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Research Article

Enhancing Agricultural Diagnostics through Linear Diophantine Multi-Fuzzy Soft Matrices with Lattice Implementation

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Abstract: A sophisticated conceptual process that is essential for system analysis and farming practice decision-making is agricultural modeling. Since, agricultural processes are rife with uncertainties and often contain untrustworthy information, applying fuzzy set theory is a beneficial strategy. A key component of improving agricultural systems is fuzzy modeling, which is renowned for producing ideal solutions. The idea of the Lattice-ordered Linear Diophantine Multi-Fuzzy Soft Matrix (LLDMFSM) and its mathematical modeling, especially suited for agricultural diagnostics, are the main topics of this paper. In addition to discussing the basic characteristics of LLDMFSM, the mean operators that apply to this novel matrix are also discussed. One unique aspect of our methodology is the use of the min-max composition strategy to handle an example model that includes multiple factors impacting agriculture. Notably, farmers can use our suggested methodology as a useful tool to help them decide which course of action is best for their farming endeavors. This work adds to the expanding research data on agricultural diagnostics by highlighting the usefulness and advantages of the LLDMFSM in assisting with well-informed agricultural decision-making.

***Keywords*:** Agriculture, Lattice ordered linear diophantine multi-fuzzy soft matrix, Mean operators, Fuzzy modeling, Min-max composition, Farming methods

MSC: 06D72,03E72, 08A72, 15B15

Abbreviation

LLDMFSM	Lattice-ordered Linear Diophantine Multi-Fuzzy Soft Matrix
LLDMFSS	Lattice-ordered Linear Diophantine Multi-Fuzzy Soft Set

1. Introduction

Agriculture is undeniably a huge livelihood provider, especially in the vast rural areas. Enhancing agricultural productivity continues to be a key focus for emerging nations. This is because it provides a substantial contribution to the farming industry as well as the economy. To achieve its objectives, agricultural research is imperative since it contributed to the development of many novel production methods and technologies that maximize agricultural productivity. The

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majority of the procedures in the agricultural industry involve ambiguity, insufficient knowledge, uncertainty, and human intuition. These processes are strongly influenced by human variables in addition to being constrained by their environment. Fuzzy set theory can manage and characterize uncertainty, ensure that incomplete information is valued, and provide a comprehensive to challenges that are crucial in agriculture.

Human reasoning and thinking, which stems from vague human notions, lead to the creation of Zadeh's fuzzy set [1]. Fuzzy sets are a rapidly expanding field that has found great application recently, especially in the agricultural industry. Its ability to support farmers in making the best choices possible when it comes to crop cultivation is evidence of this issue. Various researchers integrate fuzzy theory to solve the issues in agriculture [2]. Despite the huge response, the degree of membership is insufficient to address all circumstances. Atannosov [3] developed an intuitionistic fuzzy set(IFS) in along with non-membership grades. As a generalization of IFSs, Yager then launched the innovative interpretation of Pythagorean fuzzy sets(PFS) [4] with q-rung orthopair fuzzy sets(q-ROFS) [5]. These widely accessible fuzzy sets have some constraints on grade values and are not capable of representing some issues.

To overcome these issues, Riaz and Hashmi [6] introduced the novel theory of linear diophantine fuzzy set(LDFS) with the accumulation of reference parameters. This structure broadens the scope of existing models' valuations and incorporates control parameters to classify the issue. For MCDM issues, Aiyared Iampam [7] discussed LDFS using several Einstein aggregation methods. These operators can determine the best option as well as derive ranking knowledge. Later, Saya Ayub [8] used decision-making to derive LDF relations and associated with algebraic features. LDFS was expanded by Riaz et al. [9] by including the concept of soft rough sets with use in material handling devices. Linear Diophantine fuzzy sets with DEMATEL approach were applied to climate crisis by Jeevitha et al. [10]. In addition to describing the properties of LDFS and its applications, Vimala et al. [11] designed the MARCOS method for LDFS. Riaz [9] broadened the scope of the LDFS by introducing soft rough sets and their potential use in material handling equipment. Riaz [12] created aggregation operators (AOs) that employed linear Diophantine fuzzy numbers (LDFNs) in priority order to select third-party logistic service providers. Farid [13] proposed Einstein's prioritized linear Diophantine fuzzy AOs with applications. Frank AOs for linear Diophantine fuzzy numbers with interval values were recently developed by Riaz [14]. Using IVLDF data, Petchimuthu [15] attempted to use its AOs to solve the supplier selection problem. LDFS clustering algorithm was discussed with the help of LDF correlation coefficient by Jeevitha et al. [16].

Molodtsov [17] developed the soft set theory which is a novel method for simulating ambiguity and uncertainty. Soft set theory has a wide range of potential applications in resolving real-world issues across numerous disciplines. Maji et al. [18] derived the Fuzzy Soft set(FSS) theory, which has seen significant growth. Later, it was extended to create the Intuitionistic Fuzzy Soft Set hybrid model (IFSS) [19]. Peng [20] introduced the Pythagorean Fuzzy Soft Set (PFSS) and its uses in decision-making. q-Rung Orthopair Fuzzy Soft Set (q-ROFSS) with aggregation operators was studied by Hussain et al. [21].

Garrett Birkhoff [22] was the first to propose the lattice theory. Many researchers and decision-makers have increased their focus on the combination of fuzzy sets and lattice ordered structures since it may lead to more fresh, intriguing subjects. Because hybrid fuzzy sets lack lattice structures, Sabeena et al. [23, 24] created lattice-ordered multi-fuzzy soft set. Later, Rajareega [25] built the lattice structure in an intuitionistic fuzzy soft group and exhibited a variety of applications. As a result, we incorporate lattice structures into the hybrid linear diophantine multi-fuzzy soft set model, which combines linear diophantine multi-fuzzy set with fuzzy soft set. The study of SSs, MFSSs, and their various extensions-such as the complex linear diophantine fuzzy soft set (CLDFSS) [26], the q-rung orthopair multi-fuzzy soft set [27, 28], and the linear diophantine fuzzy multi-soft set (LDMFSS) [29] is advancing quickly. Similarity measures for LDMFFS were started and used to find an alternate fuel [30] to solve the problem of fuel deficiency.

On the other hand, Naim et al. [31] developed soft matrices and used them in problem-solving situations. Later, a new DM approach with the establishment of the fuzzy soft matrix(FSM) which symbolizes the FSS was developed. Cagman and Enginoglu [32] revised the FSM and replicated the associated operations from a unique outlook, making theoretical elements of the FSS more beneficial. Later, Harish Garg et al. [33] presented the mean operators and generalized products for FSM. The matrix representation in IFSS, and PFSS was proposed by Chetia et al. [34], and Abhishek Guleria et al. [35]. Consequently, here we introduce the notion of a matrix in LLDMFSS.

1.1 *Indent of this study*

The rationale for this work comes from the realization that parameterization and multi-dimensional methods need to work better together in the context of LDFS. These problems have major drawbacks, so an innovative approach is required. To effectively address and overcome these issues, we propose a novel concept of a lattice-ordered linear Diophantine multi-fuzzy soft set (LLDMFSM). This manuscript's main goal is to clarify the theoretical foundations of LLDMFSM and use its unique characteristics to offer an effective alternative that lessens the drawbacks of conventional parameterization and multi-dimensional approaches in the context of LDFS.

This research makes several fundamental contributions. First, we explore the usefulness of LLDMFSM in real-world settings by concentrating on the max-min composition strategy crucial component in situations involving decision-making. Our goal in highlighting the significance of this approach is to make LLDMFSM a useful instrument with a wide range of applications in making decisions. Furthermore, the study investigates the fundamental properties and theories related to the max-min composition strategy for two LLDMFSMs, offering a thorough comprehension of its operation and possible consequences.

Additionally, this work expands the use of LLDMFSM into the field of agricultural research. The main goal is to introduce LLDMFSM into a new field of Agri diagnostics and demonstrate its flexibility outside of conventional mathematical frameworks. To show its usefulness in practice, we build an example specifically for the agricultural industry. This example demonstrates how LLDMFSM can have a transformative effect on agricultural practices and is applicable in the real world by taking into account a variety of factors that influence cultivation to optimize agricultural outcomes. To put it succinctly, this study explores both theoretical and practical spheres, demonstrating the rich and varied contributions of LLDMFSM to various academic disciplines.

1.2 *Layout of this study*

The article is split into five main sections, each of which adds to a thorough examination of the suggested study. The foundational ideas are discussed in Section 2, providing the framework for a more thorough comprehension of the talks that follow. The central subject of the study is covered in detail in Section 3, which also explains the operations associated with the Lattice ordered Linear Diophantine Multi-Fuzzy Soft Matrix. Building on this theoretical framework, Section 4 investigates the novel applications of the LLDMFSM in agriculture while offering insights and performing a comparative analysis. This section provides an insightful analysis of the practical applications of the suggested model. Section 5 provides a comprehensive summary of the article's main findings and suggests possible directions for further investigation.

2. Preliminaries

Definition 2.1 [1] The fuzzy set $\mathcal{F}$ on Θ is a mapping $\mu\colon \Theta \to [0, 1]$ where $\mu(\mathfrak{x})$ reflects the degree of elements in Θ and it is represented in the form

$$\mathcal{F} = \{(\mathfrak{x}, \mu(\mathfrak{x}))\colon \mathfrak{x} \in \Theta\}$$

Definition 2.2 [36] A pair $(\mathcal{F}, \Upsilon)$ is called a multi-fuzzy soft set of dimension k over Θ, where $\mathcal{F}$ is a mapping given by $\mathcal{F}\colon \Upsilon \to M_kFS$, where Υ is a parameter set and M_kFS is a multi-fuzzy set.

$$\mathcal{M} = \{(\mathfrak{x}, \mu(\mathfrak{x})^k)\colon \mathfrak{x} \in \Theta\}$$

Definition 2.3 [3] The intuitionistic fuzzy set (IFS) on Θ is of the form

$$\mathcal{J} = \{\mathfrak{x},\ \langle\mu_{\mathcal{J}}(\mathfrak{x}),\ \nu_{\mathcal{J}}(\mathfrak{x})\rangle\colon \mathfrak{x}\in\Theta\}$$

where $\mu_{\mathcal{J}}(\mathfrak{x})$, $\nu_{\mathcal{J}}(\mathfrak{x})$ are the degree of membership and non-membership which belongs to [0,1] subject to the condition $0 \leq \mu_{\mathcal{J}}(\mathfrak{x}) + \nu_{\mathcal{J}}(\mathfrak{x}) \leq 1$.

Definition 2.4 [4] The Pythagorean fuzzy set (PFS) on Θ is in the mathematical form

$$\mathcal{P} = \{\mathfrak{x},\ \langle\mu_{\mathcal{P}}(\mathfrak{x}),\ \nu_{\mathcal{P}}(\mathfrak{x})\rangle\colon \mathfrak{x}\in\Theta\}$$

where $\mu_{\mathcal{P}}(\mathfrak{x})$, $\nu_{\mathcal{P}}(\mathfrak{x})$ are degree of membership and non-membership which belongs to [0,1] depending on the circumstance $0 \leq \mu^2_{\mathcal{P}}(\mathfrak{x}) + \nu^2_{\mathcal{P}}(\mathfrak{x}) \leq 1$.

Definition 2.5 [5] The q-rung orthopair fuzzy set (q-ROFS) on Θ is in the mathematical form

$$\mathcal{R} = \{\mathfrak{x},\ \langle\mu_{\mathcal{R}}(\mathfrak{x}),\ \nu_{\mathcal{R}}(\mathfrak{x})\rangle\colon \mathfrak{x}\in\Theta\}$$

where $\mu_{\mathcal{R}}(\mathfrak{x})$, $\nu_{\mathcal{R}}(\mathfrak{x})$ are a degree of membership and non-membership which belongs to [0,1] depending on the circumstance $0 \leq \mu^q_{\mathcal{R}}(\mathfrak{x}) + \nu^q_{\mathcal{R}}(\mathfrak{x}) \leq 1$.

Definition 2.6 [6] A linear diophantine fuzzy set $\mathcal{L}$ on Θ is an object of the form:

$$\mathcal{L} = \{(\mathfrak{x},\ \langle\mu_{\mathcal{L}}(\mathfrak{x}),\ \nu_{\mathcal{L}}(\mathfrak{x})\rangle,\ \langle\tau_{\mathcal{L}}(\mathfrak{x}),\ \eta_{\mathcal{L}}(\mathfrak{x})\rangle)\colon \mathfrak{x}\in\Theta\}$$

where, $\mu_{\mathcal{L}}(\mathfrak{x})$, $\nu_{\mathcal{L}}(\mathfrak{x})$, $\tau_{\mathcal{L}}(\mathfrak{x})$, $\eta_{\mathcal{L}}(\mathfrak{x}) \in [0,\ 1]$ are membership, non-membership and reference parameters respectively. These grades satisfy the condition $0 \leq \tau_{\mathcal{L}}(\mathfrak{x})\mu_{\mathcal{L}}(\mathfrak{x}) + \eta_{\mathcal{L}}(\mathfrak{x})\nu_{\mathcal{L}}(\mathfrak{x}) \leq 1$ for all $\mathfrak{x}\in\Theta$ with $0 \leq \tau_{\mathcal{L}}(\mathfrak{x}) + \eta_{\mathcal{L}}(\mathfrak{x}) \leq 1$.

Definition 2.7 [37] Let K be the set of indices. A linear diophantine multi-fuzzy set $\mathcal{H}$ on Θ with dimension k is the set of ordered sequences in the form

$$\mathcal{H} = \{(\mathfrak{x},\ \langle(\mu^1_{\mathcal{H}}(\mathfrak{x}),\ \mu^2_{\mathcal{H}}(\mathfrak{x}),\ \ldots,\ \mu^k_{\mathcal{H}}(\mathfrak{x})),\ \nu^1_{\mathcal{H}}(\mathfrak{x}),\ \nu^2_{\mathcal{H}}(\mathfrak{x}),\ \ldots,\ \nu^k_{\mathcal{H}}(\mathfrak{x}))\rangle,$$

$$\langle(\tau^1_{\mathcal{H}}(\mathfrak{x}),\ \tau^2_{\mathcal{H}}(\mathfrak{x}),\ \ldots,\ \tau^k_{\mathcal{H}}(\mathfrak{x})),\ (\eta^1_{\mathcal{H}}(\mathfrak{x}),\ \eta^2_{\mathcal{H}}(\mathfrak{x}),\ \ldots,\ \eta^k_{\mathcal{H}}(\mathfrak{x}))\rangle\colon \mathfrak{x}\in\Theta\}$$

where $\mu^i_{\mathcal{H}}(\mathfrak{x})$, $\nu^i_{\mathcal{H}}(\mathfrak{x})$, $\tau^i_{\mathcal{H}}(\mathfrak{x})$, $\eta^i_{\mathcal{H}}(\mathfrak{x})$ are multi membership, multi non-membership and multi-reference parameters values respectively. Along that, it satisfies the condition $0\leq \mu^i_{\mathcal{H}}(\mathfrak{x})\tau^i_{\mathcal{H}}(\mathfrak{x}) + \nu^i_{\mathcal{H}}(\mathfrak{x})\eta^i_{\mathcal{H}}(\mathfrak{x}) \leq 1$ and $0 \leq \tau^i_{\mathcal{H}}(\mathfrak{x}) + \eta^i_{\mathcal{H}}(\mathfrak{x}) \leq 1$, for every $i = 1,\ 2,\ \ldots k$. The collection of all linear diophantine multi-fuzzy sets of dimension k over Θ is denoted by $LDM_kF(\Theta)$.

