamma) = \left\{\frac{\partial H^*(AW_4^{11}, \gamma)}{\partial \gamma}\right\}\Bigg|_{\gamma=1}$$

$$W^*(AW_4^{11}) = \frac{\partial}{\partial \gamma}\begin{Bmatrix} 48\gamma + 58\gamma^2 + 76\gamma^3 + 100\gamma^4 + 124\gamma^5 \\ \\ +104\gamma^6 + 120\gamma^7 + 120\gamma^8 + 96\gamma^9 + 72\gamma^{10} \end{Bmatrix}\Bigg|_{\gamma=1}$$

The W^*-index is obtained by differentiating at $\gamma = 1$. W^*-index is as follows:

$$W^*\left(AW_4^{11}\right) = \begin{cases} 48 + 116\gamma + 228\gamma^2 + 400\gamma^3 + 620\gamma^4 \\ \\ +624\gamma^5 + 840\gamma^6 + 960\gamma^7 + 864\gamma^8 + 720\gamma^9 \end{cases}$$

$$W^*\left(AW_4^{11}\right) = 5420$$

First derivative of the H^*-polynomial is

$$(H^*)'\left(AW_4^{11}, \gamma\right)_{\gamma=1} = 5420$$

Second derivative of the H^*-polynomial is

$$W^*(AW_4^{11}) = (H^*)''(B, \gamma) = \left\{\frac{\partial^2 H^*(AW_4^{11}, \gamma)}{\partial \gamma}\right\}\Bigg|_{\gamma=1}$$

The W^*-index is obtained by second differentiation at $\gamma = 1$. W^*-index is as follows

$$W^*\left(AW_4^{11}\right) = \begin{cases} 116 + 456\gamma + 1200\gamma^2 + 2480\gamma^3 + 3120\gamma^4 \\ \\ +5040\gamma^5 + 6720\gamma^6 + 6912\gamma^7 + 6480\gamma^8 \end{cases} \quad W^*\left(AW_4^{11}\right) = 32524$$

Second derivative of the H^*-polynomial is

$$(H^*)''\left(AW_4^{11},\,\gamma\right)_{\gamma=1}=32524$$

Hyper-Wiener index is defined as

$$H^*W^*\left(B\right)=\left(H^*\right)'\left(AW_4^{11},\,\gamma\right)+\frac{1}{2}\left(H^*\right)''\left(AW_4^{11},\,\gamma\right)$$

$$H^*W^*\left(AW_4^{11}\right)=5420+\frac{32524}{2}$$

$$H^*W^*\left(AW_4^{11}\right)=21682.$$

3. Results and discussion

Theorem 3.1 Let $B=AW_m^9$ where $m\geq 1$ where m is a nonnegative integer then the H^*-polynomial of AW_m^9 is

$$H^*\left(B,\,\gamma\right)=\begin{cases}10\mathrm{m}\gamma+\left(\dfrac{m^2+21m}{2}\right)\gamma^2+(2m^2+9m)\gamma^3+(4m^2+7m)\gamma^4\\ \\ +(6m^2-4m)\gamma^5+(8m^2-8m)\gamma^6+(8m^2-8m)\gamma^7+(6m^2-6m)\gamma^8\end{cases}$$

Proof. Let AW_m^9 be a Abid Waheed's graph. There are 8 different types of distances and it consists of 9m+1 points and 10m edges.

The H^*-polynomial of AW_m^9 is

$$H^*\left(B,\,\gamma\right)=\begin{cases}\lambda(B,\,1)\gamma+\lambda(B,\,2)\gamma^2+\lambda(B,\,3)\gamma^3+\lambda(B,\,4)\gamma^4\\ \\ +(B,\,5)\gamma^5+\lambda(B,\,6)\gamma^6+\lambda(B,\,7)\gamma^7+\lambda(B,\,8)\gamma^8\end{cases}$$

By Table 5, we have

$$H^*\left(B,\,\gamma\right)=\begin{cases}10\mathrm{m}\gamma+\left(\dfrac{m^2+21m}{2}\right)\gamma^2+(2m^2+9m)\gamma^3+(4m^2+7m)\gamma^4\\ \\ +(6m^2-4m)\gamma^5+(8m^2-8m)\gamma^6+(8m^2-8m)\gamma^7+(6m^2-6m)\gamma^8\end{cases}$$

