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Research Article

Extraction of Solitons in Optical Fibers for the (2+1)-Dimensional Perturbed Nonlinear Schrödinger Equation Via the Improved Modified Extended Tanh Function Technique

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Abstract: The presented research investigates the (2+1)-dimensional perturbed nonlinear Schrödinger model. This model takes into account various effects such as fourth order dispersion, intermodal dispersion, nonlinear dispersion, group velocity dispersion, Kerr nonlinearity and self steepening effects. This model simulates the estimation of optical solitons and a variety of broadcasting networks' transmission in nonlinear fiber optics. The proposed model is studied using the improved modified extended tanh function technique. For the proposed model, several solitons and other solutions are generated. These solutions including {bright, dark and singular} solitons, singular periodic and Jacobi elliptic solutions. By introducing the two- and three-dimensional graphs, the physical behavior of the retrieved solutions is displayed.

Keywords: perturbed nonlinear Schrödinger equation, optical solitons, bright soliton, dark soliton, exact solutions, nonlinear partial differential equations

MSC: 35C05, 35Qxx, 35J10, 34A25

1. Introduction

Analytical methods for solving partial differential equations (PDEs) constitute a rich and diverse toolkit employed across various scientific and engineering disciplines. These analytical methods form the cornerstone of theoretical investigations and are essential for understanding the fundamental behavior of systems described by PDEs [1–5].

Nonlinear optics investigations encompass a multifaceted exploration of optical phenomena, delving into various intriguing aspects of light-matter interactions. At its core, nonlinear optics examines how the behavior of light propagating through a medium changes nonlinearly with respect to its intensity. This field explores phenomena such as harmonic generation, where incident light at one frequency produces new frequencies through nonlinear processes like second-harmonic generation or parametric amplification. Another facet involves nonlinear refraction, where the refractive index of a material varies with the intensity of light, leading to effects like self-focusing or self-phase modulation. Additionally,

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nonlinear optics encompasses studies of optical solitons, stable localized wave packets that maintain their shape while propagating through a nonlinear medium. Understanding these diverse phenomena is crucial for applications ranging from ultrafast optics in telecommunications to the development of advanced laser systems for scientific research and industrial applications [6–10].

The field of nonlinear optics has emerged as a captivating and technologically transformative domain, offering new horizons for the manipulation and transmission of optical signals [11–15]. Within this captivating field, the study of solitons has garnered significant attention due to their remarkable stability and capacity to maintain their shape and energy during propagation [16–20]. Solitons are robust and fascinating entities that defy the dispersive and nonlinear effects typically encountered in wave systems, and they provide a pivotal role in many applications, particularly in optical fiber communications.

In this research paper, we start investigating solitons through nonlinear optical fibers, with a specific focus on the (2+1)-dimensional perturbed nonlinear Schrödinger equation (P-NLSE) in the essence of Kerr law nonlinearity which accounts for the three-dimensional nature of pulse propagation through optical fibers, considering both spatial and temporal dimensions. Modern optical fiber research uses this equation as the foundation for the study of solitons. Solitons are considered the main carrier of information so studying and understanding the dynamic of these pulses will help to develop the telecommunications industries. (P-NLSE) is also used to model other important application in various fields such as plasma physics, semiconductor materials, solid and fluid mechanics due to their remarkable stability properties. The scientific community is interested in studying the P-NLSE. This model was studied in ref [21] by applying the Tanh-Coth method to get various traveling wave solutions. In addition, Jacobi elliptic function scheme was implemented to investigate the P-NLSE in a nano-optical fiber to derive optical solitons and analyze the effects of nonlinearity, spatial dispersion, and linear stability of the equation [22]. In this study, we investigate the (2+1)-dimensional P-NLSE model in nonlinear optical fibers. This model reads as [23]:

$$iq_t + r_1 q_{xx} + r_2 q_{yy} + q|q|^2 - i\left(aq_x - b\left(q\,|q|^2\right)_x - cq\left(\left(|q|^2\right)_x\right)\right) + q_{xxxx} - q_{yyyy} = 0, \tag{1}$$

where $q = q(x,\, y,\, t)$ represents the complex valued wave form. t is a temporal variable while x and y are spatial variables. Self-steepening terms for shorter pulses, intermodal dispersion coefficients, and coefficients of nonlinear dispersion terms are each defined by b, a, and c, respectively. The group velocity dispersion (GVD) term coefficient for the x-direction is denoted as r_1, while for the y-direction, it is represented as r_2. In addition, $q|q|^2$ indicates the normal Kerr nonlinearity whereas q_{xx} and q_{yy} respectively represent the normal GVD and paraxial diffraction.

