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Research Article

A Couple of Fresh New Perspectives on the Concatenation Model with Power-Law of Self-Phase Modulation

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Abstract: In this paper, we have secured new optical solitons for the concatenation model with power-law nonlinearity. The traveling wave hypothesis serves as the starting point. To retrieve optical soliton solutions, we have implemented two powerful techniques into the model: the Sardar Sub-Equation Method (SSEM) and the Tanh-Coth method. For power-law nonlinearity, we derived through the balancing principle that solitons would exist for different values of the power-law parameter. Therefore, we have secured a large variety of new soliton solutions for the model. This paper derives dark, bright, and singular soliton solutions for the value of n, as the first case was already covered in a previous report dedicated to addressing the model with Kerr law nonlinearity. Lastly, all the parametric existence conditions of the solitons and all solutions have been constructed.

***Keywords*:** traveling waves, Sardar's sub-equation method (SSEM), tanh-coth method, concatenation model, power law

MSC: 78A60

1. Introduction

In the last ten years or so, the concatenation model has emerged in the field of nonlinear optics [1–3]. Following its launch, a plethora of results from numerous publications is apparent everywhere, covering a broad range of topics. Both results applicable to fiber-optic systems and several results with a strong mathematical component have emerged. Conservation laws, Painleve analysis, magneto-optic solitons, quiescent solitons for nonlinear chromatic dispersion, numerical analysis, bifurcation analysis, and birefringent fibers are only a few of the subjects that have been discussed in relation to the model [4–6]. Although the Kerr law of self-phase modulation (SPM) was used in the last round of bifurcation

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analysis of the model, the current study is a generalized version of the earlier counterpart [7]. This work focuses on the bifurcation analysis of the concatenation model with power-law SPM [8–10].

Our work builds on a decade of concatenation model research, making it important in the realm of nonlinear optics [11–13]. Numerous publications on subjects including conservation laws, solitons, and fiber-optic systems have been produced by this approach [14–16]. A more comprehensive analysis is provided by our work, which focuses on the concatenation model with the SPM power-law [17–19]. We also investigate the chaotic dynamics of the model and obtain soliton solutions [20–22]. This work advances the subject by providing fresh perspectives and possible uses in nonlinear optics [23–25].

Although it has limits, our investigation improves understanding of the concatenation model with power-law SPM [26–28]. It involves simplifications and theoretical emphasis, concentrating on a particular feature of the model [29–31]. We believe that this applies practically to fiber-optic systems. These limitations should be taken into account by researchers applying our findings to practical scenarios.

This paper introduces novel optical solitons in the concatenation model with power-law nonlinearity. Utilizing the traveling wave hypothesis, we employ the Sardar Sub-Equation Method (SSEM) and the Tanh-Coth method to extract soliton solutions. While these techniques offer valuable insights into nonlinear equations, they also have limitations. SSEM's effectiveness depends on the equation's specific form, which limits its applicability. It may struggle with certain nonlinearities or boundary conditions, leading to inaccuracies. Similarly, the Tanh-Coth method requires appropriate trial functions, which may not capture highly nonlinear behavior accurately. Convergence can be sensitive to initial guesses, posing challenges in obtaining reliable solutions, particularly in complex scenarios.

2. Governing model

The concatenation model with power nonlinearity is formulated as follows [11]:

$$\begin{aligned} & i\Phi_t + a\Phi_{xx} + b\left|\Phi\right|^{2n}\Phi \\ & + c_1\left[\sigma_1\Phi_{xxxx} + \sigma_2\left(\Phi_x\right)^2\Phi^* + \sigma_3\left|\Phi_x\right|^2\Phi + \sigma_4\left|\Phi\right|^{2n}\Phi_{xx} + \sigma_5\Phi^2\Phi^*_{xx} + \sigma_6\left|\Phi\right|^{2n+2}\Phi\right] \\ & + ic_2\left[\sigma_7\Phi_{xxx} + \sigma_8\left|\Phi\right|^{2n}\Phi_x + \sigma_9\Phi^2\Phi^*_x\right] = 0. \end{aligned} \tag{1}$$

As indicated by Equation (1), the concatenation model is a composite representation of the renowned Lakshmanan-Porsezian-Daniel (LPD) model, the Sasa-Satsuma equation (SSE), and the nonlinear Schrödinger's equation (NLSE) [3–10]. In this formulation, the NLSE with power-law nonlinearity serves as the foundation for the initial three terms in Equation (1), with n representing the parameter governing power-law nonlinearity, and the coefficients c_1 and c_2 originating from the LPD model and SSE, respectively. Equation (1) simplifies to the familiar SSE when c_1 equals zero, and to the LPD model when c_2 is zero. Nevertheless, Equation (1) transforms into the well-established NLSE with power-law nonlinearity when both c_1 and c_2 are zero.

The physical reason for considering the model presented in Equation (1) is to construct a flexible and adaptable framework that integrates components from multiple equations, thereby offering an invaluable resource for investigating various aspects of nonlinear optics and examining the influence of different parameters on system behavior.

