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Research Article

A Hybrid Fuzzy Extension and Its Application in Multi-Attribute Decision Making

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Abstract: The hypersoft set theory is an extension of soft set theory. The complex non-linear diophantine fuzzy set is a hybrid fuzzy extension that serves as a generalization of the q-rung linear diophantine fuzzy set and the complex linear diophantine fuzzy set. In this paper, to tackle multi-sub-attributed real-world situations under complex non-linear diophantine fuzzy ambiance, the concept of complex q-rung linear diophantine fuzzy hypersoft set is proposed along with its score and accuracy function. Also, the idea of lattice ordered complex q-rung linear diophantine fuzzy hypersoft set is proposed in this paper, along with some of its basic algebraic operations. Furthermore, a highly effective algorithm using lattice ordered complex q-rung linear diophantine fuzzy hypersoft set is provided for handling multi-attributed decision-making issues exquisitely, along with an illustrative example in the field of vertical farming. Then, a comparative analysis between the proposed and current notions is provided to demonstrate the superiority and benefits of the suggested concepts over the current ones.

Keywords: hypersoft set, complex q-rung linear diophantine fuzzy hypersoft set, lattice, multi-attribute decision-making

MSC: 08A72, 03E72

1. Introduction

1.1 *Literature review*

The fuzzy set (FS) theory created by Zadeh [1] in 1965 is very important for addressing the challenges associated with multi-attribute decision making (MADM). Additionally, it provides a useful method for describing fuzzy data. However, the ability of FS to convey the neutral state is constrained. To get around these constraints, Atanassov [2] developed the idea of the intuitionistic fuzzy set (IFS). The two indices of IFS are membership (MS) degree and non-membership (NMS) degree. The IFS's two indices, MS degree and NMS degree, should have their total between [0, 1]. In some real-world scenarios, the total of MS and NMS is greater than 1. In these situations, the IFS failed to provide clarification for unclear information. By employing the limitation that the total of MS^2 and NMS^2 must be between 0 and 1, Yager [3] has presented the Pythagorean fuzzy set (PFS) concept and identified a solution for these kinds of ambiguous data. Due to the restrictions imposed on the MS and NMS in the IFS and PFS, these conceptions have occasionally failed to accurately reflect the information for an item. In order to expand the MS and NMS space, Yager [4] created a novel concept known as the q-rung orthopair fuzzy set (q-ROFS), which restricts the MS and NMS space to the sum of the qth power of MS and

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NMS. Since the FS, IFS, PFS, and q-ROFS concepts each have their own limits, Riaz and Hashmi [5] developed the theory of the linear diophantine fuzzy set (LDFS), which incorporates the idea of reference parameters (RPs), to address these issues. The RPs are constrained and bound in the LDFS. In order to overcome these RPs limits, Almagrabi [6] developed the q-rung linear diophantine fuzzy set (q-RLDFS), which increased the range of RPs by qth powering the RPs.

The impact of changing the FS codomain from the real unit interval [0, 1] to the unit disk has been a topic of discussion among academics. The idea of a complex fuzzy set (CFS), which is described as a complex-valued mapping with the unit circle in the complex plane as the codomain, was thus put forth by Ramot et al. [7]. In contrast to FS, CFS's range is not constrained to the range [0, 1], but instead extends into a unit disk in the complex plane. Later, the codomain of MS and NMS grades of IFS, PFS and q-ROFS were extended into unit disk in the complex plane by Alkouri and Salleh [8], Ullah et al. [9] and Liu et al. [10] respectively and obtained complex intuitionistic fuzzy set (CIFS), complex pythagorean fuzzy set (CPFS) and complex q-rung orthopair fuzzy set (Cq-ROFS). Similarly, the codomain of RPs in LDFS is also extended into unit disk in the complex plane by Kamaci [11] and obtained complex linear diophantine fuzzy set (CLDFS). Maria Shams et al. [12] extend the codomain of MS, NMS and RPs of q-RLDFS into unit disk in the complex plane and named it as complex non-linear diophantine fuzzy set (CNLDFS) because in [12] they named q-RLDFS as non-linear diophantine fuzzy set (NLDFS). Since CNLDFS is an extension of q-RLDFS, in this study, we named it as complex q-Rung linear diophantine fuzzy set (Cq-RLDFS). However, the lack of parametrization in each of these theories poses significant drawbacks. Molodtsov [13] developed the idea of SS theory, which approaches uncertainty in a parametric manner, in order to get over the limitations of parametrization. The fuzzy soft set (FSS), which is useful for encoding fuzzy data with parametric information was later introduced by Maji et al. [14] by merging FS and SS. In the same way, to illustrate other fuzzy extensions data featuring parametric information Maji et al. [15], Peng et al. [16], Hussain et al. [17], Riaz et al. [18], Thirunavukarasu et al. [19], Kumar and Bajaj [20], Mahmood and Ali [21], Xiaoming et al. [22] and Vimala et al. [23] respectively integrated SS theory with additional FS theory extensions like IFS, PFS, q-ROFS, LDFS, CFS, CIFS, CPFS, Cq-ROFS and CLDFS and acquired intuitionistic fuzzy soft set (IFSS), pythagorean fuzzy soft set (PFSS), q-rung orthopair fuzzy soft set (q-ROFSS), linear diophantine fuzzy soft set (LDFSS), complex fuzzy soft set (CFSS), complex intutionistic fuzzy soft set (CIFSS), complex pythagorean fuzzy soft set (CPFSS), complex q-rung orthopair fuzzy soft set (Cq-ROFSS) and complex linear diophantine fuzzy soft set (CLDFSS). Numerous problems in the real world have rankings of the attributes to address them. Lattice ordered soft set (LOSS), a concept that has been enormously effective in such circumstances, was first put forth by Ali et al. [24]. Lattice ordered fuzzy soft set (LOFSS) and lattice ordered intuitionistic fuzzy soft set (LOIFSS) are later concepts that were proposed by Aslam et al. [25] and Mahmood et al. [26], respectively. Each of the aforementioned studies has various limitations, such as when sub-attributes are present in attributes, in which case the aforementioned studies are unable to offer a solution. Smarandache [27] developed the idea of SS to Hypersoft set (HSS), which went beyond the restrictions by transforming the single-attributed function into a multi-sub-attributed function. The concepts of fuzzy hypersoft set (FHSS) and intuitionistic fuzzy hypersoft set (IFHSS), which describe FS and IFS data with multiple sub-parameters by combining HSS with FS and IFS, respectively, have also been proposed by Sanrandache [27]. Similar to this, HSS has been combined with other fuzzy and complex fuzzy extensions such as PFS, q-ROFS, q-RLDFS, CFS, CIFS and Cq-ROFS and obtained pythagorean fuzzy hypersoft set (PFHSS), q-rung orthopair fuzzy hypersoft set (q-ROFHSS), q-rung linear diophantine fuzzy hypersoft set (q-RLDFHSS), complex fuzzy hypersoft set (CFHSS), complex intuitionistic fuzzy hypersoft set (CIFHSS) and complex q-rung orthopair fuzzy hypersoft set (Cq-ROFHSS) by Zulqarnain et al. [28], Khan et al. [29], Surya et al. [30], Rahman et al. [31], Rahman et al. [31] and Ying et al. [32] respectively.

1.2 *Research gap and motivation*

The following are the research gaps:

• The available literature study reveals that, although multiple parametric decision making (DM) investigations have been carried out under different fuzzy structures, it is difficult to use the existing studies to exhibit Cq-RLDFS with parametric information, especially when parameters has an order among them.

The motivations of the study are as follows:

• By proposing theories that can handle parametric situations even in Cq-RLDF circumstances, the work aims to fill up research gaps.

• It is challenging to address many real-world MADM situations in the Cq-RLDFS environment using the current theories since they necessitate managing several sub-attributes simultaneously. This motivates the study to propose a MADM strategy that can manage situations even in these challenging contexts.

1.3 *Contribution and objectives*

The main objectives of this work are listed below:

• To introduce the concept of the complex q-Rung linear diophantine fuzzy hypersoft set (Cq-RLDFHSS), which has a lot of potential for handling Cq-RLDF circumstances involving several sub-attributes.

• To introduce the concept of lattice ordered Cq-RLDFHSS (LOCq-RLDFHSS), which utilizes the order of the multi-sub-attributes to handle Cq-RLDFHSS effectively.

• To present an efficient LOCq-RLDFHSS-based MADM approach.

The core contributions of the work are as follows:

• By merging Cq-RLDFS and HSS, a novel idea termed as Cq-RLDFHSS is presented in this study, along with the score and accuracy function of Cq-RLDFHS numbers.

• In this study, the concept of LOCq-RLDFHSS is put forth along with a few of its algebraic operations such as restricted intersection, extended union, AND operation, OR operation and complement.

• An MADM algorithm is described in this paper based on the proposed LOCq-RLDFHSS.

• In order to demonstrate the effectiveness of the proposed MADM algorithm, a real-world vertical farming problem is illustrated as a numerical example in this study.

• A comparative analysis is included in this study, to show the effectiveness and potency of the proposed notions and the MADM algorithm, along with the modest limitations of the proposed notions.

1.4 *Structure of the paper*

“Section 2” provides the essential fundamental definitions and notations. The proposed Cq-RLDFHSS and LOCq-RLDFHSS are defined in “Section 3”, which also includes the score and accuracy function of Cq-RLDFHS numbers as well as some of the basic algebraic operations of LOCq-RLDFHSS. “Section 4” includes a LOCq-RLDFHSS-based MADM algorithm along with a vertical farming-related MADM problem, which demonstrates the effectiveness of the suggested algorithm in solving MADM problems. A comparison analysis has been conducted in “Section 5” to describe how the suggested notions are more effective than the current notions. Lastly, the article’s conclusion is given in “Section 6”.

2. Preliminaries

This section contains the definitions and notations required for this manuscript.

Throughout this manuscript, $\mathfrak{U} = \{\mathfrak{u}_a : a = 1, 2, \ldots, m\}$ denotes the universal set.

If a binary relation $\leq$ on a non empty set A is reflexive, antisymmetric and transitive then it is called partial order. Also, $\leq$ is said to be total order on A if $a \neq b$, either $a \leq b$ or $b \leq a \forall a, b \in A$.

A lattice is a subset of set of all partial ordered sets. A lattice L is a partial order set in which $\forall a, b \in L$ the set $\{a, b\}$ has a supremum and infimum. If L contains 0 and 1 such that $\forall x \in L$, $0 \leq x \leq 1$ then L is called bounded lattice.

Definition 2.1 [2] An IFS I on $\mathfrak{U}$ is defined as

$$I = \{(\mathfrak{u}, \Psi_I(\mathfrak{u}), \Upsilon_I(\mathfrak{u})) | \mathfrak{u} \in \mathfrak{U}\}$$

where, $\Psi_I(\mathfrak{u})$ and $\Upsilon_I(\mathfrak{u}) \in [0, 1]$ are MS degree and NMS degree with the restriction that $0 \le \Psi_I(\mathfrak{u}) + \Upsilon_I(\mathfrak{u}) \le 1 \forall \mathfrak{u} \in \mathfrak{U}$.

Definition 2.2 [8] An CIFS $\mathscr{I}$ on $\mathfrak{U}$ is defined as

$$\mathscr{I} = \{(\mathfrak{u},\ \Psi_{\mathscr{I}}(\mathfrak{u})e^{i2\pi(\vartheta_{\Psi_{\mathscr{I}}}(\mathfrak{u}))},\ \Upsilon_{\mathscr{I}}(\mathfrak{u})e^{i2\pi(\vartheta_{\Upsilon_{\mathscr{I}}}(\mathfrak{u}))})|\mathfrak{u} \in \mathfrak{U}\}$$

where, $\Psi_{\mathscr{I}}(\mathfrak{u})e^{i2\pi(\vartheta_{\Psi_{\mathscr{I}}}(\mathfrak{u}))}$ and $\Upsilon_{\mathscr{I}}(\mathfrak{u})e^{i2\pi(\vartheta_{\Upsilon_{\mathscr{I}}}(\mathfrak{u}))}$ (for $\Psi_{\mathscr{Q}}(\mathfrak{u}), \Upsilon_{\mathscr{Q}}(\mathfrak{u}), \vartheta_{\Psi_{\mathscr{Q}}}(\mathfrak{u})$ and $\vartheta_{\Upsilon_{\mathscr{Q}}}(\mathfrak{u}) \in [0, 1]$) are complex valued MS(CMS) degree and complex valued NMS (CNMS) degree with the restrictions that $0 \le \Psi_{\mathscr{I}}(\mathfrak{u}) + \Upsilon_{\mathscr{I}}(\mathfrak{u}) \le 1$ and $0 \le \vartheta_{\Psi_{\mathscr{I}}}(\mathfrak{u}) + \vartheta_{\Upsilon_{\mathscr{I}}}(\mathfrak{u}) \le 1\ \forall \mathfrak{u} \in \mathfrak{U}$.

Definition 2.3 [6] An q-RLDFS Q on $\mathfrak{U}$ is described as

$$Q = \{(\mathfrak{u},\ \langle\Psi_Q(\mathfrak{u}),\ \Upsilon_Q(\mathfrak{u})\rangle,\ \langle\rho_Q(\mathfrak{u}),\ \eta_Q(\mathfrak{u})\rangle)|\mathfrak{u} \in \mathfrak{U}\}$$

where, $\Psi_Q(\mathfrak{u}), \Upsilon_Q(\mathfrak{u}), \rho_Q(\mathfrak{u})$ and $\eta_Q(\mathfrak{u}) \in [0, 1]$ are MS degree, NMS degree and their corresponding RPs respectively, with the restrictions that $0 \le \rho_Q^q(\mathfrak{u}) + \eta_Q^q(\mathfrak{u}) \le 1$ and $0 \le \rho_Q^q(\mathfrak{u})\Psi_Q(\mathfrak{u}) + \eta_Q^q(\mathfrak{u})\Upsilon_Q(\mathfrak{u}) \le 1 \forall \mathfrak{u} \in \mathfrak{U}, q \ge 1$.

Definition 2.4 [12] An Cq-RLDFS $\mathscr{Q}$ on $\mathfrak{U}$ is described as

$$\mathscr{Q} = \{(\mathfrak{u},\ \langle\Psi_{\mathscr{Q}}(\mathfrak{u})e^{i2\pi(\vartheta_{\Psi_{\mathscr{Q}}}(\mathfrak{u}))},\ \Upsilon_{\mathscr{Q}}(\mathfrak{u})e^{i2\pi(\vartheta_{\Upsilon_{\mathscr{Q}}}(\mathfrak{u}))}\rangle,\ \langle\rho_{\mathscr{Q}}(\mathfrak{u})e^{i2\pi(\vartheta_{\rho_{\mathscr{Q}}}(\mathfrak{u}))},\ \eta_{\mathscr{Q}}(\mathfrak{u})e^{i2\pi(\vartheta_{\eta_{\mathscr{Q}}}(\mathfrak{u}))}\rangle)|\mathfrak{u} \in \mathfrak{U}\}$$

where, $\Psi_{\mathscr{Q}}(\mathfrak{u})e^{i2\pi(\vartheta_{\Psi_{\mathscr{Q}}}(\mathfrak{u}))}$, $\Upsilon_{\mathscr{Q}}(\mathfrak{u})e^{i2\pi(\vartheta_{\Upsilon_{\mathscr{Q}}}(\mathfrak{u}))}$, $\rho_{\mathscr{Q}}(\mathfrak{u})e^{i2\pi(\vartheta_{\rho_{\mathscr{Q}}}(\mathfrak{u}))}$ and $\eta_{\mathscr{Q}}(\mathfrak{u})e^{i2\pi(\vartheta_{\eta_{\mathscr{Q}}}(\mathfrak{u}))}$ (for $\Psi_{\mathscr{Q}}(\mathfrak{u})$, $\Upsilon_{\mathscr{Q}}(\mathfrak{u})$, $\rho_{\mathscr{Q}}(\mathfrak{u})$, $\eta_{\mathscr{Q}}(\mathfrak{u})$, $\vartheta_{\Psi_{\mathscr{Q}}}(\mathfrak{u})$, $\vartheta_{\Upsilon_{\mathscr{Q}}}(\mathfrak{u})$, $\vartheta_{\rho_{\mathscr{Q}}}(\mathfrak{u})$ and $\vartheta_{\eta_{\mathscr{Q}}}(\mathfrak{u}) \in [0, 1]$) are CMS degree, CNMS degree and their corresponding complex valued RPs (CRPs) respectively, with the restrictions that $0 \le \rho_{\mathscr{Q}}^q(\mathfrak{u}) + \eta_{\mathscr{Q}}^q(\mathfrak{u}) \le 1$, $0 \le \vartheta_{\rho_{\mathscr{Q}}}^q(\mathfrak{u}) + \vartheta_{\eta_{\mathscr{Q}}}^q(\mathfrak{u}) \le 1$, $0 \le \rho_{\mathscr{Q}}^q(\mathfrak{u})\Psi_{\mathscr{Q}}(\mathfrak{u}) + \eta_{\mathscr{Q}}^q(\mathfrak{u})\Upsilon_{\mathscr{Q}}(\mathfrak{u}) \le 1$ and $0 \le \vartheta_{\rho_{\mathscr{Q}}}^q(\mathfrak{u})\vartheta_{\Psi_{\mathscr{Q}}}(\mathfrak{u}) + \vartheta_{\eta_{\mathscr{Q}}}^q(\mathfrak{u})\vartheta_{\Upsilon_{\mathscr{Q}}}(\mathfrak{u}) \le 1\ \forall \mathfrak{u} \in \mathfrak{U}, q \ge 1$.