Definition 2.8 [38, 39] Let Υ be the set of parameters. Define a map $\mathcal{K}\colon \Upsilon \to LDM_kF(\Theta)$. Then the ordered pair $(\mathcal{K},\ \Upsilon)$ is said to be a linear diophantine multi-fuzzy soft set of dimension k and it is of the structure $\{(\varepsilon_i,\ \mathcal{K}(\varepsilon_i))\colon \varepsilon_i\in\Upsilon\}$ where $\mathcal{K}(\varepsilon_i)$ is a $LDM_kF(\Theta)$.

3. Lattice ordered Linear Diophantine Multi-Fuzzy Soft set

Definition 3.9 Let V be the lattice of parameters. The LDM_kFSS set $(\mathscr{G},\ \Upsilon)$ over Θ is said to be Lattice ordered Linear Diophantine Multi-Fuzzy Soft set if

for all $\varepsilon_1,\ \varepsilon_2\in\Upsilon,\ \varepsilon_1\leq\varepsilon_2 \Rightarrow \mathscr{G}(\varepsilon_1)\subset\mathscr{G}(\varepsilon_2)$

i.e,

$$\mu^k_{\mathscr{G}(\varepsilon_1)}(\mathfrak{x}) \leq \mu^k_{\mathscr{G}(\varepsilon_2)}(\mathfrak{x})$$
$$\nu^k_{\mathscr{G}(\varepsilon_1)}(\mathfrak{x}) \geq \nu^k_{\mathscr{G}(\varepsilon_2)}(\mathfrak{x})$$

$$\tau^k_{\mathscr{G}(\varepsilon_1)}(\mathfrak{x}) \leq \tau^k_{\mathscr{G}(\varepsilon_2)}(\mathfrak{x})$$
$$\eta^k_{\mathscr{G}(\varepsilon_1)}(\mathfrak{x}) \geq \eta^k_{\mathscr{G}(\varepsilon_2)}(\mathfrak{x}).$$

$\forall \mathfrak{x} \in \varepsilon.$
Remark: The element is generally denoted by lattice ordered linear diophantine multi-fuzzy soft number and it is denoted by $\mathcal{LLDMFSN}$

Definition 3.10 Let $\mathcal{K}(\varepsilon_1)$ *and* $\mathcal{K}(\varepsilon_2)$ be two $\mathcal{LLDMFSN}(\Theta)$. Then the following operations are defined as:

$$(i)\ \mathcal{K}(\varepsilon_1) \cup \mathcal{K}(\varepsilon_2) = \langle max\{\mu^k_{\mathcal{K}(\varepsilon_1)}, \mu^k_{\mathcal{K}(\varepsilon_2)}\}, min\{\nu^k_{\mathcal{K}(\varepsilon_1)}, \nu_{\mathcal{K}(\varepsilon_2)}\}\rangle, \langle max\{\tau^k_{\mathcal{K}(\varepsilon_1)}, \tau^k_{\mathcal{K}(\varepsilon_2)}\}, min\{\eta^k_{\mathcal{K}(\varepsilon_1)}, \eta^k_{\mathcal{K}(\varepsilon_2)}\}\rangle \tag{1}$$

$$(ii)\ \mathcal{K}(\varepsilon_1) \cap \mathcal{K}(\varepsilon_2) = \langle min\{\mu^k_{\mathcal{K}(\varepsilon_1)}, \mu^k_{\mathcal{K}(\varepsilon_2)}\}, max\{\nu^k_{\mathcal{K}(\varepsilon_1)}, \nu^k_{\mathcal{K}(\varepsilon_2)}\}\rangle, \langle min\{\tau^k_{\mathcal{K}(\varepsilon_1)}, \tau^k_{\mathcal{K}(\varepsilon_2)}\}, max\{\eta^k_{\mathcal{K}(\varepsilon_1)}, \eta^k_{\mathcal{K}(\varepsilon_2)}\}\rangle \tag{2}$$

$$(iii)\ \mathcal{K}(\varepsilon_1) \oplus \mathcal{K}(\varepsilon_2) = \langle \mu^k_{\mathcal{K}(\varepsilon_1)} + \mu^k_{\mathcal{K}(\varepsilon_2)} - \mu^k_{\mathcal{K}(\varepsilon_1)}\mu^k_{\mathcal{K}(\varepsilon_2)}, \nu^k_{\mathcal{K}(\varepsilon_1)}\nu^k_{\mathcal{K}(\varepsilon_2)}\rangle, \langle \tau^k_{\mathcal{K}(\varepsilon_1)} + \tau^k_{\mathcal{K}(\varepsilon_2)} - \tau^k_{\mathcal{K}(\varepsilon_1)}\tau^k_{\mathcal{K}(\varepsilon_2)}, \eta^k_{\mathcal{K}(\varepsilon_1)}\eta^k_{\mathcal{K}(\varepsilon_2)}\rangle \tag{3}$$

$$(iv)\ \mathcal{K}(\varepsilon_1) \otimes \mathcal{K}(\varepsilon_2) = \langle \mu^k_{\mathcal{K}(\varepsilon_1)}\mu^k_{\mathcal{K}(\varepsilon_2)}, \nu^k_{\mathcal{K}(\varepsilon_1)} + \nu^k_{\mathcal{K}(\varepsilon_2)} - \nu^k_{\mathcal{K}(\varepsilon_1)}\nu^k_{\mathcal{K}(\varepsilon_2)}\rangle, \langle \tau^k_{\mathcal{K}(\varepsilon_1)}\tau^k_{\mathcal{K}(\varepsilon_2)}, \eta^k_{\mathcal{K}(\varepsilon_1)} + \eta^k_{\mathcal{K}(\varepsilon_2)} - \eta^k_{\mathcal{K}(\varepsilon_1)}\eta^k_{\mathcal{K}(\varepsilon_2)}\rangle \tag{4}$$

$$(v)\ \mathcal{K}(\varepsilon_1)^c = \langle \nu^k_{\mathcal{K}(\varepsilon_i)}(\mathfrak{x}), \mu^k_{\mathcal{K}(\varepsilon_i)}(\mathfrak{x})\rangle, \langle \eta^k_{\mathcal{K}(\varepsilon_i)}(\mathfrak{x}), \tau^k_{\mathcal{K}(\varepsilon_i)}(\mathfrak{x})\rangle \tag{5}$$

Example 1 Let $\mathcal{K}(\varepsilon_1) = \langle(0.6, 0.2), (0.4, 0.7)\rangle, \langle(0.5, 0.7), (0.4, 0.2)\rangle$ and
$\mathcal{K}(\varepsilon_2) = \langle(0.7, 0.5), (0.3, 0.4)\rangle, \langle(0.8, 0.6), (0.1, 0.3)\rangle$ be two $\mathcal{LLDMFSN}$. Then

1. $\mathcal{K}(\varepsilon_1) \cup \mathcal{K}(\varepsilon_2) = \langle(0.7, 0.5), (0.3, 0.4)\rangle, \langle(0.8, 0.7), (0.1, 0.2)\rangle$
2. $\mathcal{K}(\varepsilon_1) \cap \mathcal{K}(\varepsilon_2) = \langle(0.6, 0.2), (0.4, 0.7)\rangle, \langle(0.5, 0.6), (0.4, 0.3)\rangle$
3. $\mathcal{K}(\varepsilon_1)^c = \langle(0.4, 0.7), (0.6, 0.2)\rangle, \langle(0.4, 0.2), (0.5, 0.7)\rangle$
4. $\mathcal{K}(\varepsilon_1) \oplus \mathcal{K}(\varepsilon_2) = \langle(0.88, 0.6), (0.12, 0.28)\rangle, \langle(0.9, 0.88), (0.04, 0.06)\rangle$
5. $\mathcal{K}(\varepsilon_1) \otimes \mathcal{K}(\varepsilon_2) = \langle(0.42, 0.10), (0.58, 0.82)\rangle, \langle(0.4, 0.42), (0.46, 0.44)\rangle$

Definition 3.11 Let (Δ, Υ) be a lattice ordered linear diophantine multi-fuzzy soft set($\mathcal{LLDMFSS}$) on Θ. Let K be the set of indices. The relation of (Δ, Υ) is defined by

$$\Delta = \{(\theta, \varepsilon), (\langle\varsigma^k(\theta, \varepsilon), \upsilon^k(\theta, \varepsilon)\rangle, \langle\alpha^k(\theta, \varepsilon), \beta^k(\theta, \varepsilon)\rangle) : (\theta, \varepsilon) \in \Theta \times \Upsilon\}$$

with $\varsigma^k, \upsilon^k, \alpha^k, \beta^k : \Theta \times \Upsilon \to [0, 1] \forall k \in K$
If $\Theta = \{\theta_1, \theta_2, ...\theta_n\}$ and $\Upsilon = \{\varepsilon_1, \varepsilon_2, ..., \varepsilon_m\}$, then Δ is given below:
where $\lambda_{ij} = \{\langle\varsigma^k(\theta_i, \varepsilon_j), \upsilon^k(\theta_i, \varepsilon_j)\rangle, \langle\alpha^k(\theta_i, \varepsilon_j), \beta^k(\theta_i, \varepsilon_j)\rangle\}$

Δ	ε_1	ε_2	$\dots$	ε_m
θ_1	λ_{11}	λ_{12}	$\dots$	λ_{1m}
θ_2	λ_{21}	λ_{22}	$\dots$	λ_{2m}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
θ_n	λ_{n1}	λ_{n2}	$\dots$	λ_{nm}

We define a matrix form of $\mathcal{LLDMFSS}$ on Θ is given by

$$\mathcal{A} = [a_{ij}] = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1m} \\ \lambda_{21} & \lambda_{22} & \dots & \lambda_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n1} & \lambda_{n2} & \dots & \lambda_{nm} \end{bmatrix} \tag{6}$$

This is known as $\mathcal{LLDMFSM}$ on Θ.

Example 2 Consider a $\mathcal{LLDMFSS}$ as shown:

$$(\Delta, \Upsilon) = \left\{ \begin{array}{r} \Delta(\varepsilon_1) = \{\mathfrak{r}_1, \langle(0.3, 0.4), (0.6, 0.5)\rangle, \langle(0.7, 0.6), (0.2, 0.1)\rangle, \\ \\ \mathfrak{r}_2, \langle(0.7, 0.5), (0.2, 0.5)\rangle, \langle(0.3, 0.6), (0.6, 0.3)\rangle\} \\ \\ \{\Delta(\varepsilon_2) = \{\mathfrak{r}_1, \langle(0.2, 0.5), (0.7, 0.3)\rangle, \langle(0.5, 0.3), (0.4, 0.6)\rangle, \\ \\ \mathfrak{r}_2, \langle(0.4, 0.7), (0.6, 0.4)\rangle, \langle(0.6, 0.5), (0.1, 0.3)\rangle\} \end{array} \right\}$$

This is represented by the following $\mathcal{LLDMFSM}$ form:

$$\mathcal{A} = [a_{ij}] = \begin{bmatrix} \langle(0.3, 0.4), (0.6, 0.5)\rangle & \langle(0.7, 0.5), (0.2, 0.5)\rangle \\ \langle(0.7, 0.6), (0.2, 0.1)\rangle & \langle(0.3, 0.6), (0.6, 0.3)\rangle \\ & \\ \langle(0.2, 0.5), (0.7, 0.3)\rangle & \langle(0.4, 0.7), (0.6, 0.4)\rangle \\ \langle(0.5, 0.3), (0.4, 0.6)\rangle & \langle(0.6, 0.5), (0.1, 0.3)\rangle \end{bmatrix}$$

Definition 3.12 Let $U = [u_{ij}]_{m\times n} \in \mathcal{LLDMFSM}(\Theta)$, where all u_{ij}'s are lattice ordered linear diophantine multi-fuzzy soft absolute number. Then U is said to be Universal matrix.

Definition 3.13 Let $N = [n_{ij}]_{m\times n} \in \mathcal{LLDMFSM}(\Theta)$, where all n_{ij}'s are lattice ordered linear diophantine multi-fuzzy soft null number. Then U is said to be Null matrix.

Definition 3.14 Let $K = [k_{ij}]_{m\times n}$ and $L = [l_{ij}]_{m\times n}$ be two $\mathcal{LLDMFSM}$ over Θ. Then the matrix K is said to be equal matrix of L iff $k_{ij} = l_{ij}$ $\forall$ i $\in \{1, 2, ..., m\}$ and $j \in \{1, 2, ...n\}$ and it is denoted by $K = L$.