3. Travelling wave solution

The soliton solutions of Equation (1) are assumed to be:

$$\Phi(x,\ t) = \ u(\xi)\ e^{i\,\theta(x,\ t)}. \tag{2}$$

In this context, the wave variable is defined as $\xi = \ x-\gamma t$, where γ represents the speed of the soliton. Furthermore, $u(\xi)$ denotes the amplitude component of the soliton. Additionally, the phase component of the soliton is expressed as $\theta(x,\ t) = -kx+\omega\ t+\theta_0$, where k represents the soliton frequency, ω stands for the wavenumber, and θ_0 represents the phase constant. By utilizing Equation (2) and its derivatives, Equation (1) undergoes transformation to:

$$\begin{aligned}
&\left[-i\,\gamma u' - \omega u\right] + a\left[u'' - 2\,iku' - k^2\,u\right] + bu^{2n+1} \\
&+ c_1\sigma_1\left(u'''' - 4iku''' - 6k^2u'' + 4ik^3u' + k^4u\right) + c_1(\sigma_2+\sigma_3)\left(uu'^2 - 2u'kiu^2 - k^2u^3\right) \\
&+ c_1\sigma_4u^{2n}\ \left(u'' - 2\,iku' - k^2u\right) + c_1\sigma_5\left(u^2u'' - 2iku^2u' - k^2u^3\right) + c_1\sigma_6u^{2n+3} \\
&+ c_2\sigma_7\left(iu''' + 3ku'' - 3ik^2u' - k^3u\right) + c_2\sigma_8u^{2n}\left(iu' + k\,u\right) + c_2\sigma_9u^2\left(iu' + ku\right) = 0.
\end{aligned} \tag{3}$$

Equation (3) can be decomposed into real and imaginary parts, which are respectively expressed as:

$$\begin{aligned}
&c_1\sigma_1u^{(4)} + [a + 3c_2\sigma_7k - 6k^2c_1\ \sigma_1]u'' + c_1\sigma_5u^2u'' + c_1\sigma_4u^{2n}u'' \\
&+ c_1(\sigma_2+\sigma_3)uu'^2 + [c_1\sigma_1k^4 - ak^2 - c_2\sigma_7k^3 - \omega]u + bu^{2n+1} + kc_2\sigma_8u^{2n+1} \\
&+ [kc_2\sigma_9 - c_1(\sigma_2+\sigma_3+\sigma_5)k^2]u^3 - c_1\sigma_4u^{2n+1} + c_1\sigma_6u^{2n+3} = 0,
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
&[(c_2\sigma_7 - 4kc_1\sigma_1)u'''] + [4k^3c_1\sigma_1 - 3k^2c_2\sigma_7 - \gamma - 2aku'] \\
&+ [c_2\sigma_9 - 2kc_1(\sigma_2+\sigma_3+\sigma_5)]u'u^2 + [c_2\sigma_8 - 2kc_1\sigma_4]u'u^{2n} = 0.
\end{aligned} \tag{5}$$

From Equation (5), the soliton speed can be determined as follows:

$$\gamma = -2k\left(4k^2c_1\sigma_1 + a\right), \tag{6}$$

whenever

$$c_2\sigma_9 = 2kc_1(\sigma_2+\sigma_3+\sigma_5), \tag{7}$$

$$c_2\sigma_8 = 2kc_1\sigma_4, \tag{8}$$

$$c_2\sigma_7 = 4kc_1\sigma_1. \tag{9}$$

From Equations (8) and (9), we obtain the following restriction:

$$\sigma_7 = 2\sigma_8. \tag{10}$$

Equation (4) can be expressed as:

$$\begin{aligned} &c_1\sigma_1 u^{(4)} + \beta_1 u'' + c_1\sigma_5 u^2 u'' + c_1\sigma_4 u^{2n} u'' + c_1\left(\sigma_2+\sigma_3\right) u u'^2 \\ &+\beta_2 u + \beta_3 u^3 + \beta_4 u^{2n+1} + c_1\sigma_6 u^{2n+3} = 0, \end{aligned} \tag{11}$$

where

$$\beta_1 = a + 6k^2 c_1\sigma_1, \tag{12}$$

$$\beta_2 = -[ak^2 + 3c_1\sigma_1 k^4 + \omega], \tag{13}$$

$$\beta_3 = k^2 c_1\left(\sigma_2+\sigma_3+\sigma_5\right), \tag{14}$$

$$\beta_4 = \left(b + kc_2\sigma_8 - c_1\sigma_4\right). \tag{15}$$

Setting

$$u = v^{\frac{1}{n}}, \tag{16}$$

Equation (11) undergoes transformation to:

$$
c_1\sigma_1\left[\begin{array}{l} v^3 v'''' + 3\left(\frac{1-n}{n}\right) v^2 v' v''' + 2\left(\frac{1-n}{n}\right) v^2 v''^2 \\ \\ +5\left(\frac{1-2n}{n}\right)\left(\frac{1-n}{n}\right) v\, v'' v'^2 + \left(\frac{1-3n}{n}\right)\left(\frac{1-2n}{n}\right)\left(\frac{1-n}{n}\right) v'^4 \end{array}\right]
$$

