



## Research Article

# Singular Perturbations and Large Time Delays Through Accelerated Spline-Based Compression Technique

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**Abstract:** In the quest to solve the singularly perturbed delay differential equations (SPDDEs) involving large delay with integral boundary condition, the cubic spline in compression technique is explored for the study of dynamical systems to capture complex temporal phenomena in a wide range of scientific disciplines. The integral boundary condition is handled using Simpson's 1/3 rule and the scheme's applicability is validated by numerically experimenting with some problems at different values of mesh size and perturbation parameter. Numerical data are tabulated to show that the suggested approach is more accurate and is an improvement over the methods used in the literature. The insights gained from this research paper provide a foundation for further exploration and utilization of SPDDEs in understanding and predicting the behavior of complex systems across diverse scientific domains.

**Keywords:** delay differential equations, singular perturbation problems, numerical methods, cubic spline in compression, integral boundary conditions, large delay, convection-diffusion problem

**MSC:** 65L06, 65L11, 65L20, 65L50

## 1. Introduction

As time has progressed, it has become clearer that the class of problems constituted by boundary value problems with non-local boundary conditions is both intriguing and important. These issues have several uses in the scientific and technological realms. For singularly perturbed boundary value problems, a multiscale behavior emerges in the solution when the perturbation parameter is very small. While gradients in the solutions to these problems are negligible in the outer or regular region, they are quite steep in thin region known as layer region. Thus, there are substantial difficulties with numerical techniques as well as asymptotic approaches to these problems and hence these issues are widely examined in past few years. These type of problems are commonly encountered in several domains of applied mathematics, including fluid mechanics, elasticity, quantum mechanics, optimal control, chemical-reactor theory, aerodynamics, geophysics, and numerous other fields. Similarly, there has been a growing interest in the study of delay differential equations due to their occurrence in a vast array of application disciplines, including biosciences, economics, material science, medicine, robotics, etc. Delays in time are inevitable in every system with a feedback control, these occur because there is a time limit on both sensing and responding to the data. Delay models are also often used to explain how infectious diseases change over time, such as primary infection, drug treatment, etc. The works in [1-6]

can also be extended to give applications in aforementioned disciplines as they deal with fractional delay problems. Here, we are working with singularly perturbed delay differential equations (SPDDEs), that is, differential equations in which the highest derivative is multiplied by a small parameter (perturbation parameter) and which involves at least one shift term (delay term). These kind of problems arise in mathematical modelling of a variety of realistic situations such as human immunodeficiency virus (HIV) infection models [7], control theory [8], microscale heat transfer [9], etc. Furthermore, these equations find widespread application across a range of scientific disciplines, encompassing biology, chemistry, engineering, and economics, as they serve as valuable tools for modeling dynamic systems. Singularly perturbed delay differential equations (SPDDEs) include a small parameter that describes the scale difference between the system's fast and slow processes, whereas time delays represent the system's response time to external changes or stimuli. In SPDDE analysis, perturbation techniques are frequently used to approximate solutions in terms of the small parameter. These approximations play an important role in simplifying the system by putting light on the behaviour of slow dynamics that may not be seen in the actual system. A thorough grasp of how delays affect system stability and behaviour is critical. Large delays can cause instability or bifurcations, but minor delays can modify stability and create oscillations. Understanding these consequences is critical for developing effective control measures and learning about system behaviour. Researchers use SPDDEs to analyse and anticipate the long-term behaviour of various systems, particularly those with inherent delays, as part of their attempts to optimise parameters for desired outcomes in complex dynamical systems.

The authors in [10] developed a hybrid difference scheme for solving singularly perturbed delay differential equations. Kadalbajoo and Kumar [11] proposed a piecewise-uniform mesh, in the neighbourhood of the boundary layers, using the fitted mesh approach with B-spline collocation technique. The researchers in [12] suggested an upwind finite difference scheme to solve these type of problems. In [13], the authors carried out a fitted mesh approach to solve the boundary-value problem of singular perturbation problems with small delay while others looked into various concepts of singular perturbation problems involving both delay and advanced parameters [14, 15]. Subburayan and Ramanujam [16] used uniformly convergent finite difference method by incorporating piecewise linear interpolation on Shishkin meshes, as a viable approach to address singularly perturbed boundary value problem associated with second order ordinary delay differential equations of convection-diffusion type. The authors in [17] converted second order singularly perturbed delay differential equations to two first-order differential equations, one of which is singularly perturbed without delay term and the other of which is a delay differential equation. Initial value technique and fourth order Runge-Kutta method with Hermite interpolation are then applied to solve the problem. Amiraliyev and Cimen [18] employed an exponentially fitted difference scheme for the numerical solution of this problem on a uniform mesh, which was achieved by the method of integral identities and using exponential basis functions, interpolating quadrature rules with weight and remainder term in integral form. Singularly perturbed large delay differential equations are also solved by researchers in [19] both analytically and numerically. The non-uniform or adaptive mesh approach, namely the Shishkin mesh technique is used by authors in [20]. Many numerical systems obtained through finite element method [21, 22] have been developed in recent years to address the singularly perturbed delay differential equations with large delay variable, with boundary conditions. A review on the solution methodology of these kind of problems is given in [23]. Kumar and Rao [24] considered a stabilized central difference method by modifying the error terms for the boundary value problem (BVP) of singularly perturbed differential equations with a large delay. The work in [25] suggested a non polynomial spline method for solving this type of problems. An almost first order convergent finite difference scheme by using piecewise Shishkin type mesh is presented in [26] and an exponentially fitted finite difference method is suggested in [27] to tackle the problem. The works in [28-35] also give the approximate solution of these kind of problems with different numerical approaches.

Since standard or classical numerical methods are inappropriate and deliver results distant from the expectations for the singularly perturbed boundary value problems, researchers are attempting to design non-standard or non-classical numerical schemes to sort this out for small epsilon. The related SPDDES for ordinary differential equations (ODEs) with non-local boundary condition have received little attention in research and hence our plan is to develop a strategy to get an approximate solution by assessing the error analysis. Here, we are dealing with singularly perturbed delay differential equations with integral boundary conditions. Differential equations with integral boundary conditions have been shown to be a well-posed problem by the authors in [36] and Bickley [37] proposed the use of cubic splines as a means to solve linear two-point boundary value problems. The primary purpose of this work is to develop a

more precise and uniformly converging numerical method of cubic splines for solving SPDDEs subject to integral boundary conditions. Splines can provide high accuracy in approximating the solution of delay differential equations, even in the presence of large delays. Their ability to interpolate or approximate complex functions with smooth curves allows for accurate representation of the solution. Also, they offer flexibility in representing functions, allowing for the construction of piecewise polynomials that can adapt to the behaviour of the solution over different intervals. This flexibility is beneficial when dealing with singularly perturbed problems where the solution may exhibit rapid changes or have different characteristics in different regions. Despite the need to compute and store spline coefficients, spline-based methods can be computationally efficient for solving delay differential equations, particularly in comparison to methods that require dense matrices or extensive storage of historical data for large delays. Splines inherently produce smooth approximations, which can be advantageous in capturing the continuous and differentiable nature of the solution to delay differential equations. This smoothness can aid in the stability and convergence of numerical methods, especially in the presence of large delays where abrupt changes in the solution may occur.

In this work, we propose a scheme based on cubic spline in compression to singularly perturbed large delay differential equations of convection diffusion type, with integral boundary condition. The contents of the paper is organised in the following manner. In Sect. 2, we define the problem statement with some assumptions on the parameters and variables used. We discuss some properties of the solution in Sect. 3. In Sect. 4, the spline in compression method is applied and a scheme is generated using some Taylor series approximations of first order derivatives. In Sect. 5, we discuss the numerical algorithm used to find the solution and the convergence analysis is done in Sect. 6. In Sect. 7, some test problems are presented with their results, graphs and rate of convergence, which is then followed by Conclusion in Sect. 8.