Definition 3.15 Let $X = [\mathfrak{x}_{ij}]_{m\times n}$ and $Y = [\mathfrak{y}_{ij}]_{m\times n}$ be two $\mathcal{LLDMFSM}$ over Θ. Then the union of X and Y is characterized as $X \cup Y = O = [o_{ij}]_{m\times n}$
where,

$$[o_{ij}]_{m\times n} = \begin{bmatrix} \mathfrak{x}_{11}\cup\mathfrak{y}_{11} & \mathfrak{x}_{12}\cup\mathfrak{y}_{12} & \cdots & \mathfrak{x}_{1n}\cup\mathfrak{y}_{1n} \\ \mathfrak{x}_{21}\cup\mathfrak{y}_{21} & \mathfrak{x}_{22}\cup\mathfrak{y}_{22} & \cdots & \mathfrak{x}_{2n}\cup\mathfrak{y}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{x}_{m1}\cup\mathfrak{y}_{m1} & \mathfrak{x}_{m2}\cup\mathfrak{y}_{m2} & \cdots & \mathfrak{x}_{mn}\cup\mathfrak{y}_{mn} \end{bmatrix}$$

Definition 3.16 Let $X = [\mathfrak{x}_{ij}]_{m\times n}$ and $Y = [\mathfrak{y}_{ij}]_{m\times n}$ be two $\mathcal{LLDMFSM}$ over Θ. Then the intersection of X and Y is characterized as $X \cap Y = P = [p_{ij}]_{m\times n}$
where,

$$[p_{ij}]_{m\times n} = \begin{bmatrix} \mathfrak{x}_{11}\cap\mathfrak{y}_{11} & \mathfrak{x}_{12}\cap\mathfrak{y}_{12} & \cdots & \mathfrak{x}_{1n}\cap\mathfrak{y}_{1n} \\ \mathfrak{x}_{21}\cap\mathfrak{y}_{21} & \mathfrak{x}_{22}\cap\mathfrak{y}_{22} & \cdots & \mathfrak{x}_{2n}\cap\mathfrak{y}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{x}_{m1}\cap\mathfrak{y}_{m1} & \mathfrak{x}_{m2}\cap\mathfrak{y}_{m2} & \cdots & \mathfrak{x}_{mn}\cap\mathfrak{y}_{mn} \end{bmatrix}$$

Definition 3.17 Let $X = [\mathfrak{x}_{ij}]_{m\times n}$ be a $\mathcal{LLDMFSM}$ over Θ. Then the complement of X is characterized by $\mathfrak{x}^C = [\mathfrak{x}^c_{ij}]$, where

$$[\mathfrak{x}_{ij}]_{m\times n} = \begin{bmatrix} \mathfrak{x}^c_{11} & \mathfrak{x}^c_{12} & \cdots & \mathfrak{x}^c_{1n} \\ \mathfrak{x}^c_{21} & \mathfrak{x}^c_{22} & \cdots & \mathfrak{x}^c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{x}^c_{n1} & \mathfrak{x}^c_{n2} & \cdots & \mathfrak{x}^c_{mn} \end{bmatrix}$$

Proposition 3.1 Let $X = [\mathfrak{x}_{ij}]_{m\times n}$, $Y = [\mathfrak{y}_{ij}]_{m\times n}$ and $Z = [z_{ij}]_{m\times n} \in \mathcal{LLDMFSM}(\Theta)$. Then the following properties holds:

1. $[0] \subseteq [\mathfrak{x}_{ij}]$

2. $[\mathfrak{x}_{ij}] \subseteq [1]$

3. $[\mathfrak{x}_{ij}] \subseteq [\mathfrak{y}_{ij}]$ and $[\mathfrak{y}_{ij}] \subseteq [z_{ij}]$, then $[\mathfrak{x}_{ij}] \subseteq [z_{ij}]$

4. $[\mathfrak{x}_{ij}] = [\mathfrak{y}_{ij}]$ and $[\mathfrak{y}_{ij}] = [z_{ij}]$, then $[\mathfrak{x}_{ij}] = [z_{ij}]$

5. $[\mathfrak{x}_{ij}] \subseteq [\mathfrak{y}_{ij}]$ then $[\mathfrak{x}_{ij}] \cup [\mathfrak{y}_{ij}] = [\mathfrak{y}_{ij}]$ and $[\mathfrak{x}_{ij}] \cap [\mathfrak{y}_{ij}] = [\mathfrak{x}_{ij}]$

Proof. Let

$$[\mathfrak{x}_{ij}] = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \dots & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{m1} & \lambda_{m2} & \dots & \lambda_{mn} \end{bmatrix}$$

and

$$[\mathfrak{y}_{ij}] = \begin{bmatrix} \psi_{11} & \psi_{12} & \dots & \psi_{1n} \\ \psi_{21} & \psi_{22} & \dots & \psi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{m1} & \psi_{m2} & \dots & \psi_{mn} \end{bmatrix}$$

$$[\mathfrak{z}_{ij}] = \begin{bmatrix} \phi_{11} & \phi_{12} & \dots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \dots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{m1} & \phi_{m2} & \dots & \phi_{mn} \end{bmatrix}$$

(I)

$$Let [0] = [a_{ij}] = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

we know that 0 $\leq \lambda_{ij}$, $\forall$, i = {1,2,3,...n} and j = {1,2,3,...,m}.
Hence $[0] \subseteq [\mathfrak{x}_{ij}]$.
(II) Same as (I)[Since, every elements is less than or equal to 1]
(III) Let $[\mathfrak{x}_{ij}] \subseteq [\mathfrak{y}_{ij}]$ and $[\mathfrak{y}_{ij}] \subseteq [z_{ij}]$.
This implies, $\lambda_{ij} \leq \psi_{ij}$ and $\psi_{ij} \leq \phi_{ij}$, $\forall$ i = {1,2,3,...n} and j = {1,2,3,...,m}.
Thus, $\lambda_{ij} \leq \phi_{ij}$, $\forall$ i = {1,2,3,...n} and j = {1,2,3,...,m}.
Hence, $[\mathfrak{x}_{ij}] \subseteq [z_{ij}]$.
(IV) Same as (III).
(V) Let $[\mathfrak{x}_{ij}] \subseteq [\mathfrak{y}_{ij}]$, then max$\{[\mathfrak{x}_{ij}], [\mathfrak{y}_{ij}]\}$ = $[\mathfrak{y}_{ij}]$.
This implies, $[\mathfrak{x}_{ij}] \cup [\mathfrak{y}_{ij}] = [\mathfrak{y}_{ij}]$(by 1).
Similarly, $min\{[\mathfrak{x}_{ij}], [\mathfrak{y}_{ij}]\}$ = $[\mathfrak{y}_{ij}]$.
This implies, $[\mathfrak{x}_{ij}] \cap [\mathfrak{y}_{ij}] = [\mathfrak{x}_{ij}]$(by 2).

Proposition 3.2 Let $X = [\mathfrak{x}_{ij}]_{m\times n}$, $Y = [\mathfrak{y}_{ij}]_{m\times n}$ and $Z = [z_{ij}]_{m\times n} \in \mathcal{LLDMFSM}(\Theta)$. Then the following properties holds:

1. $[\mathfrak{x}_{ij}] \cap [\mathfrak{y}_{ij}] = [\mathfrak{y}_{ij}] \cap [\mathfrak{x}_{ij}]$

2. $[\mathfrak{x}_{ij}] \cup [\mathfrak{y}_{ij}] = [\mathfrak{y}_{ij}] \cup [\mathfrak{x}_{ij}]$

3. $[\mathfrak{x}_{ij}] \cap ([\mathfrak{y}_{ij}] \cap [z_{ij}]) = ([\mathfrak{x}_{ij}] \cap [\mathfrak{y}_{ij}]) \cap [z_{ij}]$

4. $[\mathfrak{x}_{ij}] \cup ([\mathfrak{y}_{ij}] \cup [z_{ij}]) = ([\mathfrak{x}_{ij}] \cup [\mathfrak{y}_{ij}]) \cup [z_{ij}]$

5. $[\mathfrak{x}_{ij}] \cup ([\mathfrak{y}_{ij}] \cap [z_{ij}]) = ([\mathfrak{x}_{ij}] \cup [\mathfrak{y}_{ij}]) \cap ([\mathfrak{x}_{ij}] \cup [z_{ij}]$

6. $[\mathfrak{x}_{ij}] \cap ([\mathfrak{y}_{ij}] \cup [z_{ij}]) = ([\mathfrak{x}_{ij}] \cap [\mathfrak{y}_{ij}]) \cup ([\mathfrak{x}_{ij}] \cap [z_{ij}]$

Proof. It is obvious from the equations 1, 2.

3.1 *Some special operators of* $\mathcal{LLDMFSM}$

Definition 3.18 Let $[\mathfrak{r}_{ij}^1], [\mathfrak{r}_{ij}^2], [\mathfrak{r}_{ij}^3], ..., [\mathfrak{r}_{ij}^r] \in \mathcal{LLDMFSM}(\Theta)$. Then the arthimetic mean($\mathscr{AM}$) of $[\mathfrak{r}_{ij}^1], [\mathfrak{r}_{ij}^2], [\mathfrak{r}_{ij}^3], ..., [\mathfrak{r}_{ij}^r]$ is defined by

$$\bigodot_{p=1}^{r}{}_{1}[\mathfrak{r}_{ij}^p] = [a_{ij}] = [\mathfrak{r}_{ij}^1]\bigodot[\mathfrak{r}_{ij}^2]\bigodot[\mathfrak{r}_{ij}^3]\bigodot...\bigodot[\mathfrak{r}_{ij}^r] \tag{7}$$

where,

$$a_{ij} = \langle\frac{1}{r}\sum_{p=1}^{r}\varsigma^{s}_{\mathfrak{r}_{ij}^p}, \frac{1}{r}\sum_{p=1}^{r}\upsilon^{s}_{\mathfrak{r}_{ij}^p}\rangle, \langle\frac{1}{r}\sum_{p=1}^{r}\alpha^{s}_{\mathfrak{r}_{ij}^p}, \frac{1}{r}\sum_{p=1}^{r}\beta^{s}_{\mathfrak{r}_{ij}^p}\rangle$$

Definition 3.19 Let $[\mathfrak{r}_{ij}^1], [\mathfrak{r}_{ij}^2], [\mathfrak{r}_{ij}^3], ..., [\mathfrak{r}_{ij}^r] \in \mathcal{LLDMFSM}(\Theta)$. Then the normalized fuzzy weighted arthimetic mean($\mathscr{NFWAM}$) of $[\mathfrak{r}_{ij}^1], [\mathfrak{r}_{ij}^2], [\mathfrak{r}_{ij}^3], ..., [\mathfrak{r}_{ij}^r]$ is defined by

$$\bigodot_{p=1}^{r}{}_{1_w}[\mathfrak{r}_{ij}^p] = [a_{ij}] = [\mathfrak{r}_{ij}^1]\bigodot[\mathfrak{r}_{ij}^2]\bigodot[\mathfrak{r}_{ij}^3]\bigodot...\bigodot[\mathfrak{r}_{ij}^r] \tag{8}$$

where,

$$a_{ij} = \langle\frac{1}{r}\sum_{p=1}^{r}w_p\varsigma^{s}_{\mathfrak{r}_{ij}^p}, \frac{1}{r}\sum_{p=1}^{r}w_p\upsilon^{s}_{\mathfrak{r}_{ij}^p}\rangle, \langle\frac{1}{r}\sum_{p=1}^{r}w_p\alpha^{s}_{\mathfrak{r}_{ij}^p}, \frac{1}{r}\sum_{p=1}^{r}w_p\beta^{s}_{\mathfrak{r}_{ij}^p}\rangle$$

where, $w_1, w_2, ..., w_r$ represent the fuzzy weight values for the $\mathcal{LLDMFSM}[\mathfrak{r}_{ij}^1], [\mathfrak{r}_{ij}^2], [\mathfrak{r}_{ij}^3], ..., [\mathfrak{r}_{ij}^r]$ respectively and also that $w_1 + w_2 + ... + w_r = 1$

Definition 3.20 Let $[\mathfrak{r}_{ij}^1], [\mathfrak{r}_{ij}^2], [\mathfrak{r}_{ij}^3], ..., [\mathfrak{r}_{ij}^r] \in \mathcal{LLDMFSM}(\Theta)$. Then the geometric mean($\mathscr{GM}$) of $[\mathfrak{r}_{ij}^1], [\mathfrak{r}_{ij}^2], [\mathfrak{r}_{ij}^3], ..., [\mathfrak{r}_{ij}^r]$ is defined by

$$\bigodot_{p=1}^{r}{}_{0}[\mathfrak{r}_{ij}^p] = [a_{ij}] = [\mathfrak{r}_{ij}^1]\bigodot[\mathfrak{r}_{ij}^2]\bigodot[\mathfrak{r}_{ij}^3]\bigodot...\bigodot[\mathfrak{r}_{ij}^r] \tag{9}$$

where

$$a_{ij} = \langle[\prod_{p=1}^{r}\varsigma^{s}_{\mathfrak{r}_{ij}^p}]^{\frac{1}{r}}, [\prod_{p=1}^{r}\upsilon^{s}_{\mathfrak{r}_{ij}^p}]^{\frac{1}{r}}\rangle, \langle[\prod_{p=1}^{r}\alpha^{s}_{\mathfrak{r}_{ij}^p}]^{\frac{1}{r}}, [\prod_{p=1}^{r}\beta^{s}_{\mathfrak{r}_{ij}^p}]^{\frac{1}{r}}\rangle$$

Definition 3.21 Let $[\mathfrak{r}_{ij}^1], [\mathfrak{r}_{ij}^2], [\mathfrak{r}_{ij}^3], ..., [\mathfrak{r}_{ij}^r] \in \mathcal{LLDMFSM}(\Theta)$. Then the normalised fuzzy weighted geometric mean($\mathscr{NFWGM}$) of $[\mathfrak{r}_{ij}^1], [\mathfrak{r}_{ij}^2], [\mathfrak{r}_{ij}^3], ..., [\mathfrak{r}_{ij}^r]$ is defined by

$$\bigodot_{p=1}^{r}{}_{0_w}[\mathfrak{r}_{ij}^p] = [a_{ij}] = [\mathfrak{r}_{ij}^1]\bigodot[\mathfrak{r}_{ij}^2]\bigodot[\mathfrak{r}_{ij}^3]\bigodot...\bigodot[\mathfrak{r}_{ij}^r] \tag{10}$$

where,

$$a_{ij} = \langle\Big[\prod_{p=1}^{r}(\varsigma^{s}_{\mathfrak{r}_{ij}^p})^{w_p}\Big], \Big[\prod_{p=1}^{r}(\upsilon^{s}_{\mathfrak{r}_{ij}^p})^{w_p}\Big]\rangle, \langle\Big[\prod_{p=1}^{r}(\alpha^{s}_{\mathfrak{r}_{ij}^p})^{w_p}\Big], \Big[\prod_{p=1}^{r}(\beta^{s}_{\mathfrak{r}_{ij}^p})^{w_p}\Big]\rangle$$

where, $w_1, w_2, ..., w_r$ represent the fuzzy weight values for the $\mathcal{LLDMFSM}[\mathfrak{r}_{ij}^1], [\mathfrak{r}_{ij}^2], [\mathfrak{r}_{ij}^3], ..., [\mathfrak{r}_{ij}^r]$ respectively and also that $w_1 + w_2 + ... + w_r = 1$