Definition 2.5 [13] Let $\mathfrak{G}$ be set of attributes associated with objects in $\mathfrak{U}$ and $\mathfrak{F} \subseteq \mathfrak{G}$. Then the pair $(\Lambda, \mathfrak{F})$ is said to be SS over $\mathfrak{U}$ defined by the mapping

$$\Lambda : \mathfrak{F} \to P(\mathfrak{U})$$

Where, $P(\mathfrak{U})$ is the collection of all subsets of $\mathfrak{U}$.

Definition 2.6 [24] Let $(\Lambda, \mathfrak{F})$ be a SS over $\mathfrak{U}$, where

$$\Lambda : \mathfrak{F} \to P(\mathfrak{U})$$

Then $(\Lambda, \mathfrak{F})$ is said to be LOSS if $\mathfrak{f}_1 \le_{\mathfrak{F}} \mathfrak{f}_2 \Rightarrow \Lambda(\mathfrak{f}_1) \subseteq \Lambda(\mathfrak{f}_2) \forall \mathfrak{f}_1, \mathfrak{f}_2 \in \mathfrak{F}$.

Definition 2.7 [15] Let $\mathfrak{G}$ be set of attributes associated with objects in $\mathfrak{U}$ and $\mathfrak{F} \subseteq \mathfrak{G}$. Then the pair $(\Lambda, \mathfrak{F})$ is said to be IFSS over $\mathfrak{U}$ defined by the mapping

$$\Lambda : \mathfrak{F} \to IFP(\mathfrak{U})$$

Where, $IFP(\mathfrak{U})$ is the collection of all IF subsets of $\mathfrak{U}$.

Definition 2.8 [26] Let $(\Lambda, \mathfrak{F})$ be a IFSS over $\mathfrak{U}$, where

$$\Lambda : \mathfrak{F} \rightarrow IFP(\mathfrak{U})$$

Then $(\Lambda, \mathfrak{F})$ is said to be LOIFSS if $\mathfrak{f}_1 \leq_{\mathfrak{F}} \mathfrak{f}_2 \Rightarrow \Lambda(\mathfrak{f}_1) \subseteq \Lambda(\mathfrak{f}_2) \forall \mathfrak{f}_1, \mathfrak{f}_2 \in \mathfrak{F}$.

Definition 2.9 [27] Let $\mathfrak{g}_1, \mathfrak{g}_2, ..., \mathfrak{g}_k$ be k different attributes and $\mathfrak{G}_1, \mathfrak{G}_2, ..., \mathfrak{G}_k$ be their corresponding attribute values such that $\mathfrak{G}_\imath \cap \mathfrak{G}_J = \emptyset$ for $\imath \neq J$ and $\mathfrak{F}_\imath \subseteq \mathfrak{G}_\imath$ for $\imath = 1, 2, ..., k$ and $\mathfrak{T}_1 = \mathfrak{F}_1 \times \mathfrak{F}_2 \times ... \times \mathfrak{F}_k \subseteq \mathfrak{G}_1 \times \mathfrak{G}_2 \times ... \times \mathfrak{G}_k$. Then HSS over $\mathfrak{U}$ is the pair $(\Lambda, \mathfrak{T}_1)$ defined by the map

$$\Lambda : \mathfrak{T}_1 \rightarrow P(\mathfrak{U})$$

It can be written as $(\Lambda, \mathfrak{T}_1) = \{(\tau, \Lambda(\tau)) : \tau \in \mathfrak{T}_1, \Lambda(\tau) \in P(\mathfrak{U})\}$.

Definition 2.10 [27] Let $\mathfrak{g}_1, \mathfrak{g}_2, ..., \mathfrak{g}_k$ be k different attributes and $\mathfrak{G}_1, \mathfrak{G}_2, ..., \mathfrak{G}_k$ be their corresponding attribute values such that $\mathfrak{G}_\imath \cap \mathfrak{G}_J = \emptyset$ for $\imath \neq J$ and $\mathfrak{F}_\imath \subseteq \mathfrak{G}_\imath$ for $\imath = 1, 2, ..., k$ and $\mathfrak{T}_1 = \mathfrak{F}_1 \times \mathfrak{F}_2 \times ... \times \mathfrak{F}_k \subseteq \mathfrak{G}_1 \times \mathfrak{G}_2 \times ... \times \mathfrak{G}_k$. Then IFHSS over $\mathfrak{U}$ is the pair $(\Lambda, \mathfrak{T}_1)$ defined by the map

$$\Lambda : \mathfrak{T}_1 \rightarrow IFP(\mathfrak{U})$$

It can be written as $(\Lambda, \mathfrak{T}_1) = \{(\tau, \Lambda(\tau)) : \tau \in \mathfrak{T}_1, \Lambda(\tau) \in IFP(\mathfrak{U})\}$.

The list of most of the abbreviations and symbols used in this study is described in Table 1 and Table 2 respectively.

Table 1. List of symbols used in the study

Symbols	Representation
$\mathfrak{U}$	Universal set
$\mathfrak{G}$	Set of attributes
$\Psi(\mathfrak{u})$	Membership degree
$\Upsilon(\mathfrak{u})$	Non membership degree
$\rho(\mathfrak{u})$	Reference parameter corresponding to membership degree
$\eta(\mathfrak{u})$	Reference parameter corresponding to non membership degree
$\Psi(\mathfrak{u})e^{i2\pi(\vartheta_\Psi(\mathfrak{u}))}$	Complex valued membership degree
$\Upsilon(\mathfrak{u})e^{i2\pi(\vartheta_\Upsilon(\mathfrak{u}))}$	Complex valued non membership degree
$\rho(\mathfrak{u})e^{i2\pi(\vartheta_\rho(\mathfrak{u}))}$	Complex valued reference parameter corresponding to complex valued membership degree
$\eta(\mathfrak{u})e^{i2\pi(\vartheta_\eta(\mathfrak{u}))}$	Complex valued reference parameter corresponding to complex valued non membership degree

Table 2. List of abbreviation used in the study

Abbreviation	Description
FS	Fuzzy set
MADM	Multi-attributed decision making
MS	Membership
IFS	Intuitionistic fuzzy set
NMS	Non membership
PFS	Pythagorean fuzzy set
q-ROFS	q-Rung orthopair fuzzy set
RPs	Reference parameters
LDFS	Linear diophantine fuzzy set
q-RLDFS	q-Rung linear diophantine fuzzy set
CFS	Complex fuzzy set
CMS	Complex valued membership
CIFS	Complex intuitionistic fuzzy set
CNMS	Complex valued non membership
CPFS	Complex pythagorean fuzzy set
Cq-ROFS	Complex q-rung orthopair fuzzy set
CRPs	Complex valued reference parameters
CLDFS	Complex linear diophantine fuzzy set
NLDFS	Non-linear diophantine fuzzy set
Cq-RLDFS	Complex q-rung linear diophantine fuzzy set
CNLDFS	Complex non-linear diophantine fuzzy set
SS	Soft set
FSS	Fuzzy soft set
IFSS	Intuitionistic fuzzy soft set
PFSS	Pythagorean fuzzy soft set
q-ROFSS	q-Rung orthopair fuzzy soft set
LDFSS	Linear diophantine fuzzy soft set
CFSS	Complex fuzzy soft set
CIFSS	Complex intuitionistic fuzzy soft set
CPFSS	Complex pythagorean fuzzy soft set
Cq-ROFSS	Complex q-rung orthopair fuzzy soft set
CLDFSS	Complex linear diophantine fuzzy soft set
LOSS	Lattice ordered soft set
LOFSS	Lattice ordered fuzzy soft set
LOIFSS	Lattice ordered intuitionistic fuzzy soft set
HSS	Hypersoft set
FHSS	Fuzzy hypersoft set
IFHSS	Intuitionistic fuzzy hypersoft set
PFHSS	Pythagorean fuzzy hypersoft set
q-ROFHSS	q-Rung orthopair fuzzy hypersoft set
q-RLDFHSS	q-Rung linear diophantine fuzzy hypersoft set
CFHSS	Complex fuzzy hypersoft set
CIFHSS	Complex intuitionistic fuzzy hypersoft set
Cq-ROFHSS	Complex q-rung orthopair fuzzy hypersoft set
Cq-RLDFHSS	Complex q-rung linear diophantine fuzzy hypersoft set
LOCq-RLDFHSS	Lattice ordered complex q-rung linear diophantine fuzzy hypersoft set

3. Lattice ordered Cq-RLDFHSS

In this section, the conceptions of Cq-RLDFHSS and LOCq-RLDFHSS are introduced along with some fundamental algebraic operations of proposed LOCq-RLDFHSS.

Definition 3.1 Let $\mathfrak{g}_1, \mathfrak{g}_2, ..., \mathfrak{g}_k$ be k different attributes and $\mathfrak{G}_1, \mathfrak{G}_2, ..., \mathfrak{G}_k$ be their corresponding attribute values such that $\mathfrak{G}_\imath \cap \mathfrak{G}_J = \emptyset$ for $\imath \neq J$ and $\mathfrak{F}_\imath \subseteq \mathfrak{G}_\imath$ for $\imath = 1, 2, ..., k$ and $\mathfrak{T}_1 = \mathfrak{F}_1 \times \mathfrak{F}_2 \times ... \times \mathfrak{F}_k \subseteq \mathfrak{G}_1 \times \mathfrak{G}_2 \times ... \times \mathfrak{G}_k$. Then Cq-RLDFHSS over $\mathfrak{U}$ (Cq-RLDFHSS($\mathfrak{U}$)) is the pair $(\Lambda, \mathfrak{T}_1)$ defined by the map

$$\Lambda : \mathfrak{T}_1 \to Cq - RLDFP(\mathfrak{U})$$

Where, Cq-RLDFP($\mathfrak{U}$) is the collection of all Cq-RLDF subsets of $\mathfrak{U}$.

It can be written as $(\Lambda, \mathfrak{T}_1) = \{(\tau, \Lambda(\tau)) : \tau \in \mathfrak{T}_1, \Lambda(\tau) \in Cq - RLDFP(\mathfrak{U})\}$ and the Cq-RLDFHS Number (Cq-RLDFHSN)

$$\Lambda_{\mathfrak{u}_a}(\tau_c) = \{\langle(\Psi_{\Lambda(\tau_c)}(\mathfrak{u}_a), \vartheta_{\Psi_{\Lambda(\tau_c)}}(\mathfrak{u}_a)), (\Upsilon_{\Lambda(\tau_c)}(\mathfrak{u}_a), \vartheta_{\Upsilon_{\Lambda(\tau_c)}}(\mathfrak{u}_a))\rangle,$$

$$\langle(\rho_{\Lambda(\tau_c)}(\mathfrak{u}_a), \vartheta_{\rho_{\Lambda(\tau_c)}}(\mathfrak{u}_a)), (\eta_{\Lambda(\tau_c)}(\mathfrak{u}_a), \vartheta_{\eta_{\Lambda(\tau_c)}}(\mathfrak{u}_a))\rangle | \mathfrak{u}_a \in \mathfrak{U} \text{ and } \tau_c \in \mathfrak{T}_1\}.$$

can be express as

$$\beth_{\tau_{ac}} = \{\langle(\Psi_{\tau_{ac}}, \vartheta_{\Psi_{\tau_{ac}}}), (\Upsilon_{\tau_{ac}}, \vartheta_{\Upsilon_{\tau_{ac}}})\rangle, \langle(\rho_{\tau_{ac}}, \vartheta_{\rho_{\tau_{ac}}}), (\eta_{\tau_{ac}}, \vartheta_{\eta_{\tau_{ac}}})\rangle\}.$$

Definition 3.2 Let $(\Lambda_1, \mathfrak{T}_1), (\Lambda_2, \mathfrak{T}_2) \in$ Cq-RLDFHSS($\mathfrak{U}$), then $(\Lambda_1, \mathfrak{T}_1)$ is said to be Cq-RLDFHS subset of $(\Lambda_2, \mathfrak{T}_2)$, if

(i) $\mathfrak{T}_1 \subseteq \mathfrak{T}_2$

(ii) $\forall \tau \in \mathfrak{T}_1,\ \Lambda_1(\tau) \subseteq \Lambda_2(\tau)$

i.e) $\Psi_{\Lambda_1(\tau)}(\mathfrak{u}_a) \leq \Psi_{\Lambda_2(\tau)}(\mathfrak{u}_a),\ \vartheta_{\Psi_{\Lambda_1(\tau)}}(\mathfrak{u}_a) \leq \vartheta_{\Psi_{\Lambda_2(\tau)}}(\mathfrak{u}_a),\ \Upsilon_{\Lambda_2(\tau)}(\mathfrak{u}_a) \leq \Upsilon_{\Lambda_1(\tau)}(\mathfrak{u}_a),$
$\vartheta_{\Upsilon_{\Lambda_2(\tau)}}(\mathfrak{u}_a) \leq \vartheta_{\Upsilon_{\Lambda_1(\tau)}}(\mathfrak{u}_a), \rho_{\Lambda_1(\tau)}(\mathfrak{u}_a) \leq \rho_{\Lambda_2(\tau)}(\mathfrak{u}_a),\ \vartheta_{\rho_{\Lambda_1(\tau)}}(\mathfrak{u}_a) \leq \vartheta_{\rho_{\Lambda_2(\tau)}}(\mathfrak{u}_a),$
$\eta_{\Lambda_2(\tau)}(\mathfrak{u}_a) \leq \eta_{\Lambda_1(\tau)}(\mathfrak{u}_a)$ and $\vartheta_{\eta_{\Lambda_2(\tau)}}(\mathfrak{u}_a) \leq \vartheta_{\eta_{\Lambda_1(\tau)}}(\mathfrak{u}_a) \forall \mathfrak{u}_a \in \mathfrak{U}$.

Definition 3.3 Let $\beth_{\tau_{ac}}$=$\{\langle(\Psi_{\tau_{ac}}, \vartheta_{\Psi_{\tau_{ac}}}), (\Upsilon_{\tau_{ac}}, \vartheta_{\Upsilon_{\tau_{ac}}})\rangle, \langle(\rho_{\tau_{ac}}, \vartheta_{\rho_{\tau_{ac}}}), (\eta_{\tau_{ac}}, \vartheta_{\eta_{\tau_{ac}}})\rangle\}$ be a Cq-RLDFHSN, then score function (SF) on $\beth_{\tau_{ac}}$ is defined by the mapping $\mathfrak{s} : Cq - RLDFHSN(\mathfrak{U}) \to$ [-1, 1] and given by

$$\mathfrak{s}(\beth_{\tau_{ac}}) = \frac{1}{4}\left[(\Psi_{\tau_{ac}} - \Upsilon_{\tau_{ac}}) + (\vartheta_{\Psi_{\tau_{ac}}} - \vartheta_{\Upsilon_{\tau_{ac}}}) + (\rho^q_{\tau_{ac}} - \eta^q_{\tau_{ac}}) + (\vartheta^q_{\rho_{\tau_{ac}}} - \vartheta^q_{\eta_{\tau_{ac}}})\right]; \quad q \geq 1$$

where, Cq-RLDFHSN($\mathfrak{U}$) is collection of Cq-RLDFHSNs on $\mathfrak{U}$.

