

Research Article

Revisitation of “Implicit Quiescent Optical Solitons with Complex Ginzburg-Landau Equation Having Nonlinear Chromatic Dispersion”: Linear Temporal Evolution

Abdullahi Rashid Adem¹, Anjan Biswas^{2,3,4,5}, Yakup Yildirim^{6,7,8*}, Ali Saleh Alshomrani³

¹Department of Mathematical Sciences, University of South Africa, South Africa

²Department of Mathematics and Physics, Grambling State University, Grambling, LA, USA

³Mathematical Modeling and Applied Computation Research Group, Center of Modern Mathematical Sciences and their Applications, Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia

⁴Department of Applied Sciences, Cross-Border Faculty of Humanities, Economics and Engineering, Dunarea de Jos University of Galati, Galati, Romania

⁵Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Medunsa, South Africa

⁶Department of Computer Engineering, Biruni University, Istanbul, Turkey

⁷Mathematics Research Center, Near East University, Nicosia, Cyprus

⁸Faculty of Arts and Sciences, University of Kyrenia, Kyrenia, Cyprus

E-mail: yyildirim@biruni.edu.tr

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Abstract: The current paper revisits the retrieval of quiescent optical solitons for the complex Ginzburg-Landau equation with nonlinear chromatic dispersion and several forms of self-phase modulation. The results obtained in this work consist of explicit or implicit quiescent optical solitons, unlike the previous report, which were expressed in terms of quadratures. Additionally, this study addresses two additional forms of self-phase modulation: the saturating law and the exponential law. This exploration yields quiescent optical solitons expressed in terms of quadratures for the first time.

Keywords: nonlinear dispersion, stationary solitons

MSC: 78A60

1. Introduction

Optical solitons are the fundamental building blocks of the telecommunications industry [1–3]. These soliton structures result from a delicate balance between chromatic dispersion (CD) and self-phase modulation (SPM) [4–6]. Catastrophic consequences can ensue if this balance is compromised [7, 8]. The global telecommunications industry would be profoundly negatively impacted if this balance were somehow affected [9, 10]. One of the negative effects would be the complete stalling of solitons or bit carriers during transcontinental or transoceanic communication [11–13]. One of the sources of losing this balance is when the CD becomes nonlinear [14–16]. This can possibly occur when optical fibers are roughly handled during installation underground or underwater [17–19]. Such mishandling would lead to the stalling of soliton transmission, thus resulting in the formation of quiescent optical solitons [20–22].

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The current paper will address the formation and mathematical structures of quiescent optical solitons resulting from nonlinear CD [23, 24]. Several models have studied the dynamics of soliton transmission across intercontinental distances [25–28]. The model focused on in this paper is the complex Ginzburg-Landau equation (CGLE). This model will be explored with a wide range of self-phase modulation (SPM) structures when nonlinear CD is considered. Incidentally, the formation of quiescent optical solitons has been studied in the past using the CGLE for nonlinear CD and with different forms of SPM structures [10]. However, the results obtained from it are expressed in terms of quadratures, and thus they are not considered quite formidable. The current paper refocuses on the same model, and the recovered results consist of explicit and/or implicit forms of quiescent optical solitons, which are also classified. The details of Lie symmetry analysis and the retrieval of quiescent solitons are exhibited in the rest of the paper.

Lie symmetry analysis offers a systematic approach for analyzing differential equations, providing insights into their solutions. Compared to recent methods, it often yields explicit or implicit solutions, allowing for clearer interpretation. However, its effectiveness heavily depends on the equation's symmetries, limiting its applicability to certain types of equations. Other recent methods may offer broader applicability but might not always provide as clear or direct solutions. Therefore, Lie symmetry analysis stands out for its rigor and clarity in specific cases but may not be as versatile as some newer approaches.

1.1 Governing model

The starting model is the CGLE, which is presented below in its dimensionless form [10, 12, 24]

$$iq_t + a(|q|^n q)_{xx} + F(|q|^2)q = \alpha \frac{|q_x|^2}{q^*} + \frac{\beta}{4|q|^2 q^*} \left[2|q|^2 (|q|^2)_{xx} - \left\{ (|q|^2)_x \right\}^2 \right] + \gamma q. \quad (1)$$

Equation (1) is a nonlinear partial differential equation with the dependent variable being $q(x, t)$, which is a complex-valued function. The independent variables are x and t , representing the spatial and temporal coordinates, respectively. The coefficient a represents nonlinear CD, with the parameter n denoting the degree of nonlinearity for CD. The functional F represents the refractive index change resulting from the intensity of light, introducing the structure of self-phase modulation (SPM). The parameters α and β represent additional nonlinear structures for the model, and γ is the detuning parameter. The coefficients a , α , β , and γ are all real-valued constants, while in the first term, $i = \sqrt{-1}$.

This work revisits a previously published study [10]. The justification for this revisitation lies in the fact that in the original publication, the results were expressed in terms of quadratures, with no recovery of functional forms. However, the current paper represents a significant improvement over the previous work, as the results are now presented in terms of explicit or implicit functions.

In the specific model being discussed, which includes nonlinear CD, soliton solutions that move or propagate, known as mobile solitons, are not supported. Instead, the model only allows for the existence of quiescent optical solitons. Quiescent optical solitons are stationary solutions of the equation, meaning they remain at a fixed position without propagating. This limitation implies that the dynamics described by the model are such that the solitons cannot move under the influence of the considered nonlinear CD. To recover the structure of such solitons, the following solution structure is chosen:

$$q(x, t) = \phi(x) e^{i\lambda t}, \quad (2)$$

where $\phi(x)$ is the amplitude component of the soliton, while λ is the wave number. Upon substituting (2) into (1), the corresponding ordinary differential equation (ODE) that emerges is:

$$a(n+1)\phi^n(x) \left[n \{ \phi'(x) \}^2 + \phi(x)\phi''(x) \right] - \phi^2(x) \{ \lambda + \gamma - F(\phi^2) \} - \alpha \{ \phi'(x) \}^2 - \beta \phi(x)\phi''(x) = 0. \quad (3)$$

It is this ODE that will be analyzed in detail for the range of SPM studied in this paper. Subsequently, these ODEs will be integrated to locate the quiescent optical solitons that emerge from them. These solitons will then be classified and presented. The details are exhibited in the subsequent section and its subsections.

2. Mathematical analysis

The formulation of quiescent optical solitons for a wide range of SPM structures will be analyzed. Both implicit and explicit quiescent optical solitons are derived using the Lie symmetry approach. The details are outlined in the subsequent subsections.

