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Research Article

Number of Solutions of Equation Over a Finite Group with Application to Graph Defined by Finite Group

Muhammad Siddiq Wira Awaldy[1*], Intan Muchtadi Alamsyah[2], Fariz Maulana[3]

[1] Master's Program in Maths, ITB, Bandung Indonesia
[2] Algebra Research Group, ITB, Bandung Indonesia
[3] Department of Mathematics, Universitas Mataram, Mataram, Indonesia
E-mail: muhammadsiddiqwa@gmail.com

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Abstract: To measure the commutativity of two elements in a group, we define the commutator by $[x, y] = x^{-1}y^{-1}xy$ for all $x, y \in G$. Two elements $x, y \in G$ commute if and only if $[x, y] = e$, where e is the identity element of G. We determine the number of pairs (x, y) in a finite group such that $[x, y] = e$. Using the free group of rank n, denoted F^n, we define for every $g \in G$ the set of n-tuples $\bar{h} \in G^n$ such that $w(\bar{h}) = g$, denoted by $N_{G, w}(g)$. Many open problems remain about the structure of $N_{G, w}(g)$ and its implications for the underlying group. One such problem is Amit's Conjecture, which states that for every $w \in F^n$, the value of $|N_{G, w}(e)|$ is at least $|G|^{n-1}$ for every finite nilpotent group G. This paper proves Amit's Conjecture for all words in two variables. Furthermore, our main theorem generalizes the result to all finite groups. As an application, we provide a bound on the number of edges in the non-braid graph of any finite nonabelian group.

Keywords: commutator of a group, verbal subgroup, number of edges

MSC: 26A25, 26A35

1. Introduction

The concept of commutativity of two elements in a finite group was first discussed by Erdös and Turan [1], where the number of pairs $(x, y) \in G \times G$, where G is a finite group, such that $xy = yx$ can be expressed as $|G|k(G)$ where $|G|$ represents the order of G and $k(G)$ represents the number conjugacy classes of G. By defining the commutator of two elements in G, namely $[x, y] = x^{-1}y^{-1}xy$, then the above problem can be rewritten as finding solutions for equation $[x, y] = e$ in a finite group G where e is the identity element in G.

What if the expression $[x, y]$ is replaced with another type of expression, such as $x^2y^{-3}xyx^6[y, x]$? By reviewing the elements of the free group generated by k variables, namely

$$w(x_1, x_2, \dots, x_k).$$

We can define a verbal mapping on the finite group, namely

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$$f_w : G^{(k)} \to G$$

$$(g_1, g_2, ..., g_k) \mapsto w(g_1, g_2, .., g_k).$$

Then the number of k-tuples $\overline{g} \in G^{(k)}$ such that $w(\overline{g}) = e$, is

$$|N_{G, w}(e)|$$

where $N_{G, w}(e) = \{\overline{g} \in G^{(k)} : w(\overline{g}) = e\}$.

The problem of finding a lower bound of $|N_{G, w}(e)|$ is still an open problem that has not been solved until now. One of the unresolved problems raised by Amit (see [2]) concerns the lower bound of $P_{G, w}(e)$ in the nilpotent group as $|N_{G, w}(e)| \geq |G|^{k-1}$ for each word w over k variable. This paper will give the lower bound of $|N_{G, w}(e)|$ for w an element of the free group with rank 2, for any finite group G. And as an application, we give a bound for the number of edges of the non-braid graph of any finite non-abelian groups.

2. Verbal mapping on a finite group

In this section, we discuss the main topic of this paper, namely verbal mapping on a finite group.

Definition 1 Let G be a finite group. Let F_n denote the free group generated by $S = \{x_1, x_2, \cdots, x_n\} \subseteq G$ and let $w(x_1, x_2, ..., x_n) \in F_n$. We define the verbal map on G associated with word w as a function

$$f_{G, w} : G^n \to G$$

$$(g_1, g_2, ..., g_n) \mapsto w(g_1, g_2, ...g_n).$$

The verbal map replaces each variable in the given word with the corresponding group element. We define

$$N_{G, w}(g) = \{(g_1, g_2, ..., g_n) \in G^n : f_{G, w}(g_1, g_2, ..., g_n) = g\},$$

and the subgroup of G generated by the image of $f_{G, w}$ is called the verbal subgroup of G, denoted by $w(G)$.