$$
+\beta_1[v^3 v'' + \left(\frac{1-n}{n}\right) v^2 v'^2] + c_1\sigma_5 v^5 v'' + c_1\sigma_5\left(\frac{1-n}{n}\right) v^{\frac{2}{n}+2} v'^2 \tag{17}
$$

$$
+c_1\sigma_4[v^5 v'' + \left(\frac{1-n}{n}\right) v^4 v'^2] + \frac{c_1(\sigma_2+\sigma_3)}{n} v^{\frac{2}{n}+2} v'^2 + n^2\beta_2 v^4
$$

$$
+n^2\beta_3 v^{\frac{2}{n}+4} + n^2\beta_4 v^6 + n^2 c_1\sigma_6 v^{\frac{2}{n}+6} = 0.
$$

For integrability, the coefficients of $v^{\frac{2}{n}+2}$, $v^{\frac{2}{n}+4}$, and $v^{\frac{2}{n}+6}$ in Equation (17) must vanish. Consequently, we derive the following nonlinear ordinary differential equation:

$$
\begin{aligned} & c_1\sigma_1[v^3 v'''' + 3M_3 v^2 v' v''' + 2M_3 v^2 v''^2 + M_1 v v'' v'^2 + M_2 v'^4] \\ & + \beta_1[v^3 v'' + M_3 v^2 v'^2] + c_1\sigma_4 v^5 v'' + c_1\sigma_4 M_3 v^4 v'^2 + n^2\beta_2 v^4 + n^2\beta_4 v^6 = 0, \end{aligned} \tag{18}
$$

where

$$
\sigma_5 = 0,\ \sigma_6 = 0,\ (\sigma_2+\sigma_3) = 0, \tag{19}
$$

$$
M_1 = \frac{5}{n^2}\left(1-3n+2n^2\right), \tag{20}
$$

$$
M_2 = \frac{1}{n^3}\left(1-6n+11n^2-6n^3\right), \tag{21}
$$

$$
M_3 = \left(\frac{1-n}{n}\right). \tag{22}
$$

4. Sardar sub-equation method (SSEM)

The main advantage of the SSEM is its ability to generate various forms of soliton solutions, ranging from dark, bright, and singular to more intricate forms such as mixed dark-bright, dark-singular, bright-singular, and mixed singular. Additionally, it offers rational, periodic, trigonometric, and other solutions.

In this method, to solve Equation (18), we assume that the solution takes the form, as proposed in references [12, 13]:

$$v(\xi) = \sum_{n=0}^{N}, \lambda_n \Psi^n(\xi) \lambda_N \neq 0, \tag{23}$$

where λ_n (for $n = 0, 1, \ldots, N$) is a constant to be calculated later. The integer number N is determined by means of the homogeneous balance method principle between the nonlinear term and the highest-order derivative in Equation (18). Additionally, the function $\Psi^n(\xi)$ in Equation (20) must satisfy the following equation:

$$\Psi'(\xi) = \sqrt{\eta_2 \Psi(\xi)^4 + \eta_1 \Psi(\xi)^2 + \eta_0}, \tag{24}$$

where η_l, with $l = 0, 1, 2$, represents constants. Correspondingly, depending on the values of the parameters η_l, Equation (1) has various known solutions, as outlined below [12, 13]:

Case 1 When $\eta_0 = 0$, $\eta_1 > 0$, and $\eta_2 \neq 0$, then we get

$$\Psi_1^{\pm}(\xi) = \pm\sqrt{-pq^{\eta_1}/_{\eta_2}} \operatorname{sech}_{pq}(\sqrt{\eta_1}\xi), \ \eta_2 < 0, \tag{25}$$

$$\Psi_2^{\pm}(\xi) = \pm\sqrt{pq^{\eta_1}/_{\eta_2}} \operatorname{csch}_{pq}(\sqrt{\eta_1}\xi), \ \eta_2 > 0, \tag{26}$$

where

$$\begin{aligned} \operatorname{sech}_{pq}(\sqrt{\eta_1}\xi) &= \frac{2}{pe^{\sqrt{\eta_1}\xi} + qe^{-\sqrt{\eta_1}\xi}}, \\ \operatorname{csch}_{pq}(\sqrt{\eta_1}\xi) &= \frac{2}{pe^{\sqrt{\eta_1}\xi} - qe^{-\sqrt{\eta_1}\xi}}. \end{aligned} \tag{27}$$

