Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

UNIVERSAL WISER PUBLISHER

Research Article

A Measure of Hyperconvexity for a Complete Ultrametric Space

Collins Amburo Agyingi

Department of Mathematical Sciences, University of South Africa, Unisa, 003, P.O. Box 392, Pretoria, South Africa
E-mail: agyinca@unisa.ac.za

Received: 31 January 2024; **Revised:** 7 April 2024; **Accepted:** 15 April 2024

Abstract: It has been shown that an ultrametric space is spherically complete if and only if it is hyperconvex. Furthermore, every spherically complete ultrametric space is complete. This, in turn, means that every hyperconvex ultrametric space is complete. However, the converse of the previous statement is not true. In this note, we show that for an ultrametric space X, its ultrametrically injective hull denoted by T_X is spherically complete. Furthermore, we study the conditions under which a complete ultrametric space will be hyperconvex. We will do this by introducing a numerical parameter that enables us to measure the lack of hyperconvexity in a complete ultrametric space.The advantage of the proposed method is that given any complete ultrametric space, we can use the method to deduce if the space is hyperconvex or not.

***Keywords*:** spherically complete, hyperconvex, ultrametrically injective, ultra-ample, ultra-extremal

MSC: 54E40, 46M10

1. Introduction

An ultrametric space is a pair (X, d_X) where d_X is a metric on X that satisfies the condition, namely that for every $x, y, z \in X$:

$$d_X(x, z) \leq \max\{d_X(x, y), d_X(y, z)\}.$$

The above condition is called the "strong triangle inequality". Throughout this note, we will always write X to mean the ultrametric space (X, u).

The following is a well-known example of an ultrametric space.

Example 1 [3, Example 1] Let $u(x_1, x_2) = \max\{x_1, x_2\}$ if x_1 is different from x_2 and $u(x_1, x_1) = 0$ for each $x_1 \geq 0$, $x_2 \geq 0$. Then $([0, \infty), u)$ defines an ultrametric space.

We denote by $B(x, r)$ the closed ball of an ultrametric space X, where

$$B(x, r) = \{x_0 \in X : u(x, x_0) \leq r\},$$

DOI: https://doi.org/10.37256/cm.6120254389

with center at x and radius r, where $r \geq 0$. Note further that if we have $B(x,\ r_1) \cap B(y,\ r_2) \neq \emptyset$ and $r_1 \leq r_2$ for some real numbers r_1 and r_2, then we must have that $B(x,\ r_1) \subseteq B(y,\ r_2)$. Furthermore, if we have $r = u(x_1,\ x_2)$, then $B(x_1,\ r) = B(x_2,\ r)$.

Let $(x_i)_i$ be points in an ultrametric space X and $(\alpha_i)_{i\geq 1} \subset [0,\ \infty)$. We will say that $(B_i)_{i\geq 1}$, $B_i := B(x_i,\ \alpha_i)$ $(i \geq 1)$ is a nested sequence of closed balls if we have that $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots$. We also have that the relation $B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$ holds.

In a complete ultrametric space, when the decreasing series of radii has a positive limit, a nested sequence of closed balls in that space may have an empty intersection. In fact, have a look at this example:

Example 2 Consider the set $\mathbb{N}$ of natural numbers that includes 0. Put $u(x,\ y) = 1 - \delta_{xy}$, for every natural number x and y. Note that $\delta_{xy} = 1$ when $x = y$ and $\delta_{xy} = 0$ when x is different from y. It is easily seen that u is a discrete ultrametric on $\mathbb{N}$ and since u is uniformly discrete, the completeness of $(\mathbb{N},\ u)$ follows. Notice that the following sequence of closed sets $F_m = [m,\ \infty)$ is decreasing and their intersection is empty. Note also that all the closed sets have a diameter equal to 1. Suppose now that we replace u by d where d is defined such that $d(x,\ y) = 2^n$ for every natural number x and y and n is an integer chosen so that $u \leq d < 2u$. Then d transforms the sets F_m into closed balls whose radii strictly decrease.

As a consequence of Example 2, we have the following definition.

Definition 3 (Compare [8, Definition 1]) We say that an ultrametric space X is spherically complete if every nested sequence of closed balls in X has a nonempty intersection.

An equivalent definition to Definition 3 is the following.