2.1 Kerr law

For Kerr law of SPM, the structure of the functional F is:

$$F(|q|^2) = b|q|^2. \quad (4)$$

Therefore, the model given by (1) reduces to:

$$iq_t + a(|q|^n q)_{xx} + |q|^2 q = \alpha \frac{|q_x|^2}{q^*} + \frac{\beta}{4|q|^2 q^*} \left[2|q|^2 (|q|^2)_{xx} - \left\{ (|q|^2)_x \right\}^2 \right] + \gamma q. \quad (5)$$

For Kerr law, the ODE reduces to

$$a(n+1)\phi^n(x) \left[n \{ \phi'(x) \}^2 + \phi(x)\phi''(x) \right] + \phi^2(x) \{ b\phi^2(x) - \gamma - \lambda \} - \alpha \{ \phi'(x) \}^2 - \beta \phi(x)\phi''(x) = 0. \quad (6)$$

For the integrability of equation (6), the following specific value of n is chosen:

$$n = -1. \quad (7)$$

This simplifies the governing model for Kerr law given by (5) to:

$$\begin{aligned} iq_t + a \left(\frac{q}{|q|} \right)_{xx} + b|q|^2 q \\ = \alpha \frac{|q_x|^2}{q^*} + \frac{\beta}{4|q|^2 q^*} \left[2|q|^2 (|q|^2)_{xx} - \left\{ (|q|^2)_x \right\}^2 \right] + \gamma q, \end{aligned} \quad (8)$$

and consequently (6) shrinks to

$$\phi^2(x) \{b\phi^2(x) - \gamma - \lambda\} - \alpha \{\phi'(x)\}^2 - \beta\phi(x)\phi''(x) = 0. \quad (9)$$

Equation (9) admits a single Lie point symmetry, namely $\partial/\partial x$. Implementing this symmetry in the integration process of the ODE (9) leads to the following solution to ODE given by (9):

$$\phi(x) = \left[\frac{(\alpha + 2\beta)(\gamma + \lambda)}{b(\alpha + \beta)} \right]^{\frac{1}{2}} \operatorname{sech} \left(x \sqrt{-\frac{\gamma + \lambda}{\alpha + \beta}} \right). \quad (10)$$

Hence, the desired bright quiescent optical 1-soliton solution to (8) is:

$$q(x, t) = \left[\frac{(\alpha + 2\beta)(\gamma + \lambda)}{b(\alpha + \beta)} \right]^{\frac{1}{2}} \operatorname{sech} \left(x \sqrt{-\frac{\gamma + \lambda}{\alpha + \beta}} \right) e^{i\lambda t}. \quad (11)$$

The existence of such bright quiescent optical solitons is guaranteed provided

$$(\alpha + \beta)(\gamma + \lambda) < 0, \quad (12)$$

and

$$b(\alpha + \beta)(\alpha + 2\beta)(\gamma + \lambda) > 0 \quad (13)$$

hold.

2.2 Power law

For power-law of SPM, the functional F is written as

$$F(|q|^2) = b|q|^{2m}, \quad (14)$$

where m represents the power-law nonlinearity parameter and b is a real-valued constant. With this structural format of the functional F , equation (1) reduces to:

$$iq_t + a(|q|^n q)_{xx} + b|q|^{2m} q = \alpha \frac{|q_x|^2}{q^*} + \frac{\beta}{4|q|^2 q^*} \left[2|q|^2 (|q|^2)_{xx} - \left\{ (|q|^2)_x \right\}^2 \right] + \gamma q. \quad (15)$$

For power-law of SPM, the ODE (3) is

$$\begin{aligned}
 & a(n+1)\phi^n(x) \left[n \{ \phi'(x) \}^2 + \phi(x)\phi''(x) \right] + b\phi^{2m+2}(x) \\
 & - \left[\alpha \{ \phi'(x) \}^2 + \beta\phi(x)\phi''(x) + (\gamma + \lambda)\phi^2(x) \right] = 0.
 \end{aligned} \tag{16}$$

For integrability of (15), the same choice of n given by (7) is selected. This modifies (15) to:

$$\begin{aligned}
 & iq_t + a \left(\frac{q}{|q|} \right)_{xx} + b|q|^{2m}q \\
 & = \alpha \frac{|q_x|^2}{q^*} + \frac{\beta}{4|q|^2 q^*} \left[2|q|^2 (|q|^2)_{xx} - \left\{ (|q|^2)_x \right\}^2 \right] + \gamma q,
 \end{aligned} \tag{17}$$

while the ODE (16) reduces to:

$$b\phi^{2m+2}(x) - \alpha \{ \phi'(x) \}^2 - \beta\phi(x)\phi''(x) - (\gamma + \lambda)\phi^2(x) = 0. \tag{18}$$

The above equation admits the same single Lie point symmetry, namely $\partial/\partial x$. This symmetry, upon its application to the integration process, leads to the following solution of the ODE

$$\phi(x) = \left[\frac{\{ \alpha + (m+1)\beta \} (\gamma + \lambda)}{b(\alpha + \beta)} \right]^{\frac{1}{2m}} \operatorname{sech}^{\frac{1}{m}} \left(mx \sqrt{-\frac{\gamma + \lambda}{\alpha + \beta}} \right), \tag{19}$$

and therefore the bright quiescent optical 1-soliton solution would be

$$q(x, t) = \left[\frac{\{ \alpha + (m+1)\beta \} (\gamma + \lambda)}{b(\alpha + \beta)} \right]^{\frac{1}{2m}} \operatorname{sech}^{\frac{1}{m}} \left(mx \sqrt{-\frac{\gamma + \lambda}{\alpha + \beta}} \right) e^{i\lambda t}. \tag{20}$$

The existence of such a bright soliton is guaranteed by virtue of (12) and

$$b(\gamma + \lambda)(\alpha + \beta) \{ \alpha + (m+1)\beta \} > 0. \tag{21}$$

It needs to be noted that upon setting $m = 1$, the results from this section collapse to the ones derived in the previous section that is with Kerr law of SPM.

2.3 Parabolic (cubic-quintic) law

For parabolic law of SPM, the functional F takes the form

$$F(|q|^2) = b_1 |q|^2 + b_2 |q|^4, \tag{22}$$

where b_j for $j = 1, 2$ are real-valued constants. Thus the CGLE (1) reduces to

$$\begin{aligned}
 & iq_t + a(|q|^n q)_{xx} + b_1 |q|^2 q + b_2 |q|^4 q \\
 &= \alpha \frac{|q_x|^2}{q^*} + \frac{\beta}{4|q|^2 q^*} \left[2|q|^2 (|q|^2)_{xx} - \left\{ (|q|^2)_x \right\}^2 \right] + \gamma q.
 \end{aligned} \tag{23}$$

The ODE (3) for parabolic law of SPM is

$$\begin{aligned}
 & a(n+1)\phi^n(x) \left[n \{ \phi'(x) \}^2 + \phi(x)\phi''(x) \right] + b_1 \phi^4(x) + b_2 \phi^6(x) \\
 & - \alpha \phi(x) \{ \phi'(x) \}^2 - \beta \phi(x)\phi''(x) - (\gamma + \lambda) \phi^2(x) = 0.
 \end{aligned} \tag{24}$$

