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Research Article

An EOQ Model for Circularity Index with Waste Minimization and Reduced Emission in Electrical and Electronic Equipment

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Abstract: A circular economy uses limited natural resources and is efficient and environmentally productive. It aims to transform from a 'take-make-waste' approach to a more restorative and regenerative one. A closed-loop system reduces waste, pollution, and carbon emissions by reusing, sharing, repairing, refurbishing, remanufacturing, and recycling. We developed a new model on reverse logistics economic order quantity with the circular economy. This paper proposes a profit-maximizing economic lot size system in which all products are at variable circularity levels. The product's degree of circularity directly influences customers' demand and gross profit per unit, and we study these effects using both linear and nonlinear forms of analysis as consumers become more socially responsible about their consumption practices. This paper aims to maximize profits from the waste of electrical and electronic equipment (WEEE) by reducing carbon emissions. Some examples are discussed to find the profit of a business with variations of parameters. The goal of this study is to help reverse the flow of products to re-manufacturers of electrical equipment, especially computer equipment manufacturers, to acquire the ideal circular index for the products they manufacture to maximize profit and preserve the environment simultaneously.

Keywords: circularity, supply chain, economic order quantity (EOQ), customer demand, E-waste, carbon emission, remanufacture, reverse logistics

MSC: 90B05, 90B06

Nomenclature

At the beginning of each period, the manager orders a fixed quantity of Q to meet the demand for the next period precisely. A circularity index measures the circularity of both standard and circular product versions. According to our assumptions, demand rates and gross profits (equal to the unit selling price minus the unit acquisition cost) are functions. To describe the production-ordered process, the following symbols are required:

θ: Index of Circularity produc $\theta \in [0,\ 1]$
Q: Total order quantity of electronic equipment
t: The time frame within which the ordered quantity is manufactured
R: Manufacturing rate

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$d(\theta)$: Customer demand rate (units/unit of time)
$p(\theta)$: Profit rate ($/unit of time)
H: Carrying cost
K: Stable set-up cost
F: Carbon dioxide emissions occur during the manufacturing process
G: Carbon emissions due to storage procedures
e: Carbon emission from the manufacturing system

1. Introduction

The concept of circularity was first championed by a man named Walter Stahel in the 1970s. According to him, extending the life of products was the logical point at which to begin a gradual shift towards a sustainable economy. The circular economy ensures sustainable growth over time by integrating production and consumption. Using a circular economy, products are more resource-efficient, reduce raw material consumption, and reuse or recycle waste. Circularity refers to the concept of designing products, processes, and systems to maximize the use of resources and minimize waste, often by reusing, recycling, or repurposing materials. In the context of business, circularity can have significant implications for both demand (the number of goods or services consumers are willing to buy at a given price) and profit (the financial gain made by a business after subtracting expenses from revenue). E-waste refers to discarded electrical or electronic devices, also known as waste electrical and electronic equipment (WEEE) or end-of-life (EOL) electronics. These used electronics may undergo refurbishment, reuse, resale, salvage recycling for material recovery, or disposal, all falling under the E-waste category. Informal handling of E-waste in developing nations can result in harmful effects on human health and environmental pollution. Globally, 50 million tonnes of E-waste, the weight of all commercial aircraft, are created annually [1]. It should be known that just 40% of EU E-waste is recycled, with the rest not sorted. In 2016, Croatia recycled 81.3% of its E-waste, while Malta only recycled 20.8%. According to the same sources, Romania had a recycling rate of 25% in 2016, ranking it second to last. Referencing 'A New Circular Vision for Electronics-WEF, 2019', it is predicted that global waste will double by 2050 if no action is taken. In real-world scenarios, manufacturers recycle the components of discarded laptops to produce new products for the market.

EMF [2] Africa is the fastest-growing laptop market, with economic and educational potential. Every year, sales of electronic equipment such as refrigerators, TVs, and mobile phones are increasing, with middle-class consumer spending reaching USD 1.3 trillion in 2010, accounting for 60% of Africa's GDP, and expected to triple by 2030. Africa produced 2.9 million metric tons of E-waste in 2019, or 2.5 kilograms per person. Backyard processing (crushing casings), hand component extraction for resale, and burning are common ways African countries recycle E-waste. Dumping cathode ray tubes in public areas poses a significant risk to both human health and the environment. Due to these issues, Ghana, Rwanda, Nigeria, and South Africa prioritize E-waste management. To handle E-waste, several nations have established extended producer responsibility (EPR) policies. A circular economy approach to E-waste management will improve resource efficiency, pollution, and waste reduction, product life, recovery of precious and rare materials, occupational and health hazards, and recycling industry formalization and job creation. CE approach requires changes in design, manufacturing, circular business models, sustainable consumption, E-waste reduction, resource recovery, and secondary resource use. Finally, CE requires a variety of new techniques, including digital tools and platforms for transparency, urban mining, secondary resource marketplaces, and value chain stakeholder connectivity.

1.1 *The traditional economic order quantity (EOQ) model incorporates circularity*

The traditional economic order quantity (EOQ) model is a fundamental tool in inventory management, aiming to optimize the balance between ordering costs and holding costs to minimize total inventory-related expenses. It calculates the optimal order quantity that minimizes the total cost of inventory, considering factors such as demand, ordering costs, holding costs, and unit costs. When integrating circularity principles into the EOQ model, the objective shifts from totally minimizing costs to optimizing sustainability and resource efficiency. Circularity integration involves considering factors such as product durability, recyclability, and end-of-life disposal strategies.

1.2 *Circular economy and the EOQ model*

The circular economy is a regenerative economic system aimed at minimizing waste and maximizing the efficient use of resources. The traditional linear economy, which follows a "take-make-dispose" model, seeks to keep products, materials, and resources in use for as long as possible through reusing, repairing, refurbishing, and recycling. It emphasizes closing loops within supply chains, reducing resource consumption, and promoting sustainable production and consumption patterns to create a more resilient and environmentally sustainable economy. This approach is guided by three design-driven principles: eliminating waste and pollution, circulating products and materials at their highest value, and regenerating nature. The circular economy with the EOQ model aims to optimize inventory levels to minimize waste and costs. While the EOQ model primarily focuses on cost minimization, the circular economy principles encourage considering broader sustainability objectives in inventory management decisions. Rabta [3] introduces an economic order quantity (EOQ) inventory model within the framework of a circular economy. We posit that products can possess varying degrees of circularity, gauged by an index. Aim to maximize profits by factoring in these considerations during decision-making processes. Consequently, we contend that the level of circularity impacts product demand, costs, and selling prices, for which we propose both linear and nonlinear relationships. Figure 1 shows how the inventory level (in time) varies with two different circularity indices. There is a change in the period length (therefore, a change in the period holding costs) as a result of variations in the slope of inventory levels R and r.

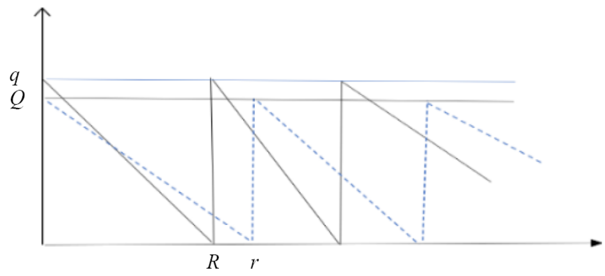

Figure 1. Variation of difference circularity index

1.3 *Circular economy and industries*

Circularity can be applied to production outside electronics. To reduce waste and maximize resource efficiency, circularity integrates production, consumption, and recycling. Industries pursuing sustainability and environmental protection can apply this approach. Circularity in automotive production might include designing vehicles using recyclable materials, remanufacturing components, and promoting end-of-life vehicle recycling. b) Eco-friendly materials, closed-loop production processes to reduce water and chemical use, and textile recycling and upcycling can promote circularity in textile manufacturing. c) Food manufacturing can be circular by improving packaging, adopting sustainable farming techniques, and creating composting or bioenergy systems for organic waste.

The circularity concept initially emerged from considerations related to environmental sustainability, but its application can extend beyond environmental concerns. In a broader sense, circularity can encompass practices that enhance efficiency, resilience, and value creation within manufacturing processes. This includes optimizing resource utilization, minimizing waste generation, promoting product longevity through repair and refurbishment, and fostering closed-loop supply chains. Initially discussed in the context of electrical and electronic equipment, the proposed EOQ model and circularity concept can adapt and extend to diverse industries with varying product characteristics. By customizing the model parameters and generalizing the concept of circularity, manufacturers can optimize inventory management practices and promote sustainable, efficient, and resilient manufacturing processes across different sectors. A methodical way to use circular economy ideas in inventory management is to use an EOQ model for the circularity index that includes reducing waste and emissions in electrical and electronic equipment. By optimizing order quantities based on circularity considerations and environmental impact, companies can reduce waste, minimize emissions, and move towards a more sustainable and resource-efficient business model.

1.4 *Supply chain management*

The supply chain is the network of organizations, resources, activities, and procedures that produce, distribute, and deliver goods and services to customers. Supply chain management reduces production costs, waste, and time. Retail sales automatically notify the manufacturer of replenishment requests in the industry standard just-in-time supply chain. Retailers may then replenish shelves almost as rapidly as they sell goods.

2. Literature review

This section presents past research that forms the basis of our study. In this article, we discuss the development of traditional inventory systems, as well as some challenges associated with sustainable inventories.

2.1 *Carbon emission*

Carbon emissions play a pivotal role in global warming. Presently, several countries have prioritized the reduction of carbon emissions. Carbon tax and cap policies are fundamental mechanisms many countries employ to accomplish this goal. Wani et al. [4] examined how to maximize the profit of apple orchards while controlling emissions and deterioration effectively. Qi et al. [5] developed a joint decision model based on the conditional value-at-risk measure to examine the impact of a firm's risk aversion and investment coefficient on decisions. Mowmita et al. [6] created the "Excessive power consumption depletes stored fossil fuels and hurts the environment in terms of CO_2 emissions" model, which emphasizes energy-efficient goods in a credit-sales-variable-production energy supply chain management model with the ordered quantity. Dwicahyani et al. [7] studied an integrated inventory model for a closed-loop supply chain (CLSC) system with a producer and retailer. We expect stochastic demand and returns for used items. A carbon price program reduces emissions from transportation, manufacturing, and storage. The firm invests in green technology to reduce emissions from operations. As a result of adjusting production rates flexibly and establishing an appropriate collection rate, the supply chain can maintain emissions and costs.

2.2 *Circular economy*

The circular economy represents a production and consumption model that emphasizes sharing, leasing, reusing, repairing, refurbishing, and recycling existing materials and products for as long as feasible. This approach aims to prolong the life cycle of products. Rabta [3] investigated the circularity index to maximize profit, using the circularity level and order quantity as decision variables in this model. Thomas and Mishra [8] proposed a circular, integrated, and sustainable model of plastic reforming that utilizes 3D printing to reduce waste, emissions, and manufacturing costs. Arash et al. [9] analyzed the effect of preservation and greenhouse gas technologies on overall profit to allow decision-makers to judge pricing and replenishment more effectively. This study endeavors to attain the manufacturer's optimal integration of circularity and production levels, with the overarching goal of environmental preservation through carbon emission reduction via appropriate operational adjustments [10]. By analytically leveraging optimal policies, it identifies the global maximum profit for production managers. Particularly beneficial for electrical equipment manufacturers, such as mobile manufacturers, this research facilitates the determination of the ideal circular index for produced items, enabling them to maximize profits while also safeguarding the environment. Vennila et al. [11] developed an inventory model adopting EOQ principles to minimize food waste through circular economy strategies. It contributes to zero waste initiatives, particularly in repurposing food, by identifying optimal circular indices for food reuse, balancing profit maximization, and environmental conservation. Rabta is the first author to introduce the EOQ Model within the context of a circular economy, but he did not investigate applications related to reducing carbon emissions. Some authors have discussed production within a circular economy framework to minimize carbon emissions. However, no one has yet introduced the EOQ Model within a circular economy focusing on waste minimization and reducing carbon emissions. In this paper, we analyze the EOQ model within the circular economy context, aiming to minimize electrical and electronic waste to reduce carbon emissions.