Table 5. Cardinality of $\lambda\,(B,\,\alpha)$for each pair of points of AW_{m}^{9} is 8, where $1 \leq \alpha \leq 8$

$\lambda\,(B,\,\alpha)$	Cardinality	$\lambda\,(B,\,\alpha)$	Cardinality
$\lambda\,(B,\,1)$	$10m$	$\lambda\,(B,\,5)$	$(6m^2-4m)$
$\lambda\,(B,\,2)$	$\left(\frac{m^2+21m}{2}\right)$	$\lambda\,(B,\,6)$	$(8m^2-8m)$
$\lambda\,(B,\,3)$	$(2m^2+9m)$	$\lambda\,(B,\,7)$	$(8m^2-8m)$
$\lambda\,(B,\,4)$	$(4m^2+7m)$	$\lambda\,(B,\,8)$	$(6m^2-6m)$

Theorem 3.2 Let $B = AW_m^{11}$ where $m \geq 1$ where m is a nonnegative integer then the H^*-polynomial of AW_m^{11} is

$$H^*\,(B,\,\gamma) = \begin{cases} 12\mathrm{m}\gamma + \left(\dfrac{m^2+25m}{2}\right)\gamma^2 + (2m^2+11m)\gamma^3 + (4m^2+9m)\gamma^4 \\ +(6m^2+7m)\gamma^5 + (8m^2-6m)\gamma^6 + (10m^2-10m)\gamma^7 \\ +(10m^2-10m)\gamma^8 + (8m^2-8m)\gamma^9 + (6m^2-6m)\gamma^{10} \end{cases}$$

Proof. Let AW_m^{11} be a Abid Waheed's graph. There are 10 different types of distances and it consists of 10m+1 points and 11m edges.

Table 6. Cardinality of $\lambda\,(B,\,\alpha)$for each pair of points of AW_m^{11} is 10, where $1 \leq \alpha \leq 10$.

The H^*-polynomial of AW_m^{11} is

Table 6. Cardinality of $\lambda\,(B,\,\alpha)$for each pair of points of AW_{m}^{11} is 10, where $1 \leq \alpha \leq 10$

$\lambda\,(B,\,\alpha)$	Cardinality	$\lambda\,(B,\,\alpha)$	Cardinality
$\lambda\,(B,\,1)$	$12m$	$\lambda\,(B,\,6)$	$(8m^2-6m)$
$\lambda\,(B,\,2)$	$\left(\frac{m^2+25m}{2}\right)$	$\lambda\,(B,\,7)$	$(10m^2-10m)$
$\lambda\,(B,\,3)$	$(2m^2+11m)$	$\lambda\,(B,\,8)$	$(10m^2-10m)$
$\lambda\,(B,\,4)$	$(4m^2+9m)$	$\lambda\,(B,\,9)$	$(8m^2-8m)$
$\lambda\,(B,\,5)$	$(6m^2+7m)$	$\lambda\,(B,\,10)$	$(6m^2-6m)$

The H^*-polynomial of AW_m^{11} is

$$H^*\,(B,\,\gamma) = \sum_{\alpha=1}^{E(B)} \lambda\,(B,\,\alpha)\gamma^{\alpha}$$

$$H^*\,(B,\,\gamma) = \begin{cases} \lambda\,(B,\,1)\gamma + \lambda\,(B,\,2)\gamma^2 + \lambda\,(B,\,3)\gamma^3 + \lambda\,(B,\,4)\gamma^4 + (B,\,5)\gamma^5 \\ +\lambda\,(B,\,6)\gamma^6 + \lambda\,(B,\,7)\gamma^7 + \lambda\,(B,\,8)\gamma^8 + (B,\,5)\gamma^9 + (B,\,5)\gamma^{10} \end{cases}$$