Definition 3.4 Let $\beth_{\tau_{ac}}$=$\{\langle(\Psi_{\tau_{ac}}, \vartheta_{\Psi_{\tau_{ac}}}), (\Upsilon_{\tau_{ac}}, \vartheta_{\Upsilon_{\tau_{ac}}})\rangle, \langle(\rho_{\tau_{ac}}, \vartheta_{\rho_{\tau_{ac}}}), (\eta_{\tau_{ac}}, \vartheta_{\eta_{\tau_{ac}}})\rangle\}$ be a Cq-RLDFHSN, then accuracy function(AF) on $\beth_{\tau_{ac}}$ is defined by the mapping $\kappa : Cq - RLDFHSN(\mathfrak{U}) \to$ [0, 1] and given as

$$\kappa(\beth_{\tau_{ac}}) = \frac{1}{4}\left[\left(\frac{(\Psi_{\tau_{ac}} + \Upsilon_{\tau_{ac}})}{2}\right) + \left(\frac{(\vartheta_{\Psi_{\tau_{ac}}} + \vartheta_{\Upsilon_{\tau_{ac}}})}{2}\right) + \left(\rho^q_{\tau_{ac}} + \eta^q_{\tau_{ac}}\right) + \left(\vartheta^q_{\rho_{\tau_{ac}}} + \vartheta^q_{\eta_{\tau_{ac}}}\right)\right]; \quad q \geq 1$$

Definition 3.5 Let

$$\beth_{\tau_{11}} = \{\langle(\Psi_{\tau_{11}}, \vartheta_{\Psi_{\tau_{11}}}), (\Upsilon_{\tau_{11}}, \vartheta_{\Upsilon_{\tau_{11}}})\rangle, \langle(\rho_{\tau_{11}}, \vartheta_{\rho_{\tau_{11}}}), (\eta_{\tau_{11}}, \vartheta_{\eta_{\tau_{11}}})\rangle\}$$

and

$$\beth_{\tau_{12}} = \{\langle(\Psi_{\tau_{12}}, \vartheta_{\Psi_{\tau_{12}}}), (\Upsilon_{\tau_{12}}, \vartheta_{\Upsilon_{\tau_{12}}})\rangle, \langle(\rho_{\tau_{12}}, \vartheta_{\rho_{\tau_{12}}}), (\eta_{\tau_{12}}, \vartheta_{\eta_{\tau_{12}}})\rangle\}$$

be two Cq-RLDFHSNs then using SF and AF we can compare these Cq-RLDFHSNs as

(i) If $\mathfrak{s}(\beth_{\tau_{11}}) > \mathfrak{s}(\beth_{\tau_{12}})$ then $\beth_{\tau_{11}} > \beth_{\tau_{12}}$

(ii) If $\mathfrak{s}(\beth_{\tau_{11}}) = \mathfrak{s}(\beth_{\tau_{12}})$ then we use AF

- If $\kappa(\beth_{\tau_{11}}) > \kappa(\beth_{\tau_{12}})$ then $\beth_{\tau_{11}} > \beth_{\tau_{12}}$
- If $\kappa(\beth_{\tau_{11}}) = \kappa(\beth_{\tau_{12}})$ then $\beth_{\tau_{11}} = \beth_{\tau_{12}}$

Definition 3.6 A Cq-RLDFHSS($\mathfrak{U}$) (Λ, $\mathfrak{F}_1 \times \mathfrak{F}_2 \times ... \times \mathfrak{F}_k = \mathfrak{T}_1$) is said to be lattice ordered Cq-RLDFHSS over $\mathfrak{U}$ (LOCq-RLDFHSS($\mathfrak{U}$)) if for mapping $\Lambda : \mathfrak{T}_1 \to Cq-RLDFP(\mathfrak{U})$,

$$\tau_1 \leq_{\mathfrak{T}_1} \tau_2 \Rightarrow \Lambda(\tau_1) \subseteq \Lambda(\tau_2) \ \forall \ \tau_1, \tau_2 \in \mathfrak{T}_1$$

i.e) $\tau_1 \leq_{\mathfrak{T}_1} \tau_2$
$\Rightarrow \Psi_{\Lambda(\tau_1)}(\mathfrak{u}_a) \leq \Psi_{\Lambda(\tau_2)}(\mathfrak{u}_a)$, $\vartheta_{\Psi_{\Lambda(\tau_1)}}(\mathfrak{u}_a) \leq \vartheta_{\Psi_{\Lambda(\tau_2)}}(\mathfrak{u}_a)$, $\Upsilon_{\Lambda(\tau_2)}(\mathfrak{u}_a) \leq \Upsilon_{\Lambda(\tau_1)}(\mathfrak{u}_a)$,
$\vartheta_{\Upsilon_{\Lambda(\tau_2)}}(\mathfrak{u}_a) \leq \vartheta_{\Upsilon_{\Lambda(\tau_1)}}(\mathfrak{u}_a)$, $\rho_{\Lambda(\tau_1)}(\mathfrak{u}_a) \leq \rho_{\Lambda(\tau_2)}(\mathfrak{u}_a)$, $\vartheta_{\rho_{\Lambda(\tau_1)}}(\mathfrak{u}_a) \leq \vartheta_{\rho_{\Lambda(\tau_2)}}(\mathfrak{u}_a)$,
$\eta_{\Lambda(\tau_2)}(\mathfrak{u}_a) \leq \eta_{\Lambda(\tau_1)}(\mathfrak{u}_a)$ and $\vartheta_{\eta_{\Lambda(\tau_2)}}(\mathfrak{u}_a) \leq \vartheta_{\eta_{\Lambda(\tau_1)}}(\mathfrak{u}_a) \forall \ \mathfrak{u}_a \in \mathfrak{U}$
where, $\tau_1 = (\tau_{1_1}, \tau_{1_2}, ..., \tau_{1_k})$, $\tau_2 = (\tau_{2_1}, \tau_{2_2}, ..., \tau_{2_k})$ and $\tau_{1_\iota}, \tau_{2_\iota} \in \mathfrak{F}_\iota$ for $\iota \in \{1, 2, ..., k\}$.

Also, each $\mathfrak{F}_\iota$ is defined by its corresponding binary relation $\leq_{\mathfrak{F}_\iota}$ and $\mathfrak{T}_1$ forms a relation defined by

$$(\tau_{1_1}, \tau_{1_2}, ..., \tau_{1_k}) \leq_{\mathfrak{T}_1} (\tau_{2_1}, \tau_{2_2}, ..., \tau_{2_k}) \Leftrightarrow \tau_{1_\iota} \leq_{\mathfrak{F}_\iota} \tau_{2_\iota}$$

in $\mathfrak{F}_\iota$ for $\iota \in \{1, 2, ..., k\}$.

Example 1 Let $\mathfrak{U} = \{\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3\}$ be the set of Tractors with their respective manufacturing date, consider the attributes $\mathfrak{g}_1 = \{\text{cost}\}$, $\mathfrak{g}_2 = \{\text{transmission systems}\}$, $\mathfrak{g}_3 = \{\text{Comfort}\}$ and $\mathfrak{G}_1 = \{\text{operational cost } (\mathfrak{g}_{11}), \text{ purchasing cost } (\mathfrak{g}_{12})\}$, $\mathfrak{G}_2 = \{\text{constant mesh } (\mathfrak{g}_{21}), \text{ partial synchromesh } (\mathfrak{g}_{22})\}$, $\mathfrak{G}_3 = \{\text{Driver's comfort } (\mathfrak{g}_{31})\}$ be their corresponding attribute values respectively.

Suppose that,

For each $\iota = 1, 2, 3$, $\mathfrak{F}_\iota = \mathfrak{G}_\iota$

The elements in each set $\mathfrak{F}_1$, $\mathfrak{F}_2$ and $\mathfrak{F}_3$ has an order among them, they are

The elements in set $\mathfrak{F}_1$ are in the order $\mathfrak{g}_{11} \leq_{\mathfrak{F}_1} \mathfrak{g}_{12}$

The elements in set $\mathfrak{F}_2$ are in the order $\mathfrak{g}_{21} \leq_{\mathfrak{F}_2} \mathfrak{g}_{22}$, $\mathfrak{F}_3$ has only one element $\mathfrak{g}_{31}$ and

$$\mathfrak{T} = \mathfrak{F}_1 \times \mathfrak{F}_2 \times \mathfrak{F}_3$$

$$= \{\tau_1 = (\mathfrak{g}_{11}, \mathfrak{g}_{21}, \mathfrak{g}_{31}), \tau_2 = (\mathfrak{g}_{11}, \mathfrak{g}_{22}, \mathfrak{g}_{31}), \tau_3 = (\mathfrak{g}_{12}, \mathfrak{g}_{21}, \mathfrak{g}_{31}), \tau_4 = (\mathfrak{g}_{12}, \mathfrak{g}_{22}, \mathfrak{g}_{31})\}$$

Then the order of elements in set $\mathfrak{T}$ is shown in Figure 1.

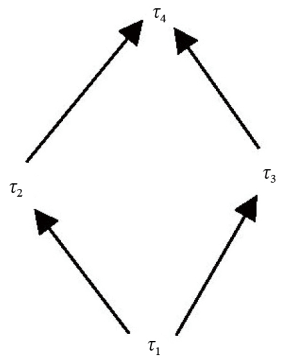

Figure 1. The order among elements in $\mathfrak{T}$

Further, the following is how the attributes are categorized

- The attribute "cost" and its attribute values manifest that the tractor is inexpensive or expensive.
- The attribute "transmission systems" and its attribute values manifest that the tractor is good or not good.
- The attribute "comfort" and its attribute values manifest that the tractor is satisfies or dissatisfies.

Then the cartesian product of attribute values manifest that the tractor is (inexpensive, good, satisfies) all together or (expensive, not good, dissatisfies) all together.

Then, Cq-RLDFHSS $(\Lambda, \mathfrak{T})$ may be expressed as

$$(\Lambda, \mathfrak{T}) = \Bigg\{ \Bigg\langle \tau_1, \left(\frac{\mathfrak{u}_1}{\langle (0.4e^{i2\pi(0.6)}, 0.9e^{i2\pi(0.8)}), (0.3e^{i2\pi(0.2)}, 0.8e^{i2\pi(0.9)}) \rangle}, \frac{\mathfrak{u}_2}{\langle (0.1e^{i2\pi(0.1)}, 0.8e^{i2\pi(0.8)}), (0.1e^{i2\pi(0.3)}, 0.9e^{i2\pi(0.7)}) \rangle}, \frac{\mathfrak{u}_3}{\langle (0.2e^{i2\pi(0.1)}, 0.9e^{i2\pi(0.9)}), (0.3e^{i2\pi(0.4)}, 0.7e^{i2\pi(0.7)}) \rangle} \right) \Bigg\rangle,$$

$$\Bigg\langle \tau_2, \left(\frac{\mathfrak{u}_1}{\langle (0.4e^{i2\pi(0.7)}, 0.8e^{i2\pi(0.8)}), (0.4e^{i2\pi(0.4)}, 0.7e^{i2\pi(0.8)}) \rangle}, \frac{\mathfrak{u}_2}{\langle (0.3e^{i2\pi(0.4)}, 0.5e^{i2\pi(0.6)}), (0.5e^{i2\pi(0.4)}, 0.7e^{i2\pi(0.6)}) \rangle}, \frac{\mathfrak{u}_3}{\langle (0.3e^{i2\pi(0.3)}, 0.7e^{i2\pi(0.7)}), (0.4e^{i2\pi(0.5)}, 0.5e^{i2\pi(0.6)}) \rangle} \right) \Bigg\rangle,$$

$$\Bigg\langle \tau_3, \left(\frac{\mathfrak{u}_1}{\langle (0.6e^{i2\pi(0.8)}, 0.6e^{i2\pi(0.6)}), (0.5e^{i2\pi(0.4)}, 0.6e^{i2\pi(0.7)}) \rangle}, \frac{\mathfrak{u}_2}{\langle (0.5e^{i2\pi(0.4)}, 0.6e^{i2\pi(0.5)}), (0.5e^{i2\pi(0.5)}, 0.8e^{i2\pi(0.7)}) \rangle}, \frac{\mathfrak{u}_3}{\langle (0.3e^{i2\pi(0.2)}, 0.8e^{i2\pi(0.6)}), (0.4e^{i2\pi(0.6)}, 0.6e^{i2\pi(0.7)}) \rangle} \right) \Bigg\rangle,$$

$$\Bigg\langle \tau_4, \left(\frac{\mathfrak{u}_1}{\langle (0.7e^{i2\pi(0.8)}, 0.6e^{i2\pi(0.6)}), (0.7e^{i2\pi(0.6)}, 0.5e^{i2\pi(0.5)}) \rangle}, \frac{\mathfrak{u}_2}{\langle (0.7e^{i2\pi(0.6)}, 0.4e^{i2\pi(0.2)}), (0.8e^{i2\pi(0.7)}, 0.6e^{i2\pi(0.3)}) \rangle}, \frac{\mathfrak{u}_3}{\langle (0.5e^{i2\pi(0.4)}, 0.5e^{i2\pi(0.5)}), (0.6e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.4)}) \rangle} \right) \Bigg\rangle \Bigg\}.$$

We will assume that $q = 3$.

The characteristics of this Cq-RLDFHSS $(\Lambda, \mathfrak{T})$ is $(\langle$CMS degree, CNMS degree$\rangle$, $\langle$(inexpensive, good, satisfies), (expensive, not good, dissatisfies)$\rangle)$ $\forall \tau_c \in \mathfrak{T}$.

Clearly $\Lambda(\tau_1) \subseteq \Lambda(\tau_2) \subseteq \Lambda(\tau_4)$ and $\Lambda(\tau_1) \subseteq \Lambda(\tau_3) \subseteq \Lambda(\tau_4)$, so $(\Lambda, \mathfrak{T})$ is a LOCq-RLDFHSS($\mathfrak{U}$).

In this LOCq-RLDFHSS, the tractor $\mathfrak{u}_1$ and the parameter $\tau_1 =$ (operational cost, constant mesh, driver's comfort) have the complex numeric value $\langle (0.4e^{i2\pi(0.6)}, 0.9e^{i2\pi(0.8)}), (0.3e^{i2\pi(0.2)}, 0.8e^{i2\pi(0.9)}) \rangle$. This value delineates that for the parameter τ_1 the tractor $\mathfrak{u}_1$ has a 40% truth value, 90% false value and it has a 60% truth value and 80% false value according to its respective manufacturing date. The pair $(0.3e^{i2\pi(0.2)}, 0.8e^{i2\pi(0.9)})$ delineate the reference parameter of the truth and false values respectively, where we can observe that for (inexpensive in operational cost, good in constant mesh, satisfies in driver's comfort) all together the tractor $\mathfrak{u}_1$ enunciate 30% and it enunciate 20% according to its respective manufacturing date and for (expensive in operational cost, not good in constant mesh, dissatisfies in driver's comfort) all together the tractor $\mathfrak{u}_1$ enunciate 80% and it enunciate 90% according to its respective manufacturing date.

Definition 3.7 Let $(\Lambda_1, \mathfrak{T}_1)$, $(\Lambda_2, \mathfrak{T}_2) \in$ LOCq-RLDFHSS$(\mathfrak{U})$. Their Restricted union is defined by $(\Lambda_1, \mathfrak{T}_1) \cup_{RES} (\Lambda_2, \mathfrak{T}_2) = (\Lambda_3, \mathfrak{T}_3)$ where $\mathfrak{T}_3 = \mathfrak{T}_1 \cap \mathfrak{T}_2$ and $\forall\ \tau \in \mathfrak{T}_3$, $\mathfrak{u} \in \mathfrak{U}$ we have $\Lambda_1(\tau) \cup \Lambda_2(\tau) = \Lambda_3(\tau)$.

$$\Psi_{\Lambda_3(\tau)}(\mathfrak{u}) = \mathrm{Max}\{\Psi_{\Lambda_1(\tau)}(\mathfrak{u}), \Psi_{\Lambda_2(\tau)}(\mathfrak{u})\},$$

$$\vartheta_{\Psi_{\Lambda_3(\tau)}}(\mathfrak{u}) = \mathrm{Max}\{\vartheta_{\Psi_{\Lambda_1(\tau)}}(\mathfrak{u}), \vartheta_{\Psi_{\Lambda_2(\tau)}}(\mathfrak{u})\},$$

$$\Upsilon_{\Lambda_3(\tau)}(\mathfrak{u}) = \mathrm{Min}\{\Upsilon_{\Lambda_1(\tau)}(\mathfrak{u}), \Upsilon_{\Lambda_2(\tau)}(\mathfrak{u})\},$$

$$\vartheta_{\Upsilon_{\Lambda_3(\tau)}}(\mathfrak{u}) = \mathrm{Min}\{\vartheta_{\Upsilon_{\Lambda_1(\tau)}}(\mathfrak{u}), \vartheta_{\Upsilon_{\Lambda_2(\tau)}}(\mathfrak{u})\},$$

$$\rho_{\Lambda_3(\tau)}(\mathfrak{u}) = \mathrm{Max}\{\rho_{\Lambda_1(\tau)}(\mathfrak{u}), \rho_{\Lambda_2(\tau)}(\mathfrak{u})\},$$

$$\vartheta_{\rho_{\Lambda_3(\tau)}}(\mathfrak{u}) = \mathrm{Max}\{\vartheta_{\rho_{\Lambda_1(\tau)}}(\mathfrak{u}), \vartheta_{\rho_{\Lambda_2(\tau)}}(\mathfrak{u})\},$$

$$\eta_{\Lambda_3(\tau)}(\mathfrak{u}) = \mathrm{Min}\{\eta_{\Lambda_1(\tau)}(\mathfrak{u}), \eta_{\Lambda_2(\tau)}(\mathfrak{u})\}$$

and

$$\vartheta_{\eta_{\Lambda_3(\tau)}}(\mathfrak{u}) = \mathrm{Min}\{\vartheta_{\eta_{\Lambda_1(\tau)}}(\mathfrak{u}), \vartheta_{\eta_{\Lambda_2(\tau)}}(\mathfrak{u})\}.$$

Proposition 3.1 Let $(\Lambda_1, \mathfrak{T}_1)$, $(\Lambda_2, \mathfrak{T}_2) \in$ LOCq-RLDFHSS$(\mathfrak{U})$. Then $(\Lambda_1, \mathfrak{T}_1) \cup_{RES} (\Lambda_2, \mathfrak{T}_2) \in$ LOCq-RLDFHSS$(\mathfrak{U})$.

Proof. Let $(\Lambda_1, \mathfrak{T}_1)$, $(\Lambda_2, \mathfrak{T}_2) \in$ LOCq-RLDFHSS$(\mathfrak{U})$. Then by Definition 3.7 $\Lambda_1(\tau) \cup \Lambda_2(\tau) = \Lambda_3(\tau)$, where $\tau \in \mathfrak{T}_3 = \mathfrak{T}_1 \cap \mathfrak{T}_2$.