Example 2 Let $G = D_6 = \{e, a, a^2, b, ab, a^2b\}$ be the dihedral group of order 6 and $w(x_1, x_2) = x_1x_2x_1$. Then

$$f_{G, w}(a, b) = aba = b.$$

Based on the above definition, the image of $f_{G, w}$ does not necessarily form a group.

Example 3 Let $G = D_6 = \{e, a, a^2, b, ab, a^2b\}$ be the dihedral group of order 6 and $w(x_1) = x_1^3$. Then the image of $f_{G, w}$ is $\{e, b, ab, a^2b\}$ which does not form a group.

3. Divisibility of $|N_{G,\ w}(e)|$ for $w \in F_2$

In [2], Amit's Conjecture is stated as follows.

Conjecture 4 For each finite nilpotent group up to G and any word $w \in F_k$, the following property holds

$$|N_{G,\ w}(e)| \geq |G|^{k-1}.$$

In this paper, we will focus on the case of $w \in F_2$, and it should also be noted that the results obtained in this paper can be applied to all finite groups. The proof written in this section is the two variable case from the proof given in [3].

Definition 5 Suppose $w \in F_2$, We define $d(w,\ x_i)$ as the degree of w in the variable x_i, which represents the total exponent sum of all occurence of x_i in w.

In other words, if

$$w = a_1^{b_1} a_2^{b_2} ... a_k^{b_k},$$

where $a_i \in \{x_1,\ x_2\}$ and $b_i \in \mathbb{Z} \setminus \{0\}$, then for each $j \in \{1,\ 2\}$, we define $d(w,\ x_j) = \sum_{a_i = x_j} b_i$.

Example 6 Let $w(x_1,\ x_2) = x_1 x_2^{-1} x_1^2 x_2^4 \in F_2$ then $d(w,\ x_1) = d(w,\ x_2) = 3$.

We define $N = \{w \in F_2 : d(w,\ x_1) = d(w,\ x_2)\}$. Note that by parsing the exponents of an element w in F_2, $d(w,\ x_i)$ can be viewed as the number of components x_i in w minus the number of components x_i^{-1} in w.

Definition 7 Let G be a group and $s,\ t \in G$. Define the following subgroup of G

$$P(s,\ t) = \bigcap_{i \in \mathbb{Z}} \left(C_G(s^i t^i) \cap C_G(t^i s^i) \right)$$

where $C_G(x) = \{g \in G : xg = gx\}$.

Lemma 8 If $b \in P(s,\ t)$ then $(bs)^i (tb^{-1})^i = s^i t^i$ and $(tb^{-1})^i (bs)^i = t^i s^i$ for $i \in \mathbb{N}$.

Proof. We prove this by induction.

For $i = 0$ we get

$$(bs)^0 (tb^{-1})^0 = e = s^0 t^0$$

$$(tb^{-1})^0 (bs)^0 = e = t^0 s^0.$$

Assuming the lemma is true for $i = k$, for $i = k+1$ we get

$$(bs)^{k+1} (tb^{-1})^{k+1} = (bs)(bs)^k (tb^{-1})^k (tb^{-1}) = bss^k t^k tb^{-1} = bs^{k+1} t^{k+1} b^{-1} = s^{k+1} t^{k+1} bb^{-1} = s^{k+1} t^{k+1}$$

$$(tb^{-1})^{k+1} (bs)^{k+1} = tb^{-1} (tb^{-1})^k (bs)^k bs = tb^{-1} t^k s^k bs = tb^{-1} bt^k s^k s = t^{k+1} s^{k+1}.$$

Since the induction hypothesis is true, then the theorem is proven. □

The idea is to construct a partition of $N_{G,\ w}(e)$ such that the cardinality of each part is divisible by $|G|$.