Definition 4 [4, Page 573] An ultrametric space X is spherically complete if for any family $(x_i)_{i\in I} \subseteq X$ and any family $(r_i)_{i\in I} \subseteq [0,\ \infty)$ that satisfies $u(x_i,\ x_j) \leq \max\{r_i,\ r_j\}$ for $i,\ j \in I$ we must have that

$$\bigcap_{i\in I} B(x_i,\ r_i) \neq \emptyset.$$

Example 5 The ultrametric space $([0,\ \infty),\ u)$, where u is the ultrametric defined in Example 1, is spherically complete since the point 0 belongs to every closed ball $B(x,\ r)$ whenever $r > 0$.

Ultrametric spaces have many properties in common with hyperconvex metric spaces, even though the two are different in many ways. Nevertheless, one of their similarities that is of interest is the injectivity property, while their most common difference is that ultrametric spaces are not metrically convex while hyperconvex metric spaces are.

Definition 6 [4, Page 571] An ultrametric space $(Y,\ u_Y)$ will be said to be ultrametrically injective if for any ultrametric space $(X,\ u_X)$ and any subspace A of X, any nonexpansive map $f:\ A \to (Y,\ u_Y)$ can be extended to a nonexpansive map $g:\ (X,\ u_X) \to (Y,\ u_Y)$.

Let us keep in mind that each spherically complete ultrametric space is complete in terms of the topology that its metric induces. This can be seen by considering the fact that the topological completeness of an ultrametric space X is equivalent to the property of Definition 4 with radii decreasing to 0. The converse, however, is not true (check for instance [12, Pages 134-145]).

In the literature, there has been no investigation on the hyperconvexity of complete ultrametric spaces. In the metric space setting, Cianciaruso and De Pascale [5] have done such an investigation. They introduced a numerical parameter that measures the lack of hyperconvexity for a complete ultrametric space.

There are many applications of ultrametric spaces in physics and also in pure mathematics (for instance, consult the survey article [11] and all the references in it to gain an understanding of the breadth and depth to which ideas in ultrametric spaces are essential to many aspects of contemporary physics). Moreover, no multipoint ultrametric space is ultrametrically injective (check [13, Example 10.2]). This does not happen with metric spaces. Therefore, it becomes important to consider these spaces very carefully. As a consequence of the above, we will restrict ourselves to the following weaker definition: The space Y will be said to be ultrametrically injective if every nonexpansive mapping from the space $A \subseteq X$ to Y can be extended to a nonexpansive mapping over X.

Definition 7 [7, Definition 4.1] The space X is said to be hyperconvex if it satisfies the following condition: for any family $(B(x_i,\ r_i))_{i\in I}$ of closed balls, we have that

$$B(x_i,\ r_i)\cap B(x_j,\ r_j)\neq\emptyset\ \forall\ i,\ j\in I\Rightarrow\bigcap_{i\in I}B(x_i,\ r_i)\neq\emptyset.$$

Theorem 8 [4] For an ultrametric space X, the following conditions are equivalent:

(a). X is spherically complete.

(b). is ultrametrically injective.

Corollary 9 [8, Corollary 3] Let A be a subspace of an ultrametric space X that is spherically complete. Then, A is a retract of X that is nonexpansive.

The goal of this paper is to initiate a thorough investigation into the hyperconvexity of complete ultrametric spaces. Our studies naturally progress alongside the well-known metric theory of hyperconvexity, as one could anticipate given the research done by Cianciaruso and De Pascale (see, for example [5]). Although these extensions are not always trivial because the ultrametric context necessitates intriguing new versions of well-known arguments, many classical notions regarding hyperconvexity have been extended from the metric setting to the ultrametric setting.

Specifically, in this study, we will provide a numerical parameter that, following in the footsteps of Cianciaruso and De Pascale, establishes whether a complete ultrametric space is hyperconvex. We'll give an example to show you how this parameter works. The construction of this parameter is essential since it enables us to tell if a complete ultrametric space is hyperconvex or not.

2. The ultrametrically injective hull of an ultrametric space

In the following, we recall certain facts about the ultrametrically injective hull of an ultrametric space from [1].

In this paper, we will always put the ultrametric n (as defined in example 1) on $[0,\ \infty)$.

A map $f:\ (X,\ u_X)\rightarrow(Y,\ u_Y)$ between two ultrametric spaces $(X,\ u_X)$ and $(Y,\ u_Y)$ is called nonexpansive provided that $u_Y(f(x),\ f(y))\leq u_X(x,\ y)$ for every $x,\ y\in X$. If the map f above satisfies $u_Y(f(x),\ f(y))=u_X(x,\ y)$ for every $x,\ y\in X$, then we say that f is an isometry. If there exists a bijective isometry between two ultrametric spaces, $(X,\ u_X)$ and $(Y,\ u_Y)$, then we will say that these spaces are isometric.