## 2. Statement of the problem

We take the following singularly perturbed large delay differential equation of convection diffusion type from [38] to illustrate the process.

$$\mathcal{H}u \equiv -\epsilon u''(\eta) + p(\eta)u'(\eta) + q(\eta)u(\eta) + r(\eta)u(\eta-1) = s(\eta), \quad \eta \in \omega = (0, 2), \quad (1)$$

together with

$$\begin{aligned} u(\eta) &= \psi(\eta), \quad \eta \in [-1, 0], \\ u(2) &= \bar{k} + \epsilon \int_0^2 g(\eta)u(\eta)d\eta = \bar{\beta}, \end{aligned} \quad (2)$$

where  $0 < \epsilon \leq 1$  and  $\bar{\beta}$  is a constant.

It is assumed that (1) and (2) have bounded, smooth and  $\epsilon$ - independent functions. Also,

$$p(\eta) \geq \bar{p} > 0,$$

$$q(\eta) \geq \bar{q} > 0,$$

$$r(\eta) \leq \bar{r} < 0,$$

$$q(\eta) + r(\eta) \geq \bar{\delta} > 0,$$

where  $\bar{p} + \bar{q} + \bar{r} > 0$  and  $\bar{q} - \bar{r} \geq 0$ . From (1) and (2), we can write as

$$\begin{aligned}
-\epsilon u''(\eta) + p(\eta)u'(\eta) + q(\eta)u(\eta) &= s(\eta) - r(\eta)\psi(\eta - 1), \quad \eta \in (0, 1], \\
-\epsilon u''(\eta) + p(\eta)u'(\eta) + q(\eta)u(\eta) + r(\eta)u(\eta - 1) &= s(\eta), \quad \eta \in (1, 2],
\end{aligned}
\tag{3}$$

satisfying,

$$u(1^-) = u(1^+), \quad u'(1^-) = u'(1^+), \quad u(2) = \bar{\beta}.$$
(4)

Equations in (3) can also be written as:

$$\mathcal{H}u \equiv \bar{f}(\eta),$$
(5)

where,

$$\mathcal{H}u = \begin{cases} \mathcal{H}_1 u(\eta) = -\epsilon u''(\eta) + p(\eta)u'(\eta) + q(\eta)u(\eta), & \eta \in \omega_1, \\ \mathcal{H}_2 u(\eta) = -\epsilon u''(\eta) + p(\eta)u'(\eta) + q(\eta)u(\eta) + r(\eta)u(\eta - 1), & \eta \in \omega_2, \end{cases}$$

$$\bar{f}(\eta) = \begin{cases} s(r) - r(\eta)\psi(\eta - 1), & \eta \in \omega_1, \\ s(\eta), & \eta \in \omega_2. \end{cases}$$

Also, let  $\kappa u(2) = u(2) - \epsilon \int_0^2 g(\eta)u(\eta)d\eta = \bar{k}$ .

Throughout the work, we take  $\omega = (0, 2)$ ,  $\bar{\omega} = [0, 2]$ ,  $\omega_1 = (0, 1)$ ,  $\omega_2 = (1, 2)$ ,  $\omega^* = \omega_1 \cup \omega_2$  and  $\varpi = C^0(\omega) \cup C^1(\omega) \cup C^2(\omega)$ .

### 3. Properties of solution

**Lemma 3.1** Let  $\zeta(\eta)$  be any function in  $\varpi$  such that  $\zeta(0) \geq 0$ ,  $\kappa\zeta(2) \geq 0$ ,  $\mathcal{H}_1\zeta(2) \geq 0$  for all  $\eta \in \omega_1$ ,  $\mathcal{H}_2\zeta(2) \geq 0$  for all  $\eta \in \omega_2$  and  $[\zeta'](1) \leq 0$ , then  $\zeta(\eta) \geq 0$ , for all  $\eta \in \bar{\omega}$ .

**Proof.** Define the function  $t(\eta)$  as

$$t(\eta) = \begin{cases} \frac{1}{8} + \frac{\eta}{2}, & \eta \in [0, 1], \\ \frac{3}{8} + \frac{\eta}{4}, & \eta \in [1, 2]. \end{cases}$$

Then  $t(\eta)$  is positive for all  $\eta \in \bar{\omega}$ . Also  $\mathcal{H}t(\eta) > 0$  for all  $\eta \in \omega_1 \cup \omega_2$ ;  $t(0)$  and  $\kappa t(2)$  are strictly positive and  $[t]'(1)$  is strictly negative. Let  $\bar{\mu}$  denotes the maximum of the set  $\left\{ \frac{-\zeta(\eta)}{t(\eta)} : \eta \in \bar{\omega} \right\}$ . Then  $\exists \eta_0 \in \bar{\omega}$  satisfying  $\zeta(\eta_0) + \bar{\mu}t(\eta_0) = 0$  and  $\zeta(\eta) + \bar{\mu}t(\eta) \geq 0$ , for all  $\eta \in \bar{\omega}$ . Hence  $(\zeta + \bar{\mu}t)(\eta_0)$  gives minimum value.

Now, assume that the theorem is not true. Then  $\bar{\mu}$  is strictly positive.

**Case (a)**  $\eta_0 = 0$ .

Then,

$$0 < (\zeta + \bar{\mu}t)(0) = \zeta(0) + \bar{\mu}t(0) = 0,$$

which is a contradiction.

**Case (b)**  $\eta_0 \in \omega_1$ .

Then,

$$0 < \mathcal{H}(\xi + \bar{\mu}t)(\eta_0) = -\epsilon(\xi + \bar{\mu}t)''(\eta_0) + a(\eta_0)(\xi + \bar{\mu}t)'(\eta_0) + b(\eta_0)(\xi + \bar{\mu}t)(\eta_0) \leq 0,$$

a contradiction.

**Case (c)**  $\eta_0 = 1$ .

Then,

$$0 \leq [(\xi + \bar{\mu}t)'](1) = [\xi'](1) + \bar{\mu}[t'](1) < 0,$$

it is a contradiction.

**Case (d)**  $\eta_0 \in \omega_2$ .

Then,

$$\begin{aligned} 0 < \mathcal{H}(\xi + \bar{\mu}t)(\eta_0) &= -\epsilon(\xi + \bar{\mu}t)''(\eta_0) + a(\eta_0)(\xi + \bar{\mu}t)'(\eta_0) \\ &+ b(\eta_0)(\xi + \bar{\mu}t)(\eta_0) + c(\eta_0)(\xi + \bar{\mu}t)(\eta_0 - 1) \leq 0, \end{aligned}$$

again a contradiction.

**Case (e)**  $\eta_0 = 2$ .

Then,

$$0 \leq \kappa(\xi + \bar{\mu}t)'(2) = (\xi + \bar{\mu}t)'(2) - \epsilon \int_0^2 g(\eta)(\xi + \bar{\mu}t)(\eta_0) d\eta \leq 0,$$

which is a contradiction.

Since we got contradiction in all the cases, our assumption was wrong and hence the proof is complete.  $\square$

**Lemma 3.2** The solution  $u(\eta)$  for the problem (1)-(2) satisfies the bound:

$$|u(\eta)| \leq \bar{C} \max \left\{ \sup_{r \in \omega^*} |\mathcal{H}u(\eta)|, |u(0)|, |\kappa u(2)| \right\}, \eta \in \bar{\omega}.$$

**Proof.** Let us consider the following functions:

$$\mathcal{G}^\pm(\eta) = \bar{C} \cdot \bar{M} \cdot t(\eta) \pm u(\eta), \eta \in \bar{\omega},$$

where  $\bar{M}$  denotes the maximum of the set  $\{|u(0)|, |\kappa u(2)|, \sup_{\eta \in \omega^*} |\mathcal{H}u(\eta)|\}$  and  $t(\eta)$  is the function in the above lemma.