If $\mathfrak{T}_1 \cap \mathfrak{T}_2 = \emptyset$, then result is trivial.

Now for $\mathfrak{T}_1 \cap \mathfrak{T}_2 \neq \emptyset$, since $\mathfrak{T}_1, \mathfrak{T}_2 \subseteq \mathfrak{G}_1 \times \mathfrak{G}_2 \times \ldots \times \mathfrak{G}_k$.

Therefore for any $\tau_c \leq_{\mathfrak{T}_1} \tau_d$ we have $\Lambda_1(\tau_c) \subseteq \Lambda_1(\tau_d), \forall \tau_c, \tau_d \in \mathfrak{T}_1$ and for any $\sigma_c \leq_{\mathfrak{T}_2} \sigma_d$ we have $\Lambda_2(\sigma_c) \subseteq \Lambda_2(\sigma_d), \forall \sigma_c, \sigma_d \in \mathfrak{T}_2$.

Now for any $\delta_c, \delta_d \in \mathfrak{T}_3$ and $\delta_c \leq_{\mathfrak{T}_3} \delta_d$

$$\Rightarrow \delta_c, \delta_d \in \mathfrak{T}_1 \cap \mathfrak{T}_2$$

$$\Rightarrow \delta_c, \delta_d \in \mathfrak{T}_1 \text{ and } \delta_c, \delta_d \in \mathfrak{T}_2$$

$$\Rightarrow \Lambda_1(\delta_c) \subseteq \Lambda_1(\delta_d) \text{ and } \Lambda_2(\delta_c) \subseteq \Lambda_2(\delta_d) \text{ whenever } \delta_c \leq_{\mathfrak{T}_1} \delta_d,\, \delta_c \leq_{\mathfrak{T}_2} \delta_d$$

$$\Rightarrow \Psi_{\Lambda_1(\delta_c)}(\mathfrak{u}) \leq \Psi_{\Lambda_1(\delta_d)}(\mathfrak{u}),\ \Psi_{\Lambda_2(\delta_c)}(\mathfrak{u}) \leq \Psi_{\Lambda_2(\delta_d)}(\mathfrak{u})$$

$$\vartheta_{\Psi_{\Lambda_1(\delta_c)}}(\mathfrak{u}) \leq \vartheta_{\Psi_{\Lambda_1(\delta_d)}}(\mathfrak{u}),\ \vartheta_{\Psi_{\Lambda_2(\delta_c)}}(\mathfrak{u}) \leq \vartheta_{\Psi_{\Lambda_2(\delta_d)}}(\mathfrak{u})$$

$$\Upsilon_{\Lambda_1(\delta_d)}(\mathfrak{u}) \leq \Upsilon_{\Lambda_1(\delta_c)}(\mathfrak{u}),\ \Upsilon_{\Lambda_2(\delta_d)}(\mathfrak{u}) \leq \Upsilon_{\Lambda_2(\delta_c)}(\mathfrak{u})$$

$$\vartheta_{\Upsilon_{\Lambda_1(\delta_d)}}(\mathfrak{u}) \leq \vartheta_{\Upsilon_{\Lambda_1(\delta_c)}}(\mathfrak{u}),\ \vartheta_{\Upsilon_{\Lambda_2(\delta_d)}}(\mathfrak{u}) \leq \vartheta_{\Upsilon_{\Lambda_2(\delta_c)}}(\mathfrak{u})$$

$$\rho_{\Lambda_1(\delta_c)}(\mathfrak{u}) \leq \rho_{\Lambda_1(\delta_d)}(\mathfrak{u}),\ \rho_{\Lambda_2(\delta_c)}(\mathfrak{u}) \leq \rho_{\Lambda_2(\delta_d)}(\mathfrak{u})$$

$$\vartheta_{\rho_{\Lambda_1(\delta_c)}}(\mathfrak{u}) \leq \vartheta_{\rho_{\Lambda_1(\delta_d)}}(\mathfrak{u}),\ \vartheta_{\rho_{\Lambda_2(\delta_c)}}(\mathfrak{u}) \leq \vartheta_{\rho_{\Lambda_2(\delta_d)}}(\mathfrak{u})$$

$$\eta_{\Lambda_1(\delta_d)}(\mathfrak{u}) \leq \eta_{\Lambda_1(\delta_c)}(\mathfrak{u}),\ \eta_{\Lambda_2(\delta_d)}(\mathfrak{u}) \leq \eta_{\Lambda_2(\delta_c)}(\mathfrak{u})$$

$$\vartheta_{\eta_{\Lambda_1(\delta_d)}}(\mathfrak{u}) \leq \vartheta_{\eta_{\Lambda_1(\delta_c)}}(\mathfrak{u}),\ \vartheta_{\eta_{\Lambda_2(\delta_d)}}(\mathfrak{u}) \leq \vartheta_{\eta_{\Lambda_2(\delta_c)}}(\mathfrak{u})$$

$$\Rightarrow \mathrm{Max}\{\Psi_{\Lambda_1(\delta_c)}(\mathfrak{u}),\ \Psi_{\Lambda_2(\delta_c)}(\mathfrak{u})\} \leq \mathrm{Max}\{\Psi_{\Lambda_1(\delta_d)}(\mathfrak{u}),\ \Psi_{\Lambda_2(\delta_d)}(\mathfrak{u})\}$$

$$\mathrm{Max}\{\vartheta_{\Psi_{\Lambda_1(\delta_c)}}(\mathfrak{u}),\ \vartheta_{\Psi_{\Lambda_2(\delta_c)}}(\mathfrak{u})\} \leq \mathrm{Max}\{\vartheta_{\Psi_{\Lambda_1(\delta_d)}}(\mathfrak{u}),\ \vartheta_{\Psi_{\Lambda_2(\delta_d)}}(\mathfrak{u})\}$$

$$\mathrm{Min}\{\Upsilon_{\Lambda_1(\delta_d)}(\mathfrak{u}),\ \Upsilon_{\Lambda_2(\delta_d)}(\mathfrak{u})\} \leq \mathrm{Min}\{\Upsilon_{\Lambda_1(\delta_c)}(\mathfrak{u}),\ \Upsilon_{\Lambda_2(\delta_c)}(\mathfrak{u})\}$$

$$\mathrm{Min}\{\vartheta_{\Upsilon_{\Lambda_1(\delta_d)}}(\mathfrak{u}),\ \vartheta_{\Upsilon_{\Lambda_2(\delta_d)}}(\mathfrak{u})\} \leq \mathrm{Min}\{\vartheta_{\Upsilon_{\Lambda_1(\delta_c)}}(\mathfrak{u}),\ \vartheta_{\Upsilon_{\Lambda_2(\delta_c)}}(\mathfrak{u})\}$$

$$\mathrm{Max}\{\rho_{\Lambda_1(\delta_c)}(\mathfrak{u}),\ \rho_{\Lambda_2(\delta_c)}(\mathfrak{u})\} \leq \mathrm{Max}\{\rho_{\Lambda_1(\delta_d)}(\mathfrak{u}),\ \rho_{\Lambda_2(\delta_d)}(\mathfrak{u})\}$$

$$\mathrm{Max}\{\vartheta_{\rho_{\Lambda_1(\delta_c)}}(\mathfrak{u}),\ \vartheta_{\rho_{\Lambda_2(\delta_c)}}(\mathfrak{u})\} \leq \mathrm{Max}\{\vartheta_{\rho_{\Lambda_1(\delta_d)}}(\mathfrak{u}),\ \vartheta_{\rho_{\Lambda_2(\delta_d)}}(\mathfrak{u})\}$$

$$\mathrm{Min}\{\eta_{\Lambda_1(\delta_d)}(\mathfrak{u}),\ \eta_{\Lambda_2(\delta_d)}(\mathfrak{u})\} \leq \mathrm{Min}\{\eta_{\Lambda_1(\delta_c)}(\mathfrak{u}),\ \eta_{\Lambda_2(\delta_c)}(\mathfrak{u})\}$$

$$\mathrm{Min}\{\vartheta_{\eta_{\Lambda_1(\delta_d)}}(\mathfrak{u}),\ \vartheta_{\eta_{\Lambda_2(\delta_d)}}(\mathfrak{u})\} \leq \mathrm{Min}\{\vartheta_{\eta_{\Lambda_1(\delta_c)}}(\mathfrak{u}),\ \vartheta_{\eta_{\Lambda_2(\delta_c)}}(\mathfrak{u})\}$$

$$\Rightarrow \Psi_{\Lambda_1(\delta_c)\cup\Lambda_2(\delta_c)}(\mathfrak{u}) \leq \Psi_{\Lambda_1(\delta_d)\cup\Lambda_2(\delta_d)}(\mathfrak{u})$$

$$\vartheta_{\Psi_{\Lambda_1(\delta_c)\cup\Lambda_2(\delta_c)}}(\mathfrak{u}) \le \vartheta_{\Psi_{\Lambda_1(\delta_d)\cup\Lambda_2(\delta_d)}}(\mathfrak{u})$$

$$\Upsilon_{\Lambda_1(\delta_d)\cup\Lambda_2(\delta_d)}(\mathfrak{u}) \le \Upsilon_{\Lambda_1(\delta_c)\cup\Lambda_2(\delta_c)}(\mathfrak{u})$$

$$\vartheta_{\Upsilon_{\Lambda_1(\delta_d)\cup\Lambda_2(\delta_d)}}(\mathfrak{u}) \le \vartheta_{\Upsilon_{\Lambda_1(\delta_c)\cup\Lambda_2(\delta_c)}}(\mathfrak{u})$$

$$\rho_{\Lambda_1(\delta_c)\cup\Lambda_2(\delta_c)}(\mathfrak{u}) \le \rho_{\Lambda_1(\delta_d)\cup\Lambda_2(\delta_d)}(\mathfrak{u})$$

$$\vartheta_{\rho_{\Lambda_1(\delta_c)\cup\Lambda_2(\delta_c)}}(\mathfrak{u}) \le \vartheta_{\rho_{\Lambda_1(\delta_d)\cup\Lambda_2(\delta_d)}}(\mathfrak{u}$$

$$\eta_{\Lambda_1(\delta_d)\cup\Lambda_2(\delta_d)}(\mathfrak{u}) \le \eta_{\Lambda_1(\delta_c)\cup\Lambda_2(\delta_c)}(\mathfrak{u})$$

$$\vartheta_{\eta_{\Lambda_1(\delta_d)\cup\Lambda_2(\delta_d)}}(\mathfrak{u}) \le \vartheta_{\eta_{\Lambda_1(\delta_c)\cup\Lambda_2(\delta_c)}}(\mathfrak{u})$$

$$\Rightarrow \Psi_{\Lambda_3(\delta_c)}(\mathfrak{u}) \le \Psi_{\Lambda_3(\delta_d)}(\mathfrak{u})$$

$$\vartheta_{\Psi_{\Lambda_3(\delta_c)}}(\mathfrak{u}) \le \vartheta_{\Psi_{\Lambda_3(\delta_d)}}(\mathfrak{u})$$

$$\Upsilon_{\Lambda_3(\delta_d)}(\mathfrak{u}) \le \Upsilon_{\Lambda_3(\delta_c)}(\mathfrak{u})$$

$$\vartheta_{\Upsilon_{\Lambda_3(\delta_d)}}(\mathfrak{u}) \le \vartheta_{\Upsilon_{\Lambda_3(\delta_c)}}(\mathfrak{u})$$

$$\rho_{\Lambda_3(\delta_c)}(\mathfrak{u}) \le \rho_{\Lambda_3(\delta_d)}(\mathfrak{u})$$

$$\vartheta_{\rho_{\Lambda_3(\delta_c)}}(\mathfrak{u}) \le \vartheta_{\rho_{\Lambda_3(\delta_d)}}(\mathfrak{u})$$

$$\eta_{\Lambda_3(\delta_d)}(\mathfrak{u}) \le \eta_{\Lambda_3(\delta_c)}(\mathfrak{u})$$

$$\vartheta_{\eta_{\Lambda_3(\delta_d)}}(\mathfrak{u}) \le \vartheta_{\eta_{\Lambda_3(\delta_c)}}(\mathfrak{u})$$

$$\Rightarrow \Lambda_3(\delta_c) \subseteq \Lambda_3(\delta_d) \text{ for } \delta_c \le_{\mathfrak{T}_3} \delta_d$$

$$\Rightarrow (\Lambda_1, \mathfrak{T}_1) \cup_{RES} (\Lambda_2, \mathfrak{T}_2) \in \text{LOCq-RLDFHSS}(\mathfrak{U})$$

□

Definition 3.8 Let $(\Lambda_1, \mathfrak{T}_1)$, $(\Lambda_2, \mathfrak{T}_2) \in$ LOCq-RLDFHSS$(\mathfrak{U})$. Their Restricted intersection is defined by $(\Lambda_1, \mathfrak{T}_1) \cap_{RES} (\Lambda_2, \mathfrak{T}_2) = (\Lambda_3, \mathfrak{T}_3)$ where $\mathfrak{T}_3 = \mathfrak{T}_1 \cap \mathfrak{T}_2$ and $\forall\ \tau \in \mathfrak{T}_3$, $\mathfrak{u} \in \mathfrak{U}$ we have $\Lambda_1(\tau) \cap \Lambda_2(\tau) = \Lambda_3(\tau)$.

$$\Psi_{\Lambda_3(\tau)}(\mathfrak{u}) = \mathrm{Min}\{\Psi_{\Lambda_1(\tau)}(\mathfrak{u}),\ \Psi_{\Lambda_2(\tau)}(\mathfrak{u})\},$$

$$\vartheta_{\Psi_{\Lambda_3(\tau)}}(\mathfrak{u}) = \mathrm{Min}\{\vartheta_{\Psi_{\Lambda_1(\tau)}}(\mathfrak{u}),\ \vartheta_{\Psi_{\Lambda_2(\tau)}}(\mathfrak{u})\}$$

$$\Upsilon_{\Lambda_3(\tau)}(\mathfrak{u}) = \mathrm{Max}\{\Upsilon_{\Lambda_1(\tau)}(\mathfrak{u}),\ \Upsilon_{\Lambda_2(\tau)}(\mathfrak{u})\},$$

$$\vartheta_{\Upsilon_{\Lambda_3(\tau)}}(\mathfrak{u}) = \mathrm{Max}\{\vartheta_{\Upsilon_{\Lambda_1(\tau)}}(\mathfrak{u}),\ \vartheta_{\Upsilon_{\Lambda_2(\tau)}}(\mathfrak{u})\},$$

$$\rho_{\Lambda_3(\tau)}(\mathfrak{u}) = \mathrm{Min}\{\rho_{\Lambda_1(\tau)}(\mathfrak{u}),\ \rho_{\Lambda_2(\tau)}(\mathfrak{u})\},$$

$$\vartheta_{\rho_{\Lambda_3(\tau)}}(\mathfrak{u}) = \mathrm{Min}\{\vartheta_{\rho_{\Lambda_1(\tau)}}(\mathfrak{u}),\ \vartheta_{\rho_{\Lambda_2(\tau)}}(\mathfrak{u})\},$$

$$\eta_{\Lambda_3(\tau)}(\mathfrak{u}) = \mathrm{Max}\{\eta_{\Lambda_1(\tau)}(\mathfrak{u}), \eta_{\Lambda_2(\tau)}(\mathfrak{u})\} \text{ and}$$

$$\vartheta_{\eta_{\Lambda_3(\tau)}}(\mathfrak{u}) = \mathrm{Max}\{\vartheta_{\eta_{\Lambda_1(\tau)}}(\mathfrak{u}),\ \vartheta_{\eta_{\Lambda_2(\tau)}}(\mathfrak{u})\}.$$

Proposition 3.2 Let $(\Lambda_1, \mathfrak{T}_1)$, $(\Lambda_2, \mathfrak{T}_2) \in$ LOCq-RLDFHSS$(\mathfrak{U})$. Then $(\Lambda_1, \mathfrak{T}_1) \cap_{RES} (\Lambda_2, \mathfrak{T}_2) \in$ LOCq-RLDFHSS$(\mathfrak{U})$.