2.3 *EPQ model*

Harris proposed the economic production quantity (EPQ) model in 1915, which extended the EOQ model by accounting for production setup costs. It determines the optimal production quantity by balancing holding costs, ordering costs, and setup costs. The EPQ model aims to minimize total production and inventory costs, offering insights into efficient production scheduling strategies. According to Khan et al. [10], manufacturers aim to assist manufacturers in achieving optimal levels of circularity and production while also protecting the environment from carbon emissions through strategic changes to their operations and maximizing global profit. Umakanta et al. [12] this study has three models: a sustainable economic production quantity carbon tax and cap model with no shortages, partial backordering, and full backordering with and without green technology investment. This methodology aims to determine the best methods for cycle time, green technology investment, and fraction period length at positive inventory levels. We explored several numerical instances to validate sustainable economic production quantity models.

2.4 *Closed loop system*

'Closed Loop Systems' or 'Circular Economy' aim to eliminate waste and pollution. It moves the traditional supply chain away from "take-make-waste" and into the 21st century. Closed-loop supply chains reduce production costs, reduce waste, improve service, keep customers coming back, and reduce pollution, which makes them good for the environment. In the 'Circular Economy', waste and pollution are designed out by using "closed loop systems". This new model brings the traditional supply chain into the twenty-first century, shifting from "take-make-waste" in a linear economy to "take-make-use-reuse-remake-recycle". Closed-loop supply chains aim to reduce and eliminate waste by creating a sustainable system. A reverse logistics process involves returns, resale, repairs, repackaging, and recycling. Liao and Deng [13] developed an optimization model to manage the unpredictable acquisition rate and market demand by coordinating forward and reverse production streams. They develop three strategies: (I) a reactive strategy to determine the optimal replenishment for the limited output of the remanufactured product; (II) a proactive strategy to implement an optimal remanufacture-up-to policy; and (III) a globally optimal strategy to maximize profitability in a closed-loop supply chain. Liao and Li [14] explained that the standard economic order quantity (EOQ) model optimizes ordering based on constant market demand, which leads to visible departures under uncertainty. This study modifies the conventional EOQ model for closed-loop supply chain (CLSC) systems. In this paper, we challenge the conventional assumption of constant market demand within the traditional EOQ model. The study delves into both cost minimization and benefit maximization challenges within the production and operational processes, taking a holistic view of the entire CLSC.

2.5 *Supply chain management*

Wahab et al. [15] presented a two-level supply chain, model to determine the optimal production-shipment policy for items with imperfect quality in three scenarios: (a) both the vendor and the buyer are in the same country, (b) the vendor and the buyer are in different countries where the stochastic exchange rate between the two countries is modelled using a mean-reverting process, and (c) environmental impact is incorporated. The goal is to reduce the overall estimated cost per unit of time. The solution procedure is proposed after the total expected cost per unit time is deduced for each scenario, assuming equal shipment size. Biswajit et al. [16] examined a green supply-chain management method for biodegradable goods that reduces pollution and uses outsourcing exclusively to regulate bioproduct quality for multi-retailers. Biswajit et al. [17] proposed eliminating supply-chain food waste. Parallel two-stage supply chains comprise the model and the linear model (produce, use, and throw away) is followed by the secondary chain, which recycles food scraps into animal feed. The recycled products are completely consumed by the secondary chain consumer, resulting in waste elimination. Taleizadeh et al. [18] explained an EOQ model that incorporates a special sale price and partial backordering. Author demonstrate the convexity of the cost-reduction function when a special order is placed at the sale price. The circularity index of the item determines the consumer demand and the unit profit. This paper examined three partial backlog scenarios: Linear demand vs linear unit profit, exponential unit profit, and logistic unit profit. Karthick et al. [19] examine the best optimal supply chain solution under uncertain demand using a genetic algorithm to minimize the total supply chain cost.

2.6 *Renewable energy*

Renewable energy refers to energy derived from naturally replenishable sources such as sunlight, wind, rain, tides, and geothermal heat, which are virtually inexhaustible and environmentally friendly. Electronics and electrical machines play a crucial role in harnessing and utilizing renewable energy effectively. One key aspect of renewable energy involves the conversion of natural resources into electricity. Electronics are utilized in devices like solar panels, which convert sunlight into electricity through the photovoltaic effect. Similarly, wind turbines harness the kinetic energy of wind and use electrical machines, such as generators, to convert it into electrical power. These generators often employ various electrical machines, like induction generators or permanent magnet synchronous generators, to convert mechanical energy into electrical energy. The collaboration among renewable energy, electronics, and electrical machines enables effective capture, conversion, and utilization of clean energy sources, promoting a sustainable energy future. Here we discussed some papers regarding how renewable energy intertwines with the concept of circularity by contributing to a more sustainable and regenerative energy system. Circular economy principles emphasize minimizing waste and maximizing the reuse of resources. Some renewable energy systems, such as bioenergy, utilize organic waste streams as feedstock. By converting waste materials into energy, these systems contribute to waste reduction and resource recovery, aligning with circular economy objectives. The integration of renewable energy with circular economy principles offers a pathway toward a more sustainable energy future by promoting resource efficiency, waste reduction, and environmental control.

The self-excited induction generator (SEIG) and the permanent magnet synchronous Generator (PMSG) are compared in this paper in terms of how they work and how well they do in small-scale, isolated applications [20]. It studies both generators under the same prime mover and load conditions, examining self-excitation and voltage regulation in SEIG operation and comparing them with PMSG performance. The findings indicate that both generators are suitable for small-scale isolated system deployment. This paper extensively examines the influence of excitation capacitance and rotor speed on the voltage and frequency of a three-phase self-excited induction generator (SEIG) [21]. It conducts numerous experimental tests to address the critical concerns of maintaining constant voltage and frequency profiles for end-users. The study proposes a method to mitigate under and over-excitation issues during SEIG excitation by operating the generator within the saturation region. Results and discussions demonstrate that maintaining excitation capacitance values within ±0.05% per phase and rotor speed within ±0.008% of optimal operational values is crucial to ensuring voltage and frequency remain within acceptable ranges during the initial start of the three-phase SEIG. This paper investigates the performance of a brushless direct current (BLDC) motor under transient and steady-state conditions using two different controllers, namely PI and ANFIS, implemented in MATLAB [22]. The findings indicate that the ANFIS controller consistently surpasses the PI controller across all speeds and diverse operational scenarios. The study suggests potential applications of BLDC motors with ANFIS controllers, particularly in electric vehicles where continuous torque is frequently needed.

2.7 *EOQ model*

Since the early 20th century, inventory management has concentrated on economics. Inventory management is a challenge across sectors due to its strong relationship with various costs. Demand, payment methods, and product deterioration affect inventory costs, which arise due to ordering, shipping, storing, quality control, and trash disposal. Business revolves around customer demand. The selling price usually determines client demand. The EOQ model, introduced by Harris in 1913 [23], is the first mathematical framework for inventory and production challenges. It determines the optimal production lot size by balancing intangible inventory costs with tangible ordering costs, assuming a continuous and constant demand rate. The primary objective of the EOQ model for circularity in electrical and electronic equipment would be to minimize total inventory costs while simultaneously maximizing circularity and minimizing waste and emissions. Mokhtari [24] determined to optimize inventory system cost and define the economic order quantity for items purchased (ordering policy) and batch amount for recovery and reuse activities (reuse policy). This study incorporates different holding costs into two inventory models for both usable and used items. Taleizadeh et al. [25] argued for EOQ models that assume two different assumptions about when the price increase will occur and that allow for partial backordering as a result. The author shows the concavity of new profit functions by placing a special order before a price rise. Taleizadeh et al. [26] proposed three distinct scenarios: no shortage, complete backordering, and partial backordering, as described for an EOQ model with multiple prepayments. Makoena et al. [27] aim to maximize the expected total profit to determine the ideal inventory policy. Mallick [28] analyzed the minimization of production cost and profit to find the EOQ formula with an optimal average cost. Vincent et al. [29] proposed a novel model that takes into account the relationship between an inventory policy (EOQ), total carbon emissions, and both price- and environmental-dependent demands. To determine a firm's ability to maximize profit while minimizing carbon emissions, Taleizadeh et al. [30] examined four cases where the products, after repair, were of imperfect quality upon arrival at the store. Mashud et al. [31] author investigated an optimized selling price model, along with investment and replenishment planning, to maximize total profit and additionally, the emission of greenhouse gases (GHGs), including CO_2, during the transportation of purchased items by the retailer was also factored into the model. This study examines Sanni et al.'s [32] proposed reverse logistics EOQ model. Profit maximization solves the reverse flow inventory issue of when and how much to order. The author developed the karush-kuhn-tucker (KKT) criteria for the objective function and the square-root equations for the firm's order size and price to solve the model's nonlinear maximization problem. Nonaka et al. [33] presented an EOQ model for reuse and recycling, which expands on the model Dobos and Richter developed in 2004. In this paper, we discussed the circularity index, in the context of the circular economy, which measures the effectiveness of a product or material's circulation within the system. It reflects the level of sustainability and efficiency in the use and reuse of resources throughout the product's life cycle. The assumption about the circularity index influencing product demand is likely based on the idea that consumers and businesses are increasingly valuing products with higher circularity index scores. The primary objective of the EOQ model for circularity in electrical and electronic equipment would be to minimize total inventory costs while simultaneously maximizing circularity and minimizing waste and emissions. Rabta [3] introduced the mathematical model for decision-making in a circular economy. Specifically, it proposes an economic order quantity inventory model for a product that has a quantifiable level of circularity. Our objective is to encourage enterprises to adopt CE practices by highlighting financial benefits and prospects. Understanding how circularity indicators affect behavior and decisions is crucial to our approach. After discussing the EOQ model in connection with waste minimization of electrical and electronics equipment to reduce carbon emissions, we examined the results regarding order quantity with circular economy. Circularity is assessed using an index that ranges between 0 and 1.

2.8 *Research limitations*

Reverse logistics in the context of waste electrical and electronic equipment (WEEE) plays a crucial role in implementing a circular economy in real life. It involves the processes and systems used to manage the collection, transportation, sorting, refurbishment, recycling, and disposal of discarded electronic devices and equipment. Rabta [3] examined the circularity index and profit using linear demand, per unit gross profit, or both, offering more realistic optimum policies for enterprises adopting Circular Economy (CE) practices. However, carbon emissions from inventory activities were not addressed. In contrast, Thomas [8] explored the circular index, incorporating customer environmental knowledge, demand, and per-unit gross profit, with a focus on reducing carbon emissions. Khan [10] aimed to fill this gap by developing a model for a sustainable production system considering circularity level, carbon emissions, demand, and gross profit, both linear and nonlinear about the circularity level. This research aims to construct a sustainable ordering system integrating production policies, circularity levels, carbon emissions, demand, and gross profit, assisting laptop manufacturers in maximizing earnings and environmental protection. The goal is to reduce carbon emissions from order quantities and inventory activities within the CE framework by creating a model that integrates linear and nonlinear circularity levels.

Rabta [3] investigated the impact of product circularity on the demand structure within an EOQ inventory model, focusing on determining the optimal inventory levels for retailers. However, this study did not explore the optimal circular economic index policy for manufacturers. It is crucial to understand the appropriate circular economy index policy for manufactured goods, as the manufacturing process establishes the circularity of the product. This leads to the research question: What is the most effective circular economy index policy for manufacturers to reduce carbon emissions while ordering items? This research aims to assist order managers and manufacturers in identifying the most suitable circularity index for their products. Solving the profit maximization problem determines the optimal order quantity and circularity level. Note that our optimization problem assumes the normalization of the circularity index. We provide numerical examples to illustrate the calculations involved. The following Figure 2 explains the structure of recycling E-waste.

Figure 2. Process of recycling E-waste

3. Assumptions

1. The circularity index determines the product's demand rate $\theta \in [0,\ 1]$ (Rabta [3], Thomas and Mishra [8]).

2. The demand and per-unit gross profit vary in correlation with the quantity of items produced, either in a linear or nonlinear manner. Consequently, both the demand and per-unit gross profit are influenced by the extent to which the produced items are available for remanufacturing purposes (Khan et al. [10]).

3. The manufacturer continually produces and sells a single item to fulfill customer demand over an indefinite period. Moreover, the remanufacture rate consistently exceeds the rate of customer demand (Khan et al. [10]).