In this work, the improved modified extended tanh-function method is applied for the proposed model. Various types of solutions are extracted such as bright solitons, dark solitons, singular solitons, singular periodic solutions, Weierstrass elliptic function solutions, hyperbolic solutions, and Jacobi elliptic solutions. The suggested method offers a greater range of options than existing approaches, namely modified simple equation approach [24], Sardar-subequation approach [25], extended modified sub-equation approach [26] and modified Kudryashov approach [27].

This paper has the following structure: In Section 2, we explain the improved modified extended tanh scheme. In Section 3, the proposed method is implemented to secure solitons and other solutions for Eq. (1). In Section 4, graphical representations are introduced for some of obtained solutions. Finally, the work is summarized in Section 5.

2. The proposed technique

This section introduced a briefly description for the improved modified extended tanh function scheme [28, 29]. We suppose a nonlinear partial differential equation (NLPDE) as below:

$$W(q,\, q_x,\, q_y,\, q_t,\, q_{yt},\, q_{xt},\, ...) = 0. \tag{2}$$

To solve Eq. (2), the procedures listed below must be followed :

Procedure (1) Once we define $q(x, y, t) = q(\xi)$ with $\xi = x + y - vt$ in which v denotes the velocity of the propagating wave (assuming $v \neq 0$) Eq. (2) subsequently transforms into an ordinary differential equation (ODE):

$$W(q, q', q'', ...) = 0. \tag{3}$$

Procedure (2) The intended solution for the resulted ordinary differential equation (ODE) is as follows:

$$q(\xi) = a_0 + \sum_{j=1}^{N} a_j z^j + a_{-j} z^{-j}, \tag{4}$$

here N is a positive number to be calculated, a_0, a_j and a_{-j} are real constants and z fulfills the following auxiliary equation:

$$z'(\xi) = \sqrt{d_0 + d_1 z(\xi) + d_2 z(\xi)^2 + d_3 z(\xi)^3 + d_4 z(\xi)^4}, \tag{5}$$

where $z' = dz/d\xi$. The constants $d_j (0 \leq j \leq 4)$ are freely chosen constants with the restriction $d_4 \neq 0$.

Procedure (3) The integer N can be evaluated by applying the balancing rule on the resulted ODE of Eq. (3).

Procedure (4) Substituting by the assumed solution of Eq. (4) along with the auxiliary equation of Eq. (5) into the derived ODE of Eq. (3), and then setting equating the coefficients of z^j to zero, a system of nonlinear equations is provided. In order to solve this system, it is possible to use software packages of Mathematica.

Procedure (5) By setting the constants d_0, d_1, d_2, d_3, d_4 with different possible values, one can get various general solutions for Eq. (5). Substituting with these general solutions along with the determined constants v and a_j into the proposed solution of Eq. (4), one can get various solitons and other solutions for the NLPDE of Eq. (2).

Comparing with other techniques such as unified Riccati equation expansion [30], Kudryashov's scheme [31, 32], the direct mapping method [33] and F-expansion method [34], this method give various types of solutions such as bright, dark, singular solitons. In addition, other mathematical solutions such as exponential, Jacobi elliptic, Weierstrass elliptic, plane wave, rational type, hyperbolic, periodic and singular periodic solutions can be extracted. However, when N, which is evaluated via the balance rule, is large, it will give a more complex system that is difficult to solve.

3. Exact solutions construction

Our goal in this section is to derive exact solutions for Eq. (1) in the form:

$$q(x, y, t) = U(\xi)\, e^{i(x+y+wt)}, \quad \xi = x + y - vt. \tag{6}$$

Substituting by Eq. (6) into Eq. (1) then separating real and imaginary parts leads to: Real part:

$$(r_1 + r_2) U(\xi)'' + (a - w - r_1 - r_2) U(\xi) + (1 - b) U(\xi)^3 = 0. \tag{7}$$

Imaginary part:

$$(2(r_1+r_2)-(a+v))U(\xi)'+(3b+2c)U(\xi)^2U(\xi)'=0. \tag{8}$$

Equating the coefficients of imaginary parts to zero, we get:

$$\begin{aligned} &2(r_1+r_2)=(a+v), \\ &3b=-2c. \end{aligned} \tag{9}$$

Let:

$$\begin{aligned} &r_1+r_2=R, \\ &a-R-w=Q. \end{aligned} \tag{10}$$

Then, the real part becomes:

$$(1-b)U(\xi)^3+QU(\xi)+RU(\xi)''=0. \tag{11}$$

To implement the proposed method on Eq. (11), N should first be evaluated by balancing U^3 with U''. We achieve a value of N that equals 1. Thus, the solution of Eq. (11) is given as follows:

$$U(\xi)=a_0+a_1z+\frac{a_{-1}}{z}. \tag{12}$$

By employing the procedures outlined in procedures (4), (5) from the preceding section, the following outcomes are obtained:

Case 1 $d_0=d_1=d_3=0$.