Proof. The proof is obvious. □

Definition 3.9 Let $(\Lambda_1, \mathfrak{T}_1)$, $(\Lambda_2, \mathfrak{T}_2) \in$ LOCq-RLDFHSS$(\mathfrak{U})$. Then their Extended union is defined by $(\Lambda_1, \mathfrak{T}_1) \cup_{EXT} (\Lambda_2, \mathfrak{T}_2) = (\Lambda_3, \mathfrak{T}_3)$ where $\mathfrak{T}_3 = \mathfrak{T}_1 \cup \mathfrak{T}_2$.

$$(\Lambda_3, \mathfrak{T}_3) = \begin{cases} \{\langle(\Psi_{\Lambda_1(\tau)}(\mathfrak{u}),\ \vartheta_{\Psi_{\Lambda_1(\tau)}}(\mathfrak{u})),\ (\Upsilon_{\Lambda_1(\tau)}(\mathfrak{u}),\ \vartheta_{\Upsilon_{\Lambda_1(\tau)}}(\mathfrak{u}))\rangle, & \text{if } \tau \in \mathfrak{T}_1 - \mathfrak{T}_2 \\ \quad \langle(\rho_{\Lambda_1(\tau)}(\mathfrak{u}),\ \vartheta_{\rho_{\Lambda_1(\tau)}}(\mathfrak{u})),\ (\eta_{\Lambda_1(\tau)}(\mathfrak{u}),\ \vartheta_{\eta_{\Lambda_1(\tau)}}(\mathfrak{u}))\rangle\} & \\ \{\langle(\Psi_{\Lambda_2(\tau)}(\mathfrak{u}),\ \vartheta_{\Psi_{\Lambda_2(\tau)}}(\mathfrak{u})),\ (\Upsilon_{\Lambda_2(\tau)}(\mathfrak{u}),\ \vartheta_{\Upsilon_{\Lambda_2(\tau)}}(\mathfrak{u}))\rangle, & \text{if } \tau \in \mathfrak{T}_2 - \mathfrak{T}_1 \\ \quad \langle(\rho_{\Lambda_2(\tau)}(\mathfrak{u}),\ \vartheta_{\rho_{\Lambda_2(\tau)}}(\mathfrak{u})),\ (\eta_{\Lambda_2(\tau)}(\mathfrak{u}),\ \vartheta_{\eta_{\Lambda_2(\tau)}}(\mathfrak{u}))\rangle\} & \\ \{\langle(\mathrm{Max}\{\Psi_{\Lambda_1(\tau)}(\mathfrak{u}), \Psi_{\Lambda_2(\tau)}(\mathfrak{u})\},\ \mathrm{Max}\{\vartheta_{\Psi_{\Lambda_1(\tau)}}(\mathfrak{u}),\ \vartheta_{\Psi_{\Lambda_2(\tau)}}(\mathfrak{u})\}), & \\ \quad (\mathrm{Min}\{\Upsilon_{\Lambda_1(\tau)}(\mathfrak{u}),\ \Upsilon_{\Lambda_2(\tau)}(\mathfrak{u})\},\ \mathrm{Min}\{\vartheta_{\Upsilon_{\Lambda_1(\tau)}}(\mathfrak{u}),\ \vartheta_{\Upsilon_{\Lambda_2(\tau)}}(\mathfrak{u})\})\rangle, & \text{if } \tau \in \mathfrak{T}_1 \cap \mathfrak{T}_2 \\ \quad \langle(\mathrm{Max}\{\rho_{\Lambda_1(\tau)}(\mathfrak{u}),\ \rho_{\Lambda_2(\tau)}(\mathfrak{u})\},\ \mathrm{Max}\{\vartheta_{\rho_{\Lambda_1(\tau)}}(\mathfrak{u}),\ \vartheta_{\rho_{\Lambda_2(\tau)}}(\mathfrak{u})\}), & \\ \quad (\mathrm{Min}\{\eta_{\Lambda_1(\tau)}(\mathfrak{u}),\ \eta_{\Lambda_2(\tau)}(\mathfrak{u})\},\ \mathrm{Min}\{\vartheta_{\eta_{\Lambda_1(\tau)}}(\mathfrak{u}),\ \vartheta_{\eta_{\Lambda_2(\tau)}}(\mathfrak{u})\})\rangle\}. & \end{cases}$$

Proposition 3.3 Let $(\Lambda_1, \mathfrak{T}_1)$, $(\Lambda_2, \mathfrak{T}_2) \in$ LOCq-RLDFHSS$(\mathfrak{U})$. Then $(\Lambda_1, \mathfrak{T}_1) \cup_{EXT} (\Lambda_2, \mathfrak{T}_2) \in$LOCq-RLDFHSS$(\mathfrak{U})$, if one of them is a LOCq-RLDFHS subset of other.

Proof. The proof is obvious. □

Definition 3.10 Let $\mathfrak{T}_1, \mathfrak{T}_2 \subseteq \mathfrak{G}_1 \times \mathfrak{G}_2 \times ... \times \mathfrak{G}_k$. Then partial order $\leq_{\mathfrak{T}_1 \times \mathfrak{T}_2}$ on $\mathfrak{T}_1 \times \mathfrak{T}_2$ is defined as for any $(\tau_1, \sigma_1), (\tau_2, \sigma_2) \in \mathfrak{T}_1 \times \mathfrak{T}_2$, $(\tau_1, \sigma_1) \leq_{\mathfrak{T}_1 \times \mathfrak{T}_2} (\tau_2, \sigma_2) \Leftrightarrow \tau_1 \leq_{\mathfrak{T}_1} \tau_2$ and $\sigma_1 \leq_{\mathfrak{T}_2} \sigma_2$.

Definition 3.11 Let $(\Lambda_1, \mathfrak{T}_1)$, $(\Lambda_2, \mathfrak{T}_2) \in$ LOCq-RLDFHSS$(\mathfrak{U})$. Then their "AND" operation is defined by $(\Lambda_1, \mathfrak{T}_1) \wedge (\Lambda_2, \mathfrak{T}_2) = (\Omega, \mathfrak{T}_1 \times \mathfrak{T}_2)$ where,

$$(\Omega, \mathfrak{T}_1 \times \mathfrak{T}_2) = \{(\tau, \sigma), (\mathfrak{u}, \Omega(\tau, \sigma)(\mathfrak{u})) : \mathfrak{u} \in \mathfrak{U}, (\tau, \sigma) \in \mathfrak{T}_1 \times \mathfrak{T}_2\}$$

and

$$\Omega(\tau, \sigma)(\mathfrak{u}) = \{\langle(\text{Min}\{\Psi_{\Lambda_1(\tau)}(\mathfrak{u}), \Psi_{\Lambda_2(\sigma)}(\mathfrak{u})\}, \text{Min}\{\vartheta_{\Psi_{\Lambda_1(\tau)}}(\mathfrak{u}), \vartheta_{\Psi_{\Lambda_2(\sigma)}}(\mathfrak{u})\}),$$

$$(\text{Max}\{\Upsilon_{\Lambda_1(\tau)}(\mathfrak{u}), \Upsilon_{\Lambda_2(\sigma)}(\mathfrak{u})\}, \text{Max}\{\vartheta_{\Upsilon_{\Lambda_1(\tau)}}(\mathfrak{u}), \vartheta_{\Upsilon_{\Lambda_2(\sigma)}}(\mathfrak{u})\})\rangle,$$

$$\langle(\text{Min}\{\rho_{\Lambda_1(\tau)}(\mathfrak{u}), \rho_{\Lambda_2(\sigma)}(\mathfrak{u})\}, \text{Min}\{\vartheta_{\rho_{\Lambda_1(\tau)}}(\mathfrak{u}), \vartheta_{\rho_{\Lambda_2(\sigma)}}(\mathfrak{u})\}),$$

$$(\text{Max}\{\eta_{\Lambda_1(\tau)}(\mathfrak{u}), \eta_{\Lambda_2(\sigma)}(\mathfrak{u})\}, \text{Max}\{\vartheta_{\eta_{\Lambda_1(\tau)}}(\mathfrak{u}), \vartheta_{\eta_{\Lambda_2(\sigma)}}(\mathfrak{u})\})\rangle\}.$$

Proposition 3.4 Let $(\Lambda_1, \mathfrak{T}_1), (\Lambda_2, \mathfrak{T}_2) \in$ LOCq-RLDFHSS$(\mathfrak{U})$. Then $(\Lambda_1, \mathfrak{T}_1) \wedge (\Lambda_2, \mathfrak{T}_2) \in$ LOCq-RLDFHSS$(\mathfrak{U})$.

Proof. The proof is obvious. □

Definition 3.12 Let $(\Lambda_1, \mathfrak{T}_1), (\Lambda_2, \mathfrak{T}_2) \in$ LOCq-RLDFHSS$(\mathfrak{u})$. Then their "OR" operation is defined by $(\Lambda_1, \mathfrak{T}_1) \vee (\Lambda_2, \mathfrak{T}_2) = (\Omega, \mathfrak{T}_1 \times \mathfrak{T}_2)$ where,

$$(\Omega, \mathfrak{T}_1 \times \mathfrak{T}_2) = \{(\tau, \sigma), (\mathfrak{u}, \Omega(\tau, \sigma)(\mathfrak{u})) : \mathfrak{u} \in \mathfrak{U}, (\tau, \sigma) \in \mathfrak{T}_1 \times \mathfrak{T}_2\}$$

and

$$\Omega(\tau, \sigma)(\mathfrak{u}) = \{\langle(\text{Max}\{\Psi_{\Lambda_1(\tau)}(\mathfrak{u}), \Psi_{\Lambda_2(\sigma)}(\mathfrak{u})\}, \text{Max}\{\vartheta_{\Psi_{\Lambda_1(\tau)}}(\mathfrak{u}), \vartheta_{\Psi_{\Lambda_2(\sigma)}}(\mathfrak{u})\}),$$

$$(\text{Min}\{\Upsilon_{\Lambda_1(\tau)}(\mathfrak{u}), \Upsilon_{\Lambda_2(\sigma)}(\mathfrak{u})\}, \text{Min}\{\vartheta_{\Upsilon_{\Lambda_1(\tau)}}(\mathfrak{u}), \vartheta_{\Upsilon_{\Lambda_2(\sigma)}}(\mathfrak{u})\})\rangle,$$

$$\langle(\text{Max}\{\rho_{\Lambda_1(\tau)}(\mathfrak{u}), \rho_{\Lambda_2(\sigma)}(\mathfrak{u})\}, \text{Max}\{\vartheta_{\rho_{\Lambda_1(\tau)}}(\mathfrak{u}), \vartheta_{\rho_{\Lambda_2(\sigma)}}(\mathfrak{u})\}),$$

$$(\text{Min}\{\eta_{\Lambda_1(\tau)}(\mathfrak{u}), \eta_{\Lambda_2(\sigma)}(\mathfrak{u})\}, \text{Min}\{\vartheta_{\eta_{\Lambda_1(\tau)}}(\mathfrak{u}), \vartheta_{\eta_{\Lambda_2(\sigma)}}(\mathfrak{u})\})\rangle\}.$$

Proposition 3.5 Let $(\Lambda_1, \mathfrak{T}_1), (\Lambda_2, \mathfrak{T}_2) \in$ LOCq-RLDFHSS$(\mathfrak{U})$. Then $(\Lambda_1, \mathfrak{T}_1) \vee (\Lambda_2, \mathfrak{T}_2) \in$ LOCq-RLDFHSS$(\mathfrak{U})$.

Proof. Let $(\Lambda_1, \mathfrak{T}_1), (\Lambda_2, \mathfrak{T}_2) \in$ LOCq-RLDFHSS$(\mathfrak{U})$. Then by Definition 3.12 $(\Lambda_1, \mathfrak{T}_1) \vee (\Lambda_2, \mathfrak{T}_2) = (\Omega, \mathfrak{T}_1 \times \mathfrak{T}_2)$ also

$$\Omega(\tau, \sigma)(\mathfrak{u}) = \{\langle(\text{Max}\{\Psi_{\Lambda_1(\tau)}(\mathfrak{u}), \Psi_{\Lambda_2(\sigma)}(\mathfrak{u})\}, \text{Max}\{\vartheta_{\Psi_{\Lambda_1(\tau)}}(\mathfrak{u}), \vartheta_{\Psi_{\Lambda_2(\sigma)}}(\mathfrak{u})\}),$$

$$(\text{Min}\{\Upsilon_{\Lambda_1(\tau)}(\mathfrak{u}), \Upsilon_{\Lambda_2(\sigma)}(\mathfrak{u})\}, \text{Min}\{\vartheta_{\Upsilon_{\Lambda_1(\tau)}}(\mathfrak{u}), \vartheta_{\Upsilon_{\Lambda_2(\sigma)}}(\mathfrak{u})\})\rangle,$$

$$\langle(\text{Max}\{\rho_{\Lambda_1(\tau)}(\mathfrak{u}),\ \rho_{\Lambda_2(\sigma)}(\mathfrak{u})\},\ \text{Max}\{\vartheta_{\rho_{\Lambda_1(\tau)}}(\mathfrak{u}),\ \vartheta_{\rho_{\Lambda_2(\sigma)}}(\mathfrak{u})\}),$$

$$(\text{Min}\{\eta_{\Lambda_1(\tau)}(\mathfrak{u}),\ \eta_{\Lambda_2(\sigma)}(\mathfrak{u})\},\ \text{Min}\{\vartheta_{\eta_{\Lambda_1(\tau)}}(\mathfrak{u}),\ \vartheta_{\eta_{\Lambda_2(\sigma)}}(\mathfrak{u})\})\rangle\}.$$

For any $\tau_c \leq_{\mathfrak{T}_1} \tau_d$ we have $\Lambda_1(\tau_c) \subseteq \Lambda_1(\tau_d)$, $\forall \tau_c,\ \tau_d \in \mathfrak{T}_1$ and for any $\sigma_c \leq_{\mathfrak{T}_2} \sigma_d$ we have $\Lambda_2(\sigma_c) \subseteq \Lambda_2(\sigma_d)$, $\forall \sigma_c,\ \sigma_d \in \mathfrak{T}_2$.

Now for any $(\tau_c,\ \sigma_c),\ (\tau_d,\ \sigma_d) \in \mathfrak{T}_1 \times \mathfrak{T}_2$. Then by Definition 3.10.

The order on $\mathfrak{T}_1 \times \mathfrak{T}_2$ is $(\tau_c,\ \sigma_c) \leq_{\mathfrak{T}_1 \times \mathfrak{T}_2} (\tau_d,\ \sigma_d) \Leftrightarrow \tau_c \leq_{\mathfrak{T}_1} \tau_d$ and $\sigma_c \leq_{\mathfrak{T}_2} \sigma_d$.

$$\Rightarrow \Lambda_1(\tau_c) \subseteq \Lambda_1(\tau_d) \text{ and } \Lambda_2(\sigma_c) \subseteq \Lambda_2(\sigma_d)$$

$$\Rightarrow \Psi_{\Lambda_1(\tau_c)}(\mathfrak{u}) \leq \Psi_{\Lambda_1(\sigma_d)}(\mathfrak{u}),\ \Psi_{\Lambda_2(\tau_c)}(\mathfrak{u}) \leq \Psi_{\Lambda_2(\sigma_d)}(\mathfrak{u})$$

$$\vartheta_{\Psi_{\Lambda_1(\tau_c)}}(\mathfrak{u}) \leq \vartheta_{\Psi_{\Lambda_1(\sigma_d)}}(\mathfrak{u}),\ \vartheta_{\Psi_{\Lambda_2(\tau_c)}}(\mathfrak{u}) \leq \vartheta_{\Psi_{\Lambda_2(\sigma_d)}}(\mathfrak{u})$$

$$\Upsilon_{\Lambda_1(\tau_d)}(\mathfrak{u}) \leq \Upsilon_{\Lambda_1(\sigma_c)}(\mathfrak{u}),\ \Upsilon_{\Lambda_2(\tau_d)}(\mathfrak{u}) \leq \Upsilon_{\Lambda_2(\sigma_c)}(\mathfrak{u})$$

$$\vartheta_{\Upsilon_{\Lambda_1(\tau_d)}}(\mathfrak{u}) \leq \vartheta_{\Upsilon_{\Lambda_1(\sigma_c)}}(\mathfrak{u}),\ \vartheta_{\Upsilon_{\Lambda_2(\tau_d)}}(\mathfrak{u}) \leq \vartheta_{\Upsilon_{\Lambda_2(\sigma_c)}}(\mathfrak{u})$$

$$\rho_{\Lambda_1(\tau_c)}(\mathfrak{u}) \leq \rho_{\Lambda_1(\sigma_d)}(\mathfrak{u}),\ \rho_{\Lambda_2(\tau_c)}(\mathfrak{u}) \leq \rho_{\Lambda_2(\sigma_d)}(\mathfrak{u})$$

$$\vartheta_{\rho_{\Lambda_1(\tau_c)}}(\mathfrak{u}) \leq \vartheta_{\rho_{\Lambda_1(\sigma_d)}}(\mathfrak{u}),\ \vartheta_{\rho_{\Lambda_2(\tau_c)}}(\mathfrak{u}) \leq \vartheta_{\rho_{\Lambda_2(\sigma_d)}}(\mathfrak{u})$$

$$\eta_{\Lambda_1(\tau_d)}(\mathfrak{u}) \leq \eta_{\Lambda_1(\sigma_c)}(\mathfrak{u}),\ \eta_{\Lambda_2(\tau_d)}(\mathfrak{u}) \leq \eta_{\Lambda_2(\sigma_c)}(\mathfrak{u})$$

$$\vartheta_{\eta_{\Lambda_1(\tau_d)}}(\mathfrak{u}) \leq \vartheta_{\eta_{\Lambda_1(\sigma_c)}}(\mathfrak{u}),\ \vartheta_{\eta_{\Lambda_2(\sigma_d)}}(\mathfrak{u}) \leq \vartheta_{\eta_{\Lambda_2(\sigma_c)}}(\mathfrak{u})$$

$$\Rightarrow \text{Max}\{\Psi_{\Lambda_1(\tau_c)}(\mathfrak{u}),\ \Psi_{\Lambda_2(\sigma_c)}(\mathfrak{u})\} \leq \text{Max}\{\Psi_{\Lambda_1(\tau_d)}(\mathfrak{u}),\ \Psi_{\Lambda_2(\sigma_d)}(\mathfrak{u})\}$$