Using these functions in above lemma, we will get the desired result.  $\square$

**Lemma 3.3** Let  $u(\eta)$  be the solution for the problem (1)-(2). Then

$$\|u^{(k)}(\eta)\|_{\omega^*} \leq C\epsilon^{-k}, \text{ for } k = 1, 2, 3.$$

**Proof.** Refer [26].  $\square$

#### 4. Derivation of the method

Let  $\eta_0 = 0$ ,  $\eta_{2N} = 2$ ,  $\eta_i = hi$ ,  $h = (\eta_{2N} - \eta_0)/N$ .

In  $[\eta_i, \eta_{i+1}]$ , a function  $A(\eta, \vartheta) = A(\eta)$  satisfying the differential equation:

$$A''(\eta) + \mathcal{G}A(\eta) = [A''(\eta_i) + \mathcal{G}A(\eta_i)] \frac{(\eta_{i+1} - \eta)}{h} + [A''(\eta_{i+1}) + \mathcal{G}A(\eta_{i+1})] \frac{(\eta - \eta_i)}{h}, \quad (6)$$

where  $A(\eta_i) = u_i$  and  $\mathcal{G} > 0$  is referred to as cubic spline in compression. Solving (6), we get,

$$A(r_i) = P \sin\left(\frac{\tau\eta_i}{h}\right) + Q \cos\left(\frac{\tau\eta_i}{h}\right) + \left(\frac{\bar{A}_i + \mathcal{G}u_i}{\mathcal{G}}\right) \left[\frac{\eta_{i+1} - \eta}{h}\right] + \left(\frac{\bar{A}_{i+1} + \mathcal{G}u_{i+1}}{\mathcal{G}}\right) \left[\frac{\eta - \eta_i}{h}\right], \quad (7)$$

Using interpolatory conditions  $A(\eta_{i+1}) = u_{i+1}$  and  $A(\eta_i) = u_i$ , we can find  $P$  and  $Q$  in the above equation (7).

Let  $A''(\eta_i) = \bar{A}_i$ ,  $\tau = h\mathcal{G}^{1/2}$ . Finally, we get,

$$\begin{aligned} A(\eta) &= \frac{-h^2}{\tau^2 \sin \tau} \left[ A_i \sin\left(\frac{\tau(\eta_{i+1} - \eta)}{h}\right) + A_{i+1} \sin\left(\frac{\tau(\eta - \eta_i)}{h}\right) \right] \\ &+ \frac{1}{\tau^2} \left[ h(\eta_{i+1} - \eta) \left( \bar{A}_i + \frac{\tau^2}{h^2} u_i \right) + (h(\eta - \eta_i)) \left( \bar{A}_{i+1} + \frac{\tau^2}{h^2} u_{i+1} \right) \right]. \end{aligned} \quad (8)$$

We will get the following equation using  $A'(\eta_i^+) = A'(\eta_i^-)$  in the equation (8).

$$\begin{aligned} \frac{u_i - u_{i-1}}{h} - \frac{h}{\tau^2} \left[ \left(1 - \frac{\tau}{\sin \tau}\right) \bar{A}_{i-1} - (1 - \tau \cot \tau) \bar{A}_i \right] \\ = \frac{u_{i+1} - u_i}{h} - \frac{h}{\tau^2} \left[ (1 - \tau \cot \tau) \bar{A}_i - \left(1 - \frac{\tau}{\sin \tau}\right) \bar{A}_{i+1} \right]. \end{aligned}$$

This implies,

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h} = \frac{h}{\tau^2} \left[ \left(1 - \frac{\tau}{\sin \tau}\right) \bar{A}_{i-1} - (2 - 2\tau \cot \tau) \bar{A}_i + \left(1 - \frac{\tau}{\sin \tau}\right) \bar{A}_{i+1} \right] \quad (9)$$

Multiplying (9) by  $-h$ , we obtain,

$$(u_{i-1} - 2u_i + u_{i+1}) = h^2 \left[ -\left(\frac{1}{\tau^2} - \frac{1}{\tau \sin \tau}\right) \bar{A}_{i-1} + 2\left(\frac{1}{\tau^2} - \frac{\cot \tau}{\tau}\right) \bar{A}_i - \left(\frac{1}{\tau^2} - \frac{1}{\tau \sin \tau}\right) \bar{A}_{i+1} \right] \quad (10)$$

Equation (10) gives a tridiagonal system,

$$u_{i+1} - 2u_i + u_{i-1} = h^2 \left( \tau_1 \bar{A}_{i-1} + 2\tau_2 \bar{A}_i + \tau_1 \bar{A}_{i+1} \right), \quad (11)$$

for  $i$  ranging from 1 to  $2N - 1$  and  $\tau_1, \tau_2$  is given by  $\tau_1 = \frac{1}{\tau \sin \tau} - \frac{1}{\tau^2}$  and  $\tau_2 = \frac{1}{\tau^2} - \frac{\cot \tau}{\tau}$ . Continuity criteria in (11) ensure the continuity of the first order derivatives of  $A(\eta, \vartheta)$  at interior points.

Consider the following approximations:

$$\left. \begin{aligned} u'_{i-1} &\approx \frac{-u_{i+1} + 4u_i - 3u_{i-1}}{2h}, \\ u'_i &\approx \frac{u_{i+1} - u_{i-1}}{2h}, \\ u'_{i+1} &\approx \frac{3u_{i+1} - 4u_i + u_{i-1}}{2h} \end{aligned} \right\} \quad (12)$$

Let  $a(r_i) = a_i, b(r_i) = b_i, c(r_i) = c_i, d(r_i) = d_i$ .

Using the approximations in (12) and  $\bar{A}_i = \frac{1}{\epsilon}(a_i u'_i + b_i u_i + c_i u(r_i - 1) - d_i)$ , we obtain the following scheme.

$$\begin{aligned} &\left( \epsilon - \tau_1 h^2 q_{i-1} + \tau_2 h q_i - \frac{\tau_1 h p_{i+1}}{2} + \frac{3\tau_1 h p_{i-1}}{2} \right) u_{i-1} - (2\epsilon + 2\tau_1 h p_{i-1} + 2\tau_2 h^2 q_i - 2\tau_1 h p_{i+1}) u_i \\ &\quad + \left( \epsilon - \tau_1 h^2 q_{i+1} + \frac{\tau_1 h p_{i-1}}{2} - \frac{3\tau_1 h p_{i+1}}{2} - \tau_2 h p_i \right) u_{i+1} \\ &\quad \approx h^2 \left[ \tau_1 (r_{i-1} u(\eta_{i-1-N}) - s_{i-1}) + \tau_1 (r_{i+1} u(\eta_{i+1-N}) - s_{i+1}) \right] \\ &\quad + h^2 \left[ 2\tau_2 (r_i u(\eta_{i-N}) - s_i) \right], \quad i = 1(1)2N - 1. \end{aligned} \quad (13)$$

The scheme (13) gives a system of  $(2N - 1)$  equations in  $(2N + 1)$  variables, which is solved using the numerical algorithm mentioned in Sect. 5, with the help of MATLAB R2022a mathematical software.

## 5. Numerical algorithm

The subsequent procedure is suggested for acquiring the numerical solution of the problem:

**Step 1** Introduce the uniform mesh by partitioning the domain  $[0, 2]$  into  $2N$  mesh intervals.

**Step 2** Compression technique is applied to the statement problem to arrive at a tridiagonal system.

**Step 3** We make use of Taylor series expansion of first order derivatives in the system obtained in Step 2 to obtain a scheme.

**Step 4** Using reduced problem and the given history function, an initial value problem is formed to find the solution at  $\eta = 1$ .