4. The setup cost for production is not influenced by the circularity level of the items being produced. Likewise, the holding cost of products over a unit of time remains unaffected by the circular index.

5. As demand is predictable, excess inventories and shortages are prohibited.

6. The manufacturer accounts for carbon emissions resulting from different activities, including setup operations, warehouse operations, and the production process, as the primary sources. Consequently, the manufacturer incurs penalties from authorities or government agencies based on the total carbon emissions.

7. The unit acquisition cost and the unit selling price (thus the unit gross profit) are functions of θ.

8. The demand rate is deterministic, known, and constant for a fixed price.

9. Figure 1 shows two distinct economic order systems that directly illustrate the effects of varying circularity levels on the overall system. Altering the circularity level of items affects the entire system in each business period, leading to fluctuations in both the duration of business cycles and the manufacturer's profits.

4. Materials and methods

The economic order quantity (EOQ) model is a traditional inventory management approach used to determine the optimal order quantity that minimizes total inventory costs. When considering electronic waste and circularity to reduce carbon emissions, we adapt the EOQ model to account for the circularity of electronic products. In this context, circularity refers to designing products and systems to keep materials in use for as long as possible through recycling, refurbishing, or reusing components. By integrating circularity into the EOQ model, we aim to optimize inventory management while promoting sustainability and reducing carbon emissions associated with electronic waste. Rabta [3] described a known and deterministic demand that is accompanied by linear costs. We adapt this model to incorporate the circularity index of the product. In this single-product, single-location inventory model, the manager replenishes inventory with a fixed quantity Q at the start of each period to meet the upcoming demand. The product is available in two versions: a standard one and a more circular version, with circularity measured on a scale from 0 to 1. Both the demand rate and the unit gross profit, calculated as the selling price minus acquisition cost per unit, vary based on the product's circularity level. Followed by Rabta [3] and Khan et al. [10] developed the production model into applications (mobile manufacturers) to earn the maximum profit and protect the environment. In this study, we aim to reduce the waste minimization and carbon emissions of electronic and electrical equipment with an EOQ model of circularity.

Here we calculate the manufacturer's total profit per unit of time for a single item in a circular economy, taking into account all of the specified assumptions. As both depend on the circularity index of the product, it is first important to describe the structures of the clients' demand and per unit gross profit. Using the linear forms is a straightforward method of including the circular index's effect on demand and profit per unit. Let $d(\theta) = a_0 + a\theta$, $P(\theta) = p_0 + b\theta$, where $a \geq 0$, $b \leq 0$, $a_0 > 0$, $p_0 > 0$ are the index level of parameters for finding the unit gross profit. Assume linear relationships where a and b are constants to account for the effect of circularity labelling on demand and unit profit. In our model, we assume linearity for those two components, but that assumption might be simplistic and unrealistic. However, their simplicity prevents these linear forms from accurately reflecting the effects of circular leveling. In this case, non-linear structures are more appropriate for exploring the consequences effectively. Evaluating and optimizing inventory management strategies can help maximize the benefits of circularity while minimizing carbon emissions associated with electronic waste.

Consider, the non-linear relationship forms to be an exponential form which is $P(\theta) = p_0 + be^{\alpha(\theta-1)}$, where $p_0, a > 0$, $b \leq 0$ and $\theta \in [0,\ 1]$. Accordingly, this characteristic of gross profit per unit is realistic in that its implementation costs increase, and then decrease more slowly. A non-linear approach is also used to study clients' demand, for example, the logarithm forms $D(\theta) = a_0 + a\log(1+\gamma\theta)$, where a_0, a, γ are constant $a_0, \gamma > 0$. A realistic approach would be to improve the circular index of the products up to a certain level, and then after that, it would become more challenging as well as more expensive. The logistics form: $P(\theta) = p_0 + \dfrac{b}{(1+e^{-\alpha(\theta-\theta_0)})}$, where p_0, $\alpha > 0$. Additionally, exploring alternative nonlinearities can help understand how circularity levels affect demand and per-unit gross profit. Linear or nonlinear demand structures are just particular cases, with linear demand being an approximation of nonlinear ones. Similarly, linear or nonlinear per-unit gross profit structures are approximations of nonlinear ones, with the exponential structure being an approximation of the logistic pattern. This study aims to provide precise optimal policies for manufacturers under various scenarios by examining nine combinations of demand and per unit gross profit. We calculate the total profit per unit time by considering general forms of demand and per unit gross profit functions using the circular index. During a half-yearly business period, manufacturers produce and then sell Q quantities of products for a gross profit per unit $P(\theta)$. As a result, Q quantity gross profit equals $Q * P(\theta)$. The manufacturer's setup cost and total holding cost are I and $\frac{1}{2}HQ\left(1-\frac{d(\theta)}{R}\right)$ respectively. Manufacturers must pay a carbon tax based on the carbon emitted during setup, warehouse operations, and production. The manufacturer is responsible for paying a carbon tax to the authorities, which is determined by the amount of carbon emissions generated during production system setup, warehouse operations, and production processes. Consequently, the manufacturer's total carbon tax for one business period is $t\left[e+Fd(\theta)+\frac{G}{2}Q\left(1-\frac{d(\theta)}{R}\right)+\left(1-\frac{d(\theta)}{R}\right)^2\right]$. Total profit per unit time is

$$\phi(Q,\ \theta)=d(\theta)p(\theta)-\frac{d(\theta)K}{Q}-\frac{1}{2}HQ\left(1-\frac{d(\theta)}{R}\right)-t\left[e+Fd(\theta)+\frac{G}{2}Q\left(1-\frac{d(\theta)}{R}\right)+\left(\left(1-\frac{d(\theta)}{R}\right)^2\right)\right]. \quad (1)$$

$$\phi(Q,\ \theta)=d(\theta)p(\theta)-tFd(\theta)-\frac{d(\theta)(K+te)}{Q}-\frac{1}{2}(H+Gt)\left(1-\frac{d(\theta)}{R}\right)Q-\left(1-\frac{d(\theta)}{R}\right)^2Q. \quad (2)$$

This objective is to determine the optimal number of produced quantities during a business period as well as the optimal circular level $\theta^* \in [0,\ 1]$ of the produced quantities so that the manufacturer can achieve the highest profit per unit of time. This results in the following optimization problems:

$$\max \pi(Q,\ \theta),\ \text{where}\ \ \sigma=\{(Q,\ \theta):0<Q<\infty\ \text{ and }\ 0\le\theta\le 1\}$$

$$(Q,\ \theta)\in\sigma \quad (3)$$

4.1 *Solution techniques*

According to the circular index of produced quantities, the optimal solution can be classified into three cases depending upon demand and gross profit per unit:

Case 1 Considering a situation in which both demand and gross profit per unit increase with respect to θ while the θ is increasing, the best solution $\theta^* = 1$ is and

$$Q=\sqrt{\frac{2(a_0+a)(K+te)}{\left((H+Gt)\left(1-\frac{a_0+a}{R}\right)\left(1-\frac{(a_0+a)}{R}\right)^2\right)}}$$

Case 2 Considering a situation in which both demand and gross profit per unit increase with respect to θ while the other is decreasing, the best solution $\theta^* = 0$ is and

$$Q=\sqrt{\frac{2a_0(K+t)}{\left((H+Gt)\left(1-\frac{a_0}{R}\right)\left(1-\frac{a_0}{R}\right)^2\right)}}$$

Case 3 It may differ from the solutions in cases 1 and 2, in this case of an increasing demand function and a decreasing per-unit gross profit function concerning that while the Lagrange multiplier technique is used in this situation to identify the optimal problem (3). When lagarangian method dealing with optimization problems subject to equality and/or inequality constraints, the Lagrangian method provides a systematic way to find the extrema (maxima or minima) of the objective function subject to these constraints. Here, the Lagrange function of the problem (3) is defined as

$$L(Q,\ \theta,\ \rho_1,\ \rho_2) = d(\theta)p(\theta) - tFd(\theta) - \frac{d(\theta)(K+te)}{Q}$$
$$-\frac{1}{2}(H+Gt)\left(1-\frac{d(\theta)}{R}\right)Q - \left(1-\frac{d(\theta)}{R}\right)^2 Q - \rho_1(\theta-1) + \rho_2\theta \tag{4}$$

An optimal necessary condition of Karush-Kuhn-Tucker (KKT) solution requires:

$$\frac{\partial L(Q,\theta,\ \rho_1,\ \rho_2)}{\partial Q} = \frac{d(\theta)(K+te)}{Q^2} - \frac{(H+Gt)}{2}\left(1-\frac{d(\theta)}{R}\right) - \left(1-\frac{d(\theta)}{R}\right)^2 = 0 \tag{5}$$

$$\frac{\partial L(Q,\ \theta,\ \rho_1,\ \rho_2)}{\partial \theta} = d'(\theta)P(\theta) + P'(\theta)d(\theta) - tFd(\theta) - \frac{d'(\theta)(K+te)}{Q}$$
$$+\frac{1}{2}(H+Gt)d'(\theta)Q + 2\left(1-(a_0+a\theta)\,a\right)Q - \rho_1 + \rho_2 = 0 \tag{6}$$

$$\rho_1(\theta-1) = 0 \tag{7}$$

$$\rho_2\theta = 0 \tag{8}$$

$$\rho_1,\ \rho_2 \geq 0 \tag{9}$$

The complementary conditions (5), (6) provides the following 3 conditions solutions of the optimization problem (3).

Case 1 ρ_1 =0, and $\theta = 0$,

$$Q = \sqrt{\frac{2d(0)(K+te)}{\left((H+Gt)\left(1-\frac{d(0)}{R}\right)\left(1-\frac{d(0)}{R}\right)^2\right)}}$$

$$\rho_2 = \left[tF + \frac{(K+te)}{Q} - \frac{Q(H+Gt)}{2} - P(0) + 2Q(1-d(0))\right]d'(0) - P'(0)d(0)$$

Case 2 $\rho_2 = 0$, and $\theta = 1$,

$$Q=\sqrt{\frac{2d(1)(K+te)}{\left((H+Gt)\left(1-\frac{d(1)}{R}\right)\left(1-\frac{d(1)}{R}\right)^2\right)}}$$

then

$$\rho_1=d'(1)P(1)+P'(1)d(1)-\left[tF-\frac{(K+te)}{Q}+\frac{Q(H+Gt)}{2}+2Q(1-d(1))\right]d'(1)$$

Case 3 $\rho_1=0$ and $\rho_2=0$ from equation (4) one has

$$Q=\sqrt{\frac{2d(\theta)(K+te)}{\left((H+Gt)\left(1-\frac{d(\theta)}{R}\right)\left(1-\frac{d(\theta)}{R}\right)^2\right)}}$$

and from (5) we get

$$d'(\theta)P(\theta)+P'(\theta)d(\theta)-tFd'(\theta)-\left[\frac{(K+te)}{Q}+\frac{Q(H+Gt)}{2}+2Q(1-d(\theta))\right]d'(\theta)=0 \tag{10}$$

$$\left[d'(\theta)P(\theta)+P'(\theta)d(\theta)-tFd'(\theta)\right]^2=\left[\frac{(K+te)}{Q}+\frac{Q(H+Gt)}{2}+2Q(1-d(\theta))\right]d'(\theta) \tag{11}$$

Now equation (above) is expressed as

$$\pi(\theta)=\left[d'(\theta)P(\theta)+P'(\theta)d(\theta)-tFd'(\theta)\right]^2-\left[\frac{(K+te)}{Q}+\frac{Q(H+Gt)}{2}+2Q(1-d(\theta))\right]d'(\theta) \tag{12}$$

There are multiple possible solutions, all of which must satisfy the feasibility requirements, such as the positivity of Q, [0, 1] and equation (11). To determine the second-order derivative constraint is evaluated for the local optimizer. The global optimizer compares local optimizer profits when the firm has many local optimizers. KKT conditions provide an optimum solution if the manufacturer's profit $\pi(Q,\ \theta)$ is concave. As a result, the concavity of $\pi(Q,\ \theta)$ is completely determined by the structure of both $d(\theta)$ and $P(\theta)$. Here we discussed the four possible combinations of the form $d(\theta)$ and $P(\theta)$.