Definition 10 [1, Definition 3.1] A map $f:\ X\rightarrow[0,\ \infty)$ will be called ultra-ample if, whenever $x,\ y\in X$, the condition that $u(x,\ y)\leq\max\{f(x),\ f(y)\}$ is satisfied. We shall use the notation $P_{(X,\ u)}$ to denote the class of all ultra-ample functions on X.

Let $g,\ g_1$, and g_2 be elements in P_X. Put $D(g,\ g)=0$ and

$$D(g_1,\ g_2)=\sup\{\max\{g_1(x),\ g_2(x)\}:\ x\in X\},$$

if g_1 is different from g_2. Then it is a straightforward exercise to check that D defines an extended ultrametric on P_X.

Lemma 11 [4, Part (a) of Lemma 3] For every element a in an ultrametric space X, the function $f_a(x):=u(x,\ a)\in P_X$ whenever $x\in X$.

Definition 12 [1, Lemma 3.2] If a function f on X is a minimum element of P_X with regard to the point-wise order on $[0,\ \infty)$, then it is considered ultra-extremal on X. The collection of all ultra-extremal functions on X shall be denoted by the notation T_X. Note that we obtain an ultrametric on T_X if we restrict D to T_X.

Proposition 13 For every x_1 and x_2 in an ultrametric space X and $g\in T_X$, the condition $g(x_1)>g(x_2)$ implies that $g(x_1)\leq u(x_1,\ x_2)$.

Proof. Suppose, on the contrary, that the conclusion above is not true. This therefore means that we can find two elements $\bar{x}$ and $\bar{y}$ in X satisfying $g(\bar{x}) > u(\bar{x},\ \bar{y})$ and $g(\bar{x}) > g(\bar{y})$. Let us set $h(x_1) = g(x_1)$ if $x_1 \in X$ and $x_1 \neq \bar{x}$ and $h(x_1) = \max\{u(\bar{x},\ \bar{y}),\ g(\bar{y})\}$ if $x_1 = \bar{x}$. Then one sees immediately that $h < g$.

Let $x_1,\ x_2 \in X$. Then

$$u(x_1,\ x_2) \leq \max\{g(x_1),\ g(x_2)\} = \max\{h(x_1),\ g(x_2)\},$$

if $x_1 \neq x_0$.

Assume now that $x_1 = \bar{x}$ and let $x_2 \in X$ be arbitrary. Then

$$\begin{aligned} u(x_1,\ x_2) = u(\bar{x},\ x_2) &\leq \max\{u(\bar{x},\ \bar{y}),\ u(\bar{y},\ x_2)\} \\ &\leq \max\{u(\bar{x},\ \bar{y}),\ g(\bar{y}),\ g(x_2)\} \\ &\leq \max\{h(\bar{x}),\ h(x_2)\}. \end{aligned}$$

The above computation shows that h is ultra-ample function. However, this is a contradiction to the fact that $g \in T_X$.□

Corollary 14 For an ultrametric space X and $g \in T_X$, it is always the case that for every $x,\ y \in X,\ g(x) \leq \max\{g(y),\ u(x,\ y)\}$. This is equivalent to saying that the map $g:\ (X,\ u) \to ([0,\ \infty),\ n)$ is a nonexpansive mapping.

Proposition 15 [1, Lemma 3.5] Let g be an ultra-extremal function on a space X. Then for every $x_0 \in X$,

$$g(x_0) = \sup\{u(x_0,\ y):\ y \in X \text{ and } u(x_0,\ y) > g(y)\}.$$

If X is a bounded ultrametric space, we will define

$$\delta(X) = \sup\{u(x,\ y):\ x,\ y \in X\}.$$

We call $\delta(X)$ the diameter of X. A direct consequence of Proposition 15 is the following proposition from [2].

Proposition 16 Let g be an ultra-extremal function on a space X. Then $g(X) \subseteq [0,\ \delta(X)]$.