Case 2 When $\eta_0 = \frac{1}{4}\frac{\eta_1^2}{\eta_2}$, $\eta_2 > 0$, and $\eta_1 < 0$, then one obtains

$$\Psi_3^{\pm}(\xi) = \pm\sqrt{-^{\eta_1}/_{2\eta_2}} \tanh_{pq}\left(\sqrt{-\frac{\eta_1}{2}}\xi\right), \tag{28}$$

$$\Psi_4^{\pm}(\xi) = \pm\sqrt{-\eta_1/2\eta_2}\coth_{pq}\left(\sqrt{-\frac{\eta_1}{2}}\xi\right), \tag{29}$$

$$\Psi_5^{\pm}(\xi) = \pm\sqrt{-\eta_1/2\eta_2}\left(\tanh_{pq}\left(\sqrt{-2\eta_1}\xi\right) \pm i\sqrt{pq}\operatorname{sech}_{pq}\left(\sqrt{-2\eta_1}\xi\right)\right), \tag{30}$$

$$\Psi_6^{\pm}(\xi) = \pm\sqrt{-\eta_1/2\eta_2}\left(\coth_{pq}\left(\sqrt{-2\eta_1}\xi\right) \pm \sqrt{pq}\operatorname{csch}_{pq}\left(\sqrt{-2\eta_1}\xi\right)\right), \tag{31}$$

$$\Psi_7^{\pm}(\xi) = \pm\frac{1}{2}\sqrt{-\eta_1/2\eta_2}\left(\tanh_{pq}\left(\sqrt{-\frac{\eta_1}{8}}\xi\right) \pm \coth_{pq}\left(\sqrt{\frac{-\eta_1}{8}}\xi\right)\right), \tag{32}$$

where

$$\begin{aligned}\tanh_{pq}(\sqrt{\eta_1}\xi) &= \frac{pe^{\sqrt{\eta_1}\xi} - qe^{-\sqrt{\eta_1}\xi}}{pe^{\sqrt{\eta_1}\xi} + qe^{-\sqrt{\eta_1}\xi}}, \\ \coth_{pq}(\sqrt{\eta_1}\xi) &= \frac{pe^{\sqrt{\eta_1}\xi} + qe^{-\sqrt{\eta_1}\xi}}{pe^{\sqrt{\eta_1}\xi} - qe^{-\sqrt{\eta_1}\xi}}.\end{aligned} \tag{33}$$

4.1 *Application of the Sardar sub-equation method*

We initiated our analysis by applying the principle of the homogeneous balance method between the nonlinear term v^3v'''' and the nonlinear term v^6 from Equation (18). This yields the equation $3N+N+4=6N$, from which we obtain N = 2. Therefore, Equation (23) transforms into:

$$v(\xi) = \left(\lambda_0 + \lambda_1\Psi(\xi) + \lambda_2\Psi^2(\xi)\right). \tag{34}$$

By substituting Equation (34) into Equation (18) and taking into account Equation (24), we obtain:

$$
\begin{aligned}
& c_1\sigma_1 \left[(\lambda_0+\lambda_1\Psi+\lambda_2\Psi^2)^3 \left\{\left(\lambda_1\eta_1+8\lambda_2\eta_1\Psi+6\lambda_1\eta_2\Psi^2+24\lambda_2\eta_2\Psi^3\right)\left(2\eta_2\Psi^3+\eta_1\Psi\right)\right.\right. \\
& \left.+\left(8\lambda_2\eta_1+12\lambda_1\eta_2\Psi+72\lambda_2\eta_2\Psi^2\right)\left(\eta_2\Psi^4+\eta_1\Psi^2+\eta_0\right)\right\} \\
& +3M_3\left\{\begin{array}{l}\left(\lambda_0+\lambda_1\Psi+\lambda_2\Psi^2\right)^2(\lambda_1+2\lambda_2\Psi) \\ \times\left(\lambda_1\eta_1+8\lambda_2\eta_1\Psi+6\,\lambda_1\eta_2\Psi^2+24\lambda_2\eta_2\Psi^3\right)\left(\eta_2\Psi^4+\eta_1\Psi^2+\eta_0\right)\end{array}\right\} \\
& +2M_3\left\{\left(\lambda_0+\lambda_1\Psi+\lambda_2\Psi^2(\xi)\right)^2\left[2\lambda_2\eta_0+\lambda_1\eta_1\Psi+4\lambda_2\eta_1\Psi^2+2\lambda_1\eta_2\Psi^3+6\lambda_2\eta_2\Psi^4\right]^2\right\} \\
& +M_1\left\{\begin{array}{l}\left(\lambda_0+\lambda_1\Psi+\lambda_2\Psi^2\right)\left(2\lambda_2\eta_0+\lambda_1\eta_1\Psi+4\lambda_2\eta_1\Psi^2+2\lambda_1\eta_2\Psi^3+6\lambda_2\eta_2\Psi^4\right) \\ \times(\lambda_1+2\lambda_2\Psi)^2\left(\eta_2\Psi^4+\eta_1\Psi^2+\eta_0\right)\end{array}\right\} \\
& \left.+M_2\left\{(\lambda_1+2\lambda_2\Psi)^4\left(\eta_2\Psi^4+\eta_1\Psi^2+\eta_0\right)^2\right\}\right] \\
& +\beta_1\left\{\left(\lambda_0+\lambda_1\Psi+\lambda_2\Psi^2\right)^3\left(2\lambda_2\eta_0+\lambda_1\eta_1\Psi+4\lambda_2\eta_1\Psi^2+2\lambda_1\eta_2\Psi^3+6\lambda_2\eta_2\Psi^4\right)\right. \\
& \left.+M_3\left(\lambda_0+\lambda_1\Psi+\lambda_2\Psi^2\right)^2(\lambda_1+2\lambda_2\Psi)^2\left(\eta_2\Psi^4+\eta_1\Psi^2+\eta_0\right)\right\} \\
& +c_1\left\{\sigma_4\left(\lambda_0+\lambda_1\Psi+\lambda_2\Psi^2\right)^5\left(2\lambda_2\eta_0+\lambda_1\eta_1\Psi+4\lambda_2\eta_1\Psi^2+2\lambda_1\eta_2\Psi^3+6\lambda_2\eta_2\Psi^4\right)\right\} \\
& +c_1\sigma_4M_3\left\{\left(\lambda_0+\lambda_1\Psi+\lambda_2\Psi^2\right)^4(\lambda_1+2\lambda_2\Psi)^2\left(\eta_2\Psi^4+\eta_1\Psi^2+\eta_0\right)\right\} \\
& +n^2\beta_2\left(\lambda_0+\lambda_1\Psi+\lambda_2\Psi^2\right)^4+n^2\beta_4\left(\lambda_0+\lambda_1\Psi+\lambda_2\Psi^2\right)^6=0.
\end{aligned} \tag{35}
$$