Definition 3.22 Let $[\mathfrak{x}_{ij}^1], [\mathfrak{x}_{ij}^2], [\mathfrak{x}_{ij}^3], ..., [\mathfrak{x}_{ij}^r] \in \mathcal{LLDMFSM}(\Theta)$. Then the harmonic mean($\mathscr{HM}$) of $[\mathfrak{x}_{ij}^1], [\mathfrak{x}_{ij}^2], [\mathfrak{x}_{ij}^3], ..., [\mathfrak{x}_{ij}^r]$ is defined by

$$\bigodot_{p=1^{-1}}^{r} [\mathfrak{x}_{ij}^p] = [a_{ij}] = [\mathfrak{x}_{ij}^1] \bigodot [\mathfrak{x}_{ij}^2] \bigodot [\mathfrak{x}_{ij}^3] \bigodot \cdots \bigodot [\mathfrak{x}_{ij}^r] \tag{11}$$

where,

$$a_{ij} = \langle \Big(\frac{1}{r}\sum_{p=1}^{r} \frac{1}{\varsigma^s_{\mathfrak{x}_{ij}^p}}\Big)^{-1}, \Big(\frac{1}{r}\sum_{p=1}^{r} \frac{1}{\upsilon^s_{\mathfrak{x}_{ij}^p}}\Big)^{-1} \rangle, \langle \Big(\frac{1}{r}\sum_{p=1}^{r} \frac{1}{\alpha^s_{\mathfrak{x}_{ij}^p}}\Big)^{-1}, \Big(\frac{1}{r}\sum_{p=1}^{r} \frac{1}{\beta^s_{\mathfrak{x}_{ij}^p}}\Big)^{-1} \rangle$$

Definition 3.23 Let $[\mathfrak{x}_{ij}^1], [\mathfrak{x}_{ij}^2], [\mathfrak{x}_{ij}^3], ..., [\mathfrak{x}_{ij}^r] \in \mathcal{LLDMFSM}(\Theta)$. Then the normalised fuzzy weighted harmonic mean($\mathscr{NFWHM}$) of $[\mathfrak{x}_{ij}^1], [\mathfrak{x}_{ij}^2], [\mathfrak{x}_{ij}^3], ..., [\mathfrak{x}_{ij}^r]$ is defined by

$$\bigodot_{p=1^{-1_w}}^{r} [\mathfrak{x}_{ij}^p] = [a_{ij}] = [\mathfrak{x}_{ij}^1] \bigodot [\mathfrak{x}_{ij}^2] \bigodot [\mathfrak{x}_{ij}^3] \bigodot \cdots \bigodot [\mathfrak{x}_{ij}^r] \tag{12}$$

where,

$$a_{ij} = \langle \Big(\frac{1}{r}\sum_{p=1}^{r} \frac{w_p}{\varsigma^s_{\mathfrak{x}_{ij}^p}}\Big)^{-1}, \Big(\frac{1}{r}\sum_{p=1}^{r} \frac{w_p}{\upsilon^s_{\mathfrak{x}_{ij}^p}}\Big)^{-1} \rangle, \langle \Big(\frac{1}{r}\sum_{p=1}^{r} \frac{w_p}{\alpha^s_{\mathfrak{x}_{ij}^p}}\Big)^{-1}, \Big(\frac{1}{r}\sum_{p=1}^{r} \frac{w_p}{\beta^s_{\mathfrak{x}_{ij}^p}}\Big)^{-1} \rangle$$

where, $w_1, w_2, ..., w_r$ represent the fuzzy weight values for the $\mathcal{LLDMFSM}[\mathfrak{x}_{ij}^1], [\mathfrak{x}_{ij}^2], [\mathfrak{x}_{ij}^3], ..., [\mathfrak{x}_{ij}^r]$ respectively and also that $w_1 + w_2 + ... + w_r = 1$

Definition 3.24 Let $[\mathfrak{x}_{ij}^1], [\mathfrak{x}_{ij}^2], [\mathfrak{x}_{ij}^3], ..., [\mathfrak{x}_{ij}^r] \in \mathcal{LLDMFSM}(\Theta)$. Then the power mean($\mathscr{PM}$) of $[\mathfrak{x}_{ij}^1], [\mathfrak{x}_{ij}^2], [\mathfrak{x}_{ij}^3], ..., [\mathfrak{x}_{ij}^r]$ is defined by

$$\bigodot_{p=1^{p}}^{r} [\mathfrak{x}_{ij}^p] = [a_{ij}] = [\mathfrak{x}_{ij}^1] \bigodot [\mathfrak{x}_{ij}^2] \bigodot [\mathfrak{x}_{ij}^3] \bigodot \cdots \bigodot [\mathfrak{x}_{ij}^r]$$

$$a_{ij} = \begin{cases} \langle \Big[\frac{1}{r}\sum_{p=1}^{r} \big(\varsigma^s_{\mathfrak{x}_{ij}^p}\big)^q\Big]^{\frac{1}{q}}, \Big[\frac{1}{r}\sum_{p=1}^{r} \upsilon^s_{\mathfrak{x}_{ij}^p}\big)^q\Big]^{\frac{1}{q}} \rangle, \langle \Big[\frac{1}{r}\sum_{p=1}^{r} \alpha^s_{\mathfrak{x}_{ij}^p}\big)^q\Big]^{\frac{1}{q}}, \Big[\frac{1}{r}\sum_{p=1}^{r} \beta^s_{\mathfrak{x}_{ij}^p}\big)^q\Big]^{\frac{1}{q}} \rangle & q \neq 0 \\ \langle \Big[\prod_{p=1}^{r} \varsigma^s_{\mathfrak{x}_{ij}^p}\Big]^{\frac{1}{r}}, \Big[\prod_{p=1}^{r} \upsilon^s_{\mathfrak{x}_{ij}^p}\Big]^{\frac{1}{r}} \rangle, \langle \Big[\prod_{p=1}^{r} \alpha^s_{\mathfrak{x}_{ij}^p}\Big]^{\frac{1}{r}}, \Big[\prod_{p=1}^{r} \beta^s_{\mathfrak{x}_{ij}^p}\Big]^{\frac{1}{r}} \rangle & q = 0 \end{cases} \tag{13}$$

Definition 3.25 Let $[\mathfrak{x}_{ij}^1], [\mathfrak{x}_{ij}^2], [\mathfrak{x}_{ij}^3], ..., [\mathfrak{x}_{ij}^r] \in \mathcal{LLDMFSM}(\Theta)$. Then the normalized fuzzy weighted power mean($\mathscr{NFWPM}$) of $[\mathfrak{x}_{ij}^1], [\mathfrak{x}_{ij}^2], [\mathfrak{x}_{ij}^3], ..., [\mathfrak{x}_{ij}^r]$ is defined by

$$\bigodot_{p=1^{p_w}}^{r} [\mathfrak{x}_{ij}^p] = [a_{ij}] = [\mathfrak{x}_{ij}^1] \bigodot [\mathfrak{x}_{ij}^2] \bigodot [\mathfrak{x}_{ij}^3] \bigodot \cdots \bigodot [\mathfrak{x}_{ij}^r]$$

$$\mathfrak{x}_{ij} = \begin{cases} \langle \Big[\sum_{p=1}^{r} w_p \big(\varsigma^s_{\mathfrak{x}_{ij}^p}\big)^q\Big]^{\frac{1}{q}}, \Big[\sum_{p=1}^{r} w_p (\upsilon^s_{\mathfrak{x}_{ij}^p}\big)^q\Big]^{\frac{1}{q}} \rangle, \langle \Big[\sum_{p=1}^{r} w_p (\alpha^s_{\mathfrak{x}_{ij}^p}\big)^q\Big]^{\frac{1}{q}}, \Big[\sum_{p=1}^{r} w_p (\beta^s_{\mathfrak{x}_{ij}^p}\big)^q\Big]^{\frac{1}{q}} \rangle & q \neq 0 \\ \langle \prod_{p=1}^{r} \big(\varsigma^s_{\mathfrak{x}_{ij}^p}\big)^{w}_{p}, \prod_{p=1}^{r} (\upsilon^s_{\mathfrak{x}_{ij}^p}\big)^{w}_{p} \rangle, \langle \prod_{p=1}^{r} (\alpha^s_{\mathfrak{x}_{ij}^p}\big)^{w}_{p}, \prod_{p=1}^{r} (\beta^s_{\mathfrak{x}_{ij}^p}\big)^{w}_{p} \rangle & q = 0 \end{cases} \tag{14}$$

where, $w_1, w_2, ..., w_r$ represent the fuzzy weight values for the $\mathcal{LLDMFSM}[\mathfrak{x}_{ij}^1], [\mathfrak{x}_{ij}^2], [\mathfrak{x}_{ij}^3], ..., [\mathfrak{x}_{ij}^r]$ respectively and also that $w_1 + w_2 + ... + w_r = 1$

Special Cases:

The pth power mean of $\mathcal{LLDMFSM}$ is considered to be

1. The arithmetic mean of $\mathcal{LLDMFSM}$ if p = 1.
2. The geometric mean of $\mathcal{LLDMFSM}$ if p = 0.
3. The harmonic mean of $\mathcal{LLDMFSM}$ if p = -1.

The normalized fuzzy weighted pth power mean of $\mathcal{LLDMFSM}$ is considered to

1. The normalized fuzzy weighted arithmetic mean of $\mathcal{LLDMFSM}$ if p = 1.
2. The normalized fuzzy weighted geometric mean of $\mathcal{LLDMFSM}$ if p = 0.
3. The normalized fuzzy weighted harmonic mean of $\mathcal{LLDMFSM}$ if p = -1.

The equation 13 becomes equation 7 when we substitute p = 1. And when p = 0, it becomes equation 9. If p = -1 in equation 13, then it becomes equation 11. Similarly for weighted mean operations.

Proposition 3.3 Let $[\mathfrak{r}_{ij}^1], [\mathfrak{r}_{ij}^2], [\mathfrak{r}_{ij}^3], ..., [\mathfrak{r}_{ij}^r] \in \mathcal{LLDMFSM}(\Theta)$. Then

(i) If $[\mathfrak{r}_{ij}^1] = [\mathfrak{r}_{ij}^2] = [\mathfrak{r}_{ij}^3] = ... = [\mathfrak{r}_{ij}^r]$ then

$$\bigodot_{p=1}^{r}{}_{1}[\mathfrak{r}_{ij}^p] = \bigodot_{p=1}^{r}{}_{-1}[\mathfrak{r}_{ij}^p] = \bigodot_{p=1}^{r}{}_{0}[\mathfrak{r}_{ij}^p]$$

(ii) If $[\mathfrak{r}_{ij}^1] = [\mathfrak{r}_{ij}^2] = [\mathfrak{r}_{ij}^3] = ... = [\mathfrak{r}_{ij}^r]$ then

$$\bigodot_{p=1}^{r}{}_{1_w}[\mathfrak{r}_{ij}^p] = \bigodot_{p=1}^{r}{}_{-1_w}[\mathfrak{r}_{ij}^p] = \bigodot_{p=1}^{r}{}_{0_w}[\mathfrak{r}_{ij}^p]$$

Proof. (i) Let $[\mathfrak{r}_{ij}^p] = [\mathfrak{r}_{ij}]$ for all $p = 1, 2, 3...r.$

i.e., for all i and j we have,

$\langle \varsigma^s_{\mathfrak{r}_{ij}^p}, \upsilon^s_{\mathfrak{r}_{ij}^p} \rangle, \langle \alpha^s_{\mathfrak{r}_{ij}^p}, \beta^s_{\mathfrak{r}_{ij}^p} \rangle = \langle \varsigma^s_{\mathfrak{r}_{ij}}, \upsilon^s_{\mathfrak{r}_{ij}} \rangle, \langle \alpha^s_{\mathfrak{r}_{ij}}, \beta^s_{\mathfrak{r}_{ij}} \rangle \;\; \forall p = 1, 2, ..., r \; \forall s$

Then,

$$\begin{aligned}\sum_{p=1}^{r} \varsigma^s_{\mathfrak{r}_{ij}^p} &= \varsigma^s_{\mathfrak{r}_{ij}^1} + \varsigma^s_{\mathfrak{r}_{ij}^2} + \cdots + \varsigma^s_{\mathfrak{r}_{ij}^r} \\ &= \varsigma^s_{\mathfrak{r}_{ij}} + \varsigma^s_{\mathfrak{r}_{ij}} + \cdots + \varsigma^s_{\mathfrak{r}_{ij}} \\ &= r\varsigma^s_{\mathfrak{r}_{ij}}\end{aligned}$$

$$\frac{1}{r}\sum_{p=1}^{r} \varsigma^s_{\mathfrak{r}_{ij}^p} = \varsigma^s_{\mathfrak{r}_{ij}}$$

Similarly,
$\frac{1}{r}\sum_{p=1}^{r}\upsilon^{s}_{\mathfrak{x}^{p}_{ij}}=\upsilon^{s}_{\mathfrak{x}_{ij}}$, $\frac{1}{r}\sum_{p=1}^{r}\alpha^{s}_{\mathfrak{x}^{p}_{ij}}=\alpha^{s}_{\mathfrak{x}_{ij}}$ and $\frac{1}{r}\sum_{p=1}^{r}\beta^{s}_{\mathfrak{x}^{p}_{ij}}=\beta^{s}_{\mathfrak{x}_{ij}}$
Therefore, by equation 7,

$$\bigodot_{p=1}^{r}{}_{1}[\mathfrak{x}^{p}_{ij}]=\langle\varsigma^{s}_{\mathfrak{x}_{ij}},\,\upsilon^{s}_{\mathfrak{x}_{ij}}\rangle,\,\langle\alpha^{s}_{\mathfrak{x}_{ij}},\,\beta^{s}_{\mathfrak{x}_{ij}}\rangle$$

$$=[\mathfrak{x}_{ij}]$$

Let us consider,

$$\prod_{p=1}^{r}\varsigma^{s}_{\mathfrak{x}^{p}_{ij}}=\varsigma^{s}_{\mathfrak{x}^{1}_{ij}}\cdot\varsigma^{s}_{\mathfrak{x}^{2}_{ij}}+\cdots+\varsigma^{s}_{\mathfrak{x}^{r}_{ij}}$$

$$=\varsigma^{s}_{\mathfrak{x}_{ij}}+\varsigma^{s}_{\mathfrak{x}_{ij}}+\cdots+\varsigma^{s}_{\mathfrak{x}_{ij}}$$

$$=(\varsigma^{s}_{\mathfrak{x}_{ij}})^{r}$$

$$[\prod_{p=1}^{r}\varsigma^{s}_{\mathfrak{x}^{p}_{ij}}]^{\frac{1}{r}}=\varsigma^{s}_{\mathfrak{x}_{ij}}$$