Definition 9 For every pair $(x, y), (w, z) \in G \times G$, we say that $(x, y) \sim (w, z)$ if there is $a \in G$ and $b \in P(x, y)$ so that

$$z = abxa^{-1}$$

$$w = ayb^{-1}a^{-1}.$$

Lemma 10 The relation $\sim$ is an equivalence relation.

Proof. First, we prove that $\sim$ is reflexive. For all $x, y \in G$, by choosing $a = b = e$ we have $(x, y) \sim (x, y)$ so that $\sim$ is reflexive.

Secondly, we prove that $\sim$ is symmetric. Suppose $x, y, w, z \in G$ with $(z, w) \sim (x, y)$ then there are $a \in G$ and $b \in P(x, y)$ such that

$$z = abxa^{-1}$$

$$w = ayb^{-1}a^{-1}$$

For every $i \in \mathbb{Z}$ we have

$$z^i w^i = (abxa^{-1})^i (ayb^{-1}a^{-1})^i = a(bx)^i a^{-1} a(yb^{-1})^i a^{-1} = a(bx)^i (yb^{-1})^i a^{-1} = ax^i y^i a^{-1}$$

$$w^i z^i = (ayb^{-1}a^{-1})^i (abxa^{-1})^i = a(yb^{-1})^i a^{-1} a(bx)^i a^{-1} = a(yb^{-1})^i (bx)^i a^{-1} = ay^i x^i$$

then

$$(ab^{-1}a^{-1})^{-1} z^i w^i ab^{-1}a^{-1} = aba^{-1} ax^i y^i a^{-1} ab^{-1}a^{-1} = abx^i y^i b^{-1}a^{-1} = ax^i y^i a^{-1} = z^i w^i$$

$$(ab^{-1}a^{-1})^{-1} w^i z^i ab^{-1}a^{-1} = aba^{-1} ay^i x^i a^{-1} ab^{-1}a^{-1} = aby^i x^i b^{-1}a^{-1} = ay^i x^i a^{-1} = w^i z^i.$$

This indicates that $ab^{-1}a^{-1} \in P(z, w)$. Obeserve that x and y can be written as:

$$x = b^{-1}a^{-1}za = a^{-1}(ab^{-1}a^{-1})za,$$

$$y = a^{-1}wab = a^{-1}waba^{-1}a = a^{-1}w(ab^{-1}a^{-1})^{-1}a.$$

Since $ab^{-1}a^{-1} \in P(x, y)$, it follows that $(x, y) \sim (z, w)$. Hence $\sim$ is symmetric.

Lastly, we prove that $\sim$ is transitive. Suppose $x, u, z, y, v, w \in G$ with $(u, v) \sim (z, w)$ and $(z, w) \sim (x, y)$ then there are $a, c \in G$, $b \in P(x, y)$ and $d \in P(z, w)$ such that

$$z = abxa^{-1}$$

$$w = ayb^{-1}a^{-1}$$

$$u = cdzc^{-1}$$

$$v = cwd^{-1}c^{-1}.$$

We prove $(u,\ v) \sim (x,\ y)$. Notice that

$$a^{-1}dabx^iy^i = a^{-1}dax^iy^ib = a^{-1}daa^{-1}z^iw^iab = a^{-1}dz^iw^iab = a^{-1}z^iw^idab = x^iy^ia^{-1}dab$$

$$a^{-1}daby^ix^i = a^{-1}day^ix^ib = a^{-1}daa^{-1}w^iz^iab = a^{-1}dw^iz^iab = a^{-1}w^iz^idab = y^ix^ia^{-1}dab.$$

This means $a^{-1}dab \in P(x,\ y)$. Since u and v can be written as

$$u = (ca)(a^{-1}dab)x(ca)^{-1}$$

$$v = (ca)y(a^{-1}dab)^{-1}(ca)^{-1}$$

$(u,\ v) \sim (x,\ y)$. Hence $\sim$ is transitive and therefore, $\sim$ is an equivalence relation. □

We define the equivalence class of $(x,\ y) \in G \times G$ as $C(x,\ y)$.

Lemma 11 Suppose $x,\ y \in G$ then x normalizes $P(x,\ y)$.