By gathering and equating the coefficients of the independent functions $\Psi^j(\xi)$ to zero, we arrive at the following set of algebraic system equations:

Family I ($\eta_0=0,\ \lambda_0=0,\ \lambda_1=0$)

Equation (35) is reduced to the following equation:

$$
\begin{aligned}
&c_1\sigma_1\left[\left\{120\eta_1\eta_2\Psi^{10}+16\eta_1{}^2\Psi^8+120\eta_2\eta_2\Psi^{12}\right\}\right.\\
&+3M_3 16\left(4\eta_1\eta_2\Psi^{10}+\eta_1\eta_1\Psi^8+3\eta_2\eta_2\Psi^{12}\right)\\
&+2M_3 4\left[\eta_1{}^2\Psi^8+\eta_1 3\eta_2\Psi^{10}+9\eta_2{}^2\Psi^{12}\right]\\
&+M_1 8\left(5\eta_1\eta_2\Psi^{10}+2\eta_1\eta_1\Psi^8+3\eta_2\eta_2\Psi^{12}\right)\\
&\left.+M_2 16\left(\eta_2{}^2\Psi^{12}+2\eta_2\Psi^{10}\eta_1+\eta_1{}^2\Psi^8\right)\right]\\
&+2\left(2\eta_1\Psi^8+3\eta_2\Psi^{10}\right)+4M_3\left(\eta_2\Psi^{10}+\eta_1\Psi^8\right)\\
&+2c_1\sigma_4\,\lambda_2{}^2\left(2\eta_1\Psi^{12}+3\,\eta_2\Psi^{14}\right)+4c_1\sigma_4 M_3\,\lambda_2{}^2\left(\eta_2\Psi^{14}+\eta_1\Psi^{12}\right)\\
&+n^2\beta_2\Psi^8+n^2\beta_4\lambda_2{}^2\Psi^{12}=0.
\end{aligned}
\tag{36}
$$

We derive the following set of algebraic system equations for the same Ψ^j, where j = 8, 10, 12:

$$
\begin{aligned}
&\Psi^8: c_1\sigma_1[16\,\eta_1{}^2+56M_3\eta_1{}^2+16M_1\eta_1{}^2]+16M_2\eta_1{}^2+4\eta_1+4M_3\eta_1+n^2\beta_2=0,\\
&\Psi^{10}: c_1\sigma_1[120+192M_3+24M_3+40M_1+32M_2]\eta_1+6+4M_3=0,\\
&\Psi^{12}: c_1\sigma_1[120+144M_3+72M_3+24M_1+16M_2]\eta_2{}^2\\
&\qquad+\left(4c_1\sigma_4\eta_1+4c_1\sigma_4M_3\eta_1+n^2\beta_4\right)\lambda_2{}^2=0.
\end{aligned}
\tag{37}
$$