Proposition 17 (Compare [10, Lemma 5]) Consider an ultrametric space X and suppose that $f_1,\ f_2 \in T_X$. Then

$$D(f_1,\ f_2) = \sup\{n(f_1(x_0),\ f_2(x_0)):\ x_0 \in X\}.$$

Proposition 18 Compare [1, lemma 3.5] If f is an ultra-extremal function on a space X, then for every $x_0 \in X$,

$$f(x_0) = \sup\{n(u(x_0,\ y),\ f(y)):\ y \in X\}.$$

Proof. The proof follows immediately from Proposition 15 and Proposition 17. □

Theorem 19 Let g be an ultra-extremal function on a space X and $a \in X$. Then $D(g, g_a) = g(a)$. The mapping $\varphi : X \to T_X$ given by $\varphi(a) = g_a$ for every $a \in X$ is an isometric embedding.

Proof. For every $g \in T_X$, we have by Proposition 18 that for every $x \in X$, $D(g, g_x) = g(x)$. Consequently, for every $x, y \in X$, $D(g_x, g_y) = g_x(y) = u(x, y)$. Thus the conclusion follows. □

Theorem 20 [4, Theorem 6] For any ultrametric space X, the space (T_X, N) is an ultrametrically injective hull (or hyperconvex hull) of X.

We will prove in Proposition 25 that T_X is spherically complete.

Lemma 21 Let g be an ultra-extremal function on a space X with the condition that $g(x_0) = 0$ for some $x_0 \in X$. Then $g = g_{x_0}$.

Proof. Since g is ultra-ampleness, we have $u(x_0, x) \leq \max\{g(x), 0\} = g(x)$ whenever $x \in X$. Thus, $g = g_{x_0}$ by ultra-extremality of f. □

Theorem 22 [1, Theorem 3.10] For every ultrametric space X, we have the following.

(a) D is an ultrametric on T_X.

(b) For every ultra-extremal function f and $a \in X$, $D(f, f_a) = f(a)$.

(c) The mapping $\varphi :\ X \to T_X$ which is given by $\varphi(a) = f_a$ is an isometric embedding $X \hookrightarrow T_X$. We call (T_X, φ) the hyperconvex hull of X.

(d) Any two hyperconvex hulls (E, φ) and (E', φ') of X are isometric. The isometry $T :\ E \to E'$ is uniquely determined by $T(\varphi(x)) = \varphi'(x)$ for every $x \in X$.

In the following, we prove some basic properties of (T_X, N) for an ultrametric space X.

Proposition 23 If $s \in T_X$, then $s \circ \varphi \in T_X$, where φ is the isometric embedding in Theorem 22.

Proof. Suppose $s \in T_X$. If $a, b \in X$, then

$$u(a, b) = D(f_a, f_b) = D(\varphi(a), \varphi(b)) \leq \max\{s(\varphi(a)), s(\varphi(b))\},$$

since $s \in T_X$. Thus $s \circ \varphi$ is ultra-ample. We need only show now that $s \circ \varphi$ is ultra-extremal on X.

Assume that is not the case. Then there is $h \in T_X$ with $h(x) \leq (s \circ \varphi)(x)$ for $x \in X$ and $h(x_0) < (s \circ \varphi)(x_0)$ for some $x_0 \in X$. Define the function ϕ on T_X by $\phi(f) = s(f)$ for every ultra-extremal function f on X and $f \neq \varphi(x_0)$ and $\phi(f) = h(x_0)$ if $f = \varphi(x_0)$.

We will prove that $D(f, g) \leq \max\{\phi(f), \phi(g)\}$ for $f, g \in T_X$. This will contradict the fact that $s \in T_X$ thus rendering our initial assumption, namely that the function h exists, false. Since s and t are equal nearly everywhere and s is ultra-ample, it will be sufficient if we prove the inequality for $g = \varphi(x_0)$ and any ultra-extremal function f on X. In essence, we will have to prove that $D(f, \varphi(x_0)) \leq \max\{\phi(f), \phi(\varphi(x_0))\}$.

Consider $f \in T_X$. Then $f(x_0) = N(f, \varphi(x_0))$ (check for instance Theorem 19). We, therefore, suppose now that $f(x_0) > 0$. Then Lemma 15 implies that

$$f(x_0) = \sup\{u(x_0, y) :\ y \in X \text{ and } u(x_0, y) > f(y)\}.$$

Thus, $\forall\, \varepsilon > 0$, $\exists\, y \in X$ such that $f(x_0) - \varepsilon \leq u(x_0, y)$ and $u(x_0, y) > f(y)$. It follows that

$$D(f, \varphi(x_0)) - \varepsilon = f(x_0) - \varepsilon \leq u(x_0, y)$$

and

$$u(x_0,\ y) \leq \max\{h(x_0),\ h(y)\} \leq \max\{s(\varphi(y)),\ \phi(\varphi(x_0))\}.$$