**Step 5** To find the solution in  $(0, 1)$ , we employ fitting factor in the scheme obtained in Step 3 and we use the value  $u(1)$  obtained in Step 4.

**Step 6** To find the solution in  $(1, 2)$ , we again use fitting factor in the scheme obtained in Step 3. But, to solve the system, we need to find the value of  $u(2)$ , which is given in terms of an integral. Hence, Simpson's 1/3 rule is used to find the same.

**Step 7** Finally, we incorporate the condition obtained in Step 6 in the scheme to obtain the solution of the problem using Gauss elimination method.

We have the following reduced problem by putting the perturbation parameter  $\epsilon = 0$  in (1),

$$p(\eta)u'_0 + q(\eta)u_0 + r(\eta)u_0(\eta - 1) = s(\eta), \quad \eta \in [0, 1],$$

subject to  $u(\eta) = \psi(\eta)$ ,  $\eta \in [-1, 0]$ .  $u_0(\eta - 1) = \psi(\eta - 1)$ , since  $u(\eta) = \psi(\eta)$  in the interval  $[-1, 0]$ .

So, by using the above condition, we get  $p(\eta)u'_0 + q(\eta)u_0 + r(\eta)\psi(\eta - 1) = s(\eta)$ , which gives,

$$u'_0 = \frac{1}{p(\eta)} [s(\eta) - q(\eta)u_0 - r(\eta)\psi(\eta - 1)],$$

with  $u_0(0) = \psi(0)$ . We make use of Runge-Kutta method to get the solution at  $\eta = 1$ , say  $\bar{\gamma}$  [ie  $u_0(1) = \bar{\gamma}$ ].

We use the following fitting factor in the numerical scheme given in (13) to find the solution in  $(0, 1)$ :

$$\sigma_{\bar{\rho}} = p(1)\bar{\rho}(\tau_1 + \tau_2) \coth\left(\frac{p(1)\bar{\rho}}{2}\right),$$

where  $h = \epsilon\bar{\rho}$ .

The scheme with the fitting factor can be described as:

$$\bar{K}_i u_{i-1} - \bar{L}_i u_i + \bar{M}_i u_{i+1} = \bar{N}_i, \quad 1 < i < N - 1,$$

where,

$$\bar{K}_i = \epsilon\sigma_{\bar{\rho}} - \tau_1 h^2 q_{i-1} + \tau_2 h q_i - \frac{\tau_1 h p_{i+1}}{2} + \frac{3\tau_1 h p_{i-1}}{2},$$

$$\bar{L}_i = 2\epsilon\sigma_{\bar{\rho}} + 2\tau_1 h p_{i-1} + 2\tau_2 h^2 q_i - 2\tau_1 h p_{i+1},$$

$$\bar{M}_i = \epsilon\sigma_{\bar{\rho}} - \tau_1 h^2 q_{i+1} + \frac{\tau_1 h p_{i-1}}{2} - \frac{3\tau_1 h p_{i+1}}{2} - \tau_2 h p_i,$$

$$\bar{N}_i = h^2 [\tau_1 (r_{i-1}\psi(\eta_{i-1-N}) - s_{i-1} + r_{i+1}\psi(\eta_{i+1-N}) - s_{i+1})] + h^2 [2\tau_2 (r_i\psi(\eta_{i-N}) - s_i)].$$

The above system can be solved with the conditions  $u_0 = \psi(0)$  and  $u_N = \bar{\gamma}$  using Gauss elimination method.

We apply Simpson's rule to find the value of  $u$  at  $r = 2$ .

$$\int_0^2 g(\eta)u(\eta)d\eta = \frac{h}{3} \left[ 2 \sum_{i=1}^{2N-1} g(\eta_{2i})u(\eta_{2i}) + 4 \sum_{i=1}^{2N} g(\eta_{2i-1})u(\eta_{2i-1}) \right] + \frac{h}{3} [g(0)u(0) + g(2)u(2)].$$

By using (2), we obtain,

$$u(2) - \frac{h}{3} \left[ 2 \sum_{i=1}^{2N-1} g(\eta_{2i})u(\eta_{2i}) + 4 \sum_{i=1}^{2N} g(\eta_{2i-1})u(\eta_{2i-1}) \right] - \frac{h}{3} [g(0)u(0) + g(2)u(2)] = \bar{k}.$$

Use  $u(0) = \psi(0)$  to get the following expression for  $u(2)$  and let  $u_{2N} = \bar{\beta}$ . Then,



$$\bar{\beta} = u(2)$$

$$= \left[ \frac{h\epsilon}{3 - \epsilon hg(2)} \left( g(0)\psi(0) + 2 \sum_{i=1}^{2N-1} g(\eta_{2i})u(\eta_{2i}) + 4 \sum_{i=1}^{2N} g(\eta_{2i-1})u(\eta_{2i-1}) \right) \right]$$

$$+ \frac{\bar{k}}{\left( 1 - \frac{\epsilon h}{3} g(2) \right)}.$$

The scheme in (1, 2) with the fitting factor can be written as:

$$\bar{K}_i u_{i-1} - \bar{L}_i u_i + \bar{M}_i u_{i+1} = \bar{N}_i, \quad N+1(1)2N-1,$$

where,

$$\bar{K}_i = \epsilon \sigma_{\bar{\rho}} - \tau_1 h^2 q_{i-1} + \tau_2 h q_i - \frac{\tau_1 h p_{i+1}}{2} + \frac{3\tau_1 h p_{i-1}}{2},$$

$$\bar{L}_i = 2\epsilon \sigma_{\bar{\rho}} + 2\tau_1 h p_{i-1} + 2\tau_2 h^2 q_i - 2\tau_1 h p_{i+1},$$

$$\bar{M}_i = \epsilon \sigma_{\bar{\rho}} - \tau_1 h^2 q_{i+1} + \frac{\tau_1 h p_{i-1}}{2} - \frac{3\tau_1 h p_{i+1}}{2} - \tau_2 h p_i,$$

$$\bar{N}_i = h^2 \left[ \tau_1 (r_{i-1} u_{i-1-N} - s_{i-1}) + 2\tau_2 (r_i u_{i-N} - s_i) + \tau_1 (r_{i+1} u_{i+1-N} - s_{i+1}) \right].$$

By employing Gauss elimination method with the conditions  $u_N = \bar{\gamma}$  and  $u_{2N} = \bar{\beta}$ , the above system can be solved.

## 6. Convergence analysis

We write the system in matrix form as:

$$TU = V, \tag{14}$$

where  $T = (t_{ij})$  is a matrix of order  $2N - 1$ .

Then for  $i = 1(1)2N - 2$ ,

$$t_{i-1} = \epsilon \sigma_{\bar{\rho}} - \tau_1 h^2 q_{i-1} + \tau_2 h q_i - \frac{\tau_1 h p_{i+1}}{2} + \frac{3\tau_1 h p_{i-1}}{2},$$

$$t_{ii} = -2\epsilon \sigma_{\bar{\rho}} - 2\tau_1 h p_{i-1} - 2\tau_2 h^2 q_i + 2\tau_1 h p_{i+1},$$

$$t_{i+1} = \epsilon \sigma_{\bar{\rho}} - \tau_1 h^2 q_{i+1} + \frac{\tau_1 h p_{i-1}}{2} - \frac{3\tau_1 h p_{i+1}}{2} - \tau_2 h p_i.$$