Proof. Suppose $c \in P(x,\ y)$, we prove that $xcx^{-1} \in P(x,\ y)$. Note that

$$xcx^{-1}x^iy^i = xc(x^{i-1}y^{i-1})y = x^iy^{i-1}cy = x^iy^{i-1}cyxx^{-1} = x^iy^ixcx^{-1}.$$

In the same way we get $xcx^{-1}y^ix^i = y^ix^ixcx^{-1}$, this means $xcx^{-1} \in P(x,\ y)$. We conclude that x normalizes $P(x,\ y)$. □

Lemma 12 Let $x,\ y \in G$. Write $G = \bigcup_{i\in\Lambda} a_iP(x,\ y)$ i.e. as a union of the disjoint cosets of $P(x,\ y)$ with $\Lambda = \{1,\ 2,\ \ldots,\ |G : P(x,\ y)|\}$. Then each element in $C(x,\ y)$ can be written as

$$(a_ibxa_i^{-1},\ a_iyb^{-1}a_i^{-1})$$

where $b \in P(x,\ y)$, $i \in \Lambda$, and $|C(x,\ y)| = |G|$.

Proof. Let $(z,\ w) \in C(x,\ y)$ then there is $a \in G$ and $b \in P(x,\ y)$ such that

$$z = abxa^{-1}$$

$$w = ayb^{-1}a^{-1}.$$

Since $a \in G$, it must belong one coset of $P(x,\ y)$ in G. That is, there exist $i \in \Lambda$ and $c \in P(x,\ y)$ such that $a = a_i c$. Because x normalizes $P(x,\ y)$, and thus

$$xc^{-1}x^{-1} \in P(x,\ y).$$

and

$$cbxc^{-1}x^{-1} \in P(x,\ y).$$

Since c is an element of the centralizer yx, and thus

$$xcx^{-1} = y^{-1}cy,$$

and hence

$$(cbxc^{-1}x^{-1})^{-1} = xcx^{-1}b^{-1}c^{-1} = y^{-1}cyb^{-1}c^{-1}.$$

Consequently, z and w can be written as

$$z = abxa^{-1} = a_icbxc^{-1}a_i^{-1} = a_i(cbxc^{-1}x^{-1})xa_i^{-1}$$

$$w = ayb^{-1}a^{-1} = a_icbxc^{-1}a_i^{-1} = a_iy(y^{-1}cyb^{-1}c^{-1})a_i^{-1}.$$

Suppose for $\mu,\ \nu \in \Lambda$ and $b,\ d \in P(x,\ y)$

$$z = a_\mu bxa_\mu^{-1} = a_\nu dxa_\nu^{-1}$$

$$w = a_\mu yb^{-1}a_\mu^{-1} = a_\nu yd^{-1}a_\nu^{-1}.$$

Then for every $i \in \mathbb{Z}$ we get

$$a_\mu x^i y^i a_\mu^{-1} = z^i w^i = a_\nu x^i y^i a_\nu^{-1}$$

$$a_\mu y^i x^i a_\mu^{-1} = w^i z^i = a_\nu y^i x^i a_\nu^{-1}.$$

This means $a_\mu^{-1} a_\nu \in P(x,\ y)$, consequently $\mu = \nu$. Then by using cancellation law, $b = d$. Now $C(x,\ y)$ can be viewed as

$$\{(a_i b x a_i^{-1},\ a_i y b^{-1} a_i^{-1}) : i \in \Lambda,\ b \in P(x,\ y)\}$$

Therefore $|C(x,\ y)| = |P(x,\ y)||G : P(x,\ y)| = |G|$. □

Lemma 13 If N is a nontrivial normal subgroup of F_2, then N is generated by elements of F_2 of the form $x_1^i x_2^i$ and $x_2^i x_1^i$ for each $i \in \mathbb{Z}$.