For integrability of (23), in addition to choosing the value of n as given by (7), one needs to select

$$\alpha = 1, \tag{25}$$

$$\beta = 1, \tag{26}$$

and

$$\gamma = -1. \tag{27}$$

Therefore, the governing model (23) modifies to

$$\begin{aligned}
 & iq_t + a \left(\frac{q}{|q|} \right)_{xx} + (b_1 |q|^2 + b_2 |q|^4) q \\
 &= \frac{|q_x|^2}{q^*} + \frac{1}{4|q|^2 q^*} \left[2|q|^2 (|q|^2)_{xx} - \left\{ (|q|^2)_x \right\}^2 \right] - q,
 \end{aligned} \tag{28}$$

and the ODE for $\phi(x)$ reduces to

$$\phi(x) \left\{ b_1 \phi^3(x) + b_2 \phi^5(x) - (\lambda - 1) \phi(x) - \phi''(x) \right\} - \{ \phi'(x) \}^2 = 0. \tag{29}$$

The above equation admits a single Lie point symmetry, namely $\partial/\partial x$ which when utilized leads to the solution:

$\phi(x) =$

$$\left[\frac{3b_1(\lambda - 1) \left\{ 2\operatorname{sech}^2(x\sqrt{2-2\lambda}) \pm \sqrt{-2\{2b_1^2 + 9b_2(\lambda - 1)\}} \operatorname{sech}(x\sqrt{2-2\lambda}) \tanh(x\sqrt{2-2\lambda}) \right\}}{2b_1^2 + 9b_2(\lambda - 1) \tanh^2(x\sqrt{2-2\lambda})} \right]^{\frac{1}{2}}, \quad (30)$$

so that the quiescent optical 1-soliton solution reads

$q(x, t) =$

$$\left[\frac{3b_1(\lambda - 1) \left\{ 2\operatorname{sech}^2(x\sqrt{2-2\lambda}) \pm \sqrt{-2\{2b_1^2 + 9b_2(\lambda - 1)\}} \operatorname{sech}(x\sqrt{2-2\lambda}) \tanh(x\sqrt{2-2\lambda}) \right\}}{2b_1^2 + 9b_2(\lambda - 1) \tanh^2(x\sqrt{2-2\lambda})} \right]^{\frac{1}{2}} e^{i\lambda t}. \quad (31)$$

This family of quiescent solitons will exist provided

$$2b_1^2 + 9b_2(\lambda - 1) < 0. \quad (32)$$

2.4 Dual-power law

For dual-power law, the structure of the SPM gives the functional F to be written as

$$F(|q|^2) = b_1 |q|^{2m} + b_2 |q|^{2m+2}, \quad (33)$$

where b_j for $j = 1, 2$ are real-valued parameters and m dictates the power-law nonlinearity. Thus, with this form of SPM, the CGLE given by (1) reduces to

$$\begin{aligned} iq_t + a(|q|^n q)_{xx} + (b_1 |q|^{2m} + b_2 |q|^{2m+2}) q \\ = \alpha \frac{|q_x|^2}{q^*} + \frac{\beta}{4|q|^2 q^*} \left[2|q|^2 (|q|^2)_{xx} - \left\{ (|q|^2)_x \right\}^2 \right] + \gamma q. \end{aligned} \quad (34)$$

The ODE (3) for $\phi(x)$ simplifies to

$$\begin{aligned} a(n+1)\phi^n(x) \left[n\{\phi'(x)\}^2 + \phi(x)\phi''(x) \right] + b_1\phi^{2m+2}(x) + b_2\phi^{2m+4}(x) \\ - \alpha\phi(x)\{\phi'(x)\}^2 - \beta\phi(x)\phi''(x) - (\gamma + \lambda)\phi^2(x) = 0. \end{aligned} \quad (35)$$

For (35) to be integrable, one must select the same value of n as given in (7) and one must additionally choose

$$\lambda + \gamma = 0. \quad (36)$$

These conditions transform the CGLE given by (34) to

$$\begin{aligned} iq_t + a \left(\frac{q}{|q|} \right)_{xx} + (b_1 |q|^{2m} + b_2 |q|^{2m+2}) q \\ = \alpha \frac{|q_x|^2}{q^*} + \frac{\beta}{4 |q|^2 q^*} \left[2 |q|^2 (|q|^2)_{xx} - \left\{ (|q|^2)_x \right\}^2 \right] - \lambda q, \end{aligned} \quad (37)$$

and the ODE (35) simplifies to

$$b_1 \phi^{2m+2}(x) + b_2 \phi^{2m+4}(x) - \alpha \phi(x) \{ \phi'(x) \}^2 - \beta \phi(x) \phi''(x) - (\gamma + \lambda) \phi^2(x) = 0. \quad (38)$$

Equation (38) admits a single Lie point symmetry, namely $\partial/\partial x$, which leads to its implicit solution in the form

$$x = \pm \frac{1}{m\phi^m} \sqrt{\frac{\alpha + (m+1)\beta}{b_1}} {}_2F_1 \left(\frac{1}{2}, -\frac{m}{2}; 1 - \frac{m}{2}; -\frac{\{\alpha + (m+1)\beta\} \phi^2 b_2}{\{\alpha + (m+2)\beta\} b_1} \right), \quad (39)$$

where the Gauss' hypergeometric function is written as:

$${}_2F_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{z^n}{n!}, \quad (40)$$

with the Pochhammer symbol being

$$(p)_n = \begin{cases} 1 & n = 0, \\ p(p+1) \cdots (p+n-1) & n > 0. \end{cases} \quad (41)$$

The condition that guarantees convergence of the hypergeometric series is

$$|z| < 1, \quad (42)$$

which for (39) implies

$$-\left[\frac{\{\alpha + (m+2)\beta\} b_1}{\{\alpha + (m+1)\beta\} b_2}\right]^{\frac{1}{2}} < \phi < \left[\frac{\{\alpha + (m+2)\beta\} b_1}{\{\alpha + (m+1)\beta\} b_2}\right]^{\frac{1}{2}}. \quad (43)$$

Also from (39), the solution existence criterion is

$$b_1 \{\alpha + (m+1)\beta\} > 0. \quad (44)$$

2.5 Log-law

For log-law of SPM, the functional F is structured as

$$F(|q|^2) = b \ln |q|^2, \quad (45)$$

for a real-valued constant b . The logarithmic term in the equation implies that the material's response to light intensity is nonlinear. In linear optics, the response of a material would be directly proportional to the intensity of light. However, in nonlinear optics, the material's behavior changes nonlinearly as the intensity increases. This nonlinearity can lead to a variety of optical phenomena that are essential for various applications. This formulates the CGLE (1) as:

$$\begin{aligned} i q_t + a (|q|^n q)_{xx} + b q \ln |q|^2 \\ = \alpha \frac{|q_x|^2}{q^*} + \frac{\beta}{4 |q|^2 q^*} \left[2 |q|^2 (|q|^2)_{xx} - \left\{ (|q|^2)_x \right\}^2 \right] + \gamma q. \end{aligned} \quad (46)$$