By Table 6, we have

$$H^*(B,\gamma)=\begin{cases} 12\text{m}\gamma+\left(\dfrac{m^2+25m}{2}\right)\gamma^2+(2m^2+11m)\gamma^3+(4m^2+9m)\gamma^4 \\ +(6m^2+7m)\gamma^5+(8m^2-6m)\gamma^6+(10m^2-10m)\gamma^7 \\ +(10m^2-10m)\gamma^8+(8m^2-8m)\gamma^9+(6m^2-6m)\gamma^{10} \end{cases}$$

Theorem 3.3 Let $B=AW_m^9$ be a simple connected graph, where $m\geq 1$, then W^*-index of AW_m^9 is $W^*(B)=205m^2-86m$

Proof. The mathematical definitions of the H^*-polynomial and W^*-index is

$$W^*(B)=\left\{\frac{\partial H^*(B,\gamma)}{\partial\gamma}\right\}\Bigg|_{\gamma=1}$$

$$W^*(B)=\left.\begin{Bmatrix} 10\text{m}+\left(\dfrac{m^2+21m}{2}\right)2\gamma+(2m^2+9m)3\gamma^2+(4m^2+7m)4\gamma^3 \\ +(6m^2-4m)5\gamma^4+(8m^2-8m)6\gamma^5+(8m^2-8m)7\gamma^6+(6m^2-6m)8\gamma^7 \end{Bmatrix}\right|_{\gamma=1}$$

After simplifications,

$$W^*(B)=205m^2-86m.$$

Theorem 3.4 Let $B=AW_m^{11}$ be a simple connected graph. Where $m\geq 1$, then W^*- index of AW_m^{11} is $W^*(B)=383m^2-177m$.

Proof. The mathematical definitions of the H^*-polynomial and W^*-index is

$$W^*(B)=\left\{\frac{\partial H^*(B,\gamma)}{\partial\gamma}\right\}\Bigg|_{\gamma=1}$$

$$H^*(B,\gamma)=\left.\begin{Bmatrix} 12\text{m}+\left(\dfrac{m^2+25m}{2}\right)2\gamma+(2m^2+11m)3\gamma^2+(4m^2+9m)4\gamma^3 \\ +(6m^2+7m)5\gamma^4+(8m^2-6m)6\gamma^5+(10m^2-10m)7\gamma^6 \\ +(10m^2-10m)8\gamma^7+(8m^2-8m)9\gamma^8+(6m^2-6m)10\gamma^9 \end{Bmatrix}\right|_{\gamma=1}$$

After simplifications,

$$W^*(B) = 383m^2 - 177m.$$

Theorem 3.5 Let $B = AW_m^9$ be a simple connected graph. Where $m \geq 1$, then the Hyper-Wiener index of B is $H^*W^*(B) = \frac{1503}{2}m^2 - \frac{1005}{2}m$

Proof. Let $B = AW_m^9$ where $m \geq 1$. Then the W^*-index of the graph is

$$W^*(B) = (H^*)'(B, \gamma) = \left\{\frac{\partial H^*(B, \gamma)}{\partial \gamma}\right\}\Big|_{\gamma=1}$$

By Theorem 3.3, We have

$$W^*(B) = 205m^2 - 86m$$

The second order derivative of the H^*-polynomial of the graph is,

$$W^*(B) = (H^*)''(B, \gamma) = \left\{\frac{\partial^2 H^*(B, \gamma)}{\partial \gamma}\right\}\Big|_{\gamma=1}$$

By Theorem 3.3, We have

$$(H^*)''(B, \gamma) = \left\{\begin{array}{l}(m^2+21m)+(2m^2+9m)6\gamma+(4m^2+7m)12\gamma^2+(6m^2-4m)20\gamma^3 \\ \\ +(8m^2-8m)30\gamma^4+(8m^2-8m)42\gamma^5+(6m^2-6m)56\gamma^6\end{array}\right\}\Bigg|_{\gamma=1}$$

$$(H^*)''(B, \gamma) = 1093m^2 - 833m$$

Hyper-Wiener index is defined as

$$H^*W^*(B) = (H^*)'\left(AW_m^9, \gamma\right) + \frac{1}{2}(H^*)''\left(AW_m^9, \gamma\right)$$

$$H^*W^*(B) = 205m^2 - 86m + \frac{1}{2}\left(1093m^2 - 833m\right)$$

$$H^*W^*(B) = \frac{1503}{2}m^2 - \frac{1005}{2}m$$

Theorem 3.6 Let $B = AW_m^{11}$ be a simple connected graph. Where $m \geq 1$, then the Hyper-Wiener Index of B is $H^*W^*(B) = \frac{3283}{2}m^2 - \frac{2291}{2}m$.