Result 1

$$a_0=0,\ a_{-1}=0,\ a_1=\sqrt{\frac{2R}{1-b}},\ Q=-d_2R.$$

Then, we have

$$q(x,\ y,\ t)=\sqrt{\frac{2d_2R}{(b-1)d_4}}\operatorname{sech}\left(\sqrt{d_2}(x+y-vt)\right)e^{i(x+y+wt)}, \tag{13}$$

$$q(x,\ y,\ t)=\sqrt{\frac{2d_2R}{(b-1)d_4}}\sec\left(\sqrt{-d_2}(x+y-vt)\right)e^{i(x+y+wt)}, \tag{14}$$

$$q(x,\ y,\ t)=\sqrt{\frac{2d_2R}{(b-1)d_4}}\csc\left(\sqrt{-d_2}(x+y-vt)\right)e^{i(x+y+wt)}. \tag{15}$$

A bright soliton is represented by Eq. (13) whereas two singular periodic solutions are represented by Eq. (14) and Eq. (15).

Case 2 $d_1=d_3=0$.

Result 1

$$a_0=0,\ a_{-1}=0,\ Q=-d_2R,\ a_1=\sqrt{\frac{2d_4R}{1-b}}.$$

So, we have :

$$q(x,\ y,\ t)=\sqrt{\frac{d_2R}{1-b}}\tan\left(\sqrt{\frac{d_2}{2}}(x+y-vt)\right)e^{i(x+y+wt)}, \tag{16}$$

$$q(x,\ y,\ t)=\sqrt{-\frac{d_2R}{1-b}}\tanh\left(\sqrt{\frac{d_2}{2}}(x+y-vt)\right)e^{i(x+y+wt)}. \tag{17}$$

A singular periodic solution is represented by Eq. (16) while a dark soliton is represented by Eq. (17).

Result 2

$$a_0=0,\ a_1=0,\ a_{-1}=\frac{d_2\sqrt{-R}}{\sqrt{2(1-b)d_4}},\ Q=-d_2R.$$

Therefore, we get the results below:

$$q(x,\ y,\ t)=\sqrt{-\frac{d_2R}{1-b}}\cot\left(\sqrt{\frac{d_2}{2}}(x+y-vt)\right)e^{i(x+y+wt)}, \tag{18}$$

$$q(x,\ y,\ t)=\sqrt{\frac{d_2R}{1-b}}\coth\left(\sqrt{\frac{d_2}{2}}(x+y-vt)\right)e^{i(x+y+wt)}. \tag{19}$$

A singular periodic solution is represented by Eq. (18) while a singular soliton is represented by Eq. (19).

Case 3 $d_1=d_3=0$.

Result 1 When $d_0 = 1$, $d_2 = -\left(m^2+1\right)$, $d_4 = m^2$.
Result 1.1

$$a_0 = 0,\ a_{-1} = 0,\ a_1 = \frac{m\sqrt{2R}}{\sqrt{b-1}},\ Q = \frac{a_1^2(b-1)\left(m^2+1\right)}{2m^2}.$$

So, the following solutions in the form of Jacobi elliptic functions are obtained:

$$q(x,y,t) = \frac{\sqrt{2}m\sqrt{R}\mathrm{cd}((x+y-vt)|m)}{\sqrt{b-1}}e^{i(x+y+wt)}, \tag{20}$$

$$q(x,\ y,\ t) = -\frac{\sqrt{2}m\sqrt{R}\mathrm{sn}((x+y-vt)|m)}{\sqrt{b-1}}e^{i(x+y+wt)}. \tag{21}$$

Result 1.2

$$a_0 = 0,\ a_1 = 0,\ a_{-1} = \frac{\sqrt{2}\sqrt{R}}{\sqrt{b-1}},\ Q = \frac{1}{2}a_{-1}^2(b-1)\left(m^2+1\right).$$

Then, we obtain the following Jacobi elliptic functions solutions as:

$$q(x,\ y,\ t) = \frac{\sqrt{2}\sqrt{R}\mathrm{dc}((x+y-vt)|m)}{\sqrt{b-1}}e^{i(x+y+wt)}, \tag{22}$$

$$q(x,\ y,\ t) = -\frac{\sqrt{2}\sqrt{R}\mathrm{ns}((x+y-vt)|m)}{\sqrt{b-1}}e^{i(x+y+wt)}. \tag{23}$$