Here the ODE for $\phi(x)$ from (3) shapes up as:

$$a(n+1)\phi^n(x) \left[n \{\phi'(x)\}^2 + \phi(x)\phi''(x) \right] + \phi(x) \left[\phi(x) \{2b \ln \phi(x) - \gamma - \lambda\} - \beta \phi''(x) \right] - \alpha \{\phi'(x)\}^2 = 0. \quad (47)$$

Integrability of (47) requires the choice of n , as given by (7), to be made. This simplifies (46) and (47) to

$$\begin{aligned} i q_t + a \left(\frac{q}{|q|} \right)_{xx} + b q \ln |q|^2 \\ = \alpha \frac{|q_x|^2}{q^*} + \frac{\beta}{4 |q|^2 q^*} \left[2 |q|^2 (|q|^2)_{xx} - \left\{ (|q|^2)_x \right\}^2 \right] + \gamma q, \end{aligned} \quad (48)$$

and

$$\phi(x) \left[\phi(x) \{2b \ln \phi(x) - \gamma - \lambda\} - \beta \phi''(x) \right] - \alpha \{\phi'(x)\}^2 = 0, \quad (49)$$

respectively. Equation (49) admits a single Lie point symmetry given by $\partial/\partial x$. This when applied to (49) integrates it to

$$\phi(x) = \sinh \left\{ \frac{\alpha\gamma + \alpha\lambda + b^2x^2 + b\beta + \beta\gamma + \beta\lambda}{2b(\alpha + \beta)} \right\} + \cosh \left\{ \frac{\alpha\gamma + \alpha\lambda + b^2x^2 + b\beta + \beta\gamma + \beta\lambda}{2b(\alpha + \beta)} \right\}. \quad (50)$$

Therefore, the straddled singular-singular quiescent optical solitons for CGLE with log-law nonlinearity is:

$$q(x, t) = \left[\sinh \left\{ \frac{\alpha\gamma + \alpha\lambda + b^2x^2 + b\beta + \beta\gamma + \beta\lambda}{2b(\alpha + \beta)} \right\} + \cosh \left\{ \frac{\alpha\gamma + \alpha\lambda + b^2x^2 + b\beta + \beta\gamma + \beta\lambda}{2b(\alpha + \beta)} \right\} \right] e^{i\lambda t}. \quad (51)$$

2.6 Anti-cubic law

The anti-cubic law of nonlinear SPM is given as

$$F(|q|^2) = \frac{b_1}{|q|^4} + b_2|q|^2 + b_3|q|^4. \quad (52)$$

By understanding this law, engineers can optimize the performance of optical systems. They can adjust the coefficients b_1 , b_2 , and b_3 to tailor the nonlinear response according to specific application requirements, such as maximizing signal-to-noise ratio, minimizing distortion, or enhancing the dynamic range of the system. This form of SPM structures the CGLE as:

$$\begin{aligned} iq_t + a(|q|^n q)_{xx} + \left(\frac{b_1}{|q|^4} + b_2|q|^2 + b_3|q|^4 \right) q \\ = \alpha \frac{|q_x|^2}{q^*} + \frac{\beta}{4|q|^2 q^*} \left[2|q|^2 (|q|^2)_{xx} - \left\{ (|q|^2)_x \right\}^2 \right] + \gamma q. \end{aligned} \quad (53)$$

The ODE (3) for $\phi(x)$ now takes the form

$$\begin{aligned} \phi^2(x) \left[a(n+1)\phi^n(x) \left\{ n \{ \phi'(x) \}^2 + \phi(x)\phi''(x) \right\} \right. \\ \left. + \phi(x) \left\{ b_2\phi^3(x) + b_3\phi^5(x) - \beta\phi''(x) - (\gamma + \lambda)\phi(x) \right\} - \alpha \{ \phi'(x) \}^2 \right] + b_1 = 0. \end{aligned} \quad (54)$$

For its integrability, one needs to have (7) and (36) to hold along with

$$b_3 = 0. \quad (55)$$

This transforms the law of SPM to

$$F(|q|^2) = \frac{b_1}{|q|^4} + b_2 |q|^2, \quad (56)$$

and therefore the CGLE reshapes up as

$$\begin{aligned} i q_t + a \left(\frac{q}{|q|} \right)_{xx} + \left(\frac{b_1}{|q|^4} + b_2 |q|^2 \right) q \\ = \alpha \frac{|q_x|^2}{q^*} + \frac{\beta}{4 |q|^2 q^*} \left[2 |q|^2 (|q|^2)_{xx} - \left\{ (|q|^2)_x \right\}^2 \right] - \lambda q, \end{aligned} \quad (57)$$

while the ODE (54) shrinks to

$$\phi^2(x) \left[\phi(x) \{ b_2 \phi^3(x) - \beta \phi''(x) \} - \alpha \{ \phi'(x) \}^2 \right] + b_1 = 0. \quad (58)$$

Equation (58) admits a single Lie point symmetry namely $\partial/\partial x$ which when applied integrates it to

$$x = \pm \frac{\phi^2}{2} \sqrt{\frac{\alpha - \beta}{b_1}} {}_2F_1 \left(\frac{1}{3}, \frac{1}{2}; \frac{4}{3}; -\frac{(\alpha - \beta)\phi^6 b_2}{(\alpha + 2\beta)b_1} \right). \quad (59)$$

The condition (42) for (59) implies

$$-\left[\frac{(\alpha + 2\beta)b_1}{(\alpha - \beta)b_2} \right]^{\frac{1}{6}} < \phi(x) < \left[\frac{(\alpha + 2\beta)b_1}{(\alpha - \beta)b_2} \right]^{\frac{1}{6}}. \quad (60)$$

Also, from (59), one must have

$$b_1(\alpha - \beta) > 0. \quad (61)$$

2.7 Generalized anti-cubic law

For generalized anti-cubic law of SPM, the functional F is given by

$$F(|q|^2) = \frac{b_1}{|q|^{2(m+1)}} + b_2 |q|^{2m} + b_3 |q|^{2(m+1)}. \quad (62)$$

This law of nonlinearity represents how the phase of a light wave changes in response to its intensity, where m is a parameter that influences the nonlinearity. The coefficients b_1 , b_2 , and b_3 determine the strength of different nonlinear effects. Understanding this law is essential for accurately modeling and predicting the behavior of optical

systems. By incorporating the nonlinear phase response described by this equation into optical models, researchers can simulate the performance of various optical components and systems more accurately. This is crucial for optimizing system performance, predicting system behavior under different conditions, and designing novel optical devices. This transforms the CGLE given by (1) to

$$\begin{aligned}
 iq_t + a(|q|^n q)_{xx} + \left\{ \frac{b_1}{|q|^{2(m+1)}} + b_2 |q|^{2m} + b_3 |q|^{2(m+1)} \right\} q \\
 = \alpha \frac{|q_x|^2}{q^*} + \frac{\beta}{4|q|^2 q^*} \left[2|q|^2 (|q|^2)_{xx} - \left\{ (|q|^2)_x \right\}^2 \right] + \gamma q.
 \end{aligned} \tag{63}$$