Proof. Let $B = AW_m^{11}$ where $m \geq 1$. Then the W^*-index of the graph is

$$W^*(B) = (H^*)'(B, \gamma) = \left\{ \frac{\partial H^*(B, \gamma)}{\partial \gamma} \right\}\Bigg|_{\gamma=1}$$

By Theorem 3.4, We have

$$W^*(B) = 383m^2 - 177m$$

The second order derivative of the H^*-polynomial of the graph is,

$$W^*(B) = (H^*)''(B, \gamma) = \left\{ \frac{\partial^2 H^*(B, \gamma)}{\partial \gamma} \right\}\Bigg|_{\gamma=1}$$

By Theorem 3.4, We have

$$(H^*)''(B, \gamma) = \left\{ \begin{array}{l} (m^2+25m)+(2m^2+11m)6\gamma+(4m^2+9m)12\gamma^2 \\ \\ +(6m^2+7m)20\gamma^3+(8m^2-6m)30\gamma^4+(10m^2-10m)42\gamma^5 \\ \\ +(10m^2-10m)56\gamma^6+(8m^2-8m)72\gamma^7+(6m^2-6m)90\gamma^8 \end{array} \right\}\Bigg|_{\gamma=1}$$

$$(H^*)''(B, \gamma) = 2517m^2 - 1937m$$

Hyper-Wiener index is defined as

$$H^*W^*(B) = (H^*)'\left(AW_m^{11}, \gamma\right) + \frac{1}{2}(H^*)''\left(AW_m^{11}, \gamma\right)$$