Proof. Define N_0 as a subgroup of F_2 generated by the elements of the form $x_1^i x_2^i$ and $x_2^i x_1^i$ for each $i \in \mathbb{Z}$. Then $N_0 \subseteq N$. Note that for every $i,\ j \in \mathbb{Z}$ we have

$$x_k^i x_k^j = x_k^{i+j} \in N_0 x_k^{i+j}\ k = 1,\ 2$$

$$x_1^i x_2^j = x_1^i x_2^i x_2^{j-i} \in N_0 x_2^{j-i}$$

$$x_2^i x_1^j = x_2^i x_1^i x_1^{j-i} \in N_0 x_1^{j-i}.$$

Suppose $w \in F$, then w can be written as the product of elements x_j^i with $j = 1,\ 2$ and $i \in \mathbb{Z}$. From the above process, w can be arranged so that $w \in N_0 x_k^l$ with $k \in \{1,\ 2\}$ and $l \in \mathbb{Z}$. If $w \in N$, since $N_0 \subseteq N$, $x_k^l \in N$ and from the definition of N, $l = 0$. Therefore $w \in N_0$, and $N \subseteq N_0$. We conclude that $N = N_0$. □

Lemma 14 Let $w \in N$. If $x,\ y \in G$ and $b \in P(x,\ y)$ then $f_{G,\ w}(bx,\ yb^{-1}) = f_{G,\ w}(x,\ y)$.

Proof. Elements of N are generated by the elements of F_2 of the form $x_1^i x_2^i$ and $x_2^i x_1^i$ for each $i \in \mathbb{Z}$. When substituted with bx and yb^{-1} we obtain

$$(bx)^i (yb^{-1})^i = x^i y^i$$

$$(yb^{-1})^i (bx)^i = y^i x^i.$$

Therefore $f_{G,\ w}(bx,\ yb^{-1}) = f_{G,\ w}(x,\ y)$. □

Lemma 15 For every group G and $w \in N$, $|N_{G,\ w}(e)|$ is divisible by $|G|$.

Proof. Suppose $(x,\ y) \in N_{G,\ w}(e)$. If $a \in G$ and $b \in P(x,\ y)$ then

$$f_{G,\ w}(abxa^{-1},\ ayb^{-1}a^{-1}) = af_{G,\ w}(bx,\ yb^{-1})a^{-1} = af_{G,\ w}(x,\ y)a^{-1} = e.$$

This means that $N_{G,\, w}(e)$ can be viewed as a composite of several equivalence classes with the relation $\sim$ in $G \times G$. Therefore $|N_{G,\, w}(e)|$ is divisible by $|G|$. $\square$

We will now prove that $|N_{G,\, w}(e)|$ is divisible by $|G|$ for every $w \in F_2$, by relating every word in F_2 to some element in N. Before presenting the main theorem, we introduce several auxiliary tools.

Definition 16 Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\, \mathbb{Z})$. We define a group homomorphism1

$$^{A} : F_2 \to F_2$$

by specifying the images of the free generators $x_1,\, x_2$ as follows:

$$x_1^A = x_1^a x_2^b, \qquad x_2^A = x_1^c x_2^d.$$

This homomorphism is then extended to all of F_2 multiplicatively.

Example 17 Let $w = x_1 x_2^2$ and $A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$. Then:

$$x_1^A = x_1 x_2^2, \qquad x_2^A = x_1^3 x_2^5,$$

so

$$w^A = x_1^A \cdot (x_2^A)^2 = (x_1 x_2^2)(x_1^3 x_2^5)^2.$$

Lemma 18 Let F' be the commutator subgroup of F_2. If $A,\, B \in GL(2,\, \mathbb{Z})$ and $w \in F_2$ then

$$(w^A)^B F' = w^{AB} F'.$$

Claim: For any $w \in F_2$, if $d(w,\, x_1) = d(w,\, x_2) = 0$ then $w \in F'$.