For this law of SPM, one arrives at the ODE for $\phi(x)$ from (3) to:

$$\begin{aligned}
 a(n+1)\phi^{n+2}(x) \left[n \{ \phi'(x) \}^2 + \phi(x)\phi''(x) \right] + b_1 \phi^{2-2m}(x) \\
 + b_2 \phi^{2m+4}(x) + b_3 \phi^{2m+6}(x) - \alpha \phi^2(x) \{ \phi'(x) \}^2 - \beta \phi^3(x)\phi''(x) - (\lambda + \gamma) \phi^4(x) = 0.
 \end{aligned} \tag{64}$$

For its integrability, one must have the criteria (7) and (36) along with the condition

$$m = -\frac{1}{2}. \tag{65}$$

These prompt the CGLE to be restructured as

$$\begin{aligned}
 iq_t + a \left(\frac{q}{|q|} \right)_{xx} + \left(\frac{b_1 + b_2}{|q|} + b_3 |q| \right) q \\
 = \alpha \frac{|q_x|^2}{q^*} + \frac{\beta}{4|q|^2 q^*} \left[2|q|^2 (|q|^2)_{xx} - \left\{ (|q|^2)_x \right\}^2 \right] - \lambda q,
 \end{aligned} \tag{66}$$

and consequently the ODE (64) simplifies to

$$(b_1 + b_2) \phi(x) + b_3 \phi^3(x) - \alpha \{ \phi'(x) \}^2 - \beta \phi(x)\phi''(x) = 0. \tag{67}$$

This ODE permits a single Lie point symmetry $\partial/\partial x$. This symmetry leads to the following implicit solution in terms of Gauss' hypergeometric function as

$$x = \pm \sqrt{\frac{2(2\alpha + \beta)\phi}{b_1 + b_2}} {}_2F_1 \left(\frac{1}{4}, \frac{1}{2}; \frac{5}{4}; -\frac{(2\alpha + \beta)\phi^2 b_3}{(2\alpha + 3\beta)(b_1 + b_2)} \right). \tag{68}$$

The convergence criteria given by (42), when applied to (69) leads to

$$-\left[\frac{(b_1 + b_2)(2\alpha + 3\beta)}{(2\alpha + \beta)b_3}\right]^{\frac{1}{2}} < \phi(x) < \left[\frac{(b_1 + b_2)(2\alpha + 3\beta)}{(2\alpha + \beta)b_3}\right]^{\frac{1}{2}}. \quad (69)$$

2.8 Quadratic-cubic law

For quadratic-cubic law of self-phase modulation, the functional F is of the form:

$$F(|q|^2) = b_1 |q| + b_2 |q|^2. \quad (70)$$

The terms b_1 and b_2 denote coefficients influencing the strength of the quadratic and cubic nonlinear effects, respectively. Understanding this law is crucial for predicting and controlling various nonlinear optical phenomena. In laser physics, nonlinear effects can significantly impact laser stability, output characteristics, and beam quality. By accurately modeling and understanding the nonlinear response described by this law, researchers can predict and mitigate unwanted nonlinear effects, ensuring optimal laser performance. This makes CGLE to be of the following form:

$$\begin{aligned} & iq_t + a(|q|^n q)_{xx} + (b_1 |q| + b_2 |q|^2) q \\ &= \alpha \frac{|q_x|^2}{q^*} + \frac{\beta}{4|q|^2 q^*} \left[2|q|^2 (|q|^2)_{xx} - \left\{ (|q|^2)_x \right\}^2 \right] + \gamma q. \end{aligned} \quad (71)$$

The transformation (2) leads to ODE for $\phi(x)$ from (71):

$$\begin{aligned} & a(n+1)\phi^n(x) \left[n \{ \phi'(x) \}^2 + \phi(x)\phi''(x) \right] + b_1 \phi^3(x) + b_2 \phi^4(x) \\ & - \alpha \phi(x) \{ \phi'(x) \}^2 - \beta \phi(x)\phi''(x) - (\gamma + \lambda) \phi^2(x) = 0. \end{aligned} \quad (72)$$

For the integrability in (72), one needs to choose the value of n as in (7). This modifies (71) to

$$\begin{aligned} & iq_t + a \left(\frac{q}{|q|} \right)_{xx} + (b_1 |q| + b_2 |q|^2) q \\ &= \alpha \frac{|q_x|^2}{q^*} + \frac{\beta}{4|q|^2 q^*} \left[2|q|^2 (|q|^2)_{xx} - \left\{ (|q|^2)_x \right\}^2 \right] + \gamma q, \end{aligned} \quad (73)$$

and consequently the ODE given by (72) condenses to

$$b_1 \phi^3(x) + b_2 \phi^4(x) - \alpha \phi(x) \{ \phi'(x) \}^2 - \beta \phi(x)\phi''(x) - (\gamma + \lambda) \phi^2(x) = 0. \quad (74)$$

With the single Lie point symmetry $\partial/\partial x$, supported by this ODE, its integral is

$$\phi(x) = \frac{A_1}{A_2}, \quad (75)$$

where

$$A_1 = -4B_1(2\alpha + 3\beta)(\gamma + \lambda) \left[4b_1^2(\alpha + \beta)^2(\alpha + 2\beta)\sqrt{-(\gamma + \lambda)(2\alpha^2 + 7\alpha\beta + 6\beta^2)} \right. \\ \left. - 4b_1B_1(\alpha + \beta)(\alpha + 2\beta)(2\alpha + 3\beta)(\gamma + \lambda) + B_2(2\alpha + 3\beta)(\gamma + \lambda)\sqrt{-(\gamma + \lambda)(2\alpha^2 + 7\alpha\beta + 6\beta^2)} \right], \quad (76)$$

$$A_2 = 16b_1^4(\alpha + \beta)^4(\alpha + 2\beta)^2 + 8b_1^2B_4(\alpha + \beta)^2(\alpha + 2\beta)(2\alpha + 3\beta)(\gamma + \lambda) + B_3^2(2\alpha + 3\beta)^2(\gamma + \lambda)^2, \quad (77)$$

$$B_1 = \sinh\left(x\sqrt{-\frac{\gamma + \lambda}{\alpha + \beta}}\right) + \cosh\left(x\sqrt{-\frac{\gamma + \lambda}{\alpha + \beta}}\right), \quad (78)$$