$$H^*W^*(B) = 383m^2 - 177m + \frac{1}{2}\left(2517m^2 - 1937m\right)$$

$$H^*W^*(B) = \frac{3283}{2}m^2 - \frac{2291}{2}m.$$

3.7 To compute the wiener index algorithm in C++ language

```
//C++ Program for to find Wiener Index
#include<iostream.h>
#include<conio.h>
long int ncr(long int);
void main(void)
```

```
{
long int n,k,i=0,j,s=0,x=0,m=0,y=0,w;
clrscr();
cout<<"\n Enter No. of vertices in a Graph :\t";
cin>>n;
cout<<"\n Enter No of Diagram in a Graph :\t";
cin>>k;
if(n%2==0)
{
cout<<"\n"<<i<<"\t";
for(j=i+1;j< = n;j++)
{
if(m<n/2+1)
{
m=m+1;
cout<<" "<<m;
x=x+m;
s=m;
}
else
{
s=s-1;
cout<<" "<<s;
x=x+s;
}
}
m=s=0;
for(i=1;i<n;i++,s=0,m=0)
{
cout<<"\n"<<i<<"\t";
for(j=i+1;j< = n;j++)
{
if(m<n/2)
{
m=m+1;
cout<<" "<<m;
x=x+m;
s=m;
}
else
{
s=s-1;
cout<<" "<<s;
x=x+s;
}
}
}
cout<<"\n\nWiener Index of G is :\t"<<x;
```

```
getch();
m=s=2;
for(i=1;i< = n;i++,m=s)
{
cout<<"\n"<<i<<"\t";
if(i< = n/2+1)
m=s-1;
else
m=s-3;
for(j=n+1;j< = 2*n;j++)
{
if(j< = 3*n/2+1)
{
m=m+1;
cout<<" "<<m;
y=y+m;
s=m;
}
else
{
s=s-1;
cout<<" "<<s;
y=y+s;
m=s;
}
}
}
cout<<"\nMulti Wiener Index of G is :\t"<<y;
if(k>1)
{
w=k*x+ncr(k)*y;
cout<<"\nFinal Wiener Index of G is :\t"<<w;
}
else
{
w=x+y;
cout<<"\nFinal Wiener Index of G is :\t"<<w;
}
getch();
}
else
{
cout<<"\n"<<i<<"\t";
for(j=i+1;j< = n;j++)
{
if(m==n/2+1)
{
cout<<" "<<m;
```

```
x=x+m;
m=m+1;
}
else if(m<n/2+1)
{
m=m+1;
cout<<" "<<m;
x=x+m;
s=m;
}
else
{
s=s-1;
cout<<" "<<s;
x=x+s;
}
}
s=m=0;
for(i=1;i<n;i++,s=0,m=0)
{
cout<<"\n"<<i<<"\t";
for(j=i+1;j< = n;j++)
{
if(m==n/2)
{
cout<<" "<<m;
x=x+m;
m=m+1;
}
else if(m<n/2)
{
m=m+1;
cout<<" "<<m;
x=x+m;
s=m;
}
else
{
s=s-1;
cout<<" "<<s;
x=x+s;
}
}
}
cout<<"\n\nWiener Index of G is :\t"<<x;
getch();
m=s=2;
for(i=1;i< = n;i++,m=s)
```

```
{
cout<<"\n"<<i<<"\t";
if(i==n/2+2)
m=s-2;
else if(i< = n/2+1)
m=s-1;
else
m=s-3;
for(j=n+1;j< = 2*n;j++)
{
if(j==3*(n+1)/2)
{
cout<<" "<<m;
y=y+m;
m=m+1;
}
else if(j< = 3*n/2+1)
{
m=m+1;
cout<<" "<<m;
y=y+m;
s=m;
}
else
{
s=s-1;
cout<<" "<<s;
y=y+s;
m=s;
}
}
}
cout<<"\nMulti Wiener Index of G is :\t"<<y;
if(k>1)
{
w=k*x+ncr(k)*y;
cout<<"\nFinal Wiener Index of G is :\t"<<w;
}
else
{
w=x+y;
cout<<"\nFinal Wiener Index of G is :\t"<<w;
}
getch();
}
}
long int ncr(long int z)
{
```

```
long int i,nf=1,nrf=1;
for(i=1;i< = z;i++)
nf*=i;
for(i=1;i< = (z-2);i++)
nrf*=i;
return(nf/(nrf*2));
}
```

4. Applications of abid-waheed graph AW_m^n

We can allocate primary sources to specific nodes in the Abid-Waheed graph by using its pattern, which ensures that the shortest path between any two nodes is inside the minimum. The following are some applications for this pattern:

An examination of social networks: The shortest path algorithms have to be used to improve supply chain routes and logistics, along with lowering costs and delivery times. Train and public transport routes are designed to maximise the shortest path possible in order to ensure efficient flow for both people and commodities.

Network routing: In computer networks, routers use shortest path algorithms to determine the fastest route for data packets to take in order to reach their destination. Telecommunication networks optimise signal routing by minimising latency and ensuring effective communication through the use of shortest path algorithms. In order to evaluate contact networks and understand the progression of illnesses, the most important paths for transmission can be found using algorithms for the shortest paths. The use of shortest path algorithms in road network design and efficient public transport route planning results in shorter travel times and less traffic.

Planned activites and Transportation: GPS navigation systems employ shortest path algorithms to find the quickest route between two sites, accounting for the present state of traffic. To minimise obstacles and reach their goal as soon as possible, robots and self-driving automobiles use shortest-path algorithms to plan their movements. Pathfinding algorithms based on the shortest route theory are necessary for characters in video games to move about and navigate.

5. Conclusion

The graph structure of the Abid-Waheed graph $B = AW_m^7$, where $m = 1, 2, 3, .., p$, has a distinct structure that is made up of the vertices $7+1$, $2(7)+1$, .., $p(7)+1$. Any vertex in the degree 3 can serve as a supply chain's or center's neutral source. This makes the pattern of the Abid-Waheed graph more flexible. Dendrimers and Abid-Waheed graphs share a pattern that is related to the strength and functioning of the medicinal product's active components inside the chemical structure. The medications in similar chemical graph topologies are examined for efficacy and durability using the graph $B = AW_m^n$, structure. This graph is used by network programmes to evaluate the effectiveness of the entire system.