Solving the set of algebraic system equations (37) yields:
Case I ($\eta_1>0$ and $\eta_2\neq 0$)

$$
\begin{aligned}
&\eta_{1,\,1}=\frac{-n^2}{2c_1\sigma_1\left(3n^3-n^2-2n+2\right)},\ \eta_{1,\,2}=-\frac{(n+2)\,n^2}{4c_1\sigma_1\left[14n^3-4n^2+n+4\right]},\ \sigma_4=0,\\
&\lambda_2=\mp\frac{2\eta_2}{n^2}\sqrt{-\frac{c_1\sigma_1[12n^3+7n^2+6n+4]}{n\beta_4}},
\end{aligned}
\tag{38}
$$

where

$$\beta_2 = 0,\ \omega = -k^2\left(1 + 3c_1\sigma_1 k^2\right).$$

The solutions of Equation (1) corresponding to Equation (38), along with solutions (25) and (26), are:

$$\Phi_1(x,\ t) = \left[\mp\frac{2\eta_2}{n^2}\sqrt{-\frac{c_1\sigma_1\left[12n^3 + 7n^2 + 6n + 4\right]}{n\beta_4}}\,\mathrm{sech}^2{}_{pq}\left(\sqrt{\eta_{1,\ j}}\xi\right)\right]^{\frac{1}{n}} \tag{39}$$

$$\times\, exp\left[i\left(-\kappa x + \omega t + \theta_0\right)\right],\quad j = 1,\ 2$$

and

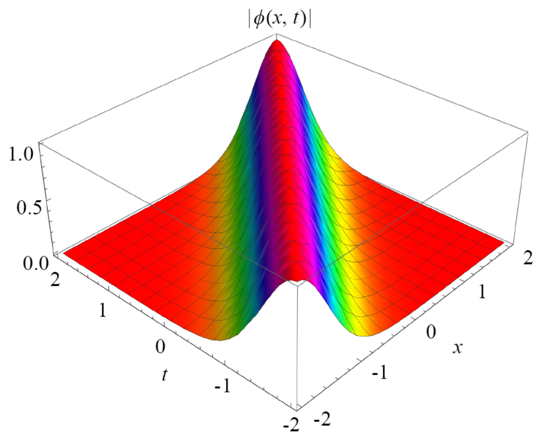
$$\Phi_2(x,\ t) = \left[\mp\frac{2\eta_2}{n^2}\sqrt{-\frac{c_1\sigma_1\left[12n^3 + 7n^2 + 6n + 4\right]}{n\beta_4}}\,\mathrm{csch}^2{}_{pq}\left(\sqrt{\eta_{1,\ j}}\xi\right)\right]^{\frac{1}{n}} \tag{40}$$

$$\times\, exp\left[i\left(-\kappa x + \omega t + \theta_0\right)\right],\quad j = 1,\ 2$$

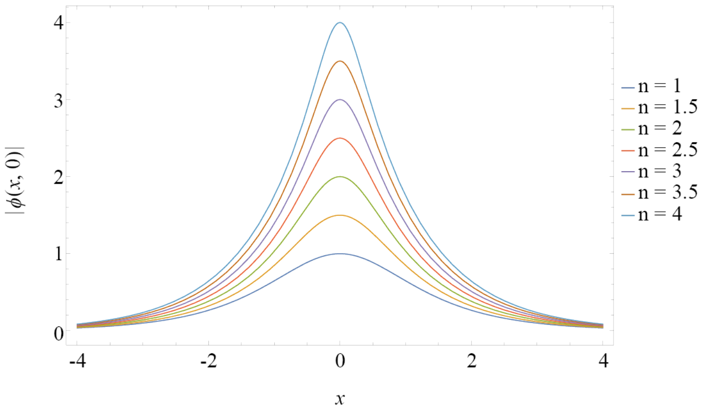
where $\beta_4 < 0$.

Solutions (39) and (40) represent the bright and singular soliton solutions, respectively.

Figure 1 illustrates the 2D and surface plots of a bright optical soliton (39), displaying the results under the following parameter settings: $\eta_2 = 1$, $\beta_4 = 1$, $c_1 = -1$, $\sigma_1 = 1$, $\gamma = 1$, $\eta_{1,\ 1} = 1$, and $\eta_{1,\ 2} = 1$.



(a) Surface plot



(b) 2D plot

Figure 1. Analysis of the individual properties showcased by a bright optical soliton

4.2 *Tanh-coth method*

Assuming $v = v(\xi)$, we utilize the ansatz proposed in reference [14]:

$$Y = \tanh(\mu\xi). \tag{41}$$

This leads to the change of variables:

$$\frac{dv}{d\xi} = \mu\left(1-Y^2\right)\frac{dv}{dY}, \tag{42}$$

and

$$\frac{d^2v}{d\xi^2} = \mu^2[-2Y\left(1-Y^2\right)\frac{dv}{dY} + \left(1-Y^2\right)^2\frac{d^2v}{dY^2}]. \tag{43}$$

For the next step, let's assume that the solution for Equation (18) is expressed in the form:

$$v(Y) = \sum_{i=0}^{p} a_i Y^i + \sum_{i=1}^{p} b_i Y^{-i}. \tag{44}$$

Applying the principle of the homogeneous balance method between the nonlinear term $v^3 v''''$ and the nonlinear term v^6 from Equation (18), we have $3N+N+4=6N$, which results in $N=2$. Therefore, Equation (44) becomes:

$$v(Y) = \left(a_0 + a_1 Y + a_2 Y^2 + \frac{b_1}{Y} + \frac{b_2}{Y^2}\right). \tag{45}$$