$$B_2 = 4b_2(2\alpha^2 + 5\alpha\beta + 3\beta^2) - \sinh\left(2x\sqrt{-\frac{\gamma + \lambda}{\alpha + \beta}}\right) - \cosh\left(2x\sqrt{-\frac{\gamma + \lambda}{\alpha + \beta}}\right), \quad (79)$$

$$B_3 = -4b_2(2\alpha^2 + 5\alpha\beta + 3\beta^2) + \sinh\left(2x\sqrt{-\frac{\gamma + \lambda}{\alpha + \beta}}\right) + \cosh\left(2x\sqrt{-\frac{\gamma + \lambda}{\alpha + \beta}}\right), \quad (80)$$

$$B_4 = 4b_2(2\alpha^2 + 5\alpha\beta + 3\beta^2) + \sinh\left(2x\sqrt{-\frac{\gamma + \lambda}{\alpha + \beta}}\right) + \cosh\left(2x\sqrt{-\frac{\gamma + \lambda}{\alpha + \beta}}\right), \quad (81)$$

provided (12) is valid along with the additional constraints

$$(\gamma + \lambda)(2\alpha^2 + 7\alpha\beta + 6\beta^2) < 0, \quad (82)$$

and

$$A_2 \neq 0, \quad (83)$$

hold true. Therefore, the quiescent optical soliton for CGLE with quadratic-cubic form of nonlinear SPM is given by (2) where $\phi(x)$ is given by (75).

2.9 Parabolic non-local law

For this law of SPM, the functional is of the form

$$F(|q|^2) = b_1 |q|^2 + b_2 |q|^4 + b_3 \left(|q|^2\right)_{xx}. \quad (84)$$

The terms b_1 , b_2 , and b_3 represent coefficients determining the strength of different nonlinear effects. Understanding this law is essential for modeling and predicting the behavior of optical systems, particularly in fields such as nonlinear optics and laser physics. Therefore, the CGLE shapes up as:

$$\begin{aligned} iq_t + a(|q|^n q)_{xx} + \left\{ b_1 |q|^2 + b_2 |q|^4 + b_3 \left(|q|^2\right)_{xx} \right\} q \\ = \alpha \frac{|q_x|^2}{q^*} + \frac{\beta}{4|q|^2 q^*} \left[2|q|^2 \left(|q|^2\right)_{xx} - \left\{ \left(|q|^2\right)_x \right\}^2 \right] + \gamma q. \end{aligned} \quad (85)$$

The transformation (2) gives the ODE in $\phi(x)$ as

$$\begin{aligned} a(n+1)\phi^n(x) \left[n \{ \phi'(x) \}^2 + \phi(x)\phi''(x) \right] + 2b_3\phi^2(x) \left[\{ \phi'(x) \}^2 + \phi(x)\phi''(x) \right] \\ + b_2\phi^6(x) + b_1\phi^4(x) - \alpha \{ \phi'(x) \}^2 - \beta\phi(x)\phi''(x) - (\gamma + \lambda)\phi^2(x) = 0. \end{aligned} \quad (86)$$

The integrability of this ODE (86) is guaranteed when (7) and (49) are valid along with the following additional parameter values:

$$\alpha = -1, \quad (87)$$

$$\beta = 1, \quad (88)$$

$$b_1 = \frac{1}{2}, \quad (89)$$

$$b_2 = -\frac{1}{4}, \quad (90)$$

and

$$b_3 = 1. \tag{91}$$

Therefore, the CGLE (85) modifies to

$$\begin{aligned} iq_t + a \left(\frac{q}{|q|} \right)_{xx} + \left\{ \frac{1}{2} |q|^2 - \frac{1}{4} |q|^4 + (|q|^2)_{xx} \right\} q \\ = -\frac{|q_x|^2}{q^*} + \frac{1}{4|q|^2 q^*} \left[2|q|^2 (|q|^2)_{xx} - \left\{ (|q|^2)_x \right\}^2 \right] - \lambda q, \end{aligned} \tag{92}$$

while the ODE (86) shrinks to

$$\{\phi'(x)\}^2 + \phi(x) \left[\phi(x) \left\{ 2 \left[\phi(x)\phi''(x) + \{\phi'(x)\}^2 \right] - \frac{1}{4}\phi^4(x) + \frac{1}{2}\phi^2(x) \right\} - \phi''(x) \right] = 0. \tag{93}$$

By the single Lie point symmetry as permitted by (93), namely $\partial/\partial x$, one arrives at the implicit solutions that are in terms of the elliptic integrals of the first and second kind:

$$\begin{aligned} x = \pm \frac{1}{16\sqrt{6}\sqrt{\phi^2(4\phi^4 - 15\phi^2 + 12)}} \left\{ \sqrt{11}A_1 \sqrt{-8\phi^2 - \sqrt{33} + 15\sqrt{\sqrt{33}\phi^2 - 15\phi^2 + 24\phi}} \right. \\ + 5\sqrt{3}A_1 \sqrt{-8\phi^2 - \sqrt{33} + 15\sqrt{\sqrt{33}\phi^2 - 15\phi^2 + 24\phi}} \\ - \sqrt{11}A_2 \sqrt{-8\phi^2 - \sqrt{33} + 15\sqrt{\sqrt{33}\phi^2 - 15\phi^2 + 24\phi}} \\ \left. + 11\sqrt{3}A_2 \sqrt{-8\phi^2 - \sqrt{33} + 15\sqrt{\sqrt{33}\phi^2 - 15\phi^2 + 24\phi}} + 64\phi^4 - 240\phi^2 + 192 \right\}, \end{aligned} \tag{94}$$

where

$$A_1 = E \left(\sin^{-1} \left(\frac{1}{2} \sqrt{\frac{1}{6} (15 + \sqrt{33})} \phi \right) \middle| \frac{1}{32} (43 - 5\sqrt{33}) \right), \tag{95}$$

and

$$A_2 = F \left(\sin^{-1} \left(\frac{1}{2} \sqrt{\frac{1}{6} (15 + \sqrt{33})} \phi \right) \middle| \frac{1}{32} (43 - 5\sqrt{33}) \right). \tag{96}$$