Conflict of interest

The authors declare there is no conflict of interest at any point with reference to research findings.

References

[1] Hosoya H. On some counting polynomials in chemistry. *Discrete Applied Mathematics*. 1988; 19(1-3): 239-257. Available from: http://doi.org/10.1016/0166-218X(88)90017-0.

[2] Knor M, Škrekovski R, Tepeh A. Mathematical aspects of wiener index. *arXiv:151000800*. 2015.

[3] Gutman I, Ashrafi AR. The edge version of the Szeged index. *Croatica Chemica Acta*. 2008; 81(2): 263-266.

[4] Gutman I, Furtula B, Petrović M. Terminal wiener index. *Journal of Mathematical Chemistry*. 2009; 46: 522-531.

[5] Cavaleri M, D'Angeli D, Donno A, Hammer S. Wiener, edge-wiener, and vertex-edge-Wiener index of Basilica graphs. *Discrete Applied Mathematics*. 2022; 307: 32-49.

[6] Farahani MR, Jamil MK, Kanna MR, Hosamani SM. The wiener index and hosoya polynomial of the subdivision graph of the wheel S (Wn) and the line graph subdivision graph of the wheel L (S (Wn)). *Applied Mathematics*. 2016; 6(2): 21-24.

[7] Stevanović D. Hosoya polynomial of composite graphs. *Discrete Mathematics*. 2001; 235(1-3): 237-244.

[8] Zaman S, Ahmed W, Sakeena A, Rasool KB, Ashebo MA. Mathematical modeling and topological graph description of dominating David derived networks based on edge partitions. *Scientific Reports*. 2023; 13(1): 15159. Available from: https://doi.org/10.1038/s41598-023-42340-6.

[9] Ullah A, Zaman S, Hamraz A, Muzammal M. On the construction of some bioconjugate networks and their structural modeling via irregularity topological indices. *The European Physical Journal E*. 2023; 46(8): 72. Available from: https://doi.org/10.1140/epje/s10189-023-00333-3.

[10] Zaman S, Ullah A, Shafaqat A. Structural modeling and topological characterization of three kinds of dendrimer networks. *The European Physical Journal E*. 2023; 46(5): 36. Available from: https://doi.org/10.1140/epje/s10189-023-00297-4.

[11] Zaman S, Jalani M, Ullah A, Ali M, Shahzadi T. On the topological descriptors and structural analysis of cerium oxide nanostructures. *Chemical Papers*. 2023; 77(5): 2917-2922. Available from: https://doi.org/10.1007/s11696-023-02675-w.

[12] Hayat S, Khan S, Khan A, Liu JB. Valency-based molecular descriptors for measuring the π-electronic energy of lower polycyclic aromatic hydrocarbons. *Polycyclic Aromatic Compounds*. 2022; 42(4): 1113-1129. Available from: https://doi.org/10.1080/10406638.2020.1768414.

[13] Ullah A, Zaman S, Hamraz A. Zagreb connection topological descriptors and structural property of the triangular chain structures. *Physica Scripta*. 2023; 98(2): 025009. Available from: https://doi.org/10.1088/1402-4896/acb327.

[14] Zaman S, Jalani M, Ullah A, Saeedi G. Structural analysis and topological characterization of sudoku nanosheet. *Journal of Mathematics*. 2022; 2022: 5915740. Available from: https://doi.org/10.1155/2022/5915740.

[15] Zaman S, Ullah A. Kemeny's constant and global mean first passage time of random walks on octagonal cell network. *Mathematical Methods in the Applied Sciences*. 2023; 46(8): 9177-9186. Available from: https://doi.org/10.1002/mma.9046.

[16] Yu X, Zaman S, Ullah A, Saeedi G, Zhang X. Matrix analysis of hexagonal model and its applications in global mean-first-passage time of random walks. *IEEE Access*. 2023; 11: 10045-10052. Available from: https://doi.org/10.1109/ACCESS.2023.3240468.