Now for  $i = 2N - 1$ ,

$$t_{2n-1,i} = \begin{cases} \frac{4g_i \bar{M}_{2N-1} \epsilon h}{(3-\epsilon h)g(2)}, & i = 1(2)N-1, \\ \frac{2g_i \bar{M}_{2N-1} \epsilon h}{(3-\epsilon h)g(2)}, & i = 2(2)N, \\ \frac{4g_i \epsilon h}{(3-\epsilon h)g(2)}, & i = N+1(2)2N-3, \\ \frac{2g_i \epsilon h}{(3-\epsilon h)g(2)}, & i = N+2(2)2N-4, \\ \frac{2g_{2N-2} \epsilon h}{(3-\epsilon h)g(2)} - \bar{K}_{2N-1}, & i = 2N-2, \\ \frac{4g_{2N-1} \epsilon h}{(3-\epsilon h)g(2)} - \bar{L}_{2N-1}, & i = 2N-1, \end{cases}$$

and  $V = (v_i)$  is a column vector, where,

$$v_i = \begin{cases} \bar{N}_1 - \bar{K}_1 \psi_0, & i = 1, \\ h^2 [\tau_1 r_{i-1} \psi_{i-1-N} + 2\tau_2 r_i \psi_{i-N} + \tau_1 r_{i+1} \psi_{i+1-N}] \\ \quad - h^2 (\tau_1 s_{i-1} + 2\tau_2 s_i + \tau_1 s_{i+1}), & i = 2(1)N-1, \\ h^2 [\tau_1 r_{i-1} u_{i-1-N} + 2\tau_2 r_i u_{i-N} + \tau_1 r_{i+1} u_{i+1-N}] \\ \quad - h^2 (\tau_1 s_{i-1} + 2\tau_2 s_i + \tau_1 s_{i+1}), & i = N(1)2N-2, \\ \bar{N}_{2n-1} - \frac{3\bar{M}_{2N-1}}{(3-\epsilon h)g(2)} \left[ \bar{k} + \frac{\epsilon h}{3} g(0) \psi_0 \right], & i = 2N-1. \end{cases}$$

Truncation error obtained is  $W_i(h) = \frac{h^4}{2} X^* + O(h^5)$ , where  $X^*$  is given by  $\frac{u_i^{(3)}}{\epsilon} \left[ \tau_1 p_{i-1} - \frac{2\tau_2 p_i}{3} + \tau_1 p_{i+1} \right]$ .  $TU = V$

can also be written as:

$$T\bar{U} - W(h) = V, \tag{15}$$

in error form, where  $\bar{U} = (\bar{u}_1 \ \bar{u}_2 \ \dots \ \bar{u}_{2N-1})^t$  denotes the exact solution and  $W(h) = (w_1(h) \ w_2(h) \ \dots \ w_{2N-1}(h))^t$  is the truncation error. Let

$$\bar{E} = \bar{U} - U = (\bar{e}_1 \ \bar{e}_2 \ \dots \ \bar{e}_{2N-1})^t.$$

denote the error of the problem (1) obtained by using the scheme in (13). From (14) and (15),

$$T\bar{E} = W(h), \tag{16}$$

Let  $S_i$  denote the  $i$ -th row sum of the matrix  $T$ . Then,

$$S_1 = \epsilon \sigma_{\bar{p}} - \frac{3\tau_1 p_{i-1} h}{2} + \frac{\tau_1 p_{i+1} h}{2} - 2h^2 \tau_2 q_i - \tau_2 p_i h - \tau_1 h^2 q_{i+1},$$

$$S_i = h^2 [-\tau_1 q_{i-1} - \tau_2 q_i - \tau_1 q_{i+1}] = h^2 (\bar{B}_i), \quad i = 2(1)2N-2,$$

$$\begin{aligned} S_{2N-1} &= \frac{2\epsilon h}{(3-\epsilon h)g(2)} \bar{M}_{2N-1} [2(g_1 + g_3 + \dots + g_{N-1}) + (g_2 + g_4 + \dots + g_N)] \\ &+ \frac{2\epsilon h}{(3-\epsilon h)g(2)} [2(g_{N+1} + g_{N+3} + \dots + g_{2N-3}) + (g_{N+2} + g_{N+4} + \dots + g_{2N-4})] \\ &+ \frac{2\epsilon h}{(3-\epsilon h)g(2)} [g_{2N-2} + 2g_{2N-1} - \bar{K}_{2N-1} - \bar{L}_{2N-1}]. \end{aligned}$$

As the mesh size gets close to zero,  $T$  is monotone and irreducible matrix and hence  $T$  is invertible and its elements are strictly positive. Then,

$$\bar{E} = T^{-1}W(h), \tag{17}$$

which implies,

$$\|\bar{E}\| \leq \|T^{-1}\| \|W(h)\|. \tag{18}$$

Let the  $(j, i)$ -th element of the inverse matrix of  $T$  be denoted by  $\bar{t}_{j,i}$ . Then, for  $j = 1(1)2N-1$ ,

$$\sum_{i=1}^{2N-1} \bar{t}_{j,i} S_i = 1.$$

So,

$$\sum_{i=1}^{2N-1} \bar{t}_{j,i} \leq \frac{1}{\min_{1 \leq i \leq 2N-1} S_i} \leq \frac{1}{h^2 |\bar{B}_i|}. \tag{19}$$

From equations (14), (17), (18) and (19), we have,

$$\bar{e}_i = \sum_{j=1}^{2N-1} \bar{t}_{j,i} w_j(h), \quad i = 1(1)2N-1,$$

which gives,

$$\begin{aligned} \bar{e}_i &\leq \left( \sum_{j=1}^{2N-1} \bar{v}_{j,i} \right) \max_{1 \leq i \leq 2N-1} |w_i(h)| \\ &\leq \frac{1}{h^2 |\bar{B}_i^*|} \times \frac{h^4 X^*}{2} = O(h^2), \end{aligned}$$

where  $X^*$  is a constant which does not depend on  $h$ . So  $\|\bar{E}\| = O(h^2)$ , which implies our approach is of second order convergent.

## 7. Numerical experiments

We examined the results of two numerical experiments for  $\tau_1 = 1/18$  and  $\tau_2 = 4/9$ , to show that the suggested strategy is applicable. The calculated solution is supplied in the form of tables for different epsilon values by varying mesh size. The double-mesh principle is applied to the examples in order to determine the maximum absolute errors.

$$E^N = \max_i |u_i^N - u_{2i}^{2N}|. \quad (20)$$

### 7.1 Rate of convergence

Using the double mesh principle, we compute the rate of convergence  $\bar{\rho}$  as follows:

$$\bar{\rho} = \frac{\log(E_h / E_{h/2})}{\log 2}.$$

#### Example 1

To examine the method, consider the following problem in [38]:

$$-\epsilon u''(\eta) + 3u'(\eta) + u(\eta) - u(\eta - 1) = 1, \quad \eta \in (0, 2),$$

with,

$$u(\eta) = 1, \quad \eta \in [-1, 0],$$

$$u(2) = 2 + \frac{\epsilon}{3} \int_0^2 \eta u(\eta) d\eta.$$

Table 1 shows the maximum absolute error that occur when the perturbation parameter is varied and and Table 2 shows the rate of convergence for  $\epsilon = 2^{-5}$  and Table 3 gives the rate of convergence for different  $\epsilon$  values. Figure 1 depicts the numerical solution for various  $\epsilon$  values and Figure 2 gives the point-wise absolute errors of this example for different  $N$  values. In addition, Figure 3 shows the maximum absolute error for various  $\epsilon$  values.