$$\text{Max}\{\vartheta_{\Psi_{\Lambda_1(\tau_c)}}(\mathfrak{u}),\ \vartheta_{\Psi_{\Lambda_2(\sigma_c)}}(\mathfrak{u})\} \leq \text{Max}\{\vartheta_{\Psi_{\Lambda_1(\tau_d)}}(\mathfrak{u}),\ \vartheta_{\Psi_{\Lambda_2(\sigma_d)}}(\mathfrak{u})\}$$

$$\text{Min}\{\Upsilon_{\Lambda_1(\tau_d)}(\mathfrak{u}),\ \Upsilon_{\Lambda_2(\sigma_d)}(\mathfrak{u})\} \leq \text{Min}\{\Upsilon_{\Lambda_1(\tau_c)}(\mathfrak{u}),\ \Upsilon_{\Lambda_2(\sigma_c)}(\mathfrak{u})\}$$

$$\text{Min}\{\vartheta_{\Upsilon_{\Lambda_1(\tau_d)}}(\mathfrak{u}),\ \vartheta_{\Upsilon_{\Lambda_2(\sigma_d)}}(\mathfrak{u})\} \leq \text{Min}\{\vartheta_{\Upsilon_{\Lambda_1(\tau_c)}}(\mathfrak{u}),\ \vartheta_{\Upsilon_{\Lambda_2(\sigma_c)}}(\mathfrak{u})\}$$

$$\text{Max}\{\rho_{\Lambda_1(\tau_c)}(\mathfrak{u}),\ \rho_{\Lambda_2(\sigma_c)}(\mathfrak{u})\} \leq \text{Max}\{\rho_{\Lambda_1(\tau_d)}(\mathfrak{u}),\ \rho_{\Lambda_2(\sigma_d)}(\mathfrak{u})\}$$

$$\text{Max}\{\vartheta_{\rho_{\Lambda_1(\tau_c)}}(\mathfrak{u}),\ \vartheta_{\rho_{\Lambda_2(\sigma_c)}}(\mathfrak{u})\} \leq \text{Max}\{\vartheta_{\rho_{\Lambda_1(\tau_d)}}(\mathfrak{u}),\ \vartheta_{\rho_{\Lambda_2(\sigma_d)}}(\mathfrak{u})\}$$

$$\text{Min}\{\eta_{\Lambda_1(\tau_d)}(\mathfrak{u}),\ \eta_{\Lambda_2(\sigma_d)}(\mathfrak{u})\} \leq \text{Min}\{\eta_{\Lambda_1(\tau_c)}(\mathfrak{u}),\ \eta_{\Lambda_2(\sigma_c)}(\mathfrak{u})\}$$

$$\text{Min}\{\vartheta_{\eta_{\Lambda_1(\tau_d)}}(\mathfrak{u}),\ \vartheta_{\eta_{\Lambda_2(\sigma_d)}}(\mathfrak{u})\} \leq \text{Min}\{\vartheta_{\eta_{\Lambda_1(\tau_c)}}(\mathfrak{u}),\ \vartheta_{\eta_{\Lambda_2(\sigma_c)}}(\mathfrak{u})\}$$

$$\Rightarrow \Psi_{\Omega(\tau_c,\ \sigma_c)}(\mathfrak{u}) \leq \Psi_{\Omega(\tau_d,\ \sigma_d)}(\mathfrak{u})$$

$$\vartheta_{\Psi_{\Omega(\tau_c,\ \sigma_c)}}(\mathfrak{u}) \leq \vartheta_{\Psi_{\Omega(\tau_d,\ \sigma_d)}}(\mathfrak{u})$$

$$\Upsilon_{\Omega(\tau_d,\ \sigma_d)}(\mathfrak{u}) \leq \Upsilon_{\Omega(\tau_c,\ \sigma_c)}(\mathfrak{u})$$

$$\vartheta_{\Upsilon_{\Omega(\tau_d,\ \sigma_d)}}(\mathfrak{u}) \leq \vartheta_{\Upsilon_{\Omega(\tau_c,\ \sigma_c)}}(\mathfrak{u})$$

$$\rho_{\Omega(\tau_c,\ \sigma_c)}(\mathfrak{u}) \leq \rho_{\Omega(\tau_d,\ \sigma_d)}(\mathfrak{u})$$

$$\vartheta_{\rho_{\Omega(\tau_c,\ \sigma_c)}}(\mathfrak{u}) \leq \vartheta_{\rho_{\Omega(\tau_d,\ \sigma_d)}}(\mathfrak{u})$$

$$\eta_{\Omega(\tau_d,\ \sigma_d)}(\mathfrak{u}) \leq \eta_{\Omega(\tau_c,\ \sigma_c)}(\mathfrak{u})$$

$$\vartheta_{\eta_{\Omega(\tau_d,\ \sigma_d)}}(\mathfrak{u}) \leq \vartheta_{\eta_{\Omega(\tau_c,\ \sigma_c)}}(\mathfrak{u})$$

$$\Rightarrow \Omega(\tau_c,\ \sigma_c) \subseteq \Omega(\tau_d,\ \sigma_d) \text{ for } (\tau_c,\ \sigma_c) \leq_{\mathfrak{T}_1 \times \mathfrak{T}_2} (\tau_d,\ \sigma_d)$$

Therefore, $(\Lambda_1,\ \mathfrak{T}_1) \vee (\Lambda_2,\ \mathfrak{T}_2) \in$ LOCq-RLDFHSS($\mathfrak{U}$). □

Definition 3.13 Let $(\Lambda_1,\ \mathfrak{T}_1) \in$ LOCq-RLDFHSS($\mathfrak{U}$).

If $\Psi_{\Lambda_1(\tau)}(\mathfrak{u}) = \rho_{\Lambda_1(\tau)}(\mathfrak{u}) = \vartheta_{\Psi_{\Lambda_1(\tau)}}(\mathfrak{u}) = \vartheta_{\rho_{\Lambda_1(\tau)}}(\mathfrak{u}) = 0,\ \Upsilon_{\Lambda_1(\tau)}(\mathfrak{u}) = \eta_{\Lambda_1(\tau)}(\mathfrak{u}) = \vartheta_{\Upsilon_{\Lambda_1(\tau)}}(\mathfrak{u}) = \vartheta_{\eta_{\Lambda_1(\tau)}}(\mathfrak{u}) = 1 \forall \tau \in \mathfrak{T}_1$ and $\mathfrak{u} \in \mathfrak{U}$, Then $(\Lambda_1,\ \mathfrak{T}_1)$ is called relative null LOCq-RLDFHSS and denoted by $\emptyset_{\mathfrak{T}_1}$.

Definition 3.14 Let $(\Lambda_1,\ \mathfrak{T}_1) \in$ LOCq-RLDFHSS($\mathfrak{U}$).

If $\Psi_{\Lambda_1(\tau)}(\mathfrak{u}) = \rho_{\Lambda_1(\tau)}(\mathfrak{u}) = \vartheta_{\Psi_{\Lambda_1(\tau)}}(\mathfrak{u}) = \vartheta_{\rho_{\Lambda_1(\tau)}}(\mathfrak{u}) = 1,\ \Upsilon_{\Lambda_1(\tau)}(\mathfrak{u}) = \eta_{\Lambda_1(\tau)}(\mathfrak{u}) = \vartheta_{\Upsilon_{\Lambda_1(\tau)}}(\mathfrak{u}) = \vartheta_{\eta_{\Lambda_1(\tau)}}(\mathfrak{u}) = 0 \forall \tau \in \mathfrak{T}_1$ and $\mathfrak{u} \in \mathfrak{U}$, Then $(\Lambda_1,\ \mathfrak{T}_1)$ is called relative universal LOCq-RLDFHSS and denoted by $\mathfrak{A}_{\mathfrak{T}_1}$.

Proposition 3.6 Let $(\Lambda_1,\ \mathfrak{T}_1) \in$ LOCq-RLDFHSS($\mathfrak{U}$). Then

1. $(\Lambda_1,\ \mathfrak{T}_1) \cup_{RES} (\Lambda_1,\ \mathfrak{T}_1) = (\Lambda_1,\ \mathfrak{T}_1)$
2. $(\Lambda_1,\ \mathfrak{T}_1) \cup_{RES} \emptyset_{\mathfrak{T}_1} = (\Lambda_1,\ \mathfrak{T}_1)$
3. $(\Lambda_1,\ \mathfrak{T}_1) \cup_{RES} \mathfrak{A}_{\mathfrak{T}_1} = \mathfrak{A}_{\mathfrak{T}_1}$
4. $(\Lambda_1,\ \mathfrak{T}_1) \cap_{RES} (\Lambda_1,\ \mathfrak{T}_1) = (\Lambda_1,\ \mathfrak{T}_1)$
5. $(\Lambda_1,\ \mathfrak{T}_1) \cap_{RES} \emptyset_{\mathfrak{T}_1} = \emptyset_{\mathfrak{T}_1}$
6. $(\Lambda_1,\ \mathfrak{T}_1) \cap_{RES} \mathfrak{A}_{\mathfrak{T}_1} = (\Lambda_1,\ \mathfrak{T}_1)$

Proof. Straightforward. □

Definition 3.15 Let $(\Lambda_1, \mathfrak{T}_1) \in$ LOCq-RLDFHSS$(\mathfrak{U})$. Then complement of $(\Lambda_1, \mathfrak{T}_1)$ denoted by $(\Lambda_1, \mathfrak{T}_1)^c$ and is defined by

$$(\Lambda_1, \mathfrak{T}_1)^c = \{(\mathfrak{u}, \{\langle \Upsilon_{\Lambda_1(\tau)}(\mathfrak{u}) e^{i2\pi(\vartheta_{\Upsilon_{\Lambda_1(\tau)}}(\mathfrak{u}))}, \Psi_{\Lambda_1(\tau)}(\mathfrak{u}) e^{i2\pi(\vartheta_{\Psi_{\Lambda_1(\tau)}}(\mathfrak{u}))}\rangle,$$

$$\langle \eta_{\Lambda_1(\tau)}(\mathfrak{u}) e^{i2\pi(\vartheta_{\eta_{\Lambda_1(\tau)}}(\mathfrak{u}))}, \rho_{\Lambda_1(\tau)}(\mathfrak{u}) e^{i2\pi(\vartheta_{\rho_{\Lambda_1(\tau)}}(\mathfrak{u}))}\rangle\}) : \tau \in \mathfrak{T}_1 \text{ and } \mathfrak{u} \in \mathfrak{U}\}.$$

Proposition 3.7 Let $(\Lambda_1, \mathfrak{T}_1) \in$ LOCq-RLDFHSS$(\mathfrak{U})$. Then $((\Lambda_1, \mathfrak{T}_1)^c)^c = (\Lambda_1, \mathfrak{T}_1)$

Proof. Let $(\Lambda_1, \mathfrak{T}_1) \in$ LOCq-RLDFHSS$(\mathfrak{U})$. Then complement of $(\Lambda_1, \mathfrak{T}_1)$ is

$$(\Lambda_1, \mathfrak{T}_1)^c = \{(\mathfrak{u}, \{\langle \Upsilon_{\Lambda_1(\tau)}(\mathfrak{u}) e^{i2\pi(\vartheta_{\Upsilon_{\Lambda_1(\tau)}}(\mathfrak{u}))}, \Psi_{\Lambda_1(\tau)}(\mathfrak{u}) e^{i2\pi(\vartheta_{\Psi_{\Lambda_1(\tau)}}(\mathfrak{u}))}\rangle,$$

$$\langle \eta_{\Lambda_1(\tau)}(\mathfrak{u}) e^{i2\pi(\vartheta_{\eta_{\Lambda_1(\tau)}}(\mathfrak{u}))}, \rho_{\Lambda_1(\tau)}(\mathfrak{u}) e^{i2\pi(\vartheta_{\rho_{\Lambda_1(\tau)}}(\mathfrak{u}))}\rangle\}) : \tau \in \mathfrak{T}_1 \text{ and } \mathfrak{u} \in \mathfrak{U}\}.$$

Now complement of $(\Lambda_1, \mathfrak{T}_1)^c$ is

$$((\Lambda_1, \mathfrak{T}_1)^c)^c = \{(\mathfrak{u}, \{\langle \Psi_{\Lambda_1(\tau)}(\mathfrak{u}) e^{i2\pi(\vartheta_{\Psi_{\Lambda_1(\tau)}}(\mathfrak{u}))}, \Upsilon_{\Lambda_1(\tau)}(\mathfrak{u}) e^{i2\pi(\vartheta_{\Upsilon_{\Lambda_1(\tau)}}(\mathfrak{u}))}\rangle,$$

$$\langle \rho_{\Lambda_1(\tau)}(\mathfrak{u}) e^{i2\pi(\vartheta_{\rho_{\Lambda_1(\tau)}}(\mathfrak{u}))}, \eta_{\Lambda_1(\tau)}(\mathfrak{u}) e^{i2\pi(\vartheta_{\eta_{\Lambda_1(\tau)}}(\mathfrak{u}))}\rangle\}) : \tau \in \mathfrak{T}_1 \text{ and } \mathfrak{u} \in \mathfrak{U}\}$$

$$= (\Lambda_1, \mathfrak{T}_1).$$

□

4. MADM technique based on LOCq-RLDFHSS

This section describes the definition of comparison table of LOCq-RLDFHSS and row sum, column sum and score value of the alternatives and a MADM method based on LOCq-RLDFHSS using the proposed definitions. Further, discusses a vertical farming MADM problem as a numerical example of the suggested MADM algorithm.

Definition 4.1 The Comparison table of LOCq-RLDFHSS $(\Lambda, \mathfrak{T}_1)$ is a square table in which the number of rows and number of columns are equal. Rows and columns both are labelled by the alternatives $\mathfrak{u}_1, \mathfrak{u}_2, ..., \mathfrak{u}_m$ of the universal set $\mathfrak{U}$ and entries $\mathfrak{e}_{ab} (a, b = 1, 2, ..., m)$ is the number of parameters fulfilling $\mathfrak{s}(\sqsupset_{\tau_{ac}}) \geq \mathfrak{s}(\sqsupset_{\tau_{bc}})$. Clearly, $0 \leq \mathfrak{e}_{ab} \leq n$ for any a, b, where n is the number of parameters in $\mathfrak{T}_1$.

Definition 4.2 The row sum of alternative $\mathfrak{u}_a$ is denoted by $\mathfrak{r}_a$ and calculated as

$$\mathfrak{r}_a = \sum_{b=1}^{m} \mathfrak{e}_{ab}$$

Definition 4.3 The column sum of alternative $\mathfrak{u}_b$ is denoted by $\mathfrak{h}_b$ and calculated as

$$\mathfrak{h}_b = \sum_{a=1}^{m} \mathfrak{e}_{ab}$$

Definition 4.4 The score value of $\mathfrak{u}_a$ is $\mathfrak{S}_a$ and calculated as

$$\mathfrak{S}_a = \mathfrak{r}_a - \mathfrak{h}_a$$

4.1 *Algorithm*

The algorithm for selecting the best alternative is as follows:

Step 1: Input the LOCq-RLDFHSSs $(\Lambda_1, \mathfrak{T}_1)$ and $(\Lambda_2, \mathfrak{T}_2)$

Step 2: Compute the resultant LOCq-RLDFHSS $(\Lambda_1, \mathfrak{T}_1) \wedge (\Lambda_2, \mathfrak{T}_2) = (\Omega, \mathfrak{T}_1 \times \mathfrak{T}_2)$ from $(\Lambda_1, \mathfrak{T}_1)$ and $(\Lambda_2, \mathfrak{T}_2)$ by using Definition 3.11

Step 3: Evaluate the score of LOCq-RLDFHSNs in $(\Omega, \mathfrak{T}_1 \times \mathfrak{T}_2)$ by using Definition 3.3

Step 4: Construct the comparison table of $(\Omega, \mathfrak{T}_1 \times \mathfrak{T}_2)$ using Definition 4.1

Step 5: Compute the score value of $\mathfrak{u}_a$ $\forall$ a, using Definition 4.4

Step 6: Find $\mathfrak{S}_l = \max_a \mathfrak{S}_a$ and select it as best alternative

Step 7: If more than one maximum was attained, select any one.

4.2 *Numerical illustration*

4.2.1 *General outlook of vertical farming*

"Vertical farming" is the practice of growing crops in vertical levels. It frequently makes use of controlled-environment agriculture, which aims to maximize plant growth, as well as soilless farming techniques including hydroponics, aquaponics, and aeroponics. Structures like as shipping containers, buildings, tunnels, and abandoned mine shafts are often utilized to accommodate vertical farming systems.

The modern concept of vertical farming was first proposed by Dickson Despommier, a professor of public and environmental health at Columbia University, in 1999. With the aid of his students, Despommier created a blueprint for a skyscraper farm that could feed 50,000 people. Despite never being put into practice, the design was successful in popularizing the idea of vertical farming.

There are a number of advantages to using vertical farming techniques, but the main one is the increased crop yield that results from requiring less land per unit. It is possible to develop a greater variety of crops at once since crops do not grow on the same parcels of land. Since crops are grown indoors, they are also resistant to weather, meaning that less crop loss occurs from unanticipated or extreme weather. Vertical farming helps to conserve the local animals and vegetation because it only takes up a limited amount of space.