Result 1.3

$$a_0 = 0,\ a_{-1} = \frac{\sqrt{2}\sqrt{R}}{\sqrt{b-1}},\ a_1 = \frac{m\sqrt{2R}}{\sqrt{b-1}},\ Q = \frac{a_1^2(b-1)\left(m^2+6m+1\right)}{2m^2}.$$

Subsequently, we attain the following Jacobi elliptic solutions:

$$q(x,\ y,\ t) = \left(\sqrt{\frac{2R}{b-1}}(m\mathrm{cd}((x+y-vt)|m)+\mathrm{dc}((x+y-vt)|m))\right)e^{i(x+y+wt)}, \tag{24}$$

$$q(x,\ y,\ t)=\left(-\sqrt{\frac{2R}{b-1}}(\mathrm{ns}((x+y-vt)|m)+m\mathrm{sn}((x+y-vt)|m))\right)e^{i(x+y+wt)}. \tag{25}$$

Result 2 When $d_0=m^2$, $d_2=-m^2-1$, $d_4=1$.
Result 2.1

$$a_0=0,\ a_{-1}=0,\ a_1=\frac{\sqrt{2}\sqrt{R}}{\sqrt{b-1}},\ Q=\frac{1}{2}a_1^2(b-1)\left(m^2+1\right).$$

So, the below Jacobi elliptic functions are achieved:

$$q(x,\ y,\ t)=-\frac{\sqrt{2}\sqrt{R}\mathrm{ns}((x+y-vt)|m)}{\sqrt{b-1}}e^{i(x+y+wt)}, \tag{26}$$

$$q(x,\ y,\ t)=\frac{\sqrt{2}\sqrt{R}\mathrm{dc}((x+y-vt)|m)}{\sqrt{b-1}}e^{i(x+y+wt)}. \tag{27}$$

Result 2.2

$$a_0=0,\ a_1=0,\ a_{-1}=\frac{m\sqrt{2R}}{\sqrt{b-1}},\ Q=\frac{a_{-1}^2(b-1)\left(m^2+1\right)}{2m^2}.$$

Then, the following Jacobi elliptic functions solutions are derived as:

$$q(x,\ y,\ t)=-\frac{\sqrt{2}m\sqrt{R}\mathrm{sn}((x+y-vt)|m)}{\sqrt{b-1}}e^{i(x+y+wt)}, \tag{28}$$

$$q(x,\ y,\ t)=\frac{\sqrt{2}m\sqrt{R}\mathrm{cd}((x+y-vt)|m)}{\sqrt{b-1}}e^{i(x+y+wt)}. \tag{29}$$

Result 2.3

$$a_0=0,\ a_{-1}=\frac{m\sqrt{2R}}{\sqrt{b-1}},\ a_1=\frac{\sqrt{2}\sqrt{R}}{\sqrt{b-1}},\ Q=\frac{1}{2}a_1^2(b-1)\left(m^2+6m+1\right).$$

As a result, we obtain the following Jacobi elliptic solutions:

$$q(x,\ y,\ t)=\left(-\sqrt{\frac{2R}{b-1}}(\mathrm{ns}((x+y-vt)|m)+m\mathrm{sn}((x+y-vt)|m))\right)e^{i(x+y+wt)}, \tag{30}$$

$$q(x,\ y,\ t)=\left(\sqrt{\frac{2R}{b-1}}(m\mathrm{cd}((x+y-vt)|m)+\mathrm{dc}((x+y-vt)|m))\right)e^{i(x+y+wt)}. \tag{31}$$

Result 3 When $d_0=m^2-1$, $d_2=2-m^2$, $d_4=-1$.
Result 3.1

$$a_0=0,\ a_{-1}=0,\ a_1=\frac{\sqrt{2}\sqrt{R}}{\sqrt{1-b}},\ Q=-\frac{1}{2}a_1^2(b-1)\left(m^2-2\right).$$

The following Jacobi elliptic solution is obtained:

$$q(x,\ y,\ t)=\frac{\sqrt{2}\sqrt{R}\mathrm{dn}((x+y-vt)|m)}{\sqrt{1-b}}e^{i(x+y+wt)}. \tag{32}$$

Result 3.2

$$a_0=0,\ a_1=0,\ a_{-1}=\frac{\sqrt{2\left(1-m^2\right)R}}{\sqrt{1-b}},\ Q=\frac{a_{-1}^2(b-1)\left(m^2-2\right)}{2\left(m^2-1\right)}.$$

So, the below Jacobi elliptic solution is achieved:

$$q(x,\ y,\ t)=\frac{\sqrt{2}\sqrt{\left(1-m^2\right)R}\mathrm{nd}((x+y-vt)|m)}{\sqrt{1-b}}e^{i(x+y+wt)}. \tag{33}$$