#### Example 2

Consider the following problem in [38] to examine the method:

$$-\epsilon u''(\eta) + (1 + \eta)u'(\eta) + (\eta + 10)u(\eta) - e^\eta u(\eta - 1) = \frac{4}{\pi^2} \eta(1 - \eta), \quad \eta \in (0, 2),$$

with,

$$u(\eta) = 2 + \eta, \quad \eta \in [-1, 0],$$

$$u(2) = 2 + \frac{\epsilon}{3} \int_0^2 \eta e^\eta \sin \eta u(\eta) d\eta.$$

**Table 1.** The maximum absolute error of Example 1 for different values of  $\epsilon$

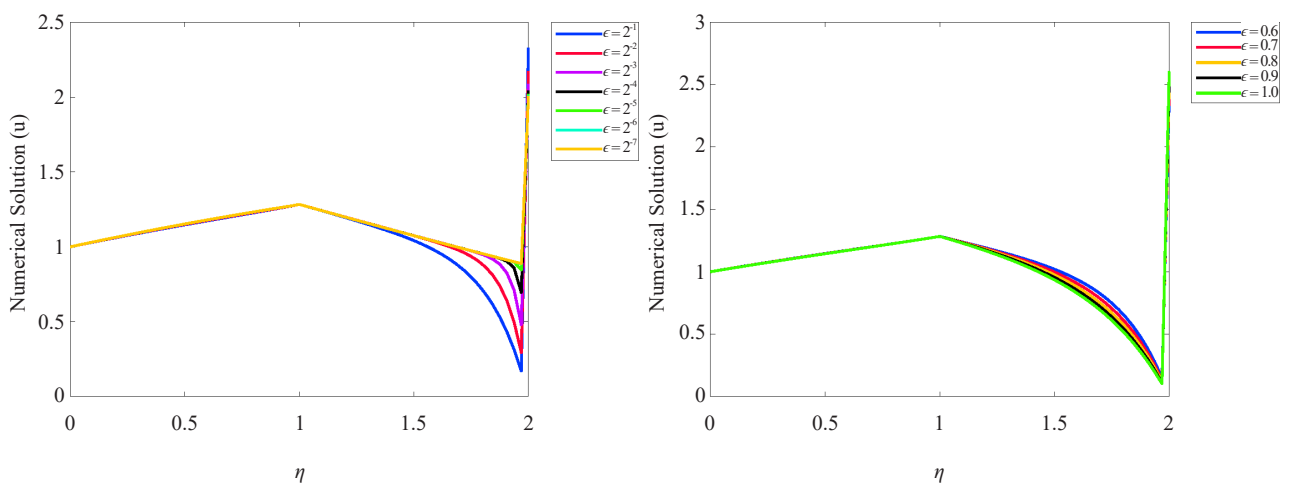
$\epsilon$	$N$						
	32	64	128	256	512	1,024	2,048
$2^{-4}$	5.84e-04	3.37e-04	1.70e-04	9.75e-05	5.21e-05	2.69e-05	1.37e-05
$2^{-8}$	6.42e-04	8.72e-04	4.39e-04	2.06e-04	7.74e-05	2.16e-05	4.87e-06
$2^{-12}$	6.42e-04	8.72e-04	4.41e-04	2.21e-04	1.11e-04	5.56e-05	2.77e-05
$2^{-16}$	6.42e-04	8.72e-04	4.41e-04	2.21e-04	1.11e-04	5.56e-05	2.78e-05
$2^{-20}$	6.42e-04	8.72e-04	4.41e-04	2.21e-04	1.11e-04	5.56e-05	2.78e-05
$2^{-24}$	6.42e-04	8.72e-04	4.41e-04	2.21e-04	1.11e-04	5.56e-05	2.78e-05
$2^{-28}$	6.42e-04	8.72e-04	4.41e-04	2.21e-04	1.11e-04	5.56e-05	2.78e-05
$2^{-32}$	6.42e-04	8.72e-04	4.41e-04	2.21e-04	1.11e-04	5.56e-05	2.78e-05
$E^N$	6.42e-04	8.72e-04	4.41e-04	2.21e-04	1.11e-04	5.56e-05	2.78e-05
CPU time	0.0781 s	0.0938 s	0.2500 s	0.3906 s	1.2813 s	6.4375 s	33.7500 s
Results in [38]							
$2^{-4}$	4.04e-01	1.19e-01	2.62e-02	6.83e-03	1.94e-03	6.09e-04	2.18e-04
$2^{-8}$	6.71e-01	3.17e-01	1.18e-01	3.51e-02	1.00e-02	2.89e-03	1.06e-03
$2^{-12}$	6.70e-01	3.18e-01	1.21e-01	3.73e-02	1.13e-02	3.26e-03	1.56e-03
$2^{-16}$	6.70e-01	3.17e-01	1.20e-01	3.73e-02	1.25e-02	4.29e-03	1.29e-03
$2^{-20}$	6.70e-01	3.17e-01	1.20e-01	3.69e-02	1.17e-02	4.20e-03	2.08e-03
$2^{-24}$	6.70e-01	3.17e-01	1.20e-01	3.69e-02	1.16e-02	3.87e-03	1.43e-03
$2^{-28}$	6.70e-01	3.17e-01	1.20e-01	3.69e-02	1.16e-02	3.84e-03	1.35e-03
$2^{-32}$	6.70e-01	3.17e-01	1.20e-01	3.69e-02	1.16e-02	3.84e-03	1.32e-03
$E^N$	6.70e-01	3.17e-01	1.20e-01	3.69e-02	1.16e-02	3.84e-03	1.32e-03

**Table 2.** Rate of convergence  $\bar{\rho}$  of Example 1 for  $\epsilon = 2^{-5}$

$h$	$\frac{h}{2}$	$E_h$	$\frac{h}{4}$	$\frac{E_h}{2}$	$\bar{\rho}$
$1/2^4$	$1/2^5$	1.59e-03	$1/2^6$	6.08e-04	1.3871
$1/2^5$	$1/2^6$	6.08e-04	$1/2^7$	1.71e-04	1.8324
$1/2^6$	$1/2^7$	1.71e-04	$1/2^8$	4.19e-05	2.0266

**Table 3.** Rate of convergence of Example 1 for different values of  $\epsilon$

$\epsilon$	$N$						
	32	64	128	256	512	1,024	2,048
$2^{-4}$	1.9905	1.7981	1.9046	1.9540	1.9775	1.9889	1.9945
$2^{-8}$	1.9945	1.0926	1.4117	1.8421	2.1463	2.0207	2.0047
$2^{-12}$	1.9854	1.9927	1.9964	1.9982	1.0042	1.0095	1.4150
$2^{-16}$	1.9854	1.9927	1.9964	1.9982	1.9991	1.9995	1.9998
$2^{-20}$	1.9854	1.9927	1.9964	1.9982	1.9991	1.9995	1.9998
$2^{-24}$	1.9854	1.9927	1.9964	1.9982	1.9991	1.9995	1.9998
$2^{-28}$	1.9854	1.9927	1.9964	1.9982	1.9991	1.9995	1.9998
$2^{-32}$	1.9854	1.9927	1.9964	1.9982	1.9991	1.9995	1.9998



**Figure 1.** The numerical solution of Example 1 for different  $\epsilon$  values

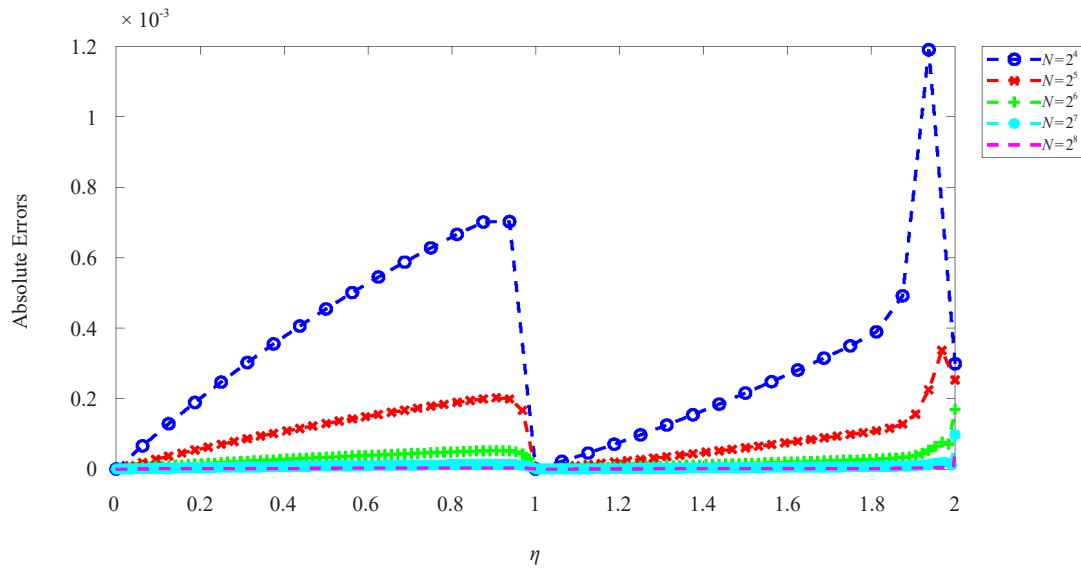


Figure 2. The point-wise absolute errors of Example 1 for different values of  $N$