Total profit per unit of time is

$$\pi(Q,\ \theta)=d(\theta)p(\theta)-\frac{d(\theta)K}{Q}-\frac{1}{2}HQ\left(1-\frac{d(\theta)}{R}\right)-t\left[e+Fd(\theta)+\frac{G}{2}Q\left(1-\frac{d(\theta)}{R}\right)+\left(\left(1-\frac{d(\theta)}{R}\right)^2\right)\right] \tag{13}$$

$$\pi(Q,\ \theta) = d(\theta)p(\theta) - tFd(\theta) - \frac{d(\theta)(K+te)}{Q} - \frac{1}{2}(H+Gt)\left(1-\frac{d(\theta)}{R}\right)Q - \left(1-\frac{d(\theta)}{R}\right)^2 \tag{14}$$

$$\max \pi(Q,\ \theta) \in \varphi, \text{ where } \varphi = \{(Q,\ \theta) : 0 < Q < \infty \text{ and } 0 \le \theta \le 1\}$$

$$\pi(Q,\ \theta) = d(\theta)p(\theta) - tFd(\theta) - \frac{d(\theta)(K+te)}{Q}$$

$$-\frac{1}{2}(H+Gt)\left(1-\frac{d(\theta)}{R}\right)Q - \left(1-\frac{d(\theta)}{R}\right)^2 Q - \rho_1(\theta-1) + \rho_2\theta \tag{15}$$

By solving the EOQ model with circularity considerations, we determine the optimal order quantity that minimizes total costs while promoting circularity in electronic products. This optimal order quantity takes into account factors such as the availability of recycled materials, refurbishing capabilities, and environmental impact.

Theorem A Let two linear relationships, if H is the negative semi-definite then prove that the objective functions is concave.

Proof. See Appendix A.

Theorem B Let linear and logistic relationships, if H is the negative semi-definite then prove that the objective functions is concave.

Proof. See Appendix B (Figure 3 and 4).

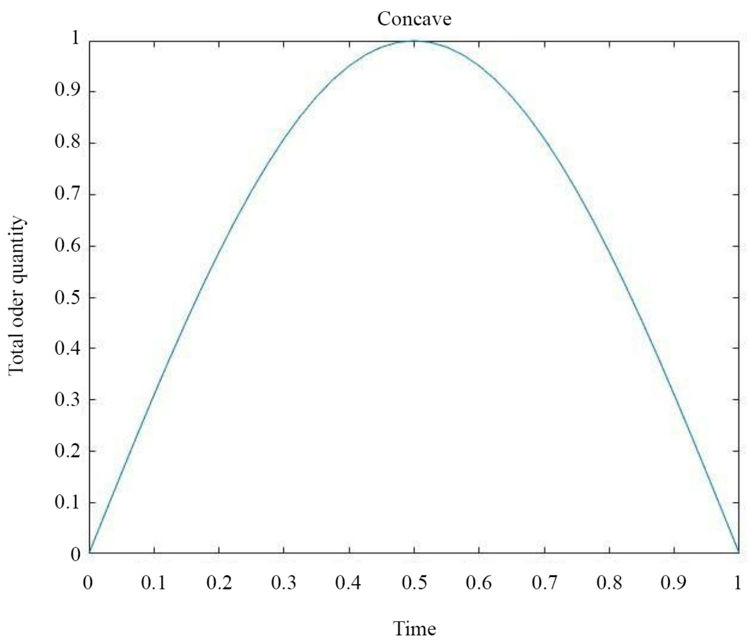

Figure 3. Curve is negative semi-definite and $\phi(Q,\ \theta)$ is concave

Theorem C Let logarithmic and exponential relationships, if H is the negative semi-definite then prove that the objective functions is concave.

Proof. See Appendix C.

Theorem D Let two logistics relationships, if H is the negative semi-definite then prove that the objective functions is concave.

Proof. See Appendix D.

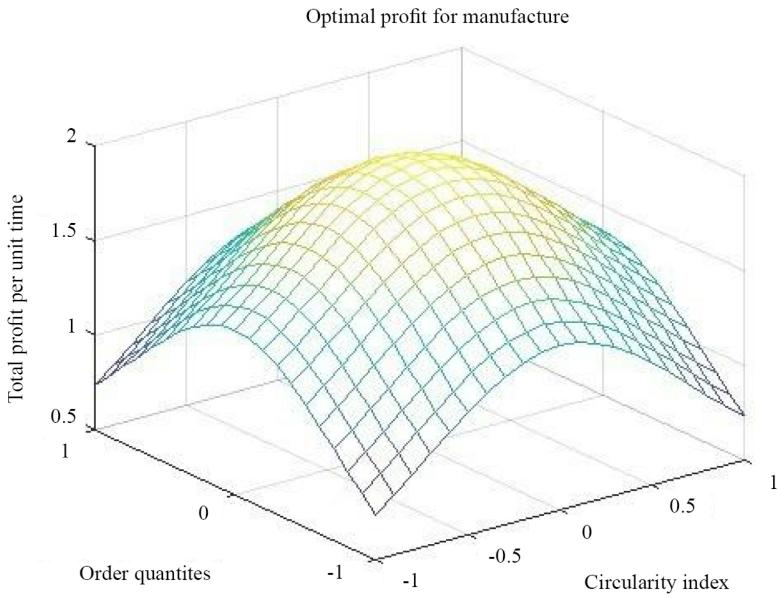

Figure 4. Shape of the objective functions non-linear relationship

5. Illustrations of linear relationship

Let's now utilize numerical illustrations to demonstrate the outcomes of the analysis discussed earlier and examine how the form of the demand and unit profit functions, along with different parameters, influence the optimal solution and profit. Let us consider the fixed order cost: according to Rabta [3] the same values are fixed in this example and comparing the values of proposed model into K = \$ 180, Holding cost: H = \$ 0.2. A linear relationship is also assumed between unit gross profit and the circularity index of the product. Furthermore, we'll assume a base demand rate a_0 = 60,000 units per year (D = 60,000) for the non-circular version of the product, and a base unit gross profit of \$ 2 (p_0 = \$ 2). The remaining parameters of the model will be varied across different scenarios to explore their impact on the optimal solution and profit.

The demand depends linearly on the circularity index of the product, expressed as: Demand $d(\theta) = a_0 + a\theta$, with a maximum additional demand factor of 8,500 units/year. Furthermore, the unit gross profit varies linearly with the circularity index according to the equation $P(\theta) = p_0 + b\theta$, where the unit premium factor $b = -0.25$. As a result, demand may increase by up to 14% while gross profit may decrease by 12.5%. Due to the higher circularity level of the product, additional costs are expected. The optimal solution is Q^* = 215,989.21 with gross profit, π^* = 18,802.

In this scenario, there's a minimal disparity between the optimal profit and the profit derived from trivial solutions θ = 0 and θ = 1. The model exhibits little sensitivity to slight changes in θ. This behavior is unique to this linear model with specific parameter values, as illustrated in other examples below. This observation is that since demand increases and unit profit decreases respectively, their impact on total profit tends to offset each other. Consequently, transitioning to a more circular product might not entail significant profit loss under certain conditions, as long as the added costs are balanced by the increased demand.

However, when one of the functions $D(\theta)$ and $P(\theta)$ changes more than the other, as demonstrated in the subsequent example, the profit function varies notably. Assuming the same parameters as before but with a higher value for a (resulting in up to a 20% increase in demand), the optimal solution now shifts to $\theta = 1$, $Q^* = 215,989.21$, $\pi^* = 18,802.0$. Ignoring the circularity index's impact and treating the problem as a classical economic order quantity (EOQ) calculation yield θ = 1, Q = 22,865.3, and π^* = 167,921, representing a 2.6% profit loss.

6. Non-linear relationships

6.1 *Illustration of linear and logistic relationships*

The model by considering nonlinear demand and unit profit functions. Embracing a non-linearity assumption not only renders the model more realistic but also reveals distinct behaviors compared to the linear case, potentially featuring multiple local extremes and greater profit variation across different circularity levels. Unit profit functions and demand functions offer numerous possibilities for combination. Our attention is directed towards a select few of these combinations. For instance, profit levels may display more variation among different circularity levels and manifest multiple nearby extrema. Here the demand is linear and a product's nonlinear (logistic) relationship determines its unit gross profit based on the circularity index $d(\theta) = a_0 + a\theta$ and $P(\theta) = p_0 + \frac{b}{(1+e^{-\alpha(\theta-\theta_0)})}$. Let $a = 8,500$, $b = -0.25$ and α = 10. The optimal solution is Q^* = 138,636.35, θ = 0.66 with gross profit, π^* = 27,348.16.

In comparison to the two straightforward feasible solutions: where θ = 0 and Q = 138,636.35 resulting in an annual profit of "\$" 27,348.16 and where θ = 1 and Q = 139,565.33 yielding an annual profit of "\$", the difference between the optimal solution and the basic ones is notable this time. This time due to the significant decrease in unit profit influenced by the exponential (representing a potential 20% rise in demand). In this scenario, the optimal solution becomes θ = 0.74 and Q = 13,365.36 with an optimal profit of \$ 130,841. This result demonstrates a respective 10% and 5% increase in profit compared to the trivial solutions where θ = 0 and θ = 1.

6.2 *Illustration of logarithmic and exponential relationship*

The demand is a logarithmic function $D(\theta) = a_0 + a\log(1+\gamma\theta)$ of the product's circularity index: whereas the unit profit function is nonlinear of the form exponential function $P(\theta) = p_0 + be^{\alpha(\theta-1)}$. Let a = 8,500, $b = -0.25$, and $\alpha = 5$. The solution overall profit, Q^* = 24,050.26 with gross profit, π^* = 16,154.28. If α = 10 then Q = 35,462.54 and gross profit π^* = 16,154.28. The clients' demand increases sharply under this combination while the gross profit per unit decreases gradually. The numerical findings demonstrate that manufacturing non-circular products becomes more profitable for the manufacturer compared to producing entirely circular products as demand follows a logarithmic growth pattern and per-unit gross profit diminishes exponentially concerning θ.

6.3 *Illustration of logarithmic and logistic relationship*

Here the demand is both a nonlinear function is logarithmic $D(\theta) = a_0 + a\log(1+\gamma\theta)$ of the product's circularity index: whereas the unit profit function is nonlinear of the form is exponential $P(\theta) = p_0 + \frac{b}{(1+e^{-\alpha(\theta-\theta_0)})}$. Once more, the objective function does not exhibit concavity. The Lagrangian's stationary points are identified by solving (nonlinear) Equation. Let $a = 8,500$, $b = -0.25$, $\theta = 0$, $\theta_0 = 0.6$, and $\alpha = 10$. The solution overall profit, $Q^* = 26,262.65$ with gross profit, $\pi^* = 18,265.36$. If $\theta = 0.78$ then $Q^* = 27,563.2$ with gross profit $\pi^* = 19,965.3$. Among these, the two local maximizers are located after verifying the second-order optimality condition. Therefore, the alternative solution stands as the global optimum, yielding a profit 1% greater than that associated with the solution equal to 0 and 2.6% higher than if it equals 1. However, the least favorable solution arises when it equals 0.39 and Q equals 26,472, as this marks a saddle point of the objective function. At this juncture, the profit diminishes by 5.6% compared to the optimal value. This analysis suggests that when logarithmic demand and logistic per-unit gross profit are combined if the threshold is approaching θ = 1, it is advisable for the manufacturer to consistently produce products with a circularity level lower than the threshold

Conversely if the threshold $\theta = 0$, the manufacturer should consistently optimal for products with the highest circularity level to maximize profit.

7. Impact on carbon emissions from the proposed order quantity system

Table 1 shows how utilizing the optimal manufacturing system in CE practices can impact profit and carbon emissions. The first column of Table 1 displays the optimal solution and greatest profit for all instances in this paper. If all examples are resolved without carbon emission ($e = f = g = 0$ and $t = 0$), the best policies and profit are listed in the second column of the table. When comparing the first and second columns, it is clear that the ideal production amount and profit are always higher in the non-carbon emission case, but the optimum circularity index is not always higher. When circularity level varies then the order quantity and profit are increased. The overall carbon emissions quantity is calculated using the optimum solution for both with and without carbon emissions. When a manufacturer ignores carbon emissions from production activities and subsequently implements eco-friendly policies, profits decrease. The proposed order system's optimal policies aid in profit growth for the management.