Similarly, $[\prod_{p=1}^{r}\upsilon^{s}_{\mathfrak{x}^{p}_{ij}}]^{\frac{1}{r}}=\upsilon^{s}_{\mathfrak{x}_{ij}}$, $[\prod_{p=1}^{r}\alpha^{s}_{\mathfrak{x}^{p}_{ij}}]^{\frac{1}{r}}=\alpha^{s}_{\mathfrak{x}_{ij}}$ and $[\prod_{p=1}^{r}\beta^{s}_{\mathfrak{x}^{p}_{ij}}]^{\frac{1}{r}}=\beta^{s}_{\mathfrak{x}_{ij}}$.
Therefore, by equation 7,

$$\bigodot_{p=1}^{r}{}_{-1}[\mathfrak{x}^{p}_{ij}]=\langle\varsigma^{s}_{\mathfrak{x}_{ij}},\,\upsilon^{s}_{\mathfrak{x}_{ij}}\rangle,\,\langle\alpha^{s}_{\mathfrak{x}_{ij}},\,\beta^{s}_{\mathfrak{x}_{ij}}\rangle$$

$$=[\mathfrak{x}_{ij}]$$

For harmonic mean let us consider,

$$\sum_{p=1}^{r} \frac{1}{\varsigma^{s}_{\mathfrak{x}^{p}_{ij}}} = \frac{1}{\varsigma^{s}_{\mathfrak{x}^{1}_{ij}}} + \frac{1}{\varsigma^{s}_{\mathfrak{x}^{2}_{ij}}} + \cdots + \frac{1}{\varsigma^{s}_{\mathfrak{x}^{r}_{ij}}}$$

$$= \frac{1}{\varsigma^{s}_{\mathfrak{x}_{ij}}} + \frac{1}{\varsigma^{s}_{\mathfrak{x}_{ij}}} + \cdots + \frac{1}{\varsigma^{s}_{\mathfrak{x}_{ij}}} (r \; times)$$

$$= \frac{r}{\varsigma^{s}_{\mathfrak{x}_{ij}}}$$

$$[\frac{1}{r} \sum_{p=1}^{r} \frac{1}{\varsigma^{s}_{\mathfrak{x}^{p}_{ij}}}]^{-1} = [\frac{1}{\varsigma^{s}_{\mathfrak{x}_{ij}}}]^{-1}$$

$$= \varsigma^{s}_{\mathfrak{x}_{ij}}$$

Similarly,
$[\frac{1}{r}\sum_{p=1}^{r}\frac{1}{\upsilon^{s}_{\mathfrak{x}^{p}_{ij}}}]^{-1} = \upsilon^{s}_{\mathfrak{x}_{ij}}$, $[\frac{1}{r}\sum_{p=1}^{r}\frac{1}{\alpha^{s}_{\mathfrak{x}^{p}_{ij}}}]^{-1} = \alpha^{s}_{\mathfrak{x}_{ij}}$ and $[\frac{1}{r}\sum_{p=1}^{r}\frac{1}{\beta^{s}_{\mathfrak{x}^{p}_{ij}}}]^{-1} = \beta^{s}_{\mathfrak{x}_{ij}}$
Therefore,

$$\bigodot_{p=1}^{r}{}_{1} [\mathfrak{x}^{p}_{ij}] = \langle \varsigma^{s}_{\mathfrak{x}_{ij}}, \upsilon^{s}_{\mathfrak{x}_{ij}} \rangle, \langle \alpha^{s}_{\mathfrak{x}_{ij}}, \beta^{s}_{\mathfrak{x}_{ij}} \rangle$$

$$= [\mathfrak{x}_{ij}]$$

(ii) Let $[\mathfrak{x}^{p}_{ij}] = [\mathfrak{x}_{ij}]$ for all $p = 1, 2, 3...r$.
i.e., for all i and j we have,
$\langle \varsigma^{s}_{\mathfrak{x}^{p}_{ij}}, \upsilon^{s}_{\mathfrak{x}^{p}_{ij}} \rangle, \langle \alpha^{s}_{\mathfrak{x}^{p}_{ij}}, \beta^{s}_{\mathfrak{x}^{p}_{ij}} \rangle = \langle \varsigma^{s}_{\mathfrak{x}_{ij}}, \upsilon^{s}_{\mathfrak{x}_{ij}} \rangle, \langle \alpha^{s}_{\mathfrak{x}_{ij}}, \beta^{s}_{\mathfrak{x}_{ij}} \rangle \forall p = 1, 2, ..., r \; \forall s$

w_1, w_2, ..., w_r be the fuzzy weight of $\mathcal{LLDMFSM}$ $[\mathfrak{r}_{ij}^1]$, $[\mathfrak{r}_{ij}^2]$, $[\mathfrak{r}_{ij}^3]$, ..., $[\mathfrak{r}_{ij}^r]$ and $w_1 + w_2 + ... + w_r = 1$.
Then,

$$\begin{aligned}\sum_{p=1}^{r}(w_p)\varsigma^s_{\mathfrak{r}_{ij}^p} =& w_1\varsigma^s_{\mathfrak{r}_{ij}^1} + w_2\varsigma^s_{\mathfrak{r}_{ij}^2} + \cdots + w_r\varsigma^s_{\mathfrak{r}_{ij}^r}\\ =& w_1\varsigma^s_{\mathfrak{r}_{ij}} + w_2\varsigma^s_{\mathfrak{r}_{ij}} + \cdots + w_r\varsigma^s_{\mathfrak{r}_{ij}}\\ =& \varsigma^s_{\mathfrak{r}_{ij}}[w_1 + w_2 + \cdots + w_r]\\ =& \varsigma^s_{\mathfrak{r}_{ij}}\end{aligned}$$

$$\sum_{p=1}^{r}\varsigma^s_{\mathfrak{r}_{ij}^p} = \varsigma^s_{\mathfrak{r}_{ij}}$$

Similarly,
$\sum_{p=1}^{r} w_p \upsilon^s_{\mathfrak{r}_{ij}^p} = \upsilon^s_{\mathfrak{r}_{ij}}$, $\sum_{p=1}^{r} w_p \alpha^s_{\mathfrak{r}_{ij}^p} = \alpha^s_{\mathfrak{r}_{ij}}$ and $\sum_{p=1}^{r} w_p \beta^s_{\mathfrak{r}_{ij}^p} = \beta^s_{\mathfrak{r}_{ij}}$
Therefore,

$$\begin{aligned}\bigodot_{p=1}^{r}{}_{1_w}[\mathfrak{r}_{ij}^p] =& \langle\varsigma^s_{\mathfrak{r}_{ij}},\, \upsilon^s_{\mathfrak{r}_{ij}}\rangle,\, \langle\alpha^s_{\mathfrak{r}_{ij}},\, \beta^s_{\mathfrak{r}_{ij}}\rangle\\ =& [\mathfrak{r}_{ij}]\end{aligned}$$

Now, consider

$$\begin{aligned}\prod_{p=1}^{r}(\varsigma^s_{\mathfrak{r}_{ij}^p})^{w_p} =& (\varsigma^s_{\mathfrak{r}_{ij}^1})^{w_1}(\varsigma^s_{\mathfrak{r}_{ij}^2})^{w_2}\ldots(\varsigma^s_{\mathfrak{r}_{ij}^r})^{w_r}\\ =& (\varsigma^s_{\mathfrak{r}_{ij}})^{w_1}(\varsigma^s_{\mathfrak{r}_{ij}})^{w_2}\cdots+(\varsigma^s_{\mathfrak{r}_{ij}})^{w_r}\\ =& (\varsigma^s_{\mathfrak{r}_{ij}})^{w_1+w_2+\ldots+w_r}\\ =& \varsigma^s_{\mathfrak{r}_{ij}}\end{aligned}$$

$$\prod_{p=1}^{r}(w_p)\varsigma^s_{\mathfrak{r}_{ij}^p} = \varsigma^s_{\mathfrak{r}_{ij}}$$

Similarly, $\prod_{p=1}^{r}(w_p)\upsilon^{s}_{\mathfrak{x}^{p}_{ij}} = \upsilon^{s}_{\mathfrak{x}_{ij}}$, $\prod_{p=1}^{r}(w_p)\alpha^{s}_{\mathfrak{x}^{p}_{ij}} = \alpha^{s}_{\mathfrak{x}_{ij}}$ and $\prod_{p=1}^{r}(w_p)\beta^{s}_{\mathfrak{x}^{p}_{ij}} = \beta^{s}_{\mathfrak{x}_{ij}}$.
Therefore,

$$\begin{aligned}\bigodot_{p=1^{-1}w}^{r}[\mathfrak{x}^{p}_{ij}] =& \langle \varsigma^{s}_{\mathfrak{x}_{ij}},\, \upsilon^{s}_{\mathfrak{x}_{ij}}\rangle,\, \langle \alpha^{s}_{\mathfrak{x}_{ij}},\, \beta^{s}_{\mathfrak{x}_{ij}}\rangle \\ =& [\mathfrak{x}_{ij}]\end{aligned}$$

For harmonic mean let us consider,

$$\begin{aligned}\sum_{p=1}^{r}\frac{w_p}{\varsigma^{s}_{\mathfrak{x}^{p}_{ij}}} =& \frac{w_1}{\varsigma^{s}_{\mathfrak{x}^{1}_{ij}}} + \frac{w_2}{\varsigma^{s}_{\mathfrak{x}^{2}_{ij}}} + \cdots + \frac{w_r}{\varsigma^{s}_{\mathfrak{x}^{r}_{ij}}} \\ =& \frac{w_1}{\varsigma^{s}_{\mathfrak{x}_{ij}}} + \frac{w_2}{\varsigma^{s}_{\mathfrak{x}_{ij}}} + \cdots + \frac{w_r}{\varsigma^{s}_{\mathfrak{x}_{ij}}} \\ =& \frac{1}{\varsigma^{s}_{\mathfrak{x}_{ij}}}[w_1 + w_2 + \ldots + w_r] \\ =& \frac{1}{\varsigma^{s}_{\mathfrak{x}_{ij}}} \\ [\sum_{p=1}^{r}\frac{1}{\varsigma^{s}_{\mathfrak{x}^{p}_{ij}}}]^{-1} =& [\frac{1}{\varsigma^{s}_{\mathfrak{x}_{ij}}}]^{-1} \\ =& \varsigma^{s}_{\mathfrak{x}_{ij}}\end{aligned}$$

Similarly,
$[\sum_{p=1}^{r}\frac{w_p}{\upsilon^{s}_{\mathfrak{x}^{p}_{ij}}}]^{-1} = \upsilon^{s}_{\mathfrak{x}_{ij}}$, $[\sum_{p=1}^{r}\frac{w_p}{\alpha^{s}_{\mathfrak{x}^{p}_{ij}}}]^{-1} = \alpha^{s}_{\mathfrak{x}_{ij}}$ and $[\sum_{p=1}^{r}\frac{w_p}{\beta^{s}_{\mathfrak{x}^{p}_{ij}}}]^{-1} = \beta^{s}_{\mathfrak{x}_{ij}}$
Therefore,

$$\begin{aligned}\bigodot_{p=1^{-1}w}^{r}[\mathfrak{x}^{p}_{ij}] =& \langle \varsigma^{s}_{\mathfrak{x}_{ij}},\, \upsilon^{s}_{\mathfrak{x}_{ij}}\rangle,\, \langle \alpha^{s}_{\mathfrak{x}_{ij}},\, \beta^{s}_{\mathfrak{x}_{ij}}\rangle \\ =& [\mathfrak{x}_{ij}]\end{aligned}$$

Proposition 3.4 Let $[\mathfrak{r}^1_{ij}]$, $[\mathfrak{r}^2_{ij}] \in \mathcal{LLDMFSM}(\Theta)$. Then the mean operations are commutative. i.e. $[\mathfrak{r}^1_{ij}] \circ [\mathfrak{r}^2_{ij}] = [\mathfrak{r}^2_{ij}] \circ [\mathfrak{r}^2_{ij}]$ for each

$$\circ \in \{\bigodot_{p=1}^{r}{}_1, \bigodot_{p=1}^{r}{}_{-1}, \bigodot_{p=1}^{r}{}_0, \bigodot_{p=1}^{r}{}_{1_w}, \bigodot_{p=1}^{r}{}_{-1_w}, \bigodot_{p=1}^{r}{}_{0_w}\}$$

Proof. The proof is obvious from the definition.

Definition 3.26 Let $P = [p_{ij}] \in \mathcal{LLDMFSM}_{m\times n}(\Theta)$ and $Q = [q_{jk}] \in \mathcal{LLDMFSM}_{n\times p}(\Theta)$. Then max-min composition fuzzy soft matrix of P and Q is defined by

$$P \star Q = [r_{ik}]_{m\times p}$$

where,

$$r_{ik} = \Big\{ \langle \max \Big\{ \min \big[\varsigma^l_P(p_{ij}), \varsigma^l_Q(q_{jk})\big]_j \Big\}, \tag{15}$$

$$\min \Big\{ \max \big[\upsilon^l_P(p_{ij}), \upsilon^l_Q(q_{jk})\big]_j \Big\} \rangle, \tag{16}$$

$$\langle \max \Big\{ \min \big[\alpha^l_P(p_{ij}), \alpha^l_Q(q_{jk})\big]_j \Big\}, \tag{17}$$

$$\min \Big\{ \max \big[\beta^l_P(p_{ij}), \beta^l_Q(q_{jk})\big]_j \Big\} \rangle \Big\} \tag{18}$$

$\forall l \in L$

Example 3 Let

$$\mathcal{P} = [p_{ij}] = \begin{bmatrix} \langle(0.3, 0.4), (0.6, 0.5)\rangle & \langle(0.7, 0.5), (0.2, 0.5)\rangle \\ \langle(0.7, 0.6), (0.2, 0.1)\rangle & \langle(0.3, 0.6), (0.6, 0.3)\rangle \\ & \\ \langle(0.2, 0.5), (0.7, 0.3)\rangle & \langle(0.4, 0.7), (0.6, 0.4)\rangle \\ \langle(0.5, 0.3), (0.4, 0.6)\rangle & \langle(0.6, 0.5), (0.1, 0.3)\rangle \end{bmatrix}$$

and

$$\mathcal{Q} = [q_{ij}] = \begin{bmatrix} \langle(0.6, 0.3), (0.4, 0.5)\rangle & \langle(0.5, 0.2), (0.6, 0.7)\rangle \\ \langle(0.7, 0.8), (0.2, 0.1)\rangle & \langle(0.6, 0.7), (0.2, 0.2)\rangle \\ & \\ \langle(0.4, 0.7), (0.6, 0.2)\rangle & \langle(0.8, 0.9), (0.3, 0.1)\rangle \\ \langle(0.2, 0.1), (0.7, 0.8)\rangle & \langle(0.4, 0.7), (0.5, 0.2)\rangle \end{bmatrix}$$