Result 3.3

$$a_0=0,\ a_{-1}=\frac{\sqrt{2\left(1-m^2\right)R}}{\sqrt{1-b}},\ a_1=\frac{\sqrt{2}\sqrt{R}}{\sqrt{1-b}},$$

$$Q=\frac{1}{2}\left(a_1^2(-b)m^2+6a_1b\sqrt{a_1^2\left(-\left(m^2-1\right)\right)}+2a_1^2b+a_1^2m^2-6a_1\sqrt{a_1^2\left(-\left(m^2-1\right)\right)}-2a_1^2\right).$$

The following Jacobi elliptic solution is obtained:

$$q(x,\ y,\ t)=\sqrt{\frac{2R}{1-b}}\left(\mathrm{dn}((x+y-vt)|m)+\sqrt{1-m^2}\mathrm{nd}((x+y-vt)|m)\right)e^{i(x+y+wt)}. \tag{34}$$

Result 4 When $d_0 = m^4 - 2m^3 + m^2$, $d_2 = -\dfrac{4}{m}$, $d_4 = -m^2 + 6m - 1$.

Result 4.1

$$a_0 = 0,\ a_{-1} = 0,\ a_1 = \frac{\sqrt{2\left(m^2 - 6m + 1\right)R}}{\sqrt{1-b}},\ Q = -\frac{2\left(a_1^2 b - a_1^2\right)}{m\left(m^2 - 6m + 1\right)}.$$

The following Jacobi elliptic solution is obtained:

$$q(x,\ y,\ t) = \frac{\sqrt{2}m\sqrt{\left(m^2 - 6m + 1\right)R}\operatorname{cn}((x+y-vt)|m)\operatorname{dn}((x+y-vt)|m)}{\sqrt{1-b}\left(m\operatorname{sn}(x+y-vt|m)^2 + 1\right)} e^{i(x+y+wt)}. \tag{35}$$

Result 4.2

$$a_0 = 0,\ a_1 = 0,\ Q = \frac{4R}{m},\ a_{-1} = \frac{\sqrt{2}\left(m^2 - m\right)\sqrt{R}}{\sqrt{b-1}}.$$

Subsequently the Jacobi elliptic solution is obtained:

$$q(x,\ y,\ t) = \frac{\sqrt{2}\left(m^2 - m\right)\sqrt{R}\operatorname{nc}((x+y-vt)|m)\operatorname{nd}((x+y-vt)|m)\left(m\operatorname{sn}((x+y-vt)|m)^2 + 1\right)}{\sqrt{b-1}m} e^{i(x+y+wt)}. \tag{36}$$

Result 4.3

$$a_0 = 0,\ a_{-1} = \frac{\sqrt{2}\left(m^2 - m\right)\sqrt{R}}{\sqrt{b-1}},$$

$$Q = \frac{2\left(-3im^5 + 21im^4 - 21im^3 + 3im^2 + 2\sqrt{m^2 - 6m + 1}\right)R}{m\sqrt{m^2 - 6m + 1}},$$

$$a_1 = \frac{\sqrt{2}\sqrt{m^2(-R) + 6mR - R}}{\sqrt{b-1}}.$$

Then, we have a Jacobi elliptic solution:

$$q(x,\ y,\ t) = \frac{\sqrt{\frac{2R}{1-b}}\left(m\sqrt{m^2 - 6m + 1}\operatorname{cn}((x+y-vt)|m)\operatorname{dn}((x+y-vt)|m)\right)}{m\operatorname{sn}((x+y-vt)|m)^2 + 1} e^{i(x+y+wt)} +$$

$$(1-m)\sqrt{\frac{2R}{1-b}}\operatorname{nc}((x+y-vt)|m)\operatorname{nd}((x+y-vt)|m)\left(m\operatorname{sn}((x+y-vt)|m)^2 + 1\right) e^{i(x+y+wt)}.$$

Result 5 When $d_0 = \frac{1}{4}\left(m^2-1\right), d_2 = \frac{1}{2}\left(m^2+1\right), d_4 = \frac{1}{4}\left(m^2-1\right)$.
Result 5.1

$$a_0 = 0,\ a_{-1} = 0,\ a_1 = \frac{\sqrt{(1-m^2)R}}{\sqrt{2(b-1)}},\ Q = -\frac{a_1^2(b-1)\left(m^2+1\right)}{m^2-1}.$$