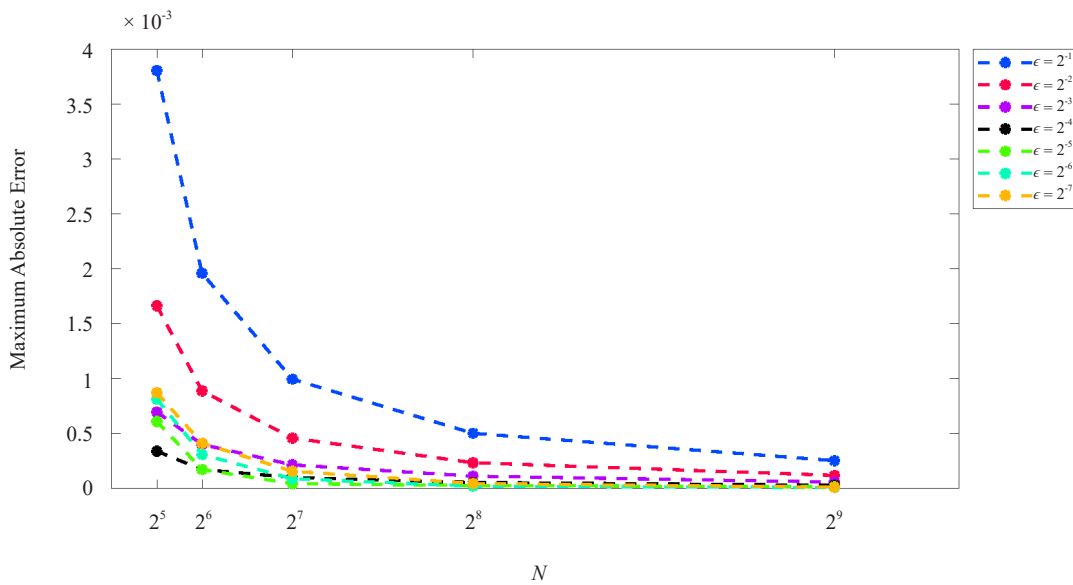


Figure 3. The maximum absolute error of Example 1 for different  $\epsilon$  values

Table 4 shows the maximum absolute error that occur when the perturbation parameter is varied and Table 5 shows the rate of convergence for  $\epsilon = 2^{-4}$  and Table 6 gives the rate of convergence for different  $\epsilon$  values. Figure 4 depicts the numerical solution for various  $\epsilon$  values and Figure 5 gives the point-wise absolute errors of this example for different  $N$  values. Also, Figure 6 shows the maximum absolute error for various  $\epsilon$  values.

**Table 4.** The maximum absolute error of Example 2 for different values of  $\epsilon$

$\epsilon$	$N$						
	32	64	128	256	512	1,024	2,048
$2^{-4}$	8.17e-03	3.17e-03	1.56e-03	8.07e-04	4.11e-04	2.07e-04	1.04e-04
$2^{-8}$	5.12e-02	1.84e-02	1.19e-02	8.94e-03	5.87e-03	3.19e-03	1.64e-03
$2^{-12}$	5.49e-02	2.94e-02	1.53e-02	7.70e-03	3.20e-03	8.93e-03	9.60e-03
$2^{-16}$	5.49e-02	2.94e-02	1.53e-02	7.83e-03	3.96e-03	1.99e-03	9.98e-04
$2^{-20}$	5.49e-02	2.94e-02	1.53e-02	7.83e-03	3.96e-03	1.99e-03	9.98e-04
$2^{-24}$	5.49e-02	2.94e-02	1.53e-02	7.83e-03	3.96e-03	1.99e-03	9.98e-04
$2^{-28}$	5.49e-02	2.94e-02	1.53e-02	7.83e-03	3.96e-03	1.99e-03	9.98e-04
$2^{-32}$	5.49e-02	2.94e-02	1.53e-02	7.83e-03	3.96e-03	1.99e-03	9.98e-04
$E^N$	5.49e-02	2.94e-02	1.53e-02	7.83e-03	3.96e-03	1.99e-03	9.98e-04
CPU time	0.1250 s	0.1781 s	0.2656 s	0.5156 s	1.3594 s	6.0156 s	33.8594 s
Results in [38]							
$2^{-4}$	6.65e-01	1.76e-01	3.55e-02	7.70e-03	1.55e-03	8.75e-04	4.24e-04
$2^{-8}$	8.77e-01	4.15e-01	1.51e-01	4.19e-02	1.27e-02	3.25e-03	1.07e-03
$2^{-12}$	8.80e-01	4.11e-01	1.51e-01	4.36e-02	1.25e-02	4.63e-03	2.17e-03
$2^{-16}$	8.80e-01	4.11e-01	1.51e-01	4.39e-02	1.29e-02	3.85e-03	1.15e-03
$2^{-20}$	8.80e-01	4.11e-01	1.51e-01	4.39e-02	1.29e-02	3.88e-03	1.15e-03
$2^{-24}$	8.80e-01	4.11e-01	1.51e-01	4.39e-02	1.29e-02	3.89e-03	1.17e-03
$2^{-28}$	8.80e-01	4.11e-01	1.51e-01	4.39e-02	1.29e-02	3.89e-03	1.17e-03
$2^{-32}$	8.80e-01	4.11e-01	1.51e-01	4.39e-02	1.29e-02	3.89e-03	1.17e-03
$E^N$	8.80e-01	4.11e-01	1.51e-01	4.39e-02	1.29e-02	3.89e-03	1.17e-03

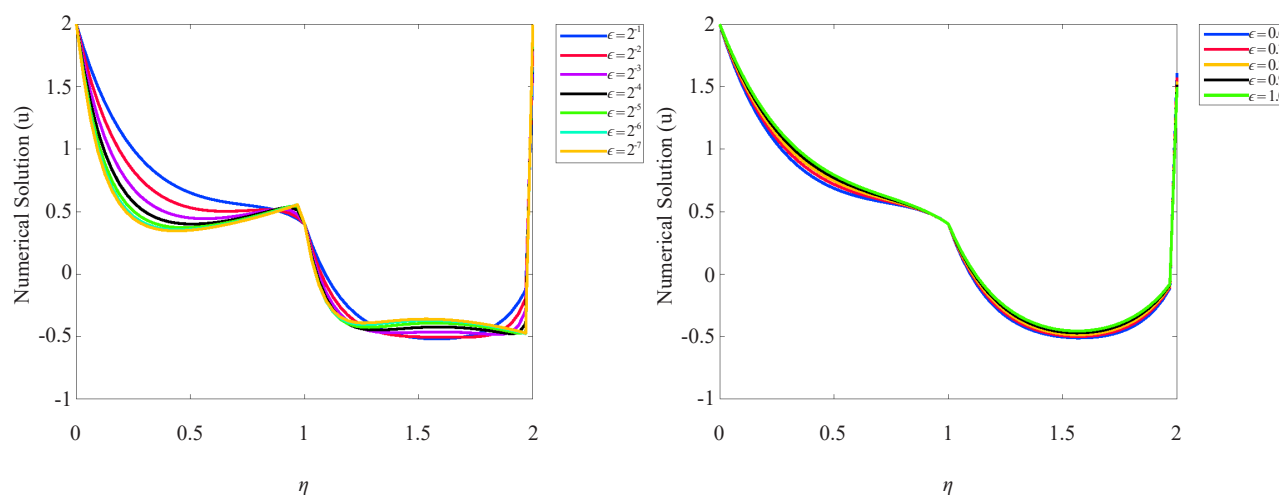
**Table 5.** Rate of convergence  $\bar{\rho}$  of Example 2 for  $\epsilon = 2^{-4}$