Then

$$
\begin{aligned}
\varsigma_{\lambda_{11}} =& max\{min(0.3, 0.6), min(0.7, 0.4)\},\\
& max\{min(0.4, 0.3), min(0.5, 0.7)\}\\
=& max(0.3, 0.4), max(0.3, 0.5)\\
=& (0.4, 0.5)\\
\upsilon_{\lambda_{11}} =& min\{max(0.6, 0.4), max(0.2, 0.6)\}\\
& min\{max(0.5, 0.5), max(0.5, 0.2)\}\\
=& min(0.6, 0.6), min(0.5, 0.5)\\
=& (0.6, 0.5)\\
\alpha_{\lambda_{11}} =& max\{min(0.7, 0.7), min(0.3, 0.2)\},\\
& max\{min(0.6, 0.8), min(0.6, 0.1)\}\\
=& max(0.7, 0.2), max(0.6, 0.1)\\
=& (0.7, 0.6)\\
\beta_{\lambda_{11}} =& min\{max(0.2, 0.2), max(0.6, 0.7)\}\\
& min\{max(0.1, 0.1), max(0.3, 0.8)\}\\
=& min(0.2, 0.7), min(0.1, 0.8)\\
=& (0.2, 0.1)
\end{aligned}
$$

$\lambda_{11} = \langle(0.4, 0.5), (0.6, 0.5)\rangle, \langle(0.7, 0.6), (0.2, 0.1)\rangle$
similarly,
$\lambda_{12} = \langle(0.7, 0.5), (0.2, 0.1)\rangle, \langle(0.6, 0.6), (0.2, 0.1)\rangle$
$\lambda_{21} = \langle(0.4, 0.7), (0.6, 0.4)\rangle, \langle(0.5, 0.3), (0.4, 0.6)\rangle$

$\lambda_{22} = \langle (04, 0.7), (0.6, 0.4) \rangle, \langle (0.5, 0.5), (0.4, 0.3) \rangle$

$$\mathcal{P} \star \mathcal{Q} = \begin{bmatrix} \langle (0.4, 0.5), (0.6, 0.5) \rangle & \langle (0.7, 0.5), (0.2, 0.1) \rangle \\ \langle (0.7, 0.6), (0.2, 0.1) \rangle & \langle (0.6, 0.6), (0.2, 0.1) \rangle \\ & \\ \langle (0.4, 0.7), (0.6, 0.4) \rangle & \langle (0.5, 0.3), (0.4, 0.6) \rangle \\ \langle (04, 0.7), (0.6, 0.4) \rangle & \langle (0.5, 0.5), (0.4, 0.3) \rangle \end{bmatrix}$$

4. Application of LLDMFSM in Agriculture

The advancement of agricultural productivity and production is largely dependent on agricultural research, which is essential to achieving the agriculture sector's enormous potential for increasing national income. Farmers, who are the main contributors to agricultural endeavors, face a multitude of challenges worldwide despite the vast opportunities. These issues require creative solutions because they involve a variety of factors that lead to agricultural losses. Because of dynamic factors like soil fertility, farming practices, and climate variability, the agricultural landscape is intrinsically complex. Given the complex ways in which these variables interact, there is an urgent need for targeted interventions that will enable farmers to choose profitable and sustainable agricultural practices.

We present an application that uses the Lattice-ordered Linear Diophantine Multi-Fuzzy Soft Matrix (LLDMFSM) to address these issues. This application functions as an advanced tool meant to help farmers get over the uncertainties related to crop selection. Through the utilization of LLDMFSM's distinct features, the application assesses and examines the intricate relationships among soil fertility, climate variability, and diverse farming techniques. The idea is to give farmers a solid, data-driven framework for making decisions about which crops to plant based on the state of the environment at the time and what works best for their particular farming situation.

This creative method not only solves the pressing issues that farmers are facing, but it also establishes agricultural research as a powerful agent of good. By incorporating cutting-edge technologies such as LLDMFSM into useful applications, we aim to empower farmers, reduce agricultural losses, and support the agriculture sector's sustainable growth. This case study demonstrates how innovative approaches have the power to transform conventional farming methods and open the door to a more resilient and successful agricultural future.

4.1 *Mathematical modelling*

This novel and hybrid approach contains the following steps:

Input:

(i) Consider Θ as a Universal set.

(ii) Consider Υ as a set of attributes.

Constructions:

(iii) Construct $\mathcal{G}$, $\mathcal{F}$: $\Theta \times \Upsilon \to \mathcal{LLDMFSN}$ with the help of decision makers.

(iv)Obtain $\mathcal{LLDMFSM}$ $\mathcal{M}$, $\mathcal{N}$ from $\mathcal{G}$, $\mathcal{F}$ respectively.

Calculations:

(v) Find max-min composition for $\mathcal{M}$ and $\mathcal{N}$ using 15.

(vi) Compute the score matrix with the help of definition.

Output:

(vii) Find the score of an object with respective output.

The schematic illustration of the algorithm is shown in the figure 1

Figure 1. The flowchart representation of algorithm

4.2 *Illustrative Example*

Suppose a farmer wishes to choose the most suitable crop that grows in all types of climatic conditions and soil types. Let us consider a set of expert systems to construct a fuzzy system. The decision-makers construct a set of alternatives K = $\{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4\}$ as four types of crops. The four climates(summer, spring, fall and winter) are taken as indexes. Now consider, D = $\{\mathscr{S}_1,\mathscr{S}_2,\mathscr{S}_3,\mathscr{S}_4\}$ be the set of soils. Here, the order of preference was given by the availability range of soil in our country. It is shown in the figure 2. Let $\mathfrak{J}$ be the LLDMFSS($\mathfrak{J}$,K) over Θ. where $\mathfrak{J}$: $K \to LLDMFSN(D)$ which has crops as alternatives and soil as attributes. The table2 shows the LLDMFSS(Θ)$\mathfrak{J}$.

Table 1. LLDMFSS(Θ)$\mathfrak{J}$

$\mathfrak{J}$	$\mathfrak{C}_1$	$\mathfrak{C}_2$	$\mathfrak{C}_3$	$\mathfrak{C}_4$
$\mathfrak{J}(S_1)$	⟨(0.6,0.4,0.7,0.5) (0.3,0.6,0.2,0.5)⟩ ⟨(0.7,0.5,0.6,0.3) (0.1,0.4,0.3,0.5)⟩	⟨(0.8,0.6,0.7,0.5) (0.2,0.3,0.4,0.4)⟩ ⟨(0.8,0.6,0.7,0.4) (0.2,0.3,0.3,0.6)⟩	⟨(0.8,0.6,0.8,0.6) (0.1,0.3,0.2,0.3)⟩ ⟨(0.9,0.7,0.8,0.4) (0.1,0.2,0.1,0.5)⟩	⟨(0.9,0.7,0.8,0.7) (0.1,0.2,0.2,0.3)⟩ ⟨(0.9,0.8,0.8,0.6) (0.1,0.1,0.1,0.4)⟩
$\mathfrak{J}(S_2)$	⟨(0.7,0.5,0.3,0.5) (0.5,0.4,0.8,0.4)⟩ ⟨(0.9,0.5,0.7,0.7) (0.1,0.4,0.2,0.3)⟩	⟨(0.7,0.6,0.7,0.6) (0.3,0.3,0.3,0.3)⟩ ⟨(0.9,0.6,0.7,0.8) (0.3,0.3,0.2,0.2)⟩	⟨(0.8,0.7,0.8,0.7) (0.2,0.2,0.3,0.2)⟩ ⟨(0.8,0.7,0.9,0.8) (0.1,0.1,0.2,0.2)⟩	⟨(0.8,0.8,0.8,0.8) (0.1,0.1,0.1,0.1)⟩ ⟨(0.8,0.8,0.9,0.8) (0.1,0.1,0.2,0.2)⟩
$\mathfrak{J}(S_3)$	⟨(0.6,0.4,0.7,0.4) (0.4,0.7,0.3,0.5)⟩ ⟨(0.7,0.6,0.4,0.3) (0.2, 0.3, 0.6, 0.4)⟩	⟨(0.6,0.7,0.8,0.8) (0.3,0.3,0.1,0.2)⟩ ⟨(0.7,0.9,0.7,0.6) (0.2, 0.1, 0.3, 0.3)⟩	⟨(0.8,0.7,0.9,0.8) (0.2,0.2,0.7,0.2)⟩ ⟨(0.8,0.9,0.8,0.7) (0.1, 0.1, 0.2, 0.2)⟩	⟨(0.8,0.8,0.9,0.8) (0.2,0.2,0.1,0.1)⟩ ⟨(0.8,0.9,0.9,0.8) (0.1, 0.1, 0.1, 0.2)⟩
$\mathfrak{J}(S_1)$	⟨(0.7, 0.5, 0.8, 0.2) (0.2, 0.4, 0.1, 0.7)⟩ ⟨(0.6, 0.2, 0.5, 0.5) (0.4, 0.5, 0.4, 0.4)⟩	⟨(0.8, 0.6, 0.9, 0.4) (0.2, 0.3, 0.1, 0.6)⟩ ⟨(0.7, 0.3, 0.6, 0.7) (0.3, 0.4, 0.3, 0.3)⟩	⟨(0.8, 0.7, 0.9, 0.6) (0.1, 0.2, 0.1, 0.5)⟩ ⟨(0.8, 0.5, 0.6, 0.8) (0.1, 0.3, 0.2, 0.2)⟩	⟨(0.8, 0.8, 0.9, 0.7) (0.1, 0.2, 0.1, 0.3)⟩ ⟨(0.8, 0.6, 0.7, 0.8) (0.1, 0.2, 0.2, 0.1)⟩

The element in table 2 $\langle(0.6, 0.4, 0.7, 0.5), (0.3, 0.6, 0.2, 0.5)\rangle, \langle(0.7, 0.5, 0.6, 0.3), (0.1, 0.4, 0.3, 0.5)\rangle$ represents the numerical value for the alternative $\mathscr{C}_1$ and the attribute $\mathscr{S}_1$. The membership value displays how much the crop is suitable for growth in soil $\mathscr{S}_1$ in four different climates, while the non-membership grade reflects how much the crop is not suitable for growth in soil $\mathscr{S}_1$. Meanwhile, the reference parameters indicate how much crop should be suitable and not suitable for soil s1.

H

Figure 2. Lattice structure for soil

Figure 3. Lattice structure for farming method

The LLDMFSS(Θ)$\mathcal{J}$ is represented in matrix form as given below.

$$
\mathcal{S}=\begin{array}{c}
\\ e_1 \\ \\ \\ \\ e_2 \\ \\ \\ \\ e_3 \\ \\ \\ \\ e_4 \\ \\ \\ \\
\end{array}
\begin{array}{cccc}
\mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 \\
\end{array}
$$

$$
\begin{array}{c}
\\ e_1 \\ \\ \\ \\ \\ e_2 \\ \\ \\ \\ \\ e_3 \\ \\ \\ \\ \\ e_4 \\ \\ \\ \\
\end{array}
\begin{pmatrix}
\langle(0.6,0.4,0.7,0.5) & \langle(0.8,0.6,0.7,0.5) & \langle(0.8,0.6,0.8,0.6) & \langle(0.9,0.7,0.8,0.7) \\
(0.3,0.6,0.2,0.5)\rangle & (0.2,0.3,0.4,0.4)\rangle & (0.1,0.3,0.2,0.3)\rangle & (0.1,0.2,0.2,0.3)\rangle \\
\langle(0.7,0.5,0.6,0.3) & \langle(0.8,0.6,0.7,0.4) & \langle(0.9,0.7,0.8,0.4) & \langle(0.9,0.8,0.8,0.6) \\
(0.1,0.4,0.3,0.5)\rangle & (0.2,0.3,0.3,0.6)\rangle & (0.1,0.2,0.1,0.5)\rangle & (0.1,0.1,0.1,0.4)\rangle \\
\\
\langle(0.7,0.5,0.3,0.5) & \langle(0.7,0.6,0.7,0.6) & \langle(0.8,0.7,0.8,0.7) & \langle(0.8,0.8,0.8,0.8) \\
(0.5,0.4,0.8,0.4)\rangle & (0.3,0.3,0.3,0.3)\rangle & (0.2,0.2,0.3,0.2)\rangle & (0.1,0.1,0.1,0.1)\rangle \\
\langle(0.9,0.5,0.7,0.7) & \langle(0.9,0.6,0.7,0.8) & \langle(0.8,0.7,0.9,0.8) & \langle(0.8,0.8,0.9,0.8) \\
(0.1,0.4,0.2,0.3)\rangle & (0.3,0.3,0.2,0.2)\rangle & (0.1,0.1,0.2,0.2)\rangle & (0.1,0.1,0.2,0.2)\rangle \\
\\
\langle(0.6,0.4,0.7,0.4) & \langle(0.6,0.7,0.8,0.8) & \langle(0.8,0.7,0.9,0.8) & \langle(0.8,0.8,0.9,0.8) \\
(0.4,0.7,0.3,0.5)\rangle & (0.3,0.3,0.1,0.2)\rangle & (0.2,0.2,0.7,0.2)\rangle & (0.2,0.2,0.1,0.1)\rangle \\
\langle(0.7,0.6,0.4,0.3) & \langle(0.7,0.9,0.7,0.6) & \langle(0.8,0.9,0.8,0.7) & \langle(0.8,0.9,0.9,0.8) \\
(0.2,0.3,0.6,0.4)\rangle & (0.2,0.1,0.3,0.3)\rangle & (0.1,0.1,0.2,0.2)\rangle & (0.1,0.1,0.1,0.2)\rangle \\
\\
\langle(0.7,0.5,0.8,0.2) & \langle(0.8,0.6,0.9,0.4) & \langle(0.8,0.7,0.9,0.6) & \langle(0.8,0.8,0.9,0.7) \\
(0.2,0.4,0.1,0.7)\rangle & (0.2,0.3,0.1,0.6)\rangle & (0.1,0.2,0.1,0.5)\rangle & (0.1,0.2,0.1,0.3)\rangle \\
\langle(0.6,0.2,0.5,0.5) & \langle(0.7,0.3,0.6,0.7) & \langle(0.8,0.5,0.6,0.8) & \langle(0.8,0.6,0.7,0.8) \\
(0.4,0.5,0.4,0.4)\rangle & (0.3,0.4,0.3,0.3)\rangle & (0.1,0.3,0.2,0.2)\rangle & (0.1,0.2,0.2,0.1)\rangle
\end{pmatrix}
$$

Now consider, R = $\{\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3, \mathscr{F}_4\}$ be the set of farming methods. Here, the order of preference was given by the amount of land required for certain farming methods. It is shown in the figure 3. Let $\mathcal{B}$ be the LLDMFSS($\mathcal{B}$,D) over Θ. where $\mathcal{B}$: $D \to LLDMFSN(R)$ which has soil as alternatives and farming methods as attributes. The table shows the LLDMFSS(Θ)$\mathcal{B}$. The element a11 $\langle(0.5, 0.4, 0.6, 0.3), (0.4, 0.6, 0.5, 0.7)\rangle$, $\langle(0.7, 0.7, 0.5, 0.6), (0.2, 0.3, 0.4, 0.3)\rangle$ represents the numerical value for the alternative $\mathscr{S}_1$ and the attribute $\mathscr{F}_1$. The membership value displays how much the soil is suitable for the farming methods $\mathscr{F}_1$ in four different climates, while the non-membership grade reflects how much the soil is not suitable for the farming method. Meanwhile, the reference parameters indicate how much soil should be suitable and not suitable for the farming method. The LLDMFSS(Θ)$\mathcal{B}$ is represented in matrix form as given below.