Proof. Let $w \in F_2$ with $d(w,\, x_1) = d(w,\, x_2)$, by applying *Hall Collecting Process* [4] there are $k,\, l \in \mathbb{Z}$ and $c \in F'$ so that $w = x_1^k x_2^l c$. Since c is generated by the commutator in F_2, $d(c,\, x_1) = d(c,\, x_2) = 0$. Hence $d(x_1^k x_2^l,\, x_1) = d(x_1^k x_2^l,\, x_2) = 0$ which means $k = l = 0$. Therefore $w \in F'$. $\square$

Proof. [Proof of Lemma 18] For every $w \in F_2$ we have $d(w^A,\, x_i) = \sum_{k=1}^{2} a_{ik} d(w,\, x_k)$. Hence for $i = 1,\, 2$

$$d((w^A)^B,\, x_i) = \sum_{k=1}^{2} b_{ik} d(w^A,\, x_k) = b_{i1} d(w^A,\, x_1) + b_{i2} d(w^A,\, x_2)$$

$$= b_{i1}(a_{11} d(w,\, x_1) + a_{21} d(w,\, x_2)) + b_{i2}(a_{12} d(w,\, x_1) + a_{22} d(w,\, x_2))$$

$$= (b_{i1} a_{11} + b_{i2} a_{21}) d(w,\, x_1) + (b_{i1} a_{12} + b_{i2} a_{22}) d(w,\, x_2) = d(w^{AB},\, x_i).$$

We obtain

$$d\left(\left(w^{A}\right)^{B},\, x_i\right) = d(w^{AB},\, x_i),$$

$$d\left(\left(w^{A}\right)^{B},\, x_i\right) - d(w^{AB},\, x_i) = 0,$$

$$d\left(\left(w^{A}\right)^{B},\, x_i\right) + d\left(\left(w^{AB}\right)^{-1},\, x_i\right) = 0,$$

$$d\left(\left(w^{A}\right)^{B}\left(w^{AB}\right)^{-1},\, x_i\right) = 0.$$

This shows that $(w^{A})^{B}\left(w^{AB}\right)^{-1} \in F'$. □

If $\overline{g} \in G \times G$ and $w \in F_2$, define $\delta(\overline{g},\, w)$ by:

$$\delta(\overline{g},\, w) = \begin{cases} 1, & \text{if } \overline{g} \in N_{G,\, w}(e), \\ 0, & \text{otherwise.} \end{cases}$$

Then, it is clear that

$$|N_{G,\, w}(e)| = \sum_{\overline{g} \in G \times G} \delta(\overline{g},\, w).$$

Definition 19 Let G be a finite group and let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\, \mathbb{Z}).$$

Define a map

$$^{A}: G \times G \to G \times G, \quad (g_1,\, g_2) \mapsto \left(g_1^{a} g_2^{b},\, g_1^{c} g_2^{d}\right).$$

Lemma 20 There is a set Δ which generates $GL(2,\, \mathbb{Z})$ so that if $A \in \Delta$ and $w \in F_2$ then

$$|N_{G,\, w^{A}}(e)| = |N_{G,\, w}(e)|.$$

Proof. Select $\Delta = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$, from [5], we get $GL(2,\, \mathbb{Z}) = \langle \Delta \rangle$.

Notice that for every $(g_1,\ g_2) \in G \times G$ we have

$$(g_1,\ g_2)^{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} = (g_2,\ g_1)$$

$$(g_1,\ g_2)^{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}} = (g_1^{-1},\ g_2)$$

$$(g_1,\ g_2)^{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}} = (g_1 g_2,\ g_2).$$

Then the map $(g_1,\ g_2) \mapsto (g_1,\ g_2)^A$ is a bijection for every $A \in \Delta$. Hence for every $w \in F_2$ we have

$$|N_{G,\ w^A}(e)| = \sum_{\overline{g} \in G \times G} \delta(\overline{g},\ w^A) = \sum_{\overline{g} \in G \times G} \delta(\overline{g}^A,\ w) = \sum_{\overline{g} \in G \times G} \delta(\overline{g},\ w) = |N_{G,\ w}(e)|.$$

□

Lemma 21 If $w \in F_2$ then there is $A \in GL(2,\ \mathbb{Z})$ such that $w^A \in N$.