[17] Hakeem A, Ullah A, Zaman S. Computation of some important degree-based topological indices for γ-graphyne and Zigzag graphyne nanoribbon. *Molecular Physics*. 2023; 121(14): e2211403. Available from: https://doi.org/10.1080/00268976.2023.2211403.

[18] Zaman S, Jalani M, Ullah A, Ahmad W, Saeedi G. Mathematical analysis and molecular descriptors of two novel metal–organic models with chemical applications. *Scientific Reports*. 2023; 13(1): 5314. Available from: https://doi.org/10.1038/s41598-023-32347-4.

[19] Ullah A, Shamsudin, Zaman S, Hamraz A, Saeedi G. Network-based modeling of the molecular topology of fuchsine acid dye with respect to some irregular molecular descriptors. *Journal of Chemistry*. 2022; 2022(1): 8131276. Available from: https://doi.org/10.1155/2022/8131276.

[20] Peng ZB, Nizami AR, Iqbal Z, Munir MM, Waqar Ahmed HM, Liu JB. Wiener and hyper-wiener indices of polygonal cylinder and torus. *Complexity*. 2021; 2021: 1-15.

[21] Islam SR, Pal M. Hyper-Wiener index for fuzzy graph and its application in share market. *Journal of Intelligent & Fuzzy Systems*. 2021; 41(1): 2073-2083.

[22] Thomas N, Mathew L, Sriram S, Subramanian K. Wiener-type indices of Parikh word representable graphs. *Ars Mathematica Contemporanea*. 2021; (2): 243-260.

[23] Cash G, Klavžar S, Petkovšek M. Three methods for calculation of the hyper-wiener index of molecular graphs. *Journal of Chemical Information and Computer Sciences*. 2002; 42(3): 571-576.

[24] Lukovits I, Linert W. A novel definition of the hyper-wiener index for cycles. *Journal of Chemical Information and Computer Sciences*. 1994; 34(4): 899-902.

[25] Meenakshi A, Bramila M. The minimum equitable dominating Seidel energy of a graph. In *AIP Conference Proceedings*. New York, NY, USA: AIP Publishing LLC; 2022. p.210020. Available from: https://doi.org/10.1063/5.0108558.

[26] Annamalai M, Adhimoolam K. A note on graceful trees. In *AIP Conference Proceedings*. New York, NY, USA: AIP Publishing; 2022. Available from: https://doi.org/10.1063/5.0108571.

[27] Meenakshi A, Mythreyi O, Bramila M, Kannan A, Senbagamalar J. Application of neutrosophic optimal network using operations. *Journal of Intelligent & Fuzzy Systems*. 2023; 45(1): 421-433. Available from: https://doi.org/10.3233/JIFS-223718.

[28] Mahboob A, Rasheed MW. Hosaya polynomial and weiner index of abid-waheed graph $(AW)_p^6$. *Applied and Computational Mathematics*. 2022; 10(3): 89-94. Available from: https://www.sciencepublishinggroup.com/article/10.11648/j.ijebo.20221003.13.

[29] Mahboob A, Rasheed MW. Hosaya polynomial and weiner index of abid-waheed graph $(AW)_p^6$. *Applied and Computational Mathematics*. 2022; 10(3): 89-94.

[30] Gutman I, Trinajstić N. Graph theory and molecular orbitals. Total φ-electron energy of alternant hydrocarbons. *Chemical Physics Letters*. 1972; 17(4): 535-538.

[31] De N, Nayeem SMA, Pal A. Reformulated first Zagreb index of some graph operations. *Mathematics*. 2015; 3(4): 945-960. Available from: https://doi.org/10.3390/math3040945.

[32] Julietraja K, Venugopal P. Computation of degree-based topological descriptors using M-polynomial for coronoid systems. *Polycyclic Aromatic Compounds*. 2022; 42(4): 1770-1793. Available from: https://doi.org/10.1080/10406638.2020.1804415.

[33] Mondal S, De N, Pal A. Topological indices of some chemical structures applied for the treatment of COVID-19 patients. *Polycyclic Aromatic Compounds*. 2022; 42(4): 1220-1234.