Table 1. Reduction of carbon emission with circularity system and profit increment

Illustration	With carbon emission			Without carbon emisision		
	θ^*	Q^*	π^*	θ^*	Q	π^*
1.	0.3388212	145,896.32	15,148.36	0.3388212	138,636.35	17,348.16
2.	1.742367	1,698,563.32	16,145.52	1.742367	24,050.26	16,154.28
3.	0.403013	182,632.2	175,846.21	0.403013	26,262.65	18,265.36
4.	0.41145	187,952.36	175,862.32	0.41145	215,989.21	18,802.06

8. Sensitivity analysis

We present a numerical analysis in this section to identify two main objectives: 1) to determine the total profit and quantities' circularity levels as a function of changes (above or below) in parameters and base values; 2) to determine how the parameters affect an optimal policy for varying types of demand and gross profit per unit. Results are tabulated after altering each parameter's value in two ways: increasing and decreasing it from the base value. A total profit can be calculated using both linear and exponential unit gross profits in MATLAB software. In our analysis, we conducted tests revealing that the total profit remains relatively stable even with slight variations in parameters a and b. In Figures 5 and 6 the blue bars show the profit variations and the yellow bars show the order Quantity variations from linear and non-linear forms.

The economic order quantity Q values for linear and nonlinear sensitivity analysis are determined from equations (17). Let's explore the impact of minor adjustments in the parameters on the optimal profit in Example 1, where linear relationships are observed. To accomplish this, we vary each parameter slightly (within a range of up to ± 10%), while holding all other parameters constant, and observe the resulting fluctuation in optimal profit. Our findings reveal that the model demonstrates remarkable resilience to variations in both the order quantity (Q) and circularity level. Even minor alterations in these parameters result in only negligible changes in total profit. This implies that managers have flexibility to deviate slightly from the optimal values if necessary. For instance, the order quantity could be adjusted to $Q = 13{,}863.35$ instead of the optimal value of $Q = 16{,}154.28$, without significantly impacting total profit. In a linear relationship, when carbon emissions per unit of production increase, the manufacturer reduces the circularity level of the product because he or she must pay greater taxes on the emissions; consequently, the investment made for circularity purposes decreases.

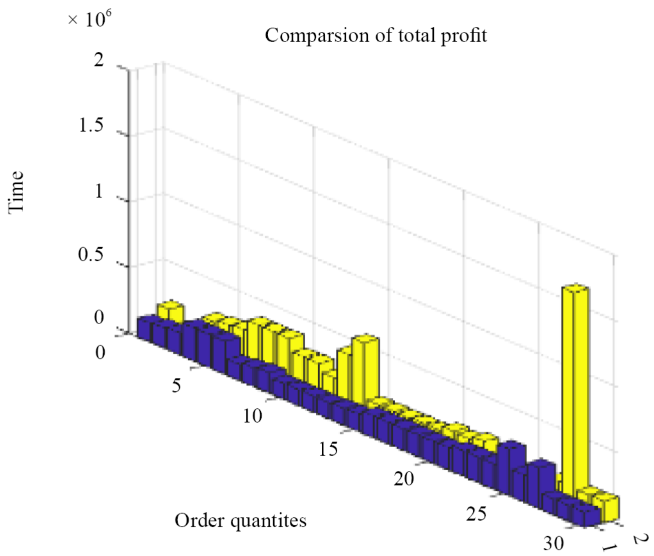

Figure 5. Analysis of Q values of linear and non-linear

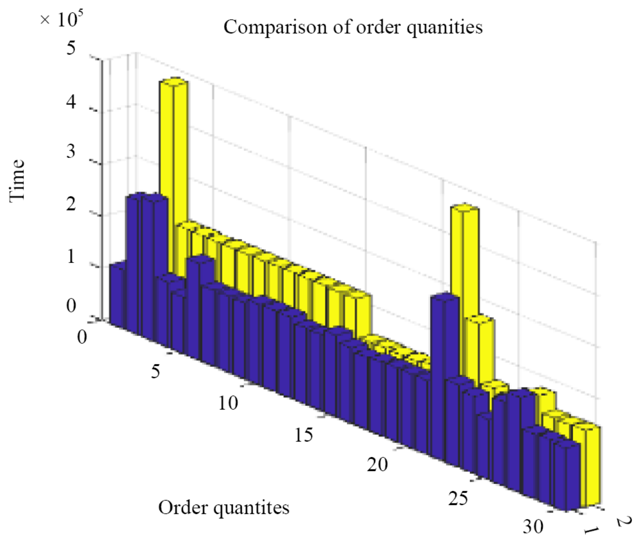

Figure 6. Analysis of π values of linear and non-linear

As illustrated in Table 2, 25% increase in emission (f) during per-unit production results in an approximate 6.77% decrease in the optimum circularity index for the linear per unit gross profit case. The outcome is comparable when the tax is increased in relation to the quantity of emissions per unit. Emissions from setup (e) and storage (G) have opposite effects on the optimal circular index compared to emissions from production (F). When emissions per unit grow, it indicates harm to the environment and reduces the circularity of the product. Table 2 shows that increasing emissions from unit item production (F) by 0.06 tons per unit under linear per unit gross profit results in a 5.7% drop in optimal circularity. While emissions from setup and storage procedures may grow or decrease, it does not indicate the item is harmful to the environment. Increased emissions from setup and storage procedures have an opposite effect on the ideal circular index. Although the optimal quantity generated. Increased setup costs for each cycle diminish the manufacturer's profit, despite favorable effects on optimal production quantity and circularity. When setup costs rise, producers want to manufacture multiple quantities in one setup to reduce the average setup cost per item. To reduce total holding costs for a large product, the production manager aims to enhance client demand by increasing the circularity of produced quantities. Increased setup costs for each cycle diminish the manufacturer's profit, despite favorable effects on optimal production quantity and circularity. When setup costs rise, producers want to manufacture multiple quantities in one setup to reduce the average setup cost per item. To reduce total holding costs for a large product, the production manager aims to enhance client demand by increasing the circularity of produced quantities. A reduction in demand from the manufacturer's primary clients results in a decrease in the overall quantity of items produced during each cycle. Furthermore, as a consequence of diminished client demand, the overall carrying cost for the manager escalates, leading to a marginal decline in profit. In order to decrease the overall carrying cost, the producer endeavors to enhance the circularity level of the manufactured items in order to augment total demand. Table 2 illustrates how, in response to a decline in base client demand from 60,000 to 59,000 units per year at a linear per unit gross profit, the manufacturer reduces the quantity produced by 8.16% while increasing the circularity level of the items by 1.6%. Moreover, the manufacturer experiences a decrease in profit of approximately 1.2%.

Table 2. Linear and exponential profits are compared

Parameter	Value	Structure of linear per unit gross profit		Structure of non-linear per unit gross profit	
		θ^*	π^*	θ^*	π^*
	60,000	113,754	143,250	171,550	136,587
d_0	63,000	258,960	156,420	211,060	145,632
	65,000	268,410	169,850	483,170	142,365
	1.1	126,250	255,000	219,040	265,770
t	1.3	109,980	252,740	218,700	287,820
	1.4	187,250	250,830	217,800	299,750
	0.01	147,010	123,650	219,040	381590
e	0.11	147,010	145,632	219,040	381,590
	0.12	147,010	156,324	219,040	381,590
	0.024	156,520	123,654	219,040	289,630
F	0.03	156,520	123,654	219,540	289,520
	0.01	156,520	123,654	219,530	289,650
	0.16	146,782	123,970	219,410	327,220
G	0.17	146,752	130,130	219,270	224,540
	0.19	158,623	133,210	219,200	191,791
	0.16	146,390	169,875	146,589	167,960
H	0.2	139,050	169,524	147,852	166,280
	0.18	142,580	169,234	146,325	164630
	190	143,650	170,230	147,380	169,830
K	210	143,660	170,420	152,020	170,610
	240	143,620	170,520	156,520	170,620
	68,000	142,563	196,520	156,326	171,218
R	67,000	16,0570	196,600	156,289	172,541
	66,000	150,230	196,760	269,000	170,212
	8,500	116,340	354,280	158,962	234,630
a	8,600	147,896	210,470	162,710	23,6120
	8,400	145,236	312,175	184,430	237,600
	-0.25	125,460	120,890	163,256	237,790
b	-0.22	126,540	120,890	175,632	235,220
	-0.24	123,650	1208,90	178,542	237,560

9. Discussion and comparative study

In this section, we cover the outcomes of numerical examples. The optimal solution table reveals that profits rise as the circularity index value of electronic product and total profit increases. The circularity index value for electronic products signifies their recycling capacity, while carbon emission denotes the reduction in environmental damage through the utilization of renewable energy resources. The optimal solution table provides clear evidence that as the circularity index value of economic order quantity rises, there is a corresponding increase in profit. Circularity integration involves considering factors such as product durability, recyclability, and end-of-life disposal strategies. This integration contributes to concavity in the optimization problem by introducing additional variables and constraints. In [3] Rabta paper, which examined the EOQ model focusing on known and deterministic demand with linear costs, we modify this model to include the product's circularity index. Within this inventory model for a single-product and single-location, the manager restocks inventory with a constant quantity Q at the beginning of each period. Rabta [3], After examining the model's reaction to minor adjustments in these parameters, we conducted a series of assessments. In the case at hand, we discovered that the overall profit demonstrates minimal fluctuation in response to alterations in parameters a and b, showcasing the model's high resilience. Even minimal fluctuation in response to alterations in parameters a and b, showcasing the model's high resilience. Even with a maximum deviation of ± 10% in either parameter, the total profit fluctuates by less than 0.6%.

In the comparative study, we adjusted the parameters a or b of our proposed model by up to ± 10%, and the resulting variation in total profit was less than 0.9% compared to the existing approach. We have managed to reduce overall profit more by using our applications. This study highlights the significance of Waste Minimization and Reduced Emission in Electrical and Electronic Equipment through an EOQ Model integrating the Circularity Index. The sustainable supply chain model integrates principles of environmental sustainability into the traditional economic order quantity (EOQ) model.

10. Conclusion

This paper presented an Economic Order Quantity inventory model within an EOQ Model for Circularity Index with Waste Minimization and Reduced Emission in Electrical and Electronic Equipment. We consider the product's manufacturing with varying circularity levels, affecting demand, costs, and selling prices. Hence, besides the order quantity, circularity level becomes a decision variable. When contrasting linear and non-linear analytical techniques, findings indicate that the non-linear approach demonstrates greater precision. Our objective is to minimize order expenses, integrate product recycling into the circular economy, diminish carbon emissions to safeguard the environment, and ultimately optimize overall business profitability. We addressed the proposed model using different demand and unit profit function combinations, determining sightly changes in optimal parameters both analytically and through illustrations. Linear relationships within the model demonstrate that higher circularity levels may incur additional costs, but these can be offset by increased demand. Nonlinear variations exhibit more intricate behavior and larger profit variations among circularity levels. This study showcases how circularity measures can inform company-level decision-making, indicating that producing circular products can be economically viable alongside environmental and social benefits. Shifting to circular product versions can often be achieved without significant profit loss by adjusting order quantities, while determining optimal circularity levels can be profits. The research study recommends adopting the E-waste nullification approach, which could serve as a vital tool for management in upholding sustainability. Among all expenses, the repair and recycling costs of end-of-life (EOL) electronic products emerge as particularly sensitive metrics. By selling scraps as raw materials to secondary manufacturers, management can alleviate the financial strain of these costs. Additionally, through the implementation of sustainable supply chain management, management can optimize both the quantity and order period, thus effectively controlling the rate of E-waste generation. In supply chain management, EOQ helps organizations balance inventory expenses and ordering costs, optimizing inventory management and supply chain efficiency.

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Conflict of interest

The authors declare no competing interest.