Hydroponics is the term for growing plants without soil. Plant roots in hydroponic systems are submerged in liquid solutions that contain macronutrients including nitrogen, phosphorus, sulfur, potassium, calcium, and magnesium, as well as trace elements like iron, chlorine, manganese, boron, zinc, copper, and molybdenum. In addition, because they are inert (chemically inactive) mediums, soil substitutes including sawdust, gravel, and sand are used to support the roots. Hydroponics can increase yield per unit area while using less water. A study found that hydroponic gardening uses 13 times less water than conventional farming and can increase lettuce yield per area by around 11 times. Because of these advantages, hydroponics is the most popular growing technique used in vertical farming.

Internet of Things (IoT) is a great tool for hydroponic farming since it enables machine-to-machine communication and remote management of the hydroponic system. Such systems don't impact the environment or the quality of the crops. IoT offers an original method for farm modernization.

In vertical farming, automation is employed in a variety of settings, including irrigation systems that regulate how much water is applied to the plants. Additionally, it controls the rates at which nutrients are provided to plants via nutrient delivery systems. Automation aids vertical farms by decreasing the need for labor and increasing output with less chances for human error.

Many studies [33–36] on DM in the context of vertical farming have been conducted. We now demonstrate the use of suggested notions and algorithm by a vertical farming MADM problem.

4.2.2 *Problem*

Let $\mathfrak{U} = \{$Automated vertical farming ($\mathfrak{u}_1$), Basic vertical farming ($\mathfrak{u}_2$), IoT vertical farming ($\mathfrak{u}_3$)$\}$ be set of hydroponic vertical farming systems. Two experts teams $\{\mathfrak{z}_1, \mathfrak{z}_2\}$ were set to evaluate the alternatives and the opinion of both teams are considered to make the final choice. To evaluate the alternatives both experts teams consider the attributes $\mathfrak{g}_1 = \{\text{cost}\}$, $\mathfrak{g}_2 = \{\text{Requirement}\}$, $\mathfrak{g}_3 = \{\text{sustainability}\}$ and $\mathfrak{G}_1 = \{$Maintenance cost ($\mathfrak{g}_{11}$), Investment cost ($\mathfrak{g}_{12}$)$\}$, $\mathfrak{G}_2 = \{$workforce required ($\mathfrak{g}_{21}$), Space required ($\mathfrak{g}_{22}$)$\}$, $\mathfrak{G}_3 = \{$Resource's usage ($\mathfrak{g}_{31}$)$\}$ be their corresponding attribute values respectively.

Further, The experts consider an order among the elements in each set $\mathfrak{G}_1$, $\mathfrak{G}_2$ and $\mathfrak{G}_3$, they are

The elements in set $\mathfrak{G}_1$ are in the order $\mathfrak{g}_{11} \leq_{\mathfrak{G}_1} \mathfrak{g}_{12}$

The elements in set $\mathfrak{G}_2$ are in the order $\mathfrak{g}_{21} \leq_{\mathfrak{G}_2} \mathfrak{g}_{22}$

$\mathfrak{G}_3$ has only one element $\mathfrak{g}_{31}$ and

$$\mathfrak{T}_1 = \mathfrak{G}_1 \times \mathfrak{G}_2 \times \mathfrak{G}_3$$

$$=\{\tau_1 = (\mathfrak{g}_{11}, \mathfrak{g}_{21}, \mathfrak{g}_{31}), \tau_2 = (\mathfrak{g}_{11}, \mathfrak{g}_{22}, \mathfrak{g}_{31}), \tau_3 = (\mathfrak{g}_{12}, \mathfrak{g}_{21}, \mathfrak{g}_{31}), \tau_4 = (\mathfrak{g}_{12}, \mathfrak{g}_{22}, \mathfrak{g}_{31})\}.$$

Then the order of elements in set $\mathfrak{T}_1$ is shown in Figure 2.

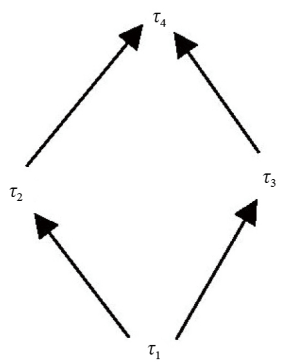

Figure 2. The order among elements in $\mathfrak{T}_1$

Further, the following is how the attributes are categorized by the experts

- The attribute "cost" and its attribute values manifest that the alternative is inexpensive or expensive.
- The attribute "requirement" and its attribute values manifest that the alternative is less or more.
- The attribute "sustainability" and its attribute values manifest that the alternative is high or low.

Then the cartesian product of attribute values manifest that the alternative is (inexpensive, less, high) all together or (expensive, more, low) all together.

Step 1 The opinion and information obtained from experts team $\mathfrak{z}_1$ is constructed as a Cq-RLDFHSS (Λ_1, $\mathfrak{T}_1$) and expressed as

$$(\Lambda_1, \mathfrak{T}_1) = \Bigg\{ \Bigg\langle \tau_1, \left(\frac{u_1}{\begin{matrix}\langle(0.41e^{i2\pi(0.45)}, 0.92e^{i2\pi(0.83)}), \\ (0.32e^{i2\pi(0.38)}, 0.79e^{i2\pi(0.8)})\rangle\end{matrix}}, \frac{u_2}{\begin{matrix}\langle(0.37e^{i2\pi(0.38)}, 0.86e^{i2\pi(0.78)}), \\ (0.45e^{i2\pi(0.48)}, 0.81e^{i2\pi(0.8)})\rangle\end{matrix}}, \frac{u_3}{\begin{matrix}\langle(0.26e^{i2\pi(0.31)}, 0.93e^{i2\pi(0.86)}), \\ (0.47e^{i2\pi(0.6)}, 0.83e^{i2\pi(0.83)})\rangle\end{matrix}} \right) \Bigg\rangle,$$

$$\Bigg\langle \tau_2, \left(\frac{u_1}{\begin{matrix}\langle(0.5e^{i2\pi(0.56)}, 0.81e^{i2\pi(0.79)}), \\ (0.41e^{i2\pi(0.43)}, 0.77e^{i2\pi(0.78)})\rangle\end{matrix}}, \frac{u_2}{\begin{matrix}\langle(0.4e^{i2\pi(0.47)}, 0.79e^{i2\pi(0.72)}), \\ (0.51e^{i2\pi(0.55)}, 0.74e^{i2\pi(0.74)})\rangle\end{matrix}}, \frac{u_3}{\begin{matrix}\langle(0.34e^{i2\pi(0.39)}, 0.86e^{i2\pi(0.86)}), \\ (0.49e^{i2\pi(0.69)}, 0.74e^{i2\pi(0.72)})\rangle\end{matrix}} \right) \Bigg\rangle,$$

$$\Bigg\langle \tau_3, \left(\frac{u_1}{\begin{matrix}\langle(0.45e^{i2\pi(0.46)}, 0.85e^{i2\pi(0.77)}), \\ (0.39e^{i2\pi(0.43)}, 0.75e^{i2\pi(0.73)})\rangle\end{matrix}}, \frac{u_2}{\begin{matrix}\langle(0.4e^{i2\pi(0.42)}, 0.82e^{i2\pi(0.76)}), \\ (0.48e^{i2\pi(0.51)}, 0.76e^{i2\pi(0.73)})\rangle\end{matrix}}, \frac{u_3}{\begin{matrix}\langle(0.47e^{i2\pi(0.46)}, 0.77e^{i2\pi(0.71)}), \\ (0.56e^{i2\pi(0.61)}, 0.69e^{i2\pi(0.71)})\rangle\end{matrix}} \right) \Bigg\rangle,$$

$$\Bigg\langle \tau_4, \left(\frac{u_1}{\begin{matrix}\langle(0.57e^{i2\pi(0.61)}, 0.79e^{i2\pi(0.71)}), \\ (0.49e^{i2\pi(0.48)}, 0.69e^{i2\pi(0.7)})\rangle\end{matrix}}, \frac{u_2}{\begin{matrix}\langle(0.51e^{i2\pi(0.55)}, 0.74e^{i2\pi(0.67)}), \\ (0.54e^{i2\pi(0.6)}, 0.68e^{i2\pi(0.66)})\rangle\end{matrix}}, \frac{u_3}{\begin{matrix}\langle(0.54e^{i2\pi(0.49)}, 0.65e^{i2\pi(0.68)}), \\ (0.61e^{i2\pi(0.72)}, 0.6e^{i2\pi(0.62)})\rangle\end{matrix}} \right) \Bigg\rangle \Bigg\}.$$

We will assume that $q = 3$.

The characteristics of this Cq-RLDFHSS $(\Lambda_1, \mathfrak{T}_1)$ is $(\langle$CMS degree, CNMS degree$\rangle$, $\langle$(inexpensive, less, high), (expensive, more, low)$\rangle)$ $\forall \tau_c \in \mathfrak{T}_1$.

Clearly $\Lambda_1(\tau_1) \subseteq \Lambda_1(\tau_2) \subseteq \Lambda_1(\tau_4)$ and $\Lambda_1(\tau_1) \subseteq \Lambda_1(\tau_3) \subseteq \Lambda_1(\tau_4)$, so $(\Lambda_1, \mathfrak{T}_1)$ is a LOCq-RLDFHSS$(\mathfrak{U})$.

Likewise, the opinion and information obtained from experts team $\mathfrak{z}_2$ is constructed as a Cq-RLDFHSS $(\Lambda_2, \mathfrak{T}_1)$ and expressed as

$$(\Lambda_2, \mathfrak{T}_1) = \Bigg\{ \Bigg\langle \tau_1, \left(\frac{u_1}{\begin{matrix}\langle(0.53e^{i2\pi(0.55)}, 0.87e^{i2\pi(0.86)}), \\ (0.36e^{i2\pi(0.42)}, 0.75e^{i2\pi(0.79)})\rangle\end{matrix}}, \frac{u_2}{\begin{matrix}\langle(0.33e^{i2\pi(0.32)}, 0.89e^{i2\pi(0.83)}), \\ (0.35e^{i2\pi(0.37)}, 0.86e^{i2\pi(0.81)})\rangle\end{matrix}}, \frac{u_3}{\begin{matrix}\langle(0.34e^{i2\pi(0.33)}, 0.9e^{i2\pi(0.88)}), \\ (0.49e^{i2\pi(0.6)}, 0.77e^{i2\pi(0.81)})\rangle\end{matrix}} \right) \Bigg\rangle,$$

$$\Bigg\langle \tau_2, \left(\frac{u_1}{\begin{matrix}\langle(0.57e^{i2\pi(0.56)}, 0.85e^{i2\pi(0.81)}), \\ (0.43e^{i2\pi(0.45)}, 0.69e^{i2\pi(0.74)})\rangle\end{matrix}}, \frac{u_2}{\begin{matrix}\langle(0.37e^{i2\pi(0.37)}, 0.84e^{i2\pi(0.78)}), \\ (0.41e^{i2\pi(0.45)}, 0.79e^{i2\pi(0.78)})\rangle\end{matrix}}, \frac{u_3}{\begin{matrix}\langle(0.37e^{i2\pi(0.38)}, 0.86e^{i2\pi(0.86)}), \\ (0.55e^{i2\pi(0.65)}, 0.72e^{i2\pi(0.78)})\rangle\end{matrix}} \right) \Bigg\rangle,$$

$$\Bigg\langle \tau_3, \left(\frac{u_1}{\begin{matrix}\langle(0.61e^{i2\pi(0.58)}, 0.73e^{i2\pi(0.71)}), \\ (0.45e^{i2\pi(0.48)}, 0.66e^{i2\pi(0.67)})\rangle\end{matrix}}, \frac{u_2}{\begin{matrix}\langle(0.42e^{i2\pi(0.4)}, 0.76e^{i2\pi(0.79)}), \\ (0.43e^{i2\pi(0.45)}, 0.79e^{i2\pi(0.77)})\rangle\end{matrix}}, \frac{u_3}{\begin{matrix}\langle(0.4e^{i2\pi(0.39)}, 0.75e^{i2\pi(0.8)}), \\ (0.55e^{i2\pi(0.65)}, 0.71e^{i2\pi(0.76)})\rangle\end{matrix}} \right) \Bigg\rangle,$$

$$\Bigg\langle \tau_4, \left(\frac{u_1}{\begin{matrix}\langle(0.68e^{i2\pi(0.6)}, 0.7e^{i2\pi(0.67)}), \\ (0.48e^{i2\pi(0.51)}, 0.63e^{i2\pi(0.67)})\rangle\end{matrix}}, \frac{u_2}{\begin{matrix}\langle(0.46e^{i2\pi(0.45)}, 0.71e^{i2\pi(0.77)}), \\ (0.49e^{i2\pi(0.5)}, 0.71e^{i2\pi(0.72)})\rangle\end{matrix}}, \frac{u_3}{\begin{matrix}\langle(0.48e^{i2\pi(0.44)}, 0.72e^{i2\pi(0.74)}), \\ (0.58e^{i2\pi(0.69)}, 0.67e^{i2\pi(0.7)})\rangle\end{matrix}} \right) \Bigg\rangle \Bigg\}.$$

We will assume that $q = 3$.

The characteristics of this Cq-RLDFHSS $(\Lambda_2, \mathfrak{T}_1)$ is $(\langle$CMS degree, CNMS degree$\rangle$, $\langle$(inexpensive, less, high), (expensive, more, low)$\rangle)$ $\forall \tau_c \in \mathfrak{T}_1$.

Clearly $\Lambda_2(\tau_1) \subseteq \Lambda_2(\tau_2) \subseteq \Lambda_2(\tau_4)$ and $\Lambda_2(\tau_1) \subseteq \Lambda_2(\tau_3) \subseteq \Lambda_2(\tau_4)$, so $(\Lambda_2, \mathfrak{T}_1)$ is a LOCq-RLDFHSS$(\mathfrak{U})$.

Step 2 Since the opinion of both the experts teams are considered to evaluate the final choice, the two LOCq-RLDFHSSs $(\Lambda_1, \mathfrak{T}_1)$ and $(\Lambda_2, \mathfrak{T}_1)$ went through "AND" operation and obtained the resultant LOCq-RLDFHSS $(\Omega, \mathfrak{T}_1 \times \mathfrak{T}_1)$ as follows.

$$(\Omega, \mathfrak{T}_1 \times \mathfrak{T}_1) = \Bigg\{ \left\langle (\tau_1, \tau_1), \left(\frac{u_1}{\langle (0.41e^{i2\pi(0.45)}, 0.92e^{i2\pi(0.86)}), (0.32e^{i2\pi(0.38)}, 0.79e^{i2\pi(0.8)}) \rangle}, \frac{u_2}{\langle (0.33e^{i2\pi(0.32)}, 0.89e^{i2\pi(0.83)}), (0.35e^{i2\pi(0.37)}, 0.86e^{i2\pi(0.81)}) \rangle}, \frac{u_3}{\langle (0.26e^{i2\pi(0.31)}, 0.93e^{i2\pi(0.86)}), (0.47e^{i2\pi(0.6)}, 0.83e^{i2\pi(0.83)}) \rangle} \right) \right\rangle,$$

$$\left\langle (\tau_1, \tau_2), \left(\frac{u_1}{\langle (0.41e^{i2\pi(0.45)}, 0.92e^{i2\pi(0.83)}), (0.32e^{i2\pi(0.38)}, 0.69e^{i2\pi(0.74)}) \rangle}, \frac{u_2}{\langle (0.37e^{i2\pi(0.37)}, 0.86e^{i2\pi(0.78)}), (0.41e^{i2\pi(0.45)}, 0.81e^{i2\pi(0.8)}) \rangle}, \frac{u_3}{\langle (0.26e^{i2\pi(0.31)}, 0.93e^{i2\pi(0.86)}), (0.47e^{i2\pi(0.6)}, 0.83e^{i2\pi(0.83)}) \rangle} \right) \right\rangle,$$

$$\left\langle (\tau_1, \tau_3), \left(\frac{u_1}{\langle (0.41e^{i2\pi(0.45)}, 0.92e^{i2\pi(0.83)}), (0.32e^{i2\pi(0.38)}, 0.79e^{i2\pi(0.8)}) \rangle}, \frac{u_2}{\langle (0.37e^{i2\pi(0.38)}, 0.86e^{i2\pi(0.79)}), (0.43e^{i2\pi(0.45)}, 0.81e^{i2\pi(0.8)}) \rangle}, \frac{u_3}{\langle (0.26e^{i2\pi(0.31)}, 0.93e^{i2\pi(0.86)}), (0.47e^{i2\pi(0.6)}, 0.83e^{i2\pi(0.83)}) \rangle} \right) \right\rangle,$$

$$\left\langle (\tau_1, \tau_4), \left(\frac{u_1}{\langle (0.41e^{i2\pi(0.45)}, 0.92e^{i2\pi(0.83)}), (0.32e^{i2\pi(0.38)}, 0.79e^{i2\pi(0.8)}) \rangle}, \frac{u_2}{\langle (0.37e^{i2\pi(0.38)}, 0.86e^{i2\pi(0.78)}), (0.45e^{i2\pi(0.48)}, 0.81e^{i2\pi(0.8)}) \rangle}, \frac{u_3}{\langle (0.26e^{i2\pi(0.31)}, 0.93e^{i2\pi(0.86)}), (0.47e^{i2\pi(0.6)}, 0.83e^{i2\pi(0.83)}) \rangle} \right) \right\rangle,$$