[34] Arockiaraj M, Clement J, Balasubramanian K. Topological indices and their applications to circumcised donut benzenoid systems, kekulenes and drugs. *Polycyclic Aromatic Compounds*. 2020; 40(2): 280-303. Available from: https://doi.org/10.1080/10406638.2017.1411958.

[35] Deo N. *Graph Theory with Applications to Engineering and Computer Science*. New York, NY, USA: Courier Dover Publications; 2017.

[36] Munir M, Nazeer W, Rafique S, Kang SM. M-polynomial and related topological indices of nanostar dendrimers. *Symmetry*. 2016; 8(9): 97.

[37] Cash GG. Relationship between the Hosoya polynomial and the hyper-wiener index. *Applied Mathematics Letters*. 2002; 15(7): 893-895. Available from: https://doi.org/10.1016/S0893-9659(02)00059-9.

[38] Chakraborty A, Mondal SP, Alam S, Mahata A. Cylindrical neutrosophic single-valued number and its application in networking problem, multi-criterion group decision-making problem and graph theory. *CAAI Transactions on Intelligence Technology*. 2020; 5(2): 68-77. Available from: https://doi.org/10.1049/trit.2019.0083.

[39] Bonchev D, Rouvray DH. *Chemical Topology: Introduction and Fundamentals*. Boca Raton, FL, USA: CRC Press; 1999.

[40] Gao W, Wang W, Farahani MR. Topological indices study of molecular structure in anticancer drugs. *Journal of Chemistry*. 2016; 2016: 3216327.

[41] Meenakshi A. Equitable domination of complement of inflated graph. *AIP Conference Proceedings*. 2019; 2112: 020079-1.

[42] Meenakshi A, Babujee JB. Paired equitable domination in graphs. *International Journal of Pure and Applied Mathematics*. 2016; 109(7): 75-81.

[43] Meenakshi A, Bramila M. The minimum maximal mean boundary dominating seidel energy of a graph. In *Proceedings of 2nd International Conference on Mathematical Modeling and Computational Science: ICMMCS 2021*. Berlin, Germany: Springer; 2022. p.541-548. Available from: https://doi.org/10.1007/978-981-19-0182-9_55.

[44] Meenakshi A, Mythreyi O. Applications of neutrosophic social network using max product networks. *Journal of Intelligent & Fuzzy Systems*. 2023; 45(1): 407-420. Available from: https://doi.org/10.3233/JIFS-223484.

[45] Meenakshi A, Kannan A, Mahdal M, Karthik K, Guras R. A comparative study of fuzzy domination and fuzzy coloring in an optimal approach. *Mathematics*. 2023; 11(18): 4019. Available from: https://doi.org/10.3390/math11184019.

[46] Meenakshi A, Kannan A, Cep R, Elangovan M. Efficient graph network using total magic labeling and its applications. *Mathematics*. 2023; 11(19): 4132. Available from: https://doi.org/10.3390/math11194132.

[47] Shao Z, Javed MS, Farahani MR. Degree based graph invariants for the molecular graph of Bismuth Tri-Iodide. *Engineering and Applied Science Letters*. 2019; 2(1): 1-11.

[48] Gao W, Asghar A, Nazeer W. Computing degree-based topological indices of Jahangir graph. *Engineering and Applied Science Letters*. 2018; 1(1): 16-22.

[49] Gao W, Asif M, Nazeer W. The study of honey comb derived network via topological indices. *Open Journal of Mathematical Analysis*. 2018; 2(2): 10-26. Available from: https://doi.org/10.30538/psrp-oma2018.0014.

[50] Deutsch E, Klavžar S. M-polynomial and degree-based topological indices. *arXiv:14071592*. 2014. Available from: https://doi.org/10.22052/IJMC.2015.10106.

[51] Gadre SR, Suresh CH, Mohan N. Electrostatic potential topology for probing molecular structure, bonding and reactivity. *Molecules*. 2021; 26(11): 3289. Available from: https://doi.org/10.3390/molecules26113289.

[52] Ibrahim H, Sharafdini R, Réti T, Akwu A. Wiener-hosoya matrix of connected graphs. *Mathematics*. 2021; 9(4): 359. Available from: https://doi.org/10.3390/math9040359.