Since s is ultra-extremal on X, by Corollary 14,

$$s(\varphi(y)) \leq \max\{s(f),\ D(\varphi(y),\ f)\} = \max\{\phi(f),\ \phi(y)\}$$

for any $f \in T_X$. It is clear that the inequalities

$$D(f,\ \varphi(x_0)) - \varepsilon \leq \max\{s(\varphi(y)),\ \phi(\varphi(x_0))\}$$

and

$$s(\varphi(y)) \leq \max\{\phi(f),\ \phi(y)\}$$

hold. Therefore

$$\begin{aligned} D(f,\ \varphi(x_0)) - \varepsilon &\leq u(x_0,\ y) \\ &\leq \max\{\phi(f),\ \phi(y),\ \phi(\varphi(x_0))\} \\ &= \max\{\phi(f),\ \phi(\varphi(x_0))\}, \end{aligned}$$

as $f(y) < u(x_0,\ y)$. Since ε was chosen arbitrarily, we have thus shown that

$$D(f,\ \varphi(x_0)) \leq \max\{\phi(f),\ \phi(\varphi(x_0))\}.$$

We conclude therefore that $s \circ \varphi$ is ultra-extremal and hence $s \circ \varphi \in T_X$. □

Lemma 24 [9, Lemma 9] Let A be a nonempty subspace of an ultrametric space X. Let $\alpha:\ A \to [0,\ \infty)$ be a mapping which is ultra-ample. Then there is a mapping $\Lambda:\ X \to [0,\ \infty)$ which is an extension of α. Furthermore, $\exists\, f \in T_X$ such that $\forall\, x \in X,\ f(x) \leq \Lambda(x)$.

Proposition 25 For any ultrametric space X, the hyperconvex hull $(T_X,\ D)$ is spherically complete.

Proof. (a) Let $(x_i)_{i\in I} \in T_X$ be a family of distinct points and $(\alpha_i)_{i\in I} \subseteq [0,\ \infty)$ be such that $D(x_i,\ x_j) \leq \max\{\alpha_i,\ \alpha_j\}$ for every $i,\ j \in I$. Let $Y = \{x_i:\ i \in I\}$ and define the mapping $\phi:\ Y \to [0,\ \infty)$ by $\phi(x_i) = \alpha_i$ whenever $i \in I$. Then ϕ is ultra-ample, consequently by Lemma 24 that there is an extension Φ of the function ϕ to the set T_X such that $D(f,\ g) \leq \max\{\Phi(f),\ \Phi(g)\}$ whenever $f,\ g \in T_X$.

Again by Lemma 24, there is $h \in T_X$ such that $h \leq \Phi$. By the property proved in Proposition 23 of ultra-extremal functions on T_X, we know that $h \circ \varphi \in T_X$. It therefore follows that

$$h \circ \varphi \in \bigcap_{f \in T_X} B(f,\ \Phi(f))$$

$$\subseteq \bigcap_{i \in I} B(x_i,\ \alpha_i).$$

For any fixed $f \in T_X$, the distance D between f and $h \circ \varphi$ is given by

$$D(h \circ \varphi,\ f) = \sup\{\max\{h(\varphi(x)),\ f(x)\} :\ x \in X\}.$$

We know from Theorem 19 for every $x \in X,\ f(x) = D(f,\ \varphi(x))$. Moreover, since h is an ultra-extremal function on T_X and using Corollary 14, we obtain that for every $x \in X$,

$$h(\varphi(x)) \leq \max\{h(f),\ N(f,\ \varphi(x))\} = \max\{h(f),\ f(x)\}.$$

Since h is ultra-ample, we have that

$$f(x) = D(\varphi(x),\ f) \leq \max\{h(\varphi(x)),\ h(f)\}$$

for each $x \in X$. By our choice of h, we have that $h(f) \leq \Phi(f)$. Hence

$$N(h \circ \varphi,\ f) = \sup\{h(\varphi(x)) :\ x \in X \text{ and } f(x) < h(\varphi(x))\}$$

whenever $f \in T_X$. □

Corollary 26 The following are equivalent for an ultrametric space X.

(a) X is spherically complete.

(b) $\forall\ \phi \in T_X,\ \exists\, x_0 \in X$ such that $\phi = \phi_{x_0}$.

(c) $\forall\ \phi \in T_X,\ \exists\, x_0 \in X$ such that $\phi(x_0) = 0$.