The elliptic integral of the first kind $F(\psi|M)$ is defined as

$$F(\psi|M) = \int_0^\psi \frac{1}{\sqrt{1-M\sin^2\theta}} d\theta, \quad (97)$$

for

$$-\frac{\pi}{2} < \psi < \frac{\pi}{2}, \quad (98)$$

and

$$M \sin^2 \psi < 1, \quad (99)$$

while the elliptic integral of the second kind $E(\psi|M)$ is

$$E(\psi|M) = \int_0^\psi \sqrt{1-M\sin^2\theta} d\theta, \quad (100)$$

where

$$-\frac{\pi}{2} < \psi < \frac{\pi}{2}, \quad (101)$$

and

$$M < 1. \quad (102)$$

2.10 Saturating law

For saturating law of SPM, the functional F takes the form

$$F(|q|^2) = \frac{b_1 |q|^2}{b_2 + b_3 |q|^2}, \quad (103)$$

where b_j for $j = 1, 2, 3$ are real-valued constants. Understanding and controlling b_1 is crucial for fine-tuning the nonlinear behavior of optical systems, especially in applications where precise control over phase modulation is required. b_2 and b_3 control how the phase modulation saturates with increasing intensity, affecting the overall dynamic range and sensitivity of the system to changes in input intensity. Therefore, the CGLE takes the form

$$\begin{aligned}
& iq_t + a(|q|^n q)_{xx} + \left(\frac{b_1 |q|^2}{b_2 + b_3 |q|^2} \right) q \\
& = \alpha \frac{|q_x|^2}{q^*} + \frac{\beta}{4|q|^2 q^*} \left[2|q|^2 (|q|^2)_{xx} - \left\{ (|q|^2)_x \right\}^2 \right] + \gamma q.
\end{aligned} \tag{104}$$

Thereafter, the transformation (2) when applied to (104) gives the ODE for $\phi(x)$ as

$$\begin{aligned}
& a(n+1) \left[n\phi^{n-1}(x) \{ \phi'(x) \}^2 + \phi^n(x) \phi''(x) \right] \\
& + \frac{b_1 \phi^3(x)}{b_3 \phi^2(x) + b_2} - \frac{\alpha \{ \phi'(x) \}^2}{\phi(x)} - \beta \phi''(x) - (\gamma + \lambda) \phi(x) = 0.
\end{aligned} \tag{105}$$

In order to be able to integrate (105), one must choose the same value of n as given by (7). With such a choice, the CGLE (104) transforms to

$$\begin{aligned}
& iq_t + a \left(\frac{q}{|q|} \right)_{xx} + \left(\frac{b_1 |q|^2}{b_2 + b_3 |q|^2} \right) q \\
& = \alpha \frac{|q_x|^2}{q^*} + \frac{\beta}{4|q|^2 q^*} \left[2|q|^2 (|q|^2)_{xx} - \left\{ (|q|^2)_x \right\}^2 \right] + \gamma q,
\end{aligned} \tag{106}$$

while the ODE (105) shrinks to

$$\frac{b_1 \phi^3(x)}{b_3 \phi^2(x) + b_2} - \frac{\alpha \{ \phi'(x) \}^2}{\phi(x)} - \beta \phi''(x) - (\gamma + \lambda) \phi(x) = 0. \tag{107}$$

Equation (107) integrates to the following implicit solution in terms of quadratures with Gauss' hypergeometric function as

$$x = \pm \sqrt{(\alpha + \beta)b_3} \int \frac{1}{\phi \sqrt{- \left[b_1 \left\{ -1 + {}_2F_1 \left(1, \frac{\alpha + \beta}{\beta}; 2 + \frac{\alpha}{\beta}; -\frac{\phi^2 b_3}{b_2} \right) \right\} + (\gamma + \lambda)b_3 \right]}} d\phi, \tag{108}$$

after implementing the single Lie point symmetry, namely $\partial/\partial x$, that it permits. This solution remains valid provided

$$(\alpha + \beta)b_3 > 0, \tag{109}$$

and

$$b_1 \left\{ -1 + {}_2F_1 \left(1, \frac{\alpha + \beta}{\beta}; 2 + \frac{\alpha}{\beta}; -\frac{\phi^2 b_3}{b_2} \right) \right\} + (\gamma + \lambda) b_3 < 0. \quad (110)$$

2.11 Exponential law

For exponential law of SPM, the functional F takes the form

$$F(|q|^2) = \frac{1}{b} \left(1 - e^{-b|q|^2} \right), \quad (111)$$

for the real-valued constant

$$b > 0. \quad (112)$$

It signifies a response that saturates at high intensities due to the exponential term. Parameter b controls the nonlinearity's strength. Understanding and controlling such responses is vital for optimizing optical devices and systems. Thus, the CGLE given by (1) reduces to

$$\begin{aligned} iq_t + a(|q|^n q)_{xx} + \frac{1}{b} \left(1 - e^{-b|q|^2} \right) q \\ = \alpha \frac{|q_x|^2}{q^*} + \frac{\beta}{4|q|^2 q^*} \left[2|q|^2 \left(|q|^2 \right)_{xx} - \left\{ \left(|q|^2 \right)_x \right\}^2 \right] + \gamma q. \end{aligned} \quad (113)$$

The transformation (2) for exponential law gives the ODE in $\phi(x)$ as

$$\begin{aligned} a(n+1) \left[n\phi^{n-1}(x) \{ \phi'(x) \}^2 + \phi^n(x) \phi''(x) \right] \\ - \frac{\phi(x) e^{-b\phi^2(x)}}{b} + \frac{\phi(x)}{b} - \frac{\alpha \{ \phi'(x) \}^2}{\phi(x)} \\ - \beta \phi''(x) - (\lambda + \gamma) \phi(x) = 0. \end{aligned} \quad (114)$$

For integrability of (114), one must chose the same value of n as in (7) that would modify the CGLE (113) to

$$\begin{aligned}
 & i q_t + a \left(\frac{q}{|q|} \right)_{xx} + \frac{1}{b} (1 - e^{-b|q|^2}) q \\
 & = \alpha \frac{|q_x|^2}{q^*} + \frac{\beta}{4|q|^2 q^*} \left[2|q|^2 (|q|^2)_{xx} - \left\{ (|q|^2)_x \right\}^2 \right] + \gamma q,
 \end{aligned} \tag{115}$$

and the ODE (114) simplifies to

$$\frac{\phi(x) \left\{ b(\gamma + \lambda) + e^{-b\phi(x)^2} - 1 \right\}}{b} + \frac{\alpha \{ \phi'(x) \}^2}{\phi(x)} + \beta \phi''(x) = 0. \tag{116}$$

The above equation admits a single Lie point symmetry, namely $\partial/\partial x$. This symmetry leads to the implicit solution in terms of quadratures of the exponential integral

$$x = \pm \sqrt{b\beta(\alpha + \beta)} \int \frac{1}{\phi \sqrt{\left\{ \beta - b\beta(\gamma + \lambda) + (\alpha + \beta) E_{-\frac{\alpha}{\beta}}(b\phi^2) \right\}}} d\phi, \tag{117}$$