Proof. Define $L = \{n(d(w,\ x_1)\ d(w,\ x_2)) : n \in \mathbb{Z}\} \subseteq \mathbb{Z}^2$. Based on *Simultaneous Basis Theorem* [6] there are a basis $u_1,\ u_2 \in \mathbb{Z}^2$, a basis $v \in L$ and $d \in \mathbb{Z}$ such that $v = du_1$. Select $B \in GL(2,\ \mathbb{Z})$ such that $u_j B = e_j$ for $j = 1,\ 2$ where $e_1 = (1\ 0),\ e_2 = (0\ 1)$. Then we get $vB = du_1 B = de_1$. Define the endomorphism C in $\mathbb{Z}^2$ with

$$e_1 C = e_1 + e_2$$

$$e_2 C = e_2.$$

It is clear that $C \in GL(2,\ \mathbb{Z})$ and $vBC = du_1 BC = de_1 C = d(e_1 + e_2)$.

Suppose $A = BC$ then $vA = d(e_1 + e_2) \in M = \{m(e_1 + e_2) \in \mathbb{Z}^2 : m \in \mathbb{Z}\}$. Suppose that $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then we get

$$(d(w,\ x_1)\ d(w,\ x_2))A = (d(w,\ x_1)(1\ 0) + d(w,\ x_2)(0,\ 1)) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = d(w,\ x_1)(a_{11}\ a_{12}) + d(w,\ x_2)(a_{21}\ a_{22})$$

$$= (d(w,\ x_1)a_{11} + d(w,\ x_2)a_{21})e_1 + (d(w,\ x_1)a_{12} + d(w,\ x_2)a_{22})e_2$$

$$= d(w^A,\ x_1)e_1 + d(w^A,\ x_2)e_2 = (d(w^A,\ x_1),\ d(w^A,\ x_2)).$$

Since $vA \in M$ and v is a basis of L, $d(w^A,\ x_1) = d(w^A,\ x_2)$. Therefore $w^A \in N$. □

4. Main result

Theorem 22 For every word $w \in F_2$, $|N_{G,\, w}(e)|$ is divisible by $|G|$.

Proof. Suppose $w \in F_2$. There is $A \in GL(2,\ \mathbb{Z})$ such that $w^A \in N$. Since $GL(2,\ \mathbb{Z}) = \langle \Delta \rangle$, A can be written as $A = A_1 A_2 \ldots A_r$ for some $A_1,\ A_2,\ \ldots,\ A_r \in \Delta$. There exist $a \in F'$ such that

$$w^A a = ((w^{A_1})^{A_2} \ldots\ldots)^{A_r}.$$

Since $F' \subseteq N$ $w^A a \in N$, hence $|N_{G,\, w^A a}(e)|$ is divisible by $|G|$. Based on Lemma 14 we have

$$|N_{G,\, w}(e)| = |N_{G,\, w^{A_1}}(e)| = |N_{G,\, (w^{A_1})^{A_2}}(e)| = \ldots = |N_{G,\, w^A a}(e)| \equiv 0\ (mod\ |G|).$$

Therefore $|N_{G,\, w}(e)|$ is divisible by $|G|$.

From the above observation, the proof of Amit's Conjecture for two variable word is stated by the theorem below.

Corollary 23 Suppose G is a finite group, then for every $w \in F_2$ we have

$$|N_{G,\, w}(e)| \geq |G|.$$

Proof. Let $w \in F_2$, by Theorem 22 there is $c \in \mathbb{N}$ such that

$$|N_{G,w}(e)| = c|G| \geq |G|.$$

$\square$

Inspired by the braid relation and as the development of a non-commuting graph, the non-braid graph $\mathcal{NB}_G$ of a finite group G is defined as the simple undirected graph where the set of vertices is G, and x is joined to y if $xyx \neq yxy$. The non-braid graph of a finite ring was studied by Cahyati et al. [7].

Example 24 Let G be a finite commutative group and $x,\ y \in G$ where $x \neq y$. If there exists no edge between x and y, then $xyx = yxy$ but then $yxx = xyx = yxy$, implies $x = y$. Hence, the non-braid graph for any finite commutative group is complete.

Example 25 For $G = D_{2n} = \langle a,\ b : a^n = e,\ b^2 = e,\ bab = a^{-1} \rangle$. Let $a^i b^j,\ a^u b^v \in G$, if there is no edge between those elements, then with an algebraic manipulation, we have $i \equiv u\ mod(n)$ and $j \equiv v\ mod(2)$. Since $i,\ u < n$ and $j,\ v < 2$ then we have $i = u$ and $j = v$, hence x and y are adjacent for all $x,\ y \in D_{2n}$ where $x \neq y$.