$h$	$\frac{h}{2}$	$E_h$	$\frac{h}{4}$	$E_{\frac{h}{2}}$	$\bar{\rho}$
$1/2^3$	$1/2^4$	8.71e-02	$1/2^5$	2.84e-02	1.6160
$1/2^4$	$1/2^5$	2.84e-02	$1/2^6$	8.17e-03	1.7976
$1/2^5$	$1/2^6$	8.17e-03	$1/2^7$	3.17e-03	1.3646



**Table 6.** Rate of convergence of Example 2 for different values of  $\epsilon$

$\epsilon$	$N$						
	32	64	128	256	512	1,024	2,048
$2^{-4}$	1.3646	1.0275	1.9468	1.9750	1.9879	1.9940	1.9971
$2^{-8}$	1.4782	1.6213	1.4180	1.6081	1.8790	1.9616	1.9865
$2^{-12}$	1.9006	1.9400	1.9923	1.2687	1.4823	1.1045	1.0562
$2^{-16}$	1.9006	1.9400	1.9691	1.9839	1.9917	1.9959	1.4116
$2^{-20}$	1.9006	1.9400	1.9691	1.9839	1.9917	1.9958	1.9979
$2^{-24}$	1.9006	1.9400	1.9691	1.9839	1.9917	1.9958	1.9979
$2^{-28}$	1.9006	1.9400	1.9691	1.9839	1.9917	1.9958	1.9979
$2^{-32}$	1.9006	1.9400	1.9691	1.9839	1.9917	1.9958	1.9979



**Figure 4.** The numerical solution of Example 2 for different  $\epsilon$  values

## 8. Conclusion

In this work, the class of singularly perturbed delay differential equations of convection diffusion type, with integral boundary condition is solved with the help of cubic spline in compression method. A fitting factor based scheme is generated for  $(0, 1)$  and  $(1, 2)$  separately to tackle the problem. The computational results by applying this strategy to two examples are given. Figure 1 and Figure 4 gives the numerical solution of Example 1 and Example 2 respectively for various  $\epsilon$  values. The tabulated values in Table 1 and Table 4 show uniform convergence to the calculated solution. From Figures 2, 3 of Example 1 and Figures 5, 6 of Example 2, we can observe that error decreases as  $N$  increases. The numerical rate of convergence is determined as two, both theoretically and numerically (See Tables 2, 3, 5, 6). Also, our findings are compared to those obtained by using established numerical techniques described in the literature and it is found that the proposed method produces numerical results that are more accurate, convergent and consistent.

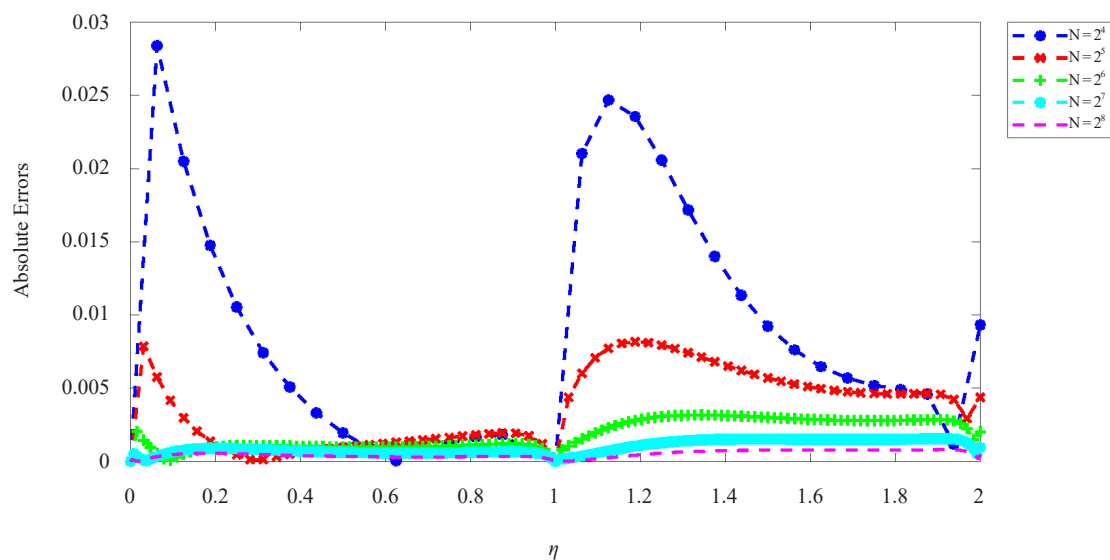


Figure 5. The point-wise absolute errors of Example 2 for different values of  $N$

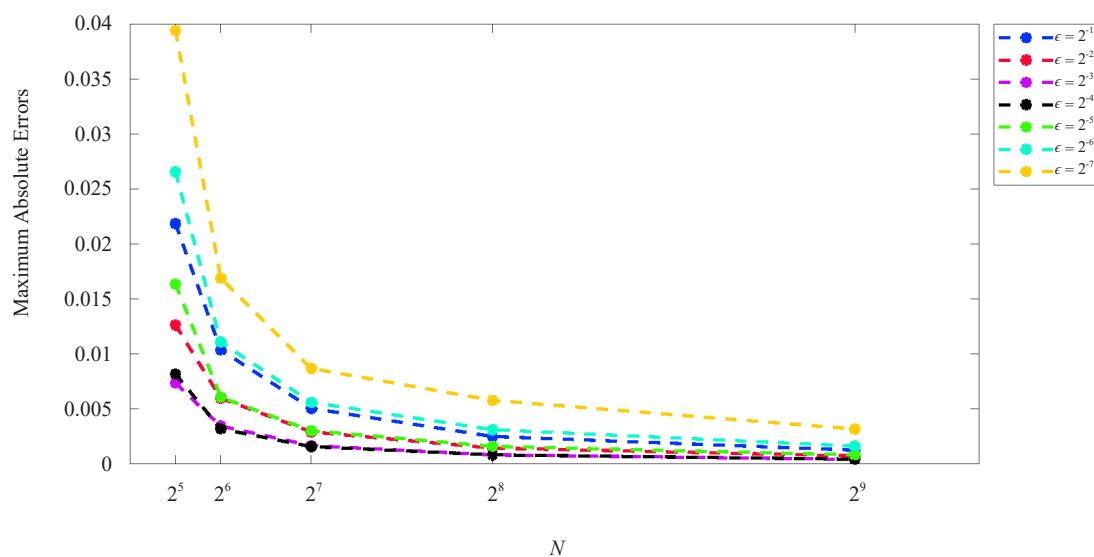


Figure 6. The maximum absolute error of Example 2 for different  $\epsilon$  values

## Data availability

No data was used for the research described in the article.

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## Authors' contributions

Both the authors have equally contributed to this work.

## Conflict of interest

The authors declare that they have no conflict of interest, relevant to the content of this article.

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