Table 2. LLDMFSS$(\Theta)\mathcal{B}$

$\mathcal{B}$	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_3$	$\mathcal{S}_4$
$\mathcal{B}(F_1)$	$\langle(0.5,0.4,0.6,0.3)$ $(0.4,0.6,0.5,0.7)\rangle$ $\langle(0.7,0.7,0.5,0.6)$ $(0.2,0.3,0.4,0.3)\rangle$	$\langle(0.6,0.5,0.6,0.4)$ $(0.3,0.5,0.4,0.6)\rangle$ $\langle(0.8,0.7,0.8,0.7)$ $(0.1,0.3,0.3,0.2)\rangle$	$\langle(0.7,0.7,0.6,0.5)$ $(0.3,0.4,0.3,0.5)\rangle$ $\langle(0.8,0.8,0.9,0.8)$ $(0.1,0.2,0.1,0.2)\rangle$	$\langle(0.8,0.7,0.8,0.7)$ $(0.2,0.3,0.1,0.2)\rangle$ $\langle(0.8,0.9,0.9,0.8)$ $(0.1,0.1,0.1,0.1)\rangle$
$\mathcal{B}(F_2)$	$\langle(0.8,0.7,0.5,0.3)$ $(0.2,0.4,0.4,0.7)\rangle$ $\langle(0.8,0.8,0.7,0.4)$ $(0.1,0.1,0.2,0.5)\rangle$	$\langle(0.8,0.7,0.6,0.5)$ $(0.2,0.3,0.3,0.6)\rangle$ $\langle(0.8,0.8,0.8,0.6)$ $(0.1,0.1,0.2,0.4)\rangle$	$\langle(0.8,0.8,0.7,0.7)$ $(0.1,0.2,0.3,0.5)\rangle$ $\langle(0.8,0.9,0.9,0.7)$ $(0.1,0.1,0.1,0.2)\rangle$	$\langle(0.8,0.9,0.8,0.7)$ $(0.1,0.1,0.2,0.3)\rangle$ $\langle(0.8,0.9,0.9,0.8)$ $(0.1,0.1,0.1,0.2)\rangle$
$\mathcal{B}(F_3)$	$\langle(0.7,0.5,0.3,0.6)$ $(0.2,0.6,0.7,0.4)\rangle$ $\langle(0.6,0.7,0.5,0.7)$ $(0.3,0.2,0.4,0.2)\rangle$	$\langle(0.7,0.6,0.5,0.7)$ $(0.2,0.5,0.6,0.3)\rangle$ $\langle(0.8,0.7,0.6,0.7)$ $(0.2,0.2,0.3,0.2)\rangle$	$\langle(0.8,0.7,0.6,0.7)$ $(0.2,0.3,0.4,0.2)\rangle$ $\langle(0.9,0.8,0.7,0.7)$ $(0.1,0.2,0.2,0.2)\rangle$	$\langle(0.9,0.8,0.6,0.8)$ $(0.1,0.2,0.3,0.2)\rangle$ $\langle(0.9,0.9,0.8,0.7)$ $(0.1,0.1,0.2,0.2)\rangle$
$\mathcal{B}(F_4)$	$\langle(0.8,0.6,0.5,0.7)$ $(0.3,0.5,0.7,0.3)\rangle$ $\langle(0.7,0.7,0.3,0.5)$ $(0.3,0.2,0.6,0.4)\rangle$	$\langle(0.8,0.7,0.6,0.8)$ $(0.2,0.2,0.3,0.2)\rangle$ $\langle(0.8,0.7,0.5,0.6)$ $(0.2,0.2,0.4,0.3)\rangle$	$\langle(0.9,0.8,0.6,0.8)$ $(0.1,0.2,0.4,0.2)\rangle$ $\langle(0.8,0.8,0.6,0.7)$ $(0.2,0.1,0.3,0.2)\rangle$	$\langle(0.9,0.8,0.9,0.8)$ $(0.1,0.2,0.1,0.2)\rangle$ $\langle(0.8,0.9,0.7,0.8)$ $(0.2,0.1,0.2,0.1)\rangle$

$$\mathcal{T}=\begin{array}{c}\\ \mathcal{S}_1\\ \\ \mathcal{S}_2\\ \\ \mathcal{S}_3\\ \\ \mathcal{S}_4\end{array}\begin{array}{c}\begin{array}{cccc}\quad\mathcal{F}_1\quad & \quad\mathcal{F}_2\quad & \quad\mathcal{F}_3\quad & \quad\mathcal{F}_4\quad\end{array}\\ \begin{pmatrix}
\begin{array}{l}\langle(0.5,0.4,0.6,0.3)\\(0.4,0.6,0.5,0.7)\rangle\\\langle(0.7,0.7,0.5,0.6)\\(0.2,0.3,0.4,0.3)\rangle\end{array} & \begin{array}{l}\langle(0.6,0.5,0.6,0.4)\\(0.3,0.5,0.4,0.6)\rangle\\\langle(0.8,0.7,0.8,0.7)\\(0.1,0.3,0.3,0.2)\rangle\end{array} & \begin{array}{l}\langle(0.7,0.7,0.6,0.5)\\(0.3,0.4,0.3,0.5)\rangle\\\langle(0.8,0.8,0.9,0.8)\\(0.1,0.2,0.1,0.2)\rangle\end{array} & \begin{array}{l}\langle(0.8,0.7,0.8,0.7)\\(0.2,0.3,0.1,0.2)\rangle\\\langle(0.8,0.9,0.9,0.8)\\(0.1,0.1,0.1,0.1)\rangle\end{array}\\
\begin{array}{l}\langle(0.8,0.7,0.5,0.3)\\(0.2,0.4,0.4,0.7)\rangle\\\langle(0.8,0.8,0.7,0.4)\\(0.1,0.1,0.2,0.5)\rangle\end{array} & \begin{array}{l}\langle(0.8,0.7,0.6,0.5)\\(0.2,0.3,0.3,0.6)\rangle\\\langle(0.8,0.8,0.8,0.6)\\(0.1,0.1,0.2,0.4)\rangle\end{array} & \begin{array}{l}\langle(0.8,0.8,0.7,0.7)\\(0.1,0.2,0.3,0.5)\rangle\\\langle(0.8,0.9,0.9,0.7)\\(0.1,0.1,0.1,0.2)\rangle\end{array} & \begin{array}{l}\langle(0.8,0.9,0.8,0.7)\\(0.1,0.1,0.2,0.3)\rangle\\\langle(0.8,0.9,0.9,0.8)\\(0.1,0.1,0.1,0.2)\rangle\end{array}\\
\begin{array}{l}\langle(0.7,0.5,0.3,0.6)\\(0.2,0.6,0.7,0.4)\rangle\\\langle(0.6,0.7,0.5,0.7)\\(0.3,0.2,0.4,0.2)\rangle\end{array} & \begin{array}{l}\langle(0.7,0.6,0.5,0.7)\\(0.2,0.5,0.6,0.3)\rangle\\\langle(0.8,0.7,0.6,0.7)\\(0.2,0.2,0.3,0.2)\rangle\end{array} & \begin{array}{l}\langle(0.8,0.7,0.6,0.7)\\(0.2,0.3,0.4,0.2)\rangle\\\langle(0.9,0.8,0.7,0.7)\\(0.1,0.2,0.2,0.2)\rangle\end{array} & \begin{array}{l}\langle(0.9,0.8,0.6,0.8)\\(0.1,0.2,0.3,0.2)\rangle\\\langle(0.9,0.9,0.8,0.7)\\(0.1,0.1,0.2,0.2)\rangle\end{array}\\
\begin{array}{l}\langle(0.8,0.6,0.5,0.7)\\(0.3,0.5,0.7,0.3)\rangle\\\langle(0.7,0.7,0.3,0.5)\\(0.3,0.2,0.6,0.4)\rangle\end{array} & \begin{array}{l}\langle(0.8,0.7,0.6,0.8)\\(0.2,0.2,0.3,0.2)\rangle\\\langle(0.8,0.7,0.5,0.6)\\(0.2,0.2,0.4,0.3)\rangle\end{array} & \begin{array}{l}\langle(0.9,0.8,0.6,0.8)\\(0.1,0.2,0.4,0.2)\rangle\\\langle(0.8,0.8,0.6,0.7)\\(0.2,0.1,0.3,0.2)\rangle\end{array} & \begin{array}{l}\langle(0.9,0.8,0.9,0.8)\\(0.1,0.2,0.1,0.2)\rangle\\\langle(0.8,0.9,0.7,0.8)\\(0.2,0.1,0.2,0.1)\rangle\end{array}
\end{pmatrix}\end{array}$$

The max-mix composition of $\mathcal{S}$ *and* $\mathcal{T}$ is calculated.

Figure 4. The diagrammatical representation of output

$$
\mathcal{S} \star \mathcal{T} = \begin{array}{c} \\ e_1 \\ \\ \\ \\ e_2 \\ \\ \\ \\ e_3 \\ \\ \\ \\ e_4 \\ \\ \\ \\ \end{array}
\begin{array}{c}
\begin{array}{cccc} \mathcal{F}_1 & \mathcal{F}_2 & \mathcal{F}_3 & \mathcal{F}_4 \end{array} \\
\begin{pmatrix}
\langle(0.8,0.6,0.6,0.7) & \langle(0.8,0.7,0.6,0.7) & \langle(0.9,0.7,0.7,0.7) & \langle(0.9,0.7,0.8,0.7) \\
(0.2,0.4,0.4,0.3)\rangle & (0.2,0.2,0.3,0.3)\rangle & (0.1,0.2,0.3,0.3)\rangle & (0.1,0.1,0.2,0.3)\rangle \\
\langle(0.8,0.7,0.7,0.5) & \langle(0.8,0.7,0.7,0.6) & \langle(0.9,0.8,0.7,0.6) & \langle(0.9,0.8,0.8,0.6) \\
(0.2,0.2,0.3,0.4)\rangle & (0.1,0.2,0.3,0.2)\rangle & (0.1,0.1,0.2,0.4)\rangle & (0.1,0.1,0.2,0.4)\rangle \\
\langle(0.8,0.6,0.6,0.7) & \langle(0.8,0.7,0.6,0.8) & \langle(0.8,0.8,0.7,0.8) & \langle(0.8,0.8,0.8,0.8) \\
(0.2,0.4,0.4,0.3)\rangle & (0.2,0.2,0.3,0.2)\rangle & (0.1,0.2,0.3,0.2)\rangle & (0.1,0.2,0.1,0.2)\rangle \\
\langle(0.8,0.7,0.7,0.5) & \langle(0.8,0.7,0.7,0.7) & \langle(0.8,0.5,0.7,0.7) & \langle(0.9,0.8,0.8,0.4) \\
(0.2,0.2,0.3,0.4)\rangle & (0.1,0.2,0.2,0.2)\rangle & (0.1,0.1,0.2,0.2)\rangle & (0.1,0.1,0.2,0.2)\rangle \\
\langle(0.8,0.7,0.6,0.7) & \langle(0.8,0.7,0.6,0.8) & \langle(0.8,0.8,0.7,0.8) & \langle(0.8,0.8,0.9,0.8) \\
(0.1,0.2,0.1,0.2)\rangle & (0.2,0.2,0.3,0.2)\rangle & (0.2,0.2,0.3,0.2)\rangle & (0.2,0.2,0.1,0.2)\rangle \\
\langle(0.9,0.8,0.8,0.4) & \langle(0.8,0.8,0.7,0.7) & \langle(0.8,0.9,0.7,0.7) & \langle(0.8,0.9,0.8,0.8) \\
(0.1,0.1,0.2,0.2)\rangle & (0.2,0.1,0.3,0.2)\rangle & (0.1,0.1,0.2,0.2)\rangle & (0.1,0.1,0.2,0.2)\rangle \\
\langle(0.8,0.6,0.6,0.7) & \langle(0.8,0.7,0.6,0.7) & \langle(0.8,0.8,0.7,0.7) & \langle(0.8,0.8,0.9,0.7) \\
(0.2,0.4,0.4,0.5)\rangle & (0.2,0.2,0.3,0.3)\rangle & (0.1,0.2,0.3,0.3)\rangle & (0.1,0.2,0.1,0.3)\rangle \\
\langle(0.7,0.6,0.6,0.7) & \langle(0.8,0.6,0.6,0.7) & \langle(0.8,0.6,0.6,0.7) & \langle(0.8,0.6,0.7,0.8) \\
(0.3,0.2,0.3,0.2)\rangle & (0.2,0.2,0.3,0.2)\rangle & (0.1,0.2,0.2,0.2)\rangle & (0.1,0.2,0.2,0.1)\rangle
\end{pmatrix}
\end{array}
$$

Then the score matrix was computed with the help of the definition.

$$
\text{Score matrix} = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array}
\begin{array}{c}
\begin{array}{cccc} \mathcal{F}_1 & \mathcal{F}_2 & \mathcal{F}_3 & \mathcal{F}_4 \end{array} \\
\begin{pmatrix}
0.37 & 0.47 & 0.53 & 0.58 \\
0.37 & 0.47 & 0.55 & 0.61 \\
0.56 & 0.52 & 0.5 & 0.6 \\
0.35 & 0.45 & 0.51 & 0.6
\end{pmatrix}
\end{array}
$$

The score values for each attribute are calculated as follows:

Score values of $\mathcal{C}_1 = 1.95$

Score values of $\mathcal{C}_2 = 2$

Score values of $\mathcal{C}_3 = 2.18$

Score values of $\mathcal{C}_4 = 1.91$

4.3 *Discussion*

The results demonstrate that crop 3 is suited for growing in all types of soils and climates, regardless of farming techniques. The diagrammatical representation of the result will shown in the figure 4.