Here a_0, a_1, a_2, b_1, and b_2 are constants to be determined. Next, substitute Equation (45) with their derivatives into Equation (18). For simplicity, let's assume $a_0 = a_1 = b_1 = 0$. We then obtain the following:

$$c_1\sigma_1\left[\left(a_2Y^2+\frac{b_2}{Y^2}\right)^3\left(-16a_2+120a_2Y^2-120a_2Y^4+\frac{120b_2}{Y^6}-\frac{108b_2}{Y^4}+\frac{12b_2}{Y^2}\right)\right.$$

$$+3M_3\left(a_2Y^2+\frac{b_2}{Y^2}\right)^2\left(2a_2Y-2\frac{b_2}{Y^3}-2a_2Y^3+\frac{2b_2}{Y}\right)$$

$$\times\left(-16a_2Y+40\,a_2Y^3-24a_2Y^5-\frac{24b_2}{Y^5}+\frac{36b_2}{Y^3}-\frac{12b_2}{Y}\right)$$

$$+2M_3\left(a_2Y^2+\frac{b_2}{Y^2}\right)^2\left(2a_2+2b_2-8a_2Y^2+6a_2Y^4+\frac{6b_2}{Y^4}-\frac{6b_2}{Y^2}\right)^2$$

$$+M_1\left(\begin{array}{l} a_2Y^2\left(2a_2+2b_2-8a_2Y^2+6a_2Y^4+\frac{6b_2}{Y^4}-\frac{6b_2}{Y^2}\right) \\ +\frac{b_2}{Y^2}\left(2a_2+2b_2-8a_2Y^2+6a_2Y^4+\frac{6b_2}{Y^4}-\frac{6b_2}{Y^2}\right) \end{array}\right)$$

$$\times\left(2a_2Y-2\frac{b_2}{Y^3}-2a_2Y^3+\frac{2b_2}{Y}\right)^2+M_2\left(2a_2Y-2\frac{b_2}{Y^3}-2a_2Y^3+\frac{2b_2}{Y}\right)^4\Bigg]$$

$$+\beta_1\left[\left(a_2Y^2+\frac{b_2}{Y^2}\right)^3\left(2a_2+2b_2-8a_2Y^2+6a_2Y^4+\frac{6b_2}{Y^4}-\frac{6b_2}{Y^2}\right)\right. \tag{46}$$

$$\left.+M_3\left(a_2Y^2+\frac{b_2}{Y^2}\right)^2\left(2a_2Y-2\frac{b_2}{Y^3}-2a_2Y^3+\frac{2b_2}{Y}\right)^2\right]$$

$$+c_1\sigma_4\left(a_2Y^2+\frac{b_2}{Y^2}\right)^5\left(2a_2+2b_2-8a_2Y^2+6\,a_2Y^4+\frac{6b_2}{Y^4}-\frac{6b_2}{Y^2}\right)$$

$$+c_1\sigma_4M_3\left(a_2Y^2+\frac{b_2}{Y^2}\right)^4\left(2a_2Y-2\frac{b_2}{Y^3}-2a_2Y^3+\frac{2b_2}{Y}\right)^2$$

$$+n^2\beta_2\left(a_2Y^2+\frac{b_2}{Y^2}\right)^4+n^2\beta_4\left(a_2Y^2+\frac{b_2}{Y^2}\right)^6=0.$$

Let $a_2=b_2$, we obtain the following equation:

$$\mu^4c_1\sigma_1\left[64\left(-2\left(-4Y+17Y^3-30Y^5+15Y^7+17\frac{1}{Y}-30\frac{1}{Y^3}+15\frac{1}{Y^5}\right)\right.\right.$$