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Appendix A

Theorem A

Proof. Linear demand and gross profit are defined as follows

$$d(\theta) = a_0 + a\theta \text{ and } p(\theta) = p_0 + b\theta$$

where $a_0 > 0,\ p_0 > 0,\ a > 0,\ b < 0$.

$$\phi(Q,\ \theta,\ \rho_1,\ \rho_2) = d(\theta)p(\theta) - tFd(\theta) - \frac{d(\theta)(K+te)}{Q}$$

$$-\frac{1}{2}(H+Gt)\left(1-\frac{d(\theta)}{R}\right)Q - \left(1-\frac{d(\theta)}{R}\right)^2 Q - \rho_1(\theta-1) + \rho_2\theta \tag{16}$$

$$\frac{\partial\phi}{\partial Q} = \frac{d(\theta)(K+te)}{Q^2} - \frac{(H+Gt)}{2}\left(1-\frac{d(\theta)}{R}\right) - \left(1-\frac{d(\theta)}{R}\right)^2 \tag{17}$$

$$\frac{\partial^2\phi}{\partial Q^2} = \frac{-2d(\theta)(K+Te)}{Q^3} < 0 \tag{18}$$

$$\frac{\partial^2\phi}{\partial Q\partial\theta} = \frac{d'(\theta)(K+te)}{Q^2} + \frac{1}{2}(H+Gt)\frac{d'(\theta)}{R} + 2\frac{d'(\theta)}{R}\left(1-\frac{d(\theta)}{R}\right) \tag{19}$$

$$\frac{\partial\phi}{\partial\theta} = ap(\theta) + bd(\theta) - tfa - \frac{a(K+te)}{Q} + \frac{1}{2}(H+GT)aQ + 2\left(1-(a_0+a\theta)a\right)Q - \rho_1 + \rho_2 \tag{20}$$

$$\frac{\partial^2\phi}{\partial\theta\partial Q} = \frac{d'(\theta)(K+te)}{Q^2} + \frac{1}{2}(H+GT)\frac{d'(\theta)}{R} + 2\frac{d^{'(\theta)}}{R}\left(1-\frac{d'(\theta)}{R}\right) \tag{21}$$

$$\frac{\partial^2\phi}{\partial\theta^2} = 2ab + 2a^2Q \tag{22}$$

Thus, the hessian matrix $\phi(Q,\ \theta)$

$$H = \begin{vmatrix} \dfrac{\partial^2\phi}{\partial Q^2} & \dfrac{\partial^2\phi}{\partial\theta\partial Q} \\ \dfrac{\partial^2\phi}{\partial Q\partial\theta} & \dfrac{\partial^2\phi}{\partial\theta^2} \end{vmatrix} \tag{23}$$

Substitute equations (18), (19), (21), (22) in equation (23) we get

$$|H| = \left(\frac{-2d(\theta)(K+te)}{Q^3}(2a(a+b))\right) - \left[\left(\frac{-d'(\theta)(K+te)}{Q^2} + \frac{1}{2}(H+Gt)\frac{d'(\theta)}{R} + 2d'(\theta)\left(1-\frac{d(\theta)}{R}\right)\right)^2\right] \tag{24}$$

The concavity of $\phi(Q,\ \theta)$ requires on the Hessian Matrix being negatively semi-definite. Furthermore

$$\text{Det } H > 0 \Leftrightarrow \left[2\left(a_0 + a\theta\right)\frac{-2(K+te)}{Q^3}\right](2a(a+b)) > -\left[\frac{-a(K+te)}{Q^2} + \frac{1}{2}(H+Gt)\frac{a}{R} + 2a\left(1-\frac{d(\theta)}{R}\right)\right] \tag{25}$$

Hence, H is negative semi-definite and

$$(a_0 + a\theta) > -\frac{Q^3}{4(K+te)(2a(a+b))}\left[\frac{-a(K+te)}{Q^2} + \frac{1}{2}(H+Gt)a + 2a\left(1-\frac{d(\theta)}{R}\right)\right] \tag{26}$$

is concave. This makes one solution possible and optimal. Possible solutions:

(i) $\theta = 0$,

$$Q = \sqrt{\frac{2a_0(K+te)}{\left((H+Gt)\left(1-\frac{a_0}{R}\right)\left(1-\frac{a_0}{R}\right)^2\right)}}$$

and

$$\rho_2 = tFa - 2ab + a\frac{(K+te)}{\sqrt{\frac{2a_0(K+te)}{\left((H+Gt)\left(1-\frac{a_0}{R}\right)\left(1-\frac{a_0}{R}\right)^2\right)}}}$$

$$+\left[2(1-a_0) - \frac{1}{2}(H+Gt)\right]a\sqrt{\frac{2a_0(K+te)}{\left((H+Gt)\left(1-\frac{a_0}{R}\right)\left(1-\frac{a_0}{R}\right)^2\right)}} \tag{27}$$

This solution is feasible and optimal if $\rho_2 \geq 0$.

(i) $\theta = 1$,

$$Q = \sqrt{\frac{2\left(a_0+a\right)\left(K+te\right)}{\left((H+Gt)\left(1-\frac{a_{0+a}}{R}\right)\left(1-\frac{a_0+a}{R}\right)^2\right)}},$$

$\rho_1 = 0$, and

$$\rho_1 = \left((P_0+b)+\frac{b}{a}(a_0+a)-tF-\frac{(K+te)}{\sqrt{\frac{2(a_0+a)(K+te)}{\left((H+Gt)\left(1-\frac{(a_0+a)}{R}\right)\left(1-\frac{(a_0+a)}{R}\right)^2\right)}}} \right.$$

$$\left. +\frac{1}{2}(H+Gt)\sqrt{\frac{2(a_0+a)(K+te)}{\left((H+Gt)\left(1-\frac{(a_0+a)}{R}\right)\left(1-\frac{(a_0+a)}{R}\right)^2\right)}} \right) \geq 0 \tag{28}$$

The above equation is optimal and feasible if $\rho_1 \geq 0$.
(iii) If $\rho_1 = 0$, $\rho_2 = 0$ and

$$Q = \sqrt{\frac{2(a_0+a\theta)(K+te)}{\left((H+Gt)\left(1-\frac{(a_0+a\theta)}{R}\right)\left(1-\frac{(a_0+a\theta)}{R}\right)^2\right)}}$$

$$[a(p_0+b\theta)+b(a_0+a\theta)-tfa]^2 = \frac{a^2(K+te)^2(H+Gt)\left(1-\left(\frac{a_0+a\theta}{R}\right)\right)\left(1-\left(\frac{a_0+a\theta}{R}\right)\right)^2}{2(a_0+a\theta)(K+te)}$$

$$+\frac{\left[\frac{a(H+Gt)}{2}-2\left(1-\left(\frac{a_0+a\theta}{R}\right)a\right)\right]^2 * 2(a_0+a\theta)(K+te)}{(H+Gt)\left(1-\left(\frac{a_0+a\theta}{R}\right)\right)\left(1-\left(\frac{a_0+a\theta}{R}\right)\right)^2} \tag{29}$$

The equation (29) is defined as $\pi_1(\theta)$

$$\pi_1(\theta) = [a(p_0 + b\theta) + b(a_0 + a\theta) - tfa]^2$$

$$- \frac{a^2(K+te)(H+Gt)\left(1 - \frac{(a_0+a\theta)}{R}\right)\left(1 - \frac{(a_0+a\theta)}{R}\right)^2}{2(a_0+a\theta)(K+te)}$$

$$+ \frac{\left[\frac{a(H+Gt)}{2} - 2\left(1 - \frac{(a_0+a\theta)}{R}a\right)\right]^2 * 2(a_0+a\theta)(K+te)}{(H+Gt)\left(1 - \frac{(a_0+a\theta)}{R}\right)\left(1 - \frac{(a_0+a\theta)}{R}\right)^2} \tag{30}$$

The expression of $\pi_1(\theta)$ is non-linear, allowing multiple solutions (not necessarily feasible) for above equation.

Appendix B

Theorem B

Proof. Let us consider the linear demand and logistic demand $d(\theta) = a_0 + a\theta$ and

$$P(\theta) = p_0 + \frac{b}{1 + e^{-\alpha(\theta - \theta_0)}}$$

$$a_0,\ p_0,\ a,\ \alpha > 0,\ \theta \in [0,\ 1]$$

$$\phi(Q,\ \theta,\ \rho_1,\ \rho_2) = (a_0 + a\theta)\left(p_0 + \frac{b}{1 + e^{-\alpha(\theta-\theta_0)}}\right) - tF(d_0 + a\theta) - \frac{(a_0 + a\theta)(K + te)}{Q}$$

$$-\frac{1}{2}(H + Gt)\left(1 - \frac{(a_0 + \mathrm{a}\theta)}{R}\right)Q - \left(1 - \frac{(a_0 + a\theta)}{R}\right)^2 Q - \rho_1(\theta - 1) + \rho_2\theta \tag{31}$$

$$\frac{\partial\phi}{\partial Q} = \frac{d(\theta)(K + te)}{Q^2} - \frac{1}{2}(H + Gt)\left(1 - \frac{(a_0 + \mathrm{a}\theta)}{R}\right) - \left(1 - \frac{(a_0 + \mathrm{a}\theta)}{R}\right)^2 \tag{32}$$

$$\frac{\partial^2\phi}{\partial Q^2} = \frac{-2d(\theta)(K + te)}{Q^3} \tag{33}$$

$$\frac{\partial^2\phi}{\partial Q\partial\theta} = \frac{a(K + te)}{Q^2} + \frac{(H + Gt)a}{2R} + 2\left(1 - \frac{(a_0 + \mathrm{a}\theta)}{R}\right)\frac{a}{R} \tag{34}$$

$$\frac{\partial\phi}{\partial\theta} = -(a_0 + \mathrm{a}\theta)\left(\frac{b\alpha\left(e^{-\alpha(\theta-\theta_0)}\right)}{\left(\left(1 + \left(e^{-\alpha(\theta-\theta_0)}\right)\right)^2\right.}\right) + \left(p_0 + \frac{b}{1 + e^{-\alpha(\theta-\theta_0)}}\right)a - tfa - \frac{a(K + te)}{Q}$$

$$+\frac{Q(H + Gt)a}{2R} - \left[2a/R\left(1 - \frac{(d_0 + \mathrm{a}\theta)}{R}\right)Q\right] - \rho_1 + \rho_2 \tag{35}$$

$$\frac{\partial^2\phi}{\partial\theta\partial Q} = \frac{a(K + te)}{Q^2} + \frac{(H + Gt)a}{2R} + 2\frac{a}{R}\left(1 - \frac{(a_0 + a\theta)}{R}\right) \tag{36}$$

$$\frac{\partial^2\phi}{\partial\theta^2} = (a_0 + \mathrm{a}\theta)\left(\frac{\begin{array}{c}\left(1+e^{-\alpha(\theta-\theta_0)}\right)^2\left(-b\alpha^2 e^{-\alpha(\theta-\theta_0)}\right)+b\alpha\left(e^{-\alpha(\theta-\theta_0)}\right)\\ \left(2\left(1+e^{-\alpha(\theta-\theta_0)}\right)\left(-\alpha\left(e^{-\alpha(\theta-\theta_0)}\right)\right)\right)\end{array}}{\left(1+e^{-\alpha(\theta-\theta_0)}\right)^2\right)^2}\right)$$

$$+\left(\frac{b\alpha\left(e^{-\alpha(\theta-\theta_0)}\right)}{\left(\left(1+e^{-\alpha(\theta-\theta_0}\right)^2}\right)a-\left(\frac{ab\alpha\left(e^{-\alpha(\theta-\theta_0)}\right)}{\left(\left(1+e^{-\alpha(\theta-\theta_0}\right)^2}\right)+2Q\frac{a^2}{R} \tag{37}$$

$$\frac{\partial^2\phi}{\partial\theta^2} = (a_0 + \mathrm{a}\theta)\left(\frac{\left(-b\alpha^2 e^{-\alpha(\theta-\theta_0)}\right)\left(1+e^{-\alpha(\theta-\theta_0)}\right)\left[\left(\left(1+e^{-\alpha(\theta-\theta_0)}\right)-\left(e^{-\alpha(\theta-\theta_0)}\right)\right)\right]}{\left(1+e^{-\alpha(\theta-\theta_0)}\right)^4}\right)$$

$$+\left(\frac{b\alpha\left(e^{-\alpha(\theta-\theta_0)}\right)}{\left(\left(1+e^{-\alpha(\theta-\theta_0}\right)^2}\right)a-\left(\frac{a^2b\alpha\left(e^{-\alpha(\theta-\theta_0)}\right)}{\left(\left(1+e^{-\alpha(\theta-\theta_0}\right)^2}\right)+2Q\frac{a^2}{R} \tag{38}$$