$$\left\langle (\tau_2, \tau_1), \left(\frac{u_1}{\langle (0.5e^{i2\pi(0.55)}, 0.87e^{i2\pi(0.86)}), (0.36e^{i2\pi(0.42)}, 0.77e^{i2\pi(0.79)}) \rangle}, \frac{u_2}{\langle (0.33e^{i2\pi(0.32)}, 0.89e^{i2\pi(0.83)}), (0.35e^{i2\pi(0.37)}, 0.86e^{i2\pi(0.81)}) \rangle}, \frac{u_3}{\langle (0.34e^{i2\pi(0.33)}, 0.9e^{i2\pi(0.88)}), (0.49e^{i2\pi(0.6)}, 0.77e^{i2\pi(0.81)}) \rangle} \right) \right\rangle,$$

$$\left\langle (\tau_2, \tau_2), \left(\frac{u_1}{\langle (0.5e^{i2\pi(0.56)}, 0.85e^{i2\pi(0.81)}), (0.41e^{i2\pi(0.43)}, 0.77e^{i2\pi(0.78)}) \rangle}, \frac{u_2}{\langle (0.37e^{i2\pi(0.37)}, 0.84e^{i2\pi(0.78)}), (0.41e^{i2\pi(0.45)}, 0.79e^{i2\pi(0.78)}) \rangle}, \frac{u_3}{\langle (0.34e^{i2\pi(0.38)}, 0.86e^{i2\pi(0.86)}), (0.49e^{i2\pi(0.65)}, 0.74e^{i2\pi(0.78)}) \rangle} \right) \right\rangle,$$

$$\left\langle (\tau_2, \tau_3), \left(\frac{u_1}{\langle (0.5e^{i2\pi(0.56)}, 0.81e^{i2\pi(0.79)}), (0.41e^{i2\pi(0.43)}, 0.77e^{i2\pi(0.78)}) \rangle}, \frac{u_2}{\langle (0.4e^{i2\pi(0.4)}, 0.79e^{i2\pi(0.79)}), (0.43e^{i2\pi(0.45)}, 0.79e^{i2\pi(0.77)}) \rangle}, \frac{u_3}{\langle (0.34e^{i2\pi(0.39)}, 0.86e^{i2\pi(0.86)}), (0.49e^{i2\pi(0.65)}, 0.74e^{i2\pi(0.78)}) \rangle} \right) \right\rangle,$$

$$\left\langle (\tau_2, \tau_4), \left(\frac{u_1}{\langle (0.5e^{i2\pi(0.56)}, 0.81e^{i2\pi(0.79)}), (0.41e^{i2\pi(0.43)}, 0.77e^{i2\pi(0.78)}) \rangle}, \frac{u_2}{\langle (0.4e^{i2\pi(0.45)}, 0.79e^{i2\pi(0.77)}), (0.49e^{i2\pi(0.5)}, 0.74e^{i2\pi(0.74)}) \rangle}, \frac{u_3}{\langle (0.34e^{i2\pi(0.39)}, 0.86e^{i2\pi(0.86)}), (0.49e^{i2\pi(0.69)}, 0.74e^{i2\pi(0.72)}) \rangle} \right) \right\rangle,$$

$$\left\langle (\tau_3, \tau_1), \left(\frac{u_1}{\langle (0.45e^{i2\pi(0.46)}, 0.87e^{i2\pi(0.86)}), (0.36e^{i2\pi(0.42)}, 0.75e^{i2\pi(0.79)}) \rangle}, \frac{u_2}{\langle (0.33e^{i2\pi(0.32)}, 0.89e^{i2\pi(0.83)}), (0.35e^{i2\pi(0.37)}, 0.86e^{i2\pi(0.81)}) \rangle}, \frac{u_3}{\langle (0.34e^{i2\pi(0.33)}, 0.9e^{i2\pi(0.88)}), (0.49e^{i2\pi(0.6)}, 0.77e^{i2\pi(0.81)}) \rangle} \right) \right\rangle,$$

$$\left\langle (\tau_3,\ \tau_2), \left(\frac{u_1}{\langle (0.45e^{i2\pi(0.46)},\ 0.85e^{i2\pi(0.81)}),\ (0.39e^{i2\pi(0.43)},\ 0.75e^{i2\pi(0.74)}) \rangle}, \frac{u_2}{\langle (0.37e^{i2\pi(0.37)},\ 0.84e^{i2\pi(0.78)}),\ (0.41e^{i2\pi(0.45)},\ 0.79e^{i2\pi(0.78)}) \rangle}, \frac{u_3}{\langle (0.37e^{i2\pi(0.38)},\ 0.86e^{i2\pi(0.86)}),\ (0.55e^{i2\pi(0.61)},\ 0.72e^{i2\pi(0.78)}) \rangle} \right) \right\rangle,$$

$$\left\langle (\tau_3,\ \tau_3), \left(\frac{u_1}{\langle (0.45e^{i2\pi(0.46)},\ 0.85e^{i2\pi(0.87)}),\ (0.39e^{i2\pi(0.43)},\ 0.75e^{i2\pi(0.73)}) \rangle}, \frac{u_2}{\langle (0.4e^{i2\pi(0.4)},\ 0.82e^{i2\pi(0.79)}),\ (0.43e^{i2\pi(0.45)},\ 0.79e^{i2\pi(0.77)}) \rangle}, \frac{u_3}{\langle (0.4e^{i2\pi(0.39)},\ 0.77e^{i2\pi(0.8)}),\ (0.55e^{i2\pi(0.61)},\ 0.71e^{i2\pi(0.76)}) \rangle} \right) \right\rangle,$$

$$\left\langle (\tau_3,\ \tau_4), \left(\frac{u_1}{\langle (0.45e^{i2\pi(0.46)},\ 0.85e^{i2\pi(0.77)}),\ (0.39e^{i2\pi(0.43)},\ 0.75e^{i2\pi(0.73)}) \rangle}, \frac{u_2}{\langle (0.4e^{i2\pi(0.42)},\ 0.82e^{i2\pi(0.77)}),\ (0.48e^{i2\pi(0.5)},\ 0.76e^{i2\pi(0.73)}) \rangle}, \frac{u_3}{\langle (0.47e^{i2\pi(0.44)},\ 0.77e^{i2\pi(0.74)}),\ (0.56e^{i2\pi(0.61)},\ 0.69e^{i2\pi(0.71)}) \rangle} \right) \right\rangle,$$

$$\left\langle (\tau_4,\ \tau_1), \left(\frac{u_1}{\langle (0.53e^{i2\pi(0.55)},\ 0.87e^{i2\pi(0.86)}),\ (0.36e^{i2\pi(0.42)},\ 0.75e^{i2\pi(0.79)}) \rangle}, \frac{u_2}{\langle (0.33e^{i2\pi(0.32)},\ 0.89e^{i2\pi(0.83)}),\ (0.35e^{i2\pi(0.37)},\ 0.86e^{i2\pi(0.81)}) \rangle}, \frac{u_3}{\langle (0.34e^{i2\pi(0.33)},\ 0.9e^{i2\pi(0.88)}),\ (0.49e^{i2\pi(0.6)},\ 0.77e^{i2\pi(0.81)}) \rangle} \right) \right\rangle,$$

$$\left\langle (\tau_4,\ \tau_2), \left(\frac{u_1}{\langle (0.57e^{i2\pi(0.56)},\ 0.85e^{i2\pi(0.81)}),\ (0.43e^{i2\pi(0.45)},\ 0.69e^{i2\pi(0.74)}) \rangle}, \frac{u_2}{\langle (0.37e^{i2\pi(0.37)},\ 0.84e^{i2\pi(0.78)}),\ (0.41e^{i2\pi(0.45)},\ 0.79e^{i2\pi(0.78)}) \rangle}, \frac{u_3}{\langle (0.37e^{i2\pi(0.38)},\ 0.86e^{i2\pi(0.86)}),\ (0.55e^{i2\pi(0.65)},\ 0.72e^{i2\pi(0.78)}) \rangle} \right) \right\rangle,$$

$$\left\langle (\tau_4,\ \tau_3), \left(\frac{u_1}{\langle (0.57e^{i2\pi(0.58)},\ 0.79e^{i2\pi(0.71)}),\ (0.45e^{i2\pi(0.48)},\ 0.69e^{i2\pi(0.7)}) \rangle}, \frac{u_2}{\langle (0.42e^{i2\pi(0.4)},\ 0.76e^{i2\pi(0.79)}),\ (0.43e^{i2\pi(0.45)},\ 0.79e^{i2\pi(0.78)}) \rangle}, \frac{u_3}{\langle (0.4e^{i2\pi(0.39)},\ 0.75e^{i2\pi(0.8)}),\ (0.55e^{i2\pi(0.65)},\ 0.71e^{i2\pi(0.76)}) \rangle} \right) \right\rangle,$$

$$\left\langle (\tau_4,\ \tau_4), \left(\frac{u_1}{\langle (0.57e^{i2\pi(0.6)},\ 0.79e^{i2\pi(0.71)}),\ (0.48e^{i2\pi(0.48)},\ 0.69e^{i2\pi(0.7)}) \rangle}, \frac{u_2}{\langle (0.46e^{i2\pi(0.45)},\ 0.74e^{i2\pi(0.77)}),\ (0.49e^{i2\pi(0.5)},\ 0.71e^{i2\pi(0.72)}) \rangle}, \frac{u_3}{\langle (0.48e^{i2\pi(0.44)},\ 0.72e^{i2\pi(0.74)}),\ (0.58e^{i2\pi(0.69)},\ 0.67e^{i2\pi(0.7)}) \rangle} \right) \right\rangle \right\}$$

Step 3 The score value of LOCq-RLDFHSNs in LOCq-RLDFHSS $(\Omega,\ \mathfrak{T}_1 \times \mathfrak{T}_1)$ is described in Table 3.

Table 3. score value of LOCq-RLDFHSNs in $(\Omega,\ \mathfrak{T}_1 \times \mathfrak{T}_1)$

$(\Omega,\ \mathfrak{T}_1 \times \mathfrak{T}_1)$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$
(τ_1, τ_1)	-0.4593	-0.5360	-0.5109
(τ_1, τ_2)	-0.3840	-0.4458	-0.5109
(τ_1, τ_3)	-0.4518	-0.4432	-0.5109
(τ_1, τ_4)	-0.4518	-0.4329	-0.5109
(τ_2, τ_1)	-0.3772	-0.5360	-0.4411
(τ_2, τ_2)	-0.3457	-0.4219	-0.3719
(τ_2, τ_3)	-0.3307	-0.3897	-0.3694
(τ_2, τ_4)	-0.3307	-0.3194	-0.3306
(τ_3, τ_1)	-0.4035	-0.5360	-0.4411
(τ_3, τ_2)	-0.3595	-0.4819	-0.3561
(τ_3, τ_3)	-0.3455	-0.3972	-0.2959
(τ_3, τ_4)	-0.3455	-0.3406	-0.2209
(τ_4, τ_1)	-0.3610	-0.5360	-0.4411
(τ_4, τ_2)	-0.2733	-0.4219	-0.3442
(τ_4, τ_3)	-0.1951	-0.3817	-0.2790
(τ_4, τ_4)	-0.1451	-0.2720	-0.1650

Step 4 From the obtained score values of LOCq-RLDFHSNs in $(\Omega,\ \mathfrak{T}_1 \times \mathfrak{T}_1)$, the comparison table of $(\Omega,\ \mathfrak{T}_1 \times \mathfrak{T}_1)$ is generated and presented in Table 4.

Table 4. Comparison table

$(\Omega,\ \mathfrak{T}_1 \times \mathfrak{T}_1)$	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$
$\mathfrak{u}_1$	16	12	12
$\mathfrak{u}_2$	4	16	4
$\mathfrak{u}_3$	4	12	16

Step 5 From the comparison table of $(\Omega,\ \mathfrak{T}_1 \times \mathfrak{T}_1)$, the row sum, column sum and score value of alternatives $\mathfrak{u}_1$, $\mathfrak{u}_2$ and $\mathfrak{u}_3$ are obtained and presented in Table 5.

Table 5. Row sum, column sum, score of the alternatives

	row sum	column sum	score
$\mathfrak{u}_1$	40	24	16
$\mathfrak{u}_2$	24	40	-16
$\mathfrak{u}_3$	32	32	0

From the obtained score value of alternatives, we can observe that the alternative $\mathfrak{u}_1$ is the most appropriate hydroponic vertical farming system and the alternatives got ranked as $\mathfrak{u}_2 < \mathfrak{u}_3 < \mathfrak{u}_1$.

5. Comparative analysis

The superiority of the proposed notions are discussed in a pointwise manner as follows:

• Since the proposed notions Cq-RLDFHSS and LOCq-RLDFHSS are handling problems with complex numbers in the unit disk of complex plane, these proposed notions obviously surpass the notions FS [1], IFS [2], PFS [3], q-ROFS [4], LDFS [5], q-RLDFS [6], SS [13], FSS [14], IFSS [15], q-ROFSS [17], LDFSS [18], HSS [27], FHSS [27], IFHSS [27], q-ROFHSS [29] and q-RLDFHSS [30] because these notions are constrained in the interval [0, 1] which is a subset of unit disk of complex plane.

• Although Cq-RLDFS [12] can manage issues that CFS [7], CIFS [8], CPFS [9], Cq-ROFS [10], and CLDFS [11] cannot, it is challenging to handle Cq-RLDF issues when they fall under many sub-attributes. When it comes to handling Cq-RLDF difficulties with multiple sub-attributes, the suggested Cq-RLDFHSS is sufficient. Furthermore, as these theories CFS, CIFS, CPFS and Cq-ROFS are unable to handle issues in a Cq-RLDF environment, the suggested Cq-RLDFHSS is superior to currently available attributed complex fuzzy theories like CFSS [19], CIFSS [20], Cq-ROFSS [21], CPFSS [22], CLDFSS [23], CFHSS [31], CIFHSS [31], and Cq-ROFHSS [32].

• Even though the current theories, LOSS [24], LOFSS [25], and LOIFSS [26], are helpful when it involves ordering among the attributes, the proposed LOCq-RLDFHSS is superior to these existing notions, since these notions are ineffective for addressing circumstances where there is an order among the attributes under Cq-RLDFS or when ordering among several sub-attributes.

It is clear from the discussion that the proposed notions outperform many of the current notions. While the suggested Cq-RLDFHSS and LOq-RLDFHSS outperform many of the current notions, they also have a few small drawbacks, like

• The suggested theories are unable to handle the case when the handling environment is not constrained by the Cq-RLDFS constraint.

Further, the described superiority of the proposed notions compared to other existing complex fuzzy and attributed complex fuzzy extensions is presented as a table in Table 6.

Table 6. Comparison of proposed notions over existing notions

Notions	CMS degree	CNMS degree	CRPs	attributes	multi sub-attributes	order among the attributes	order among the multi sub-attributes
CFS [7]	✓	×	×	×	×	×	×
CIFS [8]	✓	✓	×	×	×	×	×
Cq-ROFS [10]	✓	✓	×	×	×	×	×
CLDFS [11]	✓	✓	✓	×	×	×	×
Cq-RLDFS [12]	✓	✓	✓	×	×	×	×
CFSS [19]	✓	×	×	✓	×	×	×
CIFSS [20]	✓	✓	×	✓	×	×	×
Cq-ROFSS [22]	✓	✓	×	✓	×	×	×
CLDFSS [23]	✓	✓	✓	✓	×	×	×
CFHSS [27]	✓	×	×	✓	✓	×	×
CIFHSS [27]	✓	✓	×	✓	✓	×	×
Cq-ROFHSS [28]	✓	✓	×	✓	✓	×	×
Cq-RLDFHSS (proposed)	✓	✓	✓	✓	✓	×	×
LOCq-RLDFHSS (proposed)	✓	✓	✓	✓	✓	✓	✓

6. Conclusion

The Cq-RLDFS is a unique extension of FS theory for solving uncertain situations across a broad range. This study proposes the concepts of Cq-RLDFHSS and LOCq-RLDFHSS for addressing Cq-RLDF circumstances with multiple sub-attributes. In addition, some fundamental algebraic operations such as restricted intersection, restricted union, extended union, OR operation, AND operation and complement of LOCq-RLDFHSS and comparison table of LOCq-RLDFHSS are established and for solving MADM issues effectively, an algorithm based on LOCq-RLDFHSS comparison table and operations is defined. In combination with the real-world application of the suggested algorithm, a vertical farming MADM problem is examined. Further, in this study, the relative superiority of the suggested notions over the current notions is investigated by a comparative analysis. While the suggested theories and MADM algorithm have many benefits, they also have some minor limitations such as, the suggested theories are unable to manage scenarios in which the Cq-RLDFS constraint is not followed. In future, some aggregation operators and information measures will be developed based on proposed Cq-RLDFHSS and LOCq-RLDFHSS.

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Conflict of interest

The authors declare no competing financial interest.

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