Proof. $(a) \Rightarrow (b)$

Let $\phi \in T_X$ and $\{B(x_0,\ f(x_0))\}_{x_0 \in X}$ be in X. Since X is spherically complete,

$$\bigcap_{x_0 \in X} B(x_0,\ \phi(x_0)) \neq \emptyset.$$

Let now

$$a \in \bigcap_{x_0 \in X} B(x_0,\ \phi(x_0)).$$

This means that $u(x_0,\ a) \leq \phi(x_0)$, or, better still $\phi_a \leq \phi$ and we get directly by ultra-extremality of ϕ that $\phi = \phi_a$.

$(b) \Rightarrow (a)$

Let $(B(x_i,\ r_i))_{i\in I} \subseteq X$ be a family of closed balls possessing the binary intersection property. Assume that $x_i \neq x_j$ whenever $i \neq j$. By Lemma 24, $\exists\ \phi \in T_X$ such that $\phi(x_i) \leq r_i$ for every $i \in I$. By assumption, $\exists\ a \in X$ such that $\phi = \phi_a$. Therefore

$$a \in \bigcap_{i\in I} B(x_i,\ r_i).$$

This completes the proof that X is spherically complete.

The remaining equivalence can be deduced from the fact that $u(x,\ x) = 0$ whenever $x \in X$ and from Lemma 21. □

Remark 27 Let X be an ultrametric space and T_X be its ultrametrically injective hull. Since T_X is complete by [6, page 85], the closure of $\varphi(X)$ in T_X is as a subspace of T_X isometric to the completion of X. Moreover, an ultra-extremal function f belongs to the closure of $\varphi(X)$ if and only if there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $D(f_{x_n},\ f)$ goes to 0 as $n \to \infty$.

3. The lack of hyperconvexity for a complete ultrametric space

We begin this section with a result that characterizes hyperconvex ultrametric spaces.

Proposition 28 A space X is hyperconvex if and only if every ultra-extremal function has a minimum and this minimum value is zero.

Proof. It follows from Proposition 16 and part (c) of Corollary 26. □

Motivated by Proposition 28, we will introduce the following definition of a parameter that will help us understand the lack of hyperconvexity for complete ultrametric spaces. After presenting the definition, we will state some interesting properties of this parameter.

Definition 29 Let X be a complete ultrametric space. The parameter $h(X)$ of X is an extended real number which is defined by

$$h(X) := \sup_{f\in T_X} \inf\{f(x) :\ x \in X\}.$$

We will call this parameter the measure of lack of hyperconvexity for X.

Example 30 Let $X = \{x_1,\ x_2,\ \ldots x_n\},\ n \geq 2$ be an ultrametric space. Define $Y = \{y_1,\ y_2,\ \ldots,\ y_n\}$ to be a permutation of the set X. We define a function f by $f(y_1) \leq u(y_1,\ y_2)$ and

$$f(y_h) = \max_{1\leq k\leq h} n(u(y_h,\ y_k),\ f(y_k)).$$

Then f is ultra-ample. We will now show that $f \in T_X$.

Let g be an ultra-extremal function satisfying

$$g(y_h) \leq f(y_h) \quad \text{for} \quad h = 1,\ 2,\ \ldots,\ n.$$

We need to show that $g = f$. Suppose instead that $g(y_1) < f(y_1)$. Since

$$g(y_2) \leq f(y_2) = n(u(y_1,\ y_2),\ f(y_1)),$$

we have that

$$\max\{g(y_1),\ g(y_2)\} < \max\{f(y_1),\ f(y_2)\} \leq u(y_1,\ y_2),$$

i.e.,

$$\max\{g(y_1),\ g(y_2)\} < u(y_1,\ y_2).$$

Thus $g(y_1) = f(y_1)$ and $g(y_2) = f(y_2)$. If $g(y_j) < f(y_j)$ for some $j > 2$, then by choosing some i, we get

$$n(u(y_j,\ y_i),\ g(y_i)) \leq g(y_j) < f(y_j) = n(u(y_j,\ y_i),\ f(y_i)),$$

which contradicts the fact that $g(y_i) \leq f(y_i)$. This therefore means that

$$g(y_h) = f(y_h) \text{ for } h = 1,\ 2,\ \ldots,\ n,$$

and hence f is ultra-extremal.