Thus, the hessian matrix $\phi(Q,\ \theta)$

$$|H| = \begin{vmatrix} \frac{\partial^2\phi}{\partial\theta^2} & \frac{\partial^2\phi}{\partial\theta\partial Q} \\ \frac{\partial^2\phi}{\partial Q\partial\theta} & \frac{\partial^2\phi}{\partial Q^2} \end{vmatrix} \tag{39}$$

Substitute equations (33), (34), (36), (38) in equation (39) we have

$$|H| = \left\{\left((a_0 + a\theta)\left(\frac{\left(-b\alpha^2 e^{-\alpha(\theta-\theta_0)}\right)\left(1+e^{-\alpha(\theta-\theta_0)}\right)\left[\left(\left(1+e^{-\alpha(\theta-\theta_0)}\right)-\left(e^{-\alpha(\theta-\theta_0)}\right)\right)\right]}{\left(1+e^{-\alpha(\theta-\theta_0)}\right)^4}\right)\right.\right.$$

$$\left.+\frac{ab\alpha\left(e^{-\alpha(\theta-\theta_0)}\right)}{\left(\left(1+e^{-\alpha(\theta-\theta_0}\right)^2}-\frac{a^2b\alpha\left(e^{-\alpha(\theta-\theta_0)}\right)}{\left(\left(1+e^{-\alpha(\theta-\theta_0}\right)^2}+\ 2Qa^2\right)\left(\frac{-2d(\theta)(K+te)}{Q^3}\right)\right\}$$

$$-\left\{\left(\frac{a(K+te)}{Q^2}+\frac{(H+Gt)a}{2R}+2\frac{a}{R}\left(1-\frac{(a_0+a\theta)}{R}\right)\right)^2\right\} \tag{40}$$

The concavity of $\phi(Q,\ \theta)$ requires on the Hessian Matrix being negatively semi-definite. Furthermore

$$\text{Det}\, H > 0 \Leftrightarrow \left\{ \left((a_0 + a\theta) \left(\frac{\left(-b\alpha^2 e^{-\alpha(\theta-\theta_0)}\right)\left(1+e^{-\alpha(\theta-\theta_0)}\right)\left[\left(\left(1+e^{-\alpha(\theta-\theta_0)}\right)-\left(e^{-\alpha(\theta-\theta_0)}\right)\right)\right]}{\left(1+e^{-\alpha(\theta-\theta_0)}\right)^4} \right) \right. \right.$$

$$\left. + \left(\frac{ab\alpha\left(e^{-\alpha(\theta-\theta_0)}\right)}{\left(\left(1+e^{-\alpha(\theta-\theta_0)}\right)^2\right.} - \frac{a^2 b\alpha\left(e^{-\alpha(\theta-\theta_0)}\right)}{\left(\left(1+e^{-\alpha(\theta-\theta_0)}\right)^2\right.} + 2Qa^2) \right) \left(\frac{-2d(\theta)(K+te)}{Q^3} \right) \right\}$$

$$> \left\{ \left(\frac{a(K+te)}{Q^2} + \frac{(H+Gt)a}{2R} + 2\frac{a}{R}\left(1 - \frac{(a_0+a\theta)}{R}\right) \right)^2 \right\} \tag{41}$$

Hence, H is negative semi-definite and $\phi(Q,\ \theta)$ is a concave. The above Figure 3 and 4 shows the concavity. When one solution is feasible and optimal under this condition. We have following Possible solutions:

(i) $\theta = 0$,

$$Q = \sqrt{\frac{2a_0(K+te)}{\left((H+Gt)\left(1-\frac{a_0}{R}\right)\left(1-\frac{a_0}{R}\right)^2\right)}}$$

and

$$\rho_2 = \left\{ \left(\left(P_0 + \frac{b}{1+e^{-\delta\theta_0}}\right) - tf + \frac{(K+te)}{\sqrt{\frac{2d_0(K+te)}{\left((H+Gt)\left(1-\frac{a_0}{R}\right)\left(1-\frac{a_0}{R}\right)^2\right)}}} \right) a \right\}$$

$$+ \left[\begin{array}{l} 2\left(a_0 b\delta\theta_0 e^{\delta\theta_0}\right) - \frac{1}{2}(H+Gt)\left(1-\frac{a}{R}\right) \\ \left(1-\frac{a}{R}\right)^2 \end{array} \right] \sqrt{\frac{2\left(a_0+a\right)(K+te)}{\left((H+Gt)\left(1-\frac{a_0+a}{R}\right)\left(1-\frac{a_0+a}{R}\right)^2\right)}} \tag{42}$$

(ii) $\theta = 1$, $\rho_2 = 0$

$$\rho_1 = a\left(P_0+\frac{b}{1+e^{-\delta(1-\theta_0)}}\right)-\frac{b\delta\left((1-\theta_0)\,e^{-\delta(1-\theta_0)}\right)}{\left(1+e^{-\delta(1-\theta_0)}\right)^2}d_0-tfa-\frac{a(K+te)}{\sqrt{\dfrac{2\,(a_0+a)\,(K+te)}{\left((H+Gt)\left(1-\dfrac{a_0}{R}\right)\left(1-\dfrac{a_0}{R}\right)^2\right)}}}$$

$$-\left[\frac{(H+Gt)\left(1-\dfrac{a}{R}\right)\left(1-\dfrac{a}{R}\right)^2}{2}\right]\left[\sqrt{\frac{2\,(a_0+a)\,(K+te)}{\left((H+Gt)\left(1-\dfrac{a_0}{R}\right)\left(1-\dfrac{a_0}{R}\right)^2\right)}}\right] \tag{43}$$

(iii) $\rho_1 = 0$, $\rho_2 = 0$

$$\pi_2(\theta) = \left\{a\left(\left(P_0+\frac{b}{1+e^{-\delta(1-\theta_0)}}\right)-tf-\frac{(K+te)}{\sqrt{\dfrac{2\,(a_0+a\theta)\,(K+te)}{\left((H+Gt)\left(1-\dfrac{(a_0+a\theta)}{R}\right)\left(1-\dfrac{(a_0+a\theta)}{R}\right)^2\right)}}}\right)\right\}^2$$

$$-\left\{\left(\frac{b\delta\left((1-\theta_0)\,e^{-\delta(1-\theta_0)}\right)}{\left(1+e^{-\delta(1-\theta_0)}\right)^2}\right)(a_0+a\theta)\right.$$

$$\left.+\left[\frac{(H+Gt)\left(1-\dfrac{a}{R}\right)\left(1-\dfrac{a}{R}\right)^2}{2}\right]\left[\sqrt{\frac{2\,(a_0+a\theta)\,(K+te)}{\left((H+Gt)\left(1-\dfrac{(a_0+a\theta)}{R}\right)\left(1-\dfrac{(a_0+a\theta)}{R}\right)^2\right)}}\right]\right\} \tag{44}$$

The expression of $\pi_2(\theta)$ is non-linear, allowing multiple solutions (not necessarily feasible) for above equation.

Appendix C

Theorem C

Proof. Consider the Logarithmic and exponential demand per unit gross profit

$$D(\theta) = a_0 + a\log(1+\gamma\theta)$$

$$P(\theta) = q_0 + be^{\alpha(\theta-1)}$$

$$\begin{aligned}\phi(Q,\ \theta,\ \rho_1,\ \rho_2) = (a_0 + a\log(1+\gamma\theta))\left(P_0 + be^{\alpha(\theta-1)}\right)\\ -tf(a_0 + a\log(1+\gamma\theta)) - (a_0 + a\log(1+\gamma\theta))\frac{(K+te)}{Q}\\ -\frac{(H+Gt)}{2}\left(1-\frac{(a_0 + a\log(1+\gamma\theta))}{R}\right)Q\\ -Q\left(1-\left(\frac{(a_0 + a\log(1+\gamma\theta))}{R}\right)\right)^2 - \rho_1(\theta-1) + \rho_2\theta\end{aligned} \tag{45}$$

$$\begin{aligned}\frac{\partial\phi}{\partial Q} = \frac{(a_0 + a\log(1+\gamma\theta))(K+te)}{Q^2}\\ -\frac{(H+Gt)}{2}\left(1-\frac{(a_0 + a\log(1+\gamma\theta))}{R}\right) - \left(1-\left(\frac{(a_0 + a\log(1+\gamma\theta))}{R}\right)\right)^2\end{aligned} \tag{46}$$

$$\frac{\partial^2\phi}{\partial Q^2} = \frac{-2(a_0 + a\log(1+\gamma\theta))(K+te)}{Q^3} \tag{47}$$

$$\frac{\partial^2\phi}{\partial Q\partial\theta} = -\frac{2(K+te)a\gamma}{(1+\gamma\theta)Q^2} + \frac{(H+Gt)}{2}\frac{a\gamma}{R(1+\gamma\theta)} + 2\left(\left(1-\left(\frac{(a_0 + a\log(1+\gamma\theta))}{R}\right)\right)\right)\frac{a\gamma}{R(1+\gamma\theta)} \tag{48}$$

$$\begin{aligned}\frac{\partial\phi}{\partial\theta} = (a_0 + a\log(1+\gamma\theta))\left(\alpha b\left(e^{\alpha(\theta-1)}\right)\right) + \left(P_0 + be^{\alpha(\theta-1)}\right)\frac{a\gamma}{(1+\gamma\theta)} - \frac{tfa\gamma}{(1+\gamma\theta)}\\ -\frac{a\gamma}{(1+\gamma\theta)}\frac{(K+te)}{Q} + \frac{Q(H+Gt)}{2R}\frac{a\gamma}{(1+\gamma\theta)} + \frac{2a\gamma Q(1-(a_0 + a\log(1+\gamma\theta)))}{R(1+\gamma\theta)} - \rho_1 + \rho_2\end{aligned} \tag{49}$$

$$\frac{\partial^2\phi}{\partial\theta^2}=\left[(a_0+a\log(1+\gamma\theta))\,\alpha^2be^{\alpha(\theta-1)}+\frac{ab\gamma\alpha}{(1+\gamma\theta)}+\left(\frac{ab\gamma\alpha e^{\alpha(\theta-1)}(1+\gamma\theta)-a\gamma^2\left(P_0+be^{\alpha(\theta-1)}\right)}{(1+\gamma\theta)^2}\right)\right]$$

$$+\left[\frac{tfa\gamma^2}{(1+\gamma\theta)^2}+\frac{(K+te)}{Q}\frac{a\gamma^2}{(1+\gamma\theta)^2}-\frac{Q(H+Gt)}{2}\frac{a\gamma}{(1+\gamma\theta)^2}\right.$$

$$\left.+\frac{2Qa^2\gamma^2+2Qa\gamma(1-(a_0+a\log(1+\gamma\theta)))}{R(1+\gamma\theta)^2}\right] \tag{50}$$

$$\frac{\partial^2\phi}{\partial\theta\partial Q}=\frac{a\gamma(K+te)}{(1+\gamma\theta)Q^2}-\frac{(H+Gt)}{2}\frac{a\gamma}{(1+\gamma\theta)}+\frac{2a\gamma(a_0+a\log(1+\gamma\theta))}{R(1+\gamma\theta)} \tag{51}$$

Thus, the hessian matrix $\phi(Q,\ \theta)$

$$|H|=\begin{vmatrix}\dfrac{\partial^2\phi}{\partial\theta^2} & \dfrac{\partial^2\phi}{\partial\theta\partial Q}\\[2ex] \dfrac{\partial^2\phi}{\partial Q\partial\theta} & \dfrac{\partial^2\phi}{\partial Q^2}\end{vmatrix} \tag{52}$$

Substitute equations (47), (48), (50), (51) in equation (52) we have

$$\text{Det}\,H=\left(\left(\alpha e^{\alpha(\theta-1)}\right)\left[\alpha\,(a_0+a\log(1+\gamma\theta))+\frac{a\gamma}{(1+\gamma\theta)}-\left(P_0+be^{\alpha(\theta-1)}\right)\left(\frac{a\gamma^2}{(1+\gamma\theta)^2}\right)\left(\frac{ab}{(1+\gamma\theta)}\right)\right]\right.$$

$$+\left[tf+\frac{(K+te)}{2}+\frac{Q(H+Gt)}{2}-\frac{2Q}{\gamma}\left(a\gamma(1-\gamma(d_0+a\log(1+\gamma\theta))\right)\right]$$

$$\left.\left(\frac{-2\,(a_0+a\log(1+\gamma\theta))\,(1+\gamma\theta)(K+te)}{Q^3}\right)\right)$$

$$-\left\{\left(\frac{a\gamma(K+te)}{(1+\gamma\theta)Q^2}-\frac{(H+Gt)}{2}\frac{a\gamma}{(1+\gamma\theta)}-\frac{2a\gamma(a_0+a\log(1+\gamma\theta))}{(1+\gamma\theta)}\right)^2\right\}>0 \tag{53}$$