The following Jacobi elliptic solution is obtained:

$$q(x,\ y,\ t) = \frac{\sqrt{(1-m^2)R}\mathrm{dn}((x+y-vt)|m)}{\sqrt{2}\sqrt{b-1}(1\pm -m\mathrm{sn}((x+y-vt)|m))}e^{i(x+y+wt)}, \tag{37}$$

$$q(x,\ y,\ t) = \frac{\sqrt{(1-m^2)R}(-m\mathrm{sd}((x+y-vt)|m)\pm\mathrm{nd}((x+y-vt)|m))}{\sqrt{2}\sqrt{b-1}}e^{i(x+y+wt)}. \tag{38}$$

Result 5.2

$$a_0 = 0,\ a_1 = 0,\ a_{-1} = \frac{\sqrt{(1-m^2)R}}{\sqrt{2}\sqrt{b-1}},\ Q = -\frac{a_{-1}^2(b-1)\left(m^2+1\right)}{m^2-1}.$$

Next, we have Jacobi elliptic solutions, which are below:

$$q(x,\ y,\ t) = \frac{\sqrt{(1-m^2)R}\mathrm{nd}((x+y-vt)|m)(1\pm -m\mathrm{sn}((x+y-vt)|m))}{\sqrt{2}\sqrt{b-1}}e^{i(x+y+wt)}, \tag{39}$$

$$q(x,\ y,\ t) = \frac{\sqrt{(1-m^2)R}}{\sqrt{2}\sqrt{b-1}(-m\mathrm{sd}((x+y-vt)|m)\pm\mathrm{nd}((x+y-vt)|m))}e^{i(x+y+wt)}. \tag{40}$$

Case 4 $d_0 = d_1 = 0$.
Result 1

$$a_{-1} = 0,\ Q = a_0^2(b-1),\ a_0 = \frac{\sqrt{-d_2R}}{\sqrt{2(1-b)}},\ d_3 = \frac{2a_0a_1(b-1)}{R},\ a_1 = \frac{\sqrt{2d_4R}}{\sqrt{b-1}}.$$

Then, we get:

$$q(x,\ y,\ t) = \frac{\sqrt{d_2(-R)}\left(1-\left(\tanh\left(\frac{\sqrt{d_2}(x+y-vt)}{\sqrt{2}}\right)+1\right)\right)}{\sqrt{2}\sqrt{1-b}}e^{i(x+y+wt)}. \tag{41}$$

$$q(x,y,t)=\frac{\sqrt{d_2(-R)}\left(1-\left(\coth\left(\frac{\sqrt{d_2}(x+y-vt)}{\sqrt{2}}\right)+1\right)\right)}{\sqrt{2}\sqrt{1-b}}e^{i(x+y+wt)}. \tag{42}$$

A dark soliton is represented by Eq. (41) whereas a singular soliton is represented by Eq. (42).

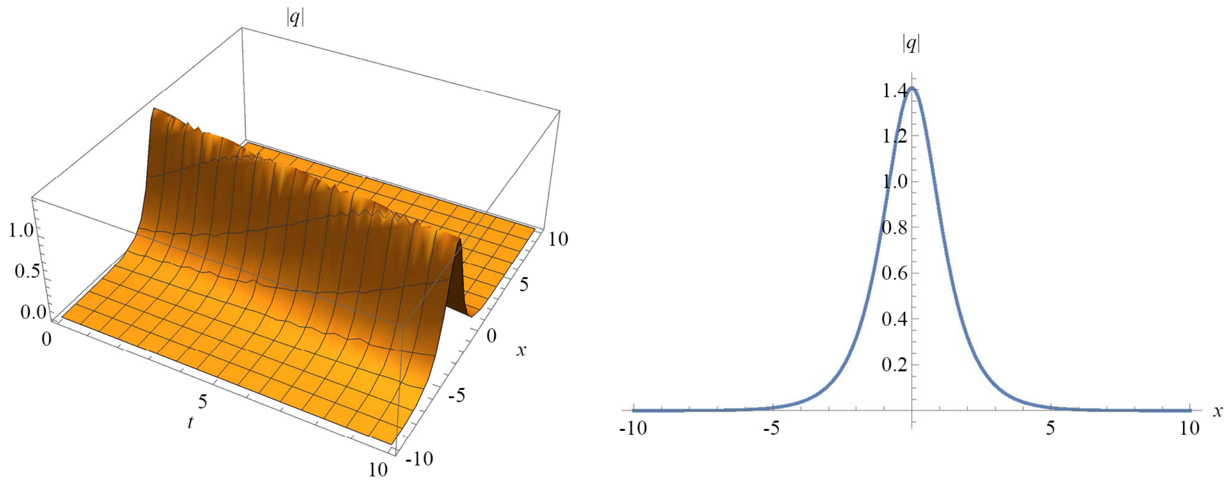

Figure 1. Bright soliton of Eq. (13) for $R=2$, $b=-0.8$, $d_2=1.17$, $v=-0.3$, $d_4=-1.3$, $y=0$