Let us choose

$$f(y_1) = \frac{1}{2} \min_{j \neq 1} u(y_j,\ y_1).$$

Then it is easily checked that the minimum of f is $f(y_1)$. By taking the supremum (which is the maximum in this case), we get that

$$h(X) \geq \frac{1}{2} \max_{j} \min_{i \neq j} u(x_i,\ x_j).$$

Remark 31 It is important to note that in the above example, if we have the discrete ultrametric, then the last inequality above becomes equality.

For the rest of this section, all our ultrametric spaces will be complete. We will assume this without mentioning it as we proceed.

Theorem 32 $h(X) = 0 \Leftrightarrow X$ is hyperconvex.

Proof. Suppose that $h(X) = 0$. Then for every $f \in T_X$, we have

$$\inf\{f(x):\ x \in X\} = 0.$$

Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be such that

$$f(x_n) \leq \frac{1}{n}.$$

Since f is ultra-extremal, one can easily show that $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Let x_0 be the limit, then we have that $f(x_0) = 0$ and hence X is hyperconvex by Proposition 28.

Assume now that X is hyperconvex with $f \in T_X$. Then by Proposition 28 $f(x_0) = 0$ for some $x_0 \in X$. It, therefore, follows that $h(X) = 0$. □

In the following, u_H will denote the Hausdorff distance. Recall that for two closed sets C_1 and C_2, we have

$$u_H(C_1,\, C_2) = \inf\{r > 0:\ C_1 \subset B_r(C_2) \text{ and } C_2 \subset B_r(C_1)\},$$

where for instance,

$$B_r(C_1) = \bigcup_{c\in C_1} B_r(c).$$

Theorem 33 If $(E,\, \varphi)$ is a hyperconvex hull of an ultrametric space X, then $h(X) = u_H(\varphi(X),\, E)$.

Proof. This follows immediately from parts (b) and (c) of Theorem 22. □

Theorem 34 For a bounded space X, we have that

$$h(X) \leq \operatorname{diam} X = \operatorname{diam} T_X.$$

Proof. Consider the map g where $g(x) = \delta(X)$ for every $x \in X$ and

$$\delta(X) = \sup\{u(x,\, y):\ y \in X\}.$$

Then g is ultra-ample and satisfies $f(x) \leq g(x) = \delta(X)$ for some point $x \in X$ and every $f \in T_X$. Hence $\inf\{f(x):\ x \in X\} \leq \delta(X)$ and the same holds for $h(X)$. □

Remark 35 In the proof of Theorem 34, the only constant function that can be ultra-extremal is precisely $g(x) = \delta(X)$. This function is indeed ultra-extremal if and only if every point of X is a diametral point, i.e., $\forall\, x \in X$, we have $\sup\{u(x,\, y):\ y \in X\} = \delta(X)$.

Theorem 36 For two spaces X and Y with $X \subset Y$, we have $h(X) \leq \max\{h(Y),\, u_H(X,\, Y)\}$.

Proof. Since X lies in Y, it can be isometrically embedded in T_Y and so also is the case for T_X namely $X \hookrightarrow T_X \hookrightarrow T_Y$. Consequently

$$
\begin{aligned}
h(X) &= u_H(X,\ T_X) \\
&\le u_H(X,\ T_Y) \\
&\le \max\{u_H(X,\ Y),\ u_H(Y,\ T_Y)\} \\
&= \max\{u_H(X,\ Y),\ h(Y)\}.
\end{aligned}
$$

□

Theorem 37 Let X be an ultrametric space. If for every $n \in \mathbb{N}$, we have that $X \subset S_n$, $u_H(S_n,\ X) \to 0$, and $h(S_n) \to 0$, then we must have that $h(X) = 0$.

Proof. It follows directly from Theorem 36. □

Theorem 38 For every natural number n, let $S_n \subset X$ with $u_H(S_n,\ X) \to 0$. Then $h(S_n) \to 0$ if and only if $h(X) = 0$.

Proof. Let S_n be the sequence in the statement of the theorem. It suffices to show that $h(S_n)$ goes to 0 as n goes to ∞ implies that $h(X) = 0$. This is because the reverse implication is a consequence of Theorem 36.