4.4 *Superiority and Novelty of our proposed method*

Table 3 demonstrates the analysis of comparison of our suggested approach with previous concepts. The implementation of matrix theory in LLDMFSS demonstrates the novel aspect of our suggested theory since the literature study reveals the fact that there isn't any research that applies the principle of a matrix in a linear diophantine soft environment. The intervals for grade values become larger by the linear diophantine ambiance. Also, Our suggested theory is emphasized by the incorporation of lattice theory and soft theory.

Table 3. Analysis in Comparison with Current Methodologies

Reference	Data Information	MCDM Approaches	Limitations
C. Naim et al. [31]	Soft values	Product Operation	It does not hold fuzzy values.
N. Cagmam et al. [32]	Fuzzy soft values	Max-min composition method	It doesn't handle the problems having non-membership grade values.
B. Chetia et al. [34]	Intuitionistic soft values	-	It fails in the cases $0.6 + 0.5 \nleq 1$.
J.I. Mondal et al. [40]	Intuitionistic soft values	Weighted Arithmetic mean operator	It fails in the cases where the sum of the grade values exists 1.
R. Rathika et al. [41]	Intuitionistic soft values	Score matrix & Value Matrix	It fails in the cases where the sum of the grade values exists 1.
Guleria et al. [35]	Pythagorean soft values	Score matrix & Utility Matrix	It fails in the cases where the sum of the squares of the grade values exists 1.
M. Kirisci et al. [42]	Pythagorean soft values	Choice matrix	It fails in the cases where the sum of the squares of the grade values exists 1.
Sabeena et al. [43]	Lattice ordered multi-fuzzy soft values	Min-max, Max-Product, Max- Average Compositions	It doesn't contain non-memberaship values.
Our Proposed method	LLDMFS values	Max-Min Composition	It fails in the cases where the sum of reference parameter exists 1.

4.5 *Economic impact of this research*

Agricultural research has a significant influence on altering future worldwide patterns of poverty and other outcomes. The paper presents a model intended to depict and tackle issues in agriculture. Through the provision of insightful information, this research plays a critical role in supporting farmers in their decision-making regarding farming practices. Reducing agricultural losses and increasing farmer income are the ultimate goals. The model is made to give decision-makers flexibility in determining the grade values due to the presence of reference parameters. A noteworthy characteristic of the model is its flexibility in meeting various requirements, as it provides a variety of solutions customized to address various issues encountered by farmers. In essence, the system offers a customizable approach that enables farmers to customize solutions according to their particular needs and circumstances. Several of the ambiguity issues that farmers frequently face could be addressed and resolved by this research. It seeks to greatly enhance agricultural practices, lower uncertainty, and improve farmers' financial well-being by providing customized and workable solutions through the model.

5. Conclusion

In this manuscript, the hypothesis of LLDMFSM was initiated with the formation of LLDMFSN. The basic operations and properties of LLDMFSM are explored. And some of its mean operators with propositions are investigated. The implementation of the max-min composition for LLDMFSM was made for algorithmic reasons. A concrete illustration is given to demonstrate the viability of our idea. The outcome will give farmers the finest option for better farming. To further demonstrate the superiority of our research, the economic impact was also provided. In the future, our work will aim to explore the similarity measure and entropy measure for LLDMFSS.

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Conflict of Interest

All authors declare no conflicts of interest in this paper.

Disclosure

Disclosure.

Confilict of Interest

The authors declare no competing financial interest.

References

[1] Zadeh LA. Fuzzy sets. *Information and Control*. 1965; 8(3): 338-353. Available from: https://www.sciencedirect.com/science/article/pii/S001999586590241X.

[2] Pandey P, Litoriya R, Tiwari A. A framework for fuzzy modelling in agricultural diagnostics. *Journal Européen des Systèmes Automatisés*. 2018. Available from: https://api.semanticscholar.org/CorpusID:67926481.

[3] Atanassov KT. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*. 1986 Aug; 20(1): 87 – 96. Available from: http://dx.doi.org/10.1016/s0165-0114(86)80034-3.

[4] Yager RR. Pythagorean Membership Grades in Multicriteria Decision Making. *IEEE Transactions on Fuzzy Systems*. 2014 Aug; 22(4): 958–965. Available from: http://dx.doi.org/10.1109/tfuzz.2013.2278989.

[5] Yager RR. Generalized Orthopair Fuzzy Sets. *IEEE Transactions on Fuzzy Systems*. 2017 Oct; 25(5): 1222–1230. Available from: http://dx.doi.org/10.1109/tfuzz.2016.2604005.

[6] Riaz M, Hashmi MR. Linear Diophantine fuzzy set and its applications towards multi-attribute decision-making problems. *Journal of Intelligent Fuzzy Systems*. 2019 Oct; 37(4): 5417–5439. Available from: http://dx.doi.org/10.3233/jifs-190550.

[7] Iampan A, García GS, Riaz M, Athar Farid HM, Chinram R. Linear Diophantine Fuzzy Einstein Aggregation Operators for Multi-Criteria Decision-Making Problems. *Journal of Mathematics*. 2021 Jul; 2021: 1–31. Available from: http://dx.doi.org/10.1155/2021/5548033.

[8] Ayub S, Shabir M, Riaz M, Aslam M, Chinram R. Linear Diophantine Fuzzy Relations and Their Algebraic Properties with Decision Making. *Symmetry*. 2021 May; 13(6): 945. Available from: http://dx.doi.org/10.3390/sym13060945.

[9] Riaz M, Hashmi MR, Kalsoom H, Pamucar D, Chu YM. Linear Diophantine Fuzzy Soft Rough Sets for the Selection of Sustainable Material Handling Equipment. *Symmetry*. 2020 Jul; 12(8): 1215. Available from: http://dx.doi.org/10.3390/sym12081215.

[10] Kannan J, Jayakumar V, Pethaperumal M, Kather Mohideen AB. An intensified linear diophantine fuzzy combined DEMATEL framework for the assessment of climate crisis. *Stochastic Environmental Research and Risk Assessment*. 2024 Jan. Available from: http://dx.doi.org/10.1007/s00477-023-02618-7.

[11] Jayakumar V, Kannan J, Kausar N, Deveci M, Wen X. Multicriteria group decision making for prioritizing IoT risk factors with linear diophantine fuzzy sets and MARCOS method. *Granular Computing*. 2024 May; 9(3). Available from: http://dx.doi.org/10.1007/s41066-024-00480-8.

[12] Riaz M, Farid HMA, Aslam M, Pamucar D, Bozanić D. Novel Approach for Third-Party Reverse Logistic Provider Selection Process under Linear Diophantine Fuzzy Prioritized Aggregation Operators. *Symmetry*. 2021 Jun; 13(7): 1152. Available from: http://dx.doi.org/10.3390/sym13071152.

[13] Farid HMA, Riaz M, Khan MJ, Kumam P, Sitthithakerngkiet K. Sustainable thermal power equipment supplier selection by Einstein prioritized linear Diophantine fuzzy aggregation operators. *AIMS Mathematics*. 2022; 7(6): 11201–11242. Available from: http://dx.doi.org/10.3934/math.2022627.

[14] Riaz M, Farid HMA, Wang W, Pamucar D. Interval-Valued Linear Diophantine Fuzzy Frank Aggregation Operators with Multi-Criteria Decision-Making. *Mathematics*. 2022 May; 10(11): 1811. Available from: http://dx.doi.org/10.3390/math10111811.

[15] Petchimuthu S, Riaz M, Kamacı H. Correlation coefficient measures and aggregation operators on interval-valued linear Diophantine fuzzy sets and their applications. *Computational and Applied Mathematics*. 2022 Nov; 41(8). Available from: http://dx.doi.org/10.1007/s40314-022-02077-w.

[16] Kannan J, Jayakumar V, Saeed M, Alballa T, Khalifa HAEW, Gomaa HG. Linear Diophantine Fuzzy Clustering Algorithm Based on Correlation Coefficient and Analysis on Logistic Efficiency of Food Products. *IEEE Access*. 2024; 12: 34889–34902. Available from: http://dx.doi.org/10.1109/access.2024.3371986.

[17] Molodtsov D. Soft Set theory. *computers and Mathematics with applications*. 1999; 37: 19-31.

[18] Maji K P, Biswas R, Roy AR. Fuzzy soft Set. *Journal of Fuzzy Mathematics*. 2001; 9: 589-602.

[19] Maji K P, Biswas R, Roy AR. Intuitionistic Fuzzy soft Set. *Journal of Fuzzy Mathematics*. 2001; 9: 677-692.

[20] Peng X, Yang Y, Song J, Yun J. Pythagorean Fuzzy Soft Set and Its Application. *Computer Engineering*. 2015 07; 41: 224-229.

[21] Hussain A, Ali MI, Mahmood T, Munir M. q-Rung orthopair fuzzy soft average aggregation operators and their application in multicriteria decision-making. *International Journal of Intelligent Systems*. 2020 Jan; 35(4): 571–599. Available from: http://dx.doi.org/10.1002/int.22217.

[22] Albert A. Lattice Theory: Lattice Theory . Vol. XXV. Colloquium Publications. By Garrett Birkhoff. v + 155 pp. New York: American Mathematical Society. 1940. *Science*. 1940 Dec; 92(2400): 606–606. Available from: http://dx.doi.org/10.1126/science.92.2400.606-a.

[23] Begam SS, J DV. Application of lattice ordered multi-fuzzy soft set in forecasting process. *Journal of Intelligent Fuzzy Systems*. 2019 03; 36: 2323-2331.

[24] Begam SS, J DV, D P. A novel study on the algebraic applications of special class of lattice ordered multi-fuzzy soft sets. *Journal of Discrete Mathematical Sciences and Cryptography*. 2019 07; 22: 883-899.

[25] S R, J DV. Operations on complex intuitionistic fuzzy soft lattice ordered group and CIFS-COPRAS method for equipment selection process. *Journal of Intelligent Fuzzy Systems*. 2021 04; 41: 1-10.

[26] Jayakumar V, Mohideen ABK, Saeed MH, Alsulami H, Hussain A, Saeed M. Development of Complex Linear Diophantine Fuzzy Soft Set in Determining a Suitable Agri-Drone for Spraying Fertilizers and Pesticides. *IEEE Access*. 2023; 11: 9031–9041. Available from: http://dx.doi.org/10.1109/access.2023.3239675.

[27] Pethaperumal M, Jeyakumar V, Kannan J, Banu A. An algebraic analysis on exploring q-Rung orthopair multi-fuzzy sets. *Journal of Fuzzy Extension and Applications*. 2023; 4(3): 235-245.

[28] Vimala J, Mahalakshmi P, Rahman AU, Saeed M. A customized TOPSIS method to rank the best airlines to fly during COVID-19 pandemic with q-rung orthopair multi-fuzzy soft information. *Soft Computing*. 2023 Aug; 27(20): 14571–14584. Available from: http://dx.doi.org/10.1007/s00500-023-08976-2.

[29] Vimala J, Garg H, Jeevitha K. Prognostication of Myocardial Infarction Using Lattice Ordered Linear Diophantine Multi-fuzzy Soft Set. *International Journal of Fuzzy Systems*. 2023 Aug; 26(1): 44–59. Available from: http://dx.doi.org/10.1007/s40815-023-01574-2.

[30] Kannan J, Jayakumar V, Pethaperumal M, Shanmugam NS. Linear diophantine multi-fuzzy soft similarity measures: An analysis on alternative-fuel. *Journal of Intelligent Fuzzy Systems*. 2024 Apr: 1–13. Available from: http://dx.doi.org/10.3233/jifs-219415.

[31] Çağman N, Enginoğlu S. Soft matrix theory and its decision making. *Computers Mathematics with Applications*. 2010; 59(10): 3308-3314. Available from: https://www.sciencedirect.com/science/article/pii/S0898122110001914.

[32] Çağman N, Enginoğlu S. Fuzzy soft matrix theory and its application in decision making. *Iranian journal of fuzzy systems*. 2012 02; 9: 109-119.

[33] Petchimuthu S, Garg H, Kamacı H, Atagün A. The mean operators and generalized products of fuzzy soft matrices and their applications in MCGDM. 2020 05; 39: 68.

[34] Chetia B, Das P. Some Results of Intuitionistic Fuzzy Soft Matrix Theory.

[35] Guleria A, Bajaj R. On Pythagorean fuzzy soft matrices, operations and their applications in decision making and medical diagnosis. *Soft Computing*. 2019 09; 23.

[36] Yang Y, Tan X, Meng C. The multi-fuzzy soft set and its application in decision making. *Applied Mathematical Modelling*. 2013; 37(7): 4915-4923. Available from: https://www.sciencedirect.com/science/article/pii/S0307904X12006208.

[37] Jeevitha K, Garg H, Vimala J, Aljuaid H, Abdel-Aty AH. Linear diophantine multi-fuzzy aggregation operators and its application in digital transformation. *Journal of Intelligent Fuzzy Systems*. 2023 Aug; 45(2): 3097–3107. Available from: http://dx.doi.org/10.3233/jifs-223844.

[38] Kannan J, Jayakumar V, Saeid AB. Lattice Algebraic Structures on LDMFS Domains. *New Mathematics and Natural Computation*. 2024 Mar: 1–21. Available from: http://dx.doi.org/10.1142/s1793005725500218.

[39] KANNAN J, JAYAKUMAR V. Sustainable method for tender selection using linear Diophantine multi-fuzzy soft set. *Communications Faculty Of Science University of Ankara Series A1Mathematics and Statistics*. 2023 Jul; 72(4): 976–991. Available from: http://dx.doi.org/10.31801/cfsuasmas.1255830.

[40] Islam Mondal MJ, Kumar Roy T. Intuitionistic Fuzzy Soft Matrix Theory. *Mathematics and Statistics*. 2013 Aug; 1(2): 43–49. Available from: http://dx.doi.org/10.13189/ms.2013.010205.

[41] Rathika R, Subramanian S. An application of intuitionistic fuzzy soft matrix theory based on the reference function in decision making problem. *International Journal of Pure and Applied Mathematics*. 2018 01; 119: 219-233.

[42] Kirişci M, Simsek N. Decision Making Method related to Pythagorean Fuzzy Soft Sets with Infectious Diseases Application. *Journal of King Saud University - Computer and Information Sciences*. 2021 08.

[43] Begam SS, J DV. Compositions on lattice ordered multi-fuzzy soft matrix and its simulated application in medical diagnosis. *Mathematics and Computers in Simulation*. 2021 03; 201.