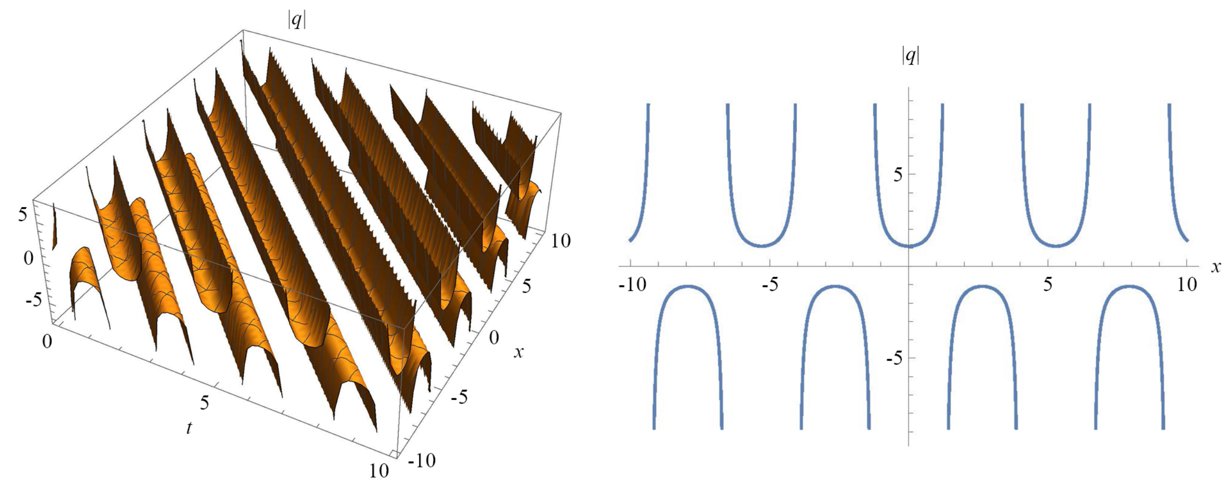

Figure 2. Singular periodic solution of Eq. (14) for $R=-0.5$, $b=-0.3$, $d_2=-1.41$, $v=-2$, $d_4=-0.9$, $y=0$

4. Illustrations of the solutions graphically

Graphs in both two-dimensional (2D) and three-dimensional (3D) formats are presented to illustrate the physical attributes of various specific solutions. Figure 1 illustrates a bright soliton of Eq. (13) for the parameters $R=2$, $b=-0.8$,

$d_2 = 1.17$, $v = -0.3$, $d_4 = -1.3$, $y = 0$. The bright soliton solution is characterized by high intensity peak at the center. This type of solution is very important in nonlinear fiber optics and optical telecommunication systems. Figure 2 displays a singular periodic solution of Eq. (14) for the parameters $R = -0.5$, $b = -0.3$, $d_2 = -1.41$, $v = -2$, $d_4 = -0.9$, $y = 0$. This solution displays a periodically repeating wave with a point of singularity, showing valuable insights into the behavior of nonlinear systems with recurring singularities. Figure 3 demonstrates a dark soliton solution of Eq. (17) for the parameters $R = -0.4$, $b = -0.9$, $d_2 = 0.8$, $v = -0.4$, $y = 0$. The dark soliton solution is characterized by low intensity peak at the center. This type of solution is very important in fiber lasers as they are stable in noise circumstances and not much susceptible to loss. It was found that dark soliton is more suitable than bright soliton when used in optical communications. Figure 4 illustrates a singular soliton of Eq. (19) for the parameters $R = 0.3$, $b = -0.9$, $d_2 = 0.6$, $v = 0.05$, $y = 0$. This solution represents a rare phenomenon in nonlinear physics, characterized by a point of singularity or divergence in intensity. It captures the abrupt change at the point, offering insight into the interplay of nonlinearity and dispersion in forming exotic solitary waves.