Let now $f \in T_X$ and $g_k \in T_{S_n}$ be such that for every $n \in \mathbb{N}$, $g_n(s) \le f(s)$ for $s \in S_n$. Let $a_n > \max\{u_H(S_n,\ X),\ h(S_n)\}$ with $a_n \to 0$ as $n \to \infty$; we can find $p_n \in S_n$ such that $u(p_n,\ s) \le \max\{a_n,\ g_n(s)\}$ for each $s \in S_n$. For each $y \in X$, choose $s \in S_n$ such that $u(y,\ s) \le a_n$. Consequently,

$$
\begin{aligned}
u(p_n,\ y) &\le \max\{u(p_n,\ s),\ u(s,\ y)\} \\
&\le \max\{a_n,\ g_n(s),\ a_n\} \\
&\le \max\{f(s),\ a_n\} \\
&\le \max\{f(y),\ u(y,\ s),\ a_n\} \\
&\le \max\{f(y),\ a_n\}.
\end{aligned}
$$

It therefore follows that

$$
f(p_n) = \sup\{u(p_n,\ y) :\ y \in Y \text{ and } u(p_n,\ y) > f(y)\} \le a_n
$$

and

$$
u(p_m,\ p_n) \le \max\{f(p_m),\ a_n\} \le \max\{a_m,\ a_n\} \to 0.
$$

The above shows that the sequence $(p_n)_{n\in\mathbb{N}}$ Cauchy. Thus, if we let p be the limit of the sequence, then we have that $f(p) = 0$. □

4. Conclusion

We continued studies about hyperconvexity in the category of (complete) ultrametric spaces and nonexpansive maps. In particular, we proved that for any ultrametric space X, its ultrametrically injective hull denoted by T_X is spherically complete. Furthermore, we introduced a parameter that measures the lack of hyperconvexity for a complete ultrametric space (see Definition 29). An example to illustrate Definition 29 is presented (Example 30). Finally, we present some properties of this parameter (check for instance Theorems 32-34, 36-38).

The parameter h introduced above will be very vital in the study of the hyperconvexity of a nonArchimedean Banach space. In its present form, it doesn't seem significant in nonArchimedean Banach spaces, thus, there will be a need to define a new parameter by modifying the existing one.

The parameter h introduced above will be useful in the study of the hyperconvexity of nonArchimedean banach spaces. In its present form, it doesn't seem significant in nonArchimedean banach spaces, thus, a new parameter have to be defined by modifying the existing one.

Acknowledgement

The author would like to thank the referees for the several valuable suggestions that have helped to improve the presentation of this article. The author would also like to thank the national Research Foundation of South Africa for financial support under the Competitive Support for Y-rated researchers.

Conflict of interest

We declare that we have no personal relationships or financial interests that could have influence the work presented in this manuscript.

References

[1] Agyingi CA. Extensions of ultrametric spaces. *Topology Proceedings.* 2016; 47: 207-220.

[2] Agyingi CA. When separability of the space of ultra-extremal functions is preserved. *Quaestiones Mathematicae.* 2019; 42(6): 718-801.

[3] Agyingi CA. Some characterizations of ultrametrically injective ultrametric spaces. *Afrika Matematika.* 2021; 32: 517-529.

[4] Bayod JM, Martínez-Maurica J. Ultrametrically injective spaces. *Proceedings of the American Mathematical Society.* 1987; 101(3): 571-576.

[5] Cianciaruso F, De Pascale E. The hausdorff measure of non hyperconvexity. *Atti Del Seminario Matematico E Fisico Dell'universita'di Modena [Proceedings of the Mathematical and Physical Seminar of the University of Modena].* 1999; 47: 261-267.

[6] Heckmanns U. Aspects of ultrametric spaces. In: *Queen's Papers in Pure and Applied Mathematics.* Queen's University, Kingston; 1998. p.109.

[7] Khamsi MA, Kirk WA. *An Introduction to Metric Spaces and Fixed Point Theory.* John Wiley, New York; 2001.

[8] Kirk W, Shahzad N. Some fixed point results in ultrametric spaces. *Topology and its Applications.* 2012; 159(15): 3327-3334.

[9] Kemajou E, Künzi H-PA, Otafudu OO. The Isbell-hull of a di-space. *Topology and its Applications.* 2012; 159: 2463-2475.

[10] Künzi H-PA, Otafudu OO. The Ultra-quasi-metrically injective hull of a T_0-ultra-quasi-metric space. *Applied Categorical Structures.* 2013; 21(6): 651-670.

[11] Rammal R, Toulouse G, Virasoro MA. Ultrametricity for physicists. *Reviews of Modern Physics.* 1986; 58: 765-788.

[12] Robert AM. *A Course in p-Adic Analysis.* Springer-Verlag, New York; 2000.

[13] Wells JH, Williams LR. *Extensions and Embeddings in Analysis.* Springer, Berlin, New York; 1975.