where the exponential integral $E_m(z)$ is defined as

$$E_m(z) = \int_1^\infty \frac{e^{t(-z)}}{t^m} dt. \tag{118}$$

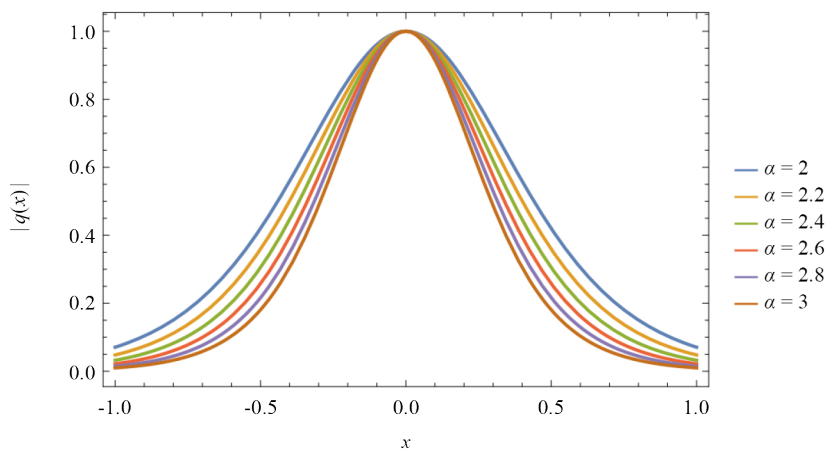
The solution existence criterion is

$$\beta(\alpha + \beta) > 0. \tag{119}$$

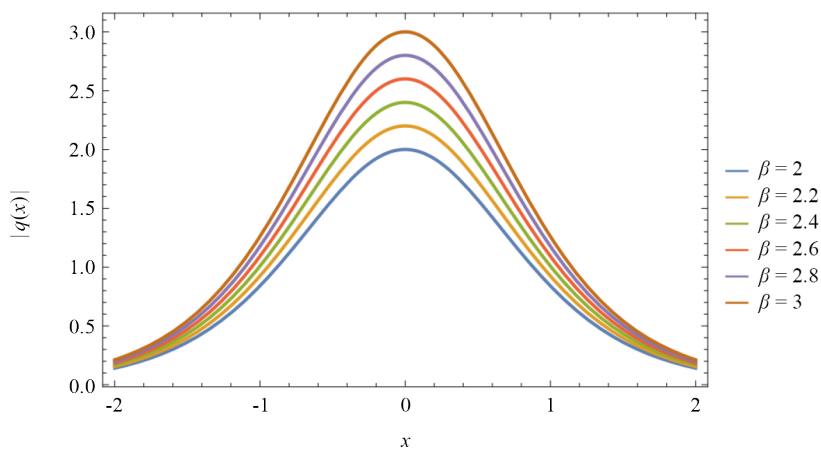
3. Results and discussion

Figure 1 illustrates the behavior of an optical bright soliton solution as described by Eq. (11) with specific parameter values: $\lambda = -3.1$, $b = -1$ and $\alpha = \beta = \gamma = 2, 2.2, 2.4, 2.6, 2.8, 3$. The characteristics of this solution are analyzed through various representations including 2D plots. Figure 1 employs the modulus of the complex-valued soliton profile $q(x, t)$, providing a comprehensive view of the soliton's amplitude without regard to phase information. The 2D plots in Figures 1(a), 1(b), and 1(c) focus solely on the amplitude profile of the bright soliton solution, allowing for a detailed examination of the effects of various parameters. Specifically, Figures 1(a), 1(b), and 1(c) explore the influence of the nonlinear term (α), perturbation term (β), and detuning parameter (γ), respectively. By varying these parameters, we observe changes in the soliton's shape, width, and propagation dynamics. For instance, increasing the nonlinear coefficient (α) leads to soliton compression, while variations in the perturbation term (β) induces soliton broadening in the soliton profile. Similarly, adjustments to the detuning parameter (γ) affects soliton compression and broadening. As a result, the results presented in Figure 1 provide valuable insights into the characteristics and behavior of optical bright soliton solutions,

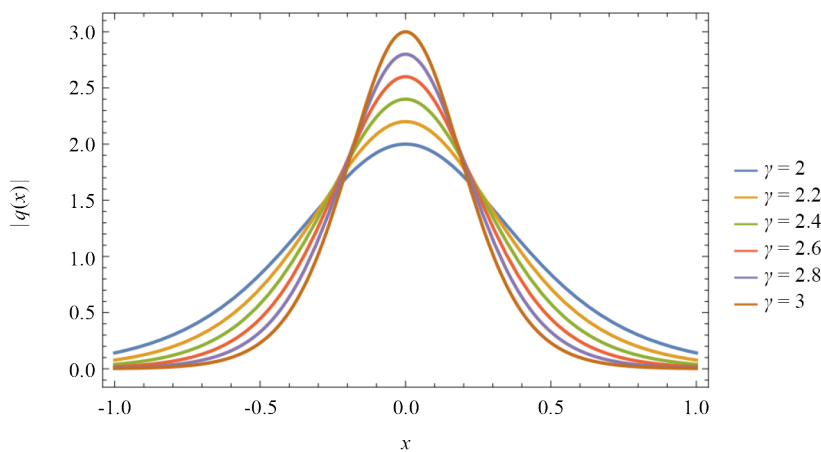
offering a comprehensive understanding of their dynamics in nonlinear optical systems. These findings have implications for various applications, including optical communication, signal processing, and nonlinear optics.



(a) 2D plot with the effect of α



(b) 2D plot with the effect of β



(c) 2D plot with the effect of γ

Figure 1. Profile of a bright soliton solution

4. Conclusions

The current paper studied and analyzed in detail the emergence of quiescent optical solitons for the CGLE with nonlinear CD and eleven structural forms of SPM. The Lie symmetry method has made this analysis and the retrieval of quiescent optical solitons possible. Solutions presented in Sections 2.1 and 2.3 are special cases derived from the general solutions in Sections 2.2 and 2.4 by setting $m = 1$. However, we also provide soliton solutions for arbitrary values of m , extending beyond the restricted case of $m = 1$ outlined in Sections 2.2 and 2.4. This paper thus represents a significant improvement over previously reported results with the same model in 2022, where the solutions were in terms of quadratures [10]. Additionally, the current paper addresses two new forms of SPM for the first time: the saturating law and the exponential law. However, in both of these new cases, the results are in terms of quadratures with special functions, specifically Gauss' hypergeometric functions and the exponential integral. Nevertheless, the results are indeed very encouraging in this regard.

An important milestone of optical solitons is its stability analysis. This would mean addressing the Benjamin-Fier stability analysis or the Vakhitkov-Kolokolov stability analysis. However, it is important to note that stability analysis is conducted only for mobile solitons. It is consequently redundant to address stability analysis for quiescent solitons in this work. Therefore, what enters the bucket list is the stability analysis when the mobile solitons will be recovered for these models with linear CD in presence of Hamiltonian perturbation terms. This analysis would thus be an icing on the cake for mobile solitons as always.

While the current paper has studied the CGLE with linear temporal evolution, the follow-up work will involve generalized temporal evolution. The results of these research activities will be disseminated over time once they are aligned with the results of existing works [25–28]. Subsequently, this analysis will be applied to various additional models studied in optics, encompassing both linear and generalized temporal evolution. This is just the tip of the iceberg.

Conflict of interest

The authors claim that there is no conflict of interest.

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