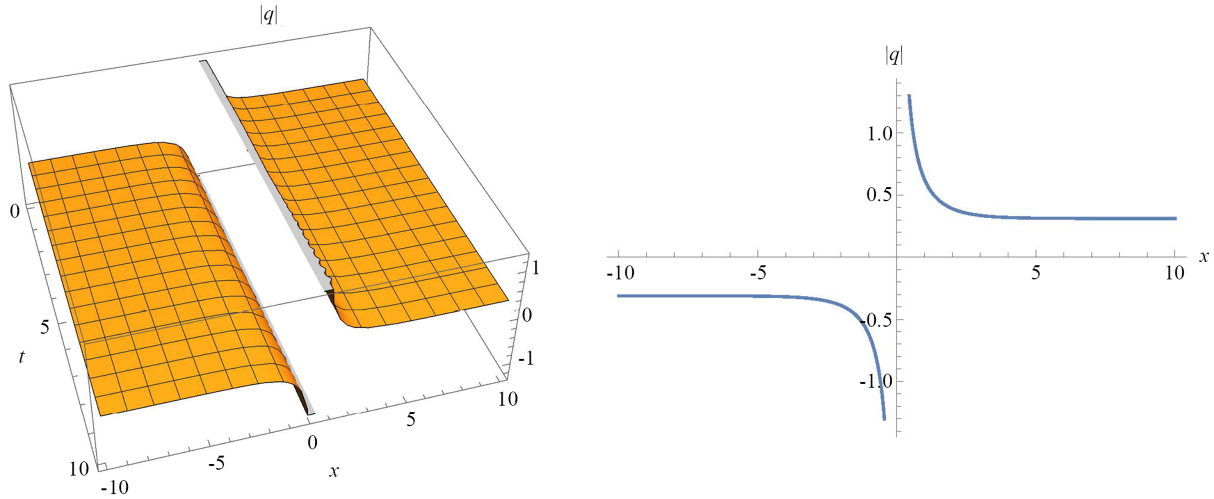

Figure 3. Dark soliton of Eq. (17) for $R = -0.4$, $b = -0.9$, $d_2 = 0.8$, $v = -0.4$, $y = 0$

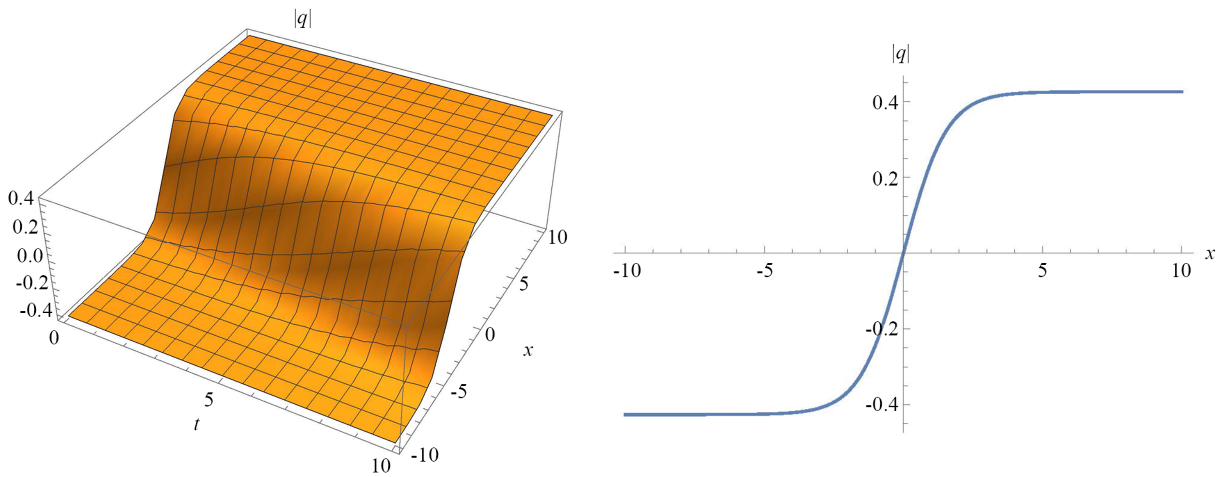

Figure 4. Singular soliton of Eq. (19) for $R = 0.3$, $b = -0.9$, $d_2 = 0.6$, $v = 0.05$, $y = 0$

5. Conclusion

In our study, we employed the improved modified extended tanh function method to effectively analyze the (2+1)-dimensional perturbed nonlinear Schrödinger equation incorporating Kerr law nonlinearity and fourth-order dispersion. Through this method, we were able to effectively secure a various types of solutions. These solutions including {bright, dark and singular} solitons, singular periodic and Jacobi elliptic solutions. To visually depict the characteristics of some of these exact solutions, 3D and 2D graphical representations are introduced. The extracted solitons play an important role in the development of the telecommunication industry, as this type of wave can propagate over very long distances while retaining its shape and speed, and this proof that a dedicated balance has occurred between the dispersion and the nonlinear effects. In the future work, stochastic P-NLSE can be studied to consider the effect of the noise on the extracted solutions.

Confilict of interest

The authors declare no competing financial interest.

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