We conclude that the non braid graph for D_{2n} is a complete graph.

The set of edges of the non-braid graph $\mathcal{NB}_G$ is $E(\mathcal{NB}_G) = \{(x,\ y) : xyx \neq yxy\}$. By taking $w \in F_2$ as $w = xyxy^{-1}x^{-1}y^{-1}$, it is clear that

$$|E(\mathcal{NB}_G)| = \frac{|G|^2 - |N_{G,\, w}(e)|}{2}$$

since we consider $(x,\ y)$ and $(y,\ x)$ as the same edge. By Theorem 22 we get

$$|E(\mathscr{N}\mathscr{B}_G)| \leq \frac{|G|^2 - |G|}{2}.$$

Example 26 For $G = S_3$, there are no edges between (12) and (13), (12) and (23), and (13) and (23); and for each other pair of vertices there is an edge. Therefore, the number of edges is 12. By Theorem 22, $E(\mathscr{N}\mathscr{B}_{S_3}) \leq \frac{|G|^2 - |G|}{2} = 15$.

Example 27 For $G = S_4$, there are no edges between:

• (12) and (13), (12) and (23), (12) and (24), (12) and (14), (13) and (23), (23) and (34), (23) and (24), (13) and (34), (13) and (14), (34) and (24), (34) and (14), (24) and (14),

• (1432) and (1243), (1432) and (1342), (1432) and (1423), (1432) and (1324), (1243) and (1423), (1243) and (1234), (1243) and (1324), (1342) and (1423), (1342) and (1234), (1342) and (1324), (1234) and (1324),

• (132) and (143), (132) and (124), (123) and (243), (123) and (134), (243) and (142), (243) and (134), (143) and (234), (143) and (124), and, (234) and (124); and for other pairs of vertices, there is an edge. Therefore, the number of edges is 244. By Theorem 22, $|E(\mathscr{N}\mathscr{B}_{S_4})| \leq \frac{|G|^2 - |G|}{2} = 276$.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Erdos P, Turan P. On some problems of a statistical group theory IV. *Acta Math Hungarica*. 1968; 19(3-4): 413-435.

[2] Camina RD, Cocke WL, Thillaisundaram A. The amit-ashurst conjecture for finite metacyclic p-group. *European Journal of Mathematics*. 2023; 9(3): 46. Available from: https://doi.org/10.1007/s40879-023-00644-x.

[3] Solomon L. The solution of equations in groups. *Archiv Der Mathematik [Archive for Mathematics]*. 1969; 20(3): 241-247. Available from: https://doi.org/10.1007/BF01899292.

[4] Hall P. *The Theory of Groups*. New York: Dover Publication, Inc; 1959.

[5] Coxeter HSM, Moses WOJ. *Generators and Relations for Discrete Groups*. Heidelberg: Springer; 1980. Available from: https://doi.org/10.1007/978-3-662-21943-0.

[6] Ash RB. *Basic Abstract Algebra: For Graduate Students and Advanced Undergraduates*. USA: Dover Publications; 2006.

[7] Cahyati ES, Fadhiilah RA, Putra ADCB, Wijayanti IE. Non-braid graphs of ring $\mathbb{Z}_n$. *Jurnal Teori dan Aplikasi Matematika [Journal of Theory and Applications of Mathematics]*. 2022; 6(1): 106-116. Available from: https://doi.org/10.31764/jtam.v6i1.5559.

[8] Wang S, Farahani MR, Kanna M, Jamil MK, Kumar RP. The Wiener index and the hosoya polynomial of the Jahangir graphs. *Applied and Computational Mathematics*. 2016; 5(3): 138-141.

[9] Xu K, Das KC. On Harary index of graphs. *Discrete Applied Mathematics*. 2011; 159(15): 1631-1640. Available from: https://doi.org/10.1016/j.dam.2011.06.003.