The concavity of $\phi(Q,\ \theta)$ requires on the Hessian Matrix being negatively semi-definite. Furthermore

$$\text{Det } H > 0 \Leftrightarrow \left(\alpha e^{\alpha(\theta-1)} \left[\alpha\left(a_0 + a\log(1+\gamma\theta)\right) + \frac{a\gamma}{(1+\gamma\theta)} - \left(P_0 + be^{\alpha(\theta-1)}\right)\left(\frac{a\gamma^2}{(1+\gamma\theta)^2}\right)\left(\frac{ab}{(1+\gamma\theta)}\right)\right]\right.$$

$$\left. + \left[tf + \frac{(K+te)}{2} + \frac{Q(H+Gt)}{2} - \frac{2Q}{\gamma}\left(a\gamma(1-\gamma(a_0+a\log(1+\gamma\theta))\right)\right]\right)$$

$$\left\{ \frac{-2\left(a_0 + a\log(1+\gamma\theta)\right)(1+\gamma\theta)(K+te)}{Q^3} \right\}$$

$$> -\left\{ \left(\frac{a\gamma(K+te)}{(1+\gamma\theta)Q^2} - \frac{(H+Gt)}{2}\frac{a\gamma}{(1+\gamma\theta)} - \frac{2a\gamma\left(a_0 + a\log(1+\gamma\theta)\right)}{(1+\gamma\theta)} \right)^2 \right\} \tag{54}$$

Hence, H is negative semi-definite and $\phi(Q,\ \theta)$ is a concave.
When one solution is feasible and optimal under this condition. We have the following Possible solutions:
(i) $\theta = 0$, $\rho_1 = 0$ and

$$\rho_2 = tfa\gamma + \frac{(a_0 + a\gamma)(K+te)}{\sqrt{\dfrac{2a_0(K+te)}{\left((H+Gt)\left(1-\dfrac{a_0}{R}\right)\left(1-\dfrac{a_0}{R}\right)^2\right)}}} + \sqrt{\frac{2a_0(K+te)}{\left((H+Gt)\left(1-\dfrac{a_0}{R}\right)\left(1-\dfrac{a_0}{R}\right)^2\right)}}$$

$$\left\{ \frac{(H+Gt)}{2}\left(1 - \left(1 - \frac{a_0 + a\gamma}{R}\right) + 2\left(1 - \frac{a_0}{R}\right)\right)a\gamma \right\} - \left(a_0 b\delta e^{-\delta}\right) - \left(P_0 + be^{-\delta}\right)a\gamma \tag{55}$$

(ii) $\theta = 1$, $\rho_2 = 0$ and

$$\rho_1 = \frac{tfa\gamma}{1+\gamma} + \frac{\left(a_0 + \dfrac{a\gamma}{1+\gamma}\right)(K+te)}{\sqrt{\dfrac{2a_0(K+te)}{\left((H+Gt)\left(1-\dfrac{a_0}{R}\right)\left(1-\dfrac{a_0}{R}\right)^2\right)}}} + \sqrt{\frac{2a_0(K+te)}{\left((H+Gt)\left(1-\dfrac{a_0}{R}\right)\left(1-\dfrac{a_0}{R}\right)^2\right)}}\left\{\left(\frac{(H+Gt)}{2}\right)\right.$$

$$\left.\left(1 - \left(a_0 + \frac{a\gamma}{1+\gamma}\right) + 2\big(1 - (a_0 + a\log(1+\gamma)\big) - \big(1 - (a_0 + a\log(1+\gamma))(b\delta)\big)\right)\right\}(P_0 + b)\left(\frac{a\gamma}{1+\gamma}\right) \tag{56}$$

(iii) $\rho_1 = 0$, $\rho_2 = 0$ and

$$\pi_3(\theta) = \left(\frac{tf\gamma a}{1+\gamma} + \frac{\left(d_0 + \frac{a\gamma}{1+\gamma}\right)(K+te)}{\sqrt{\frac{2a_0(K+te)}{\left((H+Gt)\left(1-\frac{a_0}{R}\right)\left(1-\frac{a_0}{R}\right)^2\right)}}} \right)^2$$

$$+ \left[\left(\frac{H+Gt}{2}\right) \left(1 - \left(a_0 + \frac{a\gamma}{1+\gamma}\right)\right) \left(2(1-(a\log(1+\gamma)))\right) \right]$$

$$\left(\sqrt{\frac{2a_0(K+te)}{\left((H+Gt)\left(1-\frac{a_0}{R}\right)\left(1-\frac{a_0}{R}\right)^2\right)}} \right) \tag{57}$$

The expression of $\pi_3(\theta)$ is non-linear, allowing multiple solutions (not necessarily feasible) for above equation.

Appendix D

Theorem D

Proof. Let the logistics demand and logistic gross profit per unit are logistics form,

$$D(\theta) = a_0 + \frac{a}{1+e^{-\alpha(\theta-\theta_0)}}$$

$$P(\theta) = p_0 + \frac{b}{1+e^{-\gamma(\theta-\theta_0)}}$$

$$\begin{aligned}\phi\left(Q,\ \theta,\ \rho_1,\ \rho_2\right) = &\left(a_0 + \frac{a}{1+e^{-\alpha(\theta-\theta_0)}}\right)\left(p_0 + \frac{b}{1+e^{-\gamma(\theta-\theta_0)}}\right) - tf\left(a_0 + \frac{a}{1+e^{-\alpha(\theta-\theta_0)}}\right) \\ &- \frac{\left(a_0 + \frac{a}{1+e^{-\alpha(\theta-\theta_0)}}\right)(K+te)}{Q} - \frac{Q(H+Gt)}{2}\left(1-\left(a_0 + \frac{a}{1+e^{-\alpha(\theta-\theta_0)}}\right)/R\right) \\ &- \left(1-\left(a_0 + \frac{a}{1+e^{-\alpha(\theta-\theta_0)}}\right)/R\right)^2 Q - \rho_1(\theta-1) + \rho_2 \end{aligned} \tag{58}$$

$$\frac{\partial^2\phi}{\partial Q^2} = \frac{-2\left(a_0 + \frac{a}{1+e^{-\alpha(\theta-\theta_0)}}\right)(K+te)}{Q^3} \tag{59}$$

$$\begin{aligned}\frac{\partial^2\phi}{\partial Q\partial\theta} = &\frac{(K+te)}{Q^2}\frac{a}{1-\alpha e^{-\alpha(\theta-\theta_0)}} \\ &- \frac{(H+Gt)}{2R}\left(\frac{a}{1-\alpha e^{-\alpha(\theta-\theta_0)}}\right) + \frac{2a\left(1-\left(a_0 + \frac{a}{1+e^{-\alpha(\theta-\theta_0)}}\right)\right)}{R(1+\gamma\theta)\left(1-\alpha e^{-\alpha(\theta-\theta_0)}\right)}\end{aligned} \tag{60}$$

$$\frac{\partial^2\phi}{\partial\theta\partial Q} = \frac{(K+te)}{Q^2}\frac{a}{1-\alpha e^{-\alpha(\theta-\theta_0)}} - \frac{(H+Gt)}{2R}\left(\frac{a}{1-\alpha e^{-\alpha(\theta-\theta_0)}}\right) + \frac{2a\left(1-\left(a_0 + \frac{a}{1+e^{-\alpha(\theta-\theta_0)}}\right)\right)}{R(1+\gamma\theta)\left(1-\alpha e^{-\alpha(\theta-\theta_0)}\right)} \tag{61}$$

$$\frac{\partial^2\phi}{\partial\theta^2} = \left(a_0 + \frac{a}{1+e^{-\alpha(\theta-\theta_0)}}\right)\left(\frac{b}{1+\gamma^2 e^{-\gamma(\theta-\theta_0)}}\right) + \left(\frac{b}{1-\gamma e^{-\gamma(\theta-\theta_0)}}\right)\left(\frac{a}{1-\alpha e^{-\alpha(\theta-\theta_0)}}\right)$$

$$+\left(p_0 + \frac{b}{1+e^{-\gamma(\theta-\theta_0)}}\right)\left(\frac{a}{1+\alpha^2 e^{-\alpha(\theta-\theta_0)}}\right) + \left(\frac{a}{1-\alpha e^{-\alpha(\theta-\theta_0)}}\right)\left(\frac{b}{1-\gamma e^{-\gamma(\theta-\theta_0)}}\right)$$

$$-\left(\frac{tfa}{1+\alpha^2 e^{-\alpha(\theta-\theta_0)}}\right) - \frac{(K+te)}{Q}\left(\frac{a}{1+\alpha^2 e^{-\alpha(\theta-\theta_0)}}\right) + \frac{(H+Gt)Q}{2R}\left(\frac{a}{1+\alpha^2 e^{-\alpha(\theta-\theta_0)}}\right)$$

$$-2\left[\left(\left(\frac{-a}{1-\alpha e^{-\alpha(\theta-\theta_0)}}\right)\left(\frac{-a}{1-\alpha e^{-\alpha(\theta-\theta_0)}}\right)\right)\right.$$

$$\left.+\left(1-\left(a_0 + \frac{a}{1+e^{-\alpha(\theta-\theta_0)}}\right)/R\right)\left(\left(\frac{-a}{1+\alpha^2 e^{-\alpha(\theta-\theta_0)}}\right)\right)\right] \tag{62}$$

Thus, the hessian matrix $\phi(Q,\ \theta)$

$$\text{Det } H = \left(\frac{-2\left(a_0 + \frac{a}{1+e^{-\alpha(\theta-\theta_0))}}\right)(K+te)}{Q^3}\right)\left\{\left(a_0 + \frac{a}{1+e^{-\alpha(\theta-\theta_0)}}\right)\left(\frac{b}{1+\gamma^2 e^{-\gamma(\theta-\theta_0)}}\right)\right.$$

$$+\left(\frac{b}{1-\gamma e^{-\gamma(\theta-\theta_0)}}\right)\left(\frac{a}{1-\alpha e^{-\alpha(\theta-\theta_0)}}\right)\left(p_0 + \frac{b}{1+e^{-\gamma(\theta-\theta_0)}}\right)\left(\frac{a}{1+\alpha^2 e^{-\alpha(\theta-\theta_0)}}\right)$$

$$+\left(\frac{a}{1-\alpha e^{-\alpha(\theta-\theta_0)}}\right)\left(\frac{b}{1-\gamma e^{-\gamma(\theta-\theta_0)}}\right) - \left(\frac{tfa}{1+\alpha^2 e^{-\alpha(\theta-\theta_0)}}\right)$$

$$-2\left[\left(\left(\frac{-a}{1-\alpha e^{-\alpha(\theta-\theta_0)}}\right)\left(\frac{-a}{1-\alpha e^{-\alpha(\theta-\theta_0)}}\right)\right)\right.$$

$$\left.\left.\left(1-\left(a_0 + \frac{a}{1+e^{-\alpha(\theta-\theta_0)}}\right)/R\right)\left(\left(\frac{-a}{1+\alpha^2 e^{-\alpha(\theta-\theta_0)}}\right)\right)\right]\right\}$$

$$-\left\{\frac{(K+te)}{Q^2}\frac{a}{1-\alpha e^{-\alpha(\theta-\theta_0)}} - \frac{(H+Gt)}{2R}\right.$$

$$\left(\frac{a}{1-\alpha e^{-\alpha(\theta-\theta_0)}}\right)+\frac{2a\left(1-\left(a_0+\frac{a}{1+e^{-\alpha(\theta-\theta_0)}}\right)\right)}{R(1+\gamma\theta)\left(1-\alpha e^{-\alpha(\theta-\theta_0)}\right)}\Bigg\}^2>0 \tag{63}$$

Hence, H is negative semi-definite and $\phi(Q,\ \theta)$ is a concave.