$$+5\left(17Y-4Y^3+17Y^5-30Y^7+15Y^9-30\frac{1}{Y}+15\frac{1}{Y^3}\right)$$

$$-3\left(-30Y+17Y^3-4Y^5+17Y^7-30Y^9+15Y^{11}+15\frac{1}{Y}\right)$$

$$
\begin{aligned}
& -2\left(17Y-30Y^3+15Y^5-4\frac{1}{Y}+17\frac{1}{Y^3}-30\frac{1}{Y^5}+15\frac{1}{Y^7}\right)\\
& +5\left(-30Y+15Y^3+17\frac{1}{Y}-4\frac{1}{Y^3}++17\frac{1}{Y^5}-30\frac{1}{Y^7}+15\frac{1}{Y^9}\right)\\
& -3\left(-4\frac{1}{Y^5}+17\frac{1}{Y^3}-15\frac{1}{Y}-15Y+17Y^3\right)\Bigg)\\
& -72\left(\begin{array}{l}-20+16Y^2-3Y^4-8Y^6+10Y^8-8Y^{10}+3Y^{12}\\ +16\frac{1}{Y^2}-3\frac{1}{Y^4}-8\frac{1}{Y^6}+10\frac{1}{Y^8}-8\frac{1}{Y^{10}}+3\frac{1}{Y^{12}}\end{array}\right)\\
& -12\left(\begin{array}{l}8Y^{12}-24Y^{10}+46Y^8-88\,Y^6+119Y^4-144Y^2\\ +164-144\frac{1}{Y^2}+119\frac{1}{Y^4}-88\frac{1}{Y^6}+46\frac{1}{Y^8}-24\frac{1}{Y^{10}}+8\frac{1}{Y^{12}}\end{array}\right)\\
& +8M_1\left(\begin{array}{l}3Y^{12}-10Y^{10}+10Y^8-2Y^6-3Y^4\\ +4Y^2-20+4\frac{1}{Y^2}-3\frac{1}{Y^4}-2\frac{1}{Y^6}+10\frac{1}{Y^8}-10\frac{1}{Y^{10}}+3\frac{1}{Y^{12}}\end{array}\right) \qquad (47)\\
& +M_2 16\left(\begin{array}{l}Y^{12}-4Y^{10}+2Y^8+12Y^6-17Y^4-8Y^2\\ +28-8\frac{1}{Y^2}-17\frac{1}{Y^4}+12\frac{1}{Y^6}+2\frac{1}{Y^8}-4\frac{1}{Y^{10}}+\frac{1}{Y^{12}}\end{array}\right)\Bigg]\\
& +2\mu^2\beta_1\left[\begin{array}{l}\left(\begin{array}{l}3Y^{10}-4Y^8+11Y^6-16Y^4+18Y^2-24\\ +18\frac{1}{Y^2}-16\frac{1}{Y^4}+11\frac{1}{Y^6}-4\frac{1}{Y^8}+3\frac{1}{Y^{10}}\end{array}\right)\\ -3\left(Y^{10}-2Y^8+Y^6-2Y^2+4-2\frac{1}{Y^2}+\frac{1}{Y^6}-2\frac{1}{Y^8}+\frac{1}{Y^{10}}\right)\end{array}\right]\\
& +n^2\beta_2\left(Y^8+4Y^4+6+4\frac{1}{Y^4}+\frac{1}{Y^8}\right)\\
& +2\mu^2c_1\sigma_4a_2^2\left(2\left(Y^{10}+5Y^6+10Y^2+10\,\frac{1}{Y^2}+5\frac{1}{Y^6}+\frac{1}{Y^{10}}\right)\right.
\end{aligned}
$$

$$-4\left(Y^{12}+5Y^{8}+10Y^{4}+10+5\ \frac{1}{Y^{4}}+\frac{1}{Y^{8}}\right)$$

$$+3\left(Y^{14}+5Y^{10}+10\,Y^{6}+10Y^{2}+5\frac{1}{Y^{2}}+\frac{1}{Y^{6}}\right)$$

$$-4\left(Y^{8}+5Y^{4}+10+10\frac{1}{Y^{4}}+5\frac{1}{Y^{8}}+\frac{1}{Y^{12}}\right)$$

$$+3\left(Y^{6}+5Y^{2}+10\ \frac{1}{Y^{2}}+10\frac{1}{Y^{6}}+5\frac{1}{Y^{10}}+\frac{1}{Y^{14}}\right)\Bigg)$$

$$-6\mu^{2}\,c_{1}\sigma_{4}a_{2}^{2}\left(\begin{array}{l} Y^{14}-2Y^{12}+3Y^{10}+Y^{6}+2Y^{4}-5Y^{2}+4 \\ -5\frac{1}{Y^{2}}-6\frac{1}{Y^{4}}+\frac{1}{Y^{6}}-4\frac{1}{Y^{8}}+3\frac{1}{Y^{10}}-2\frac{1}{Y^{12}}+\frac{1}{Y^{14}} \end{array}\right)$$

$$+n^{2}\beta_{4}a_{2}^{2}\left(Y^{12}+6Y^{8}+15Y^{4}+20+15\frac{1}{Y^{4}}+6\frac{1}{Y^{8}}+\frac{1}{Y^{12}}\right)=0.$$

Thus, a set of algebraic equations is obtained as follows:

$$\begin{aligned}
&\left(Y^{12}+\frac{1}{Y^{12}}\right): 8\mu^{4}c_{1}\sigma_{1}[49n^{3}+128n^{2}+38n+2]+4\mu^{2}n^{3}c_{1}\sigma_{4}a_{2}^{2}+n^{5}\beta_{4}a_{2}^{2}=0,\\
&\left(Y^{10}+\frac{1}{Y^{10}}\right): 8\mu^{2}c_{1}\sigma_{1}[56n^{3}+62n^{2}-2n-8]+n^{3}c_{1}\sigma_{4}a_{2}^{2}=0,\\
&\left(Y^{8}+\frac{1}{Y^{8}}\right): \begin{array}{l} 8\mu^{4}c_{1}\sigma_{1}[-83n^{3}-106n^{2}+26n+4] \\ +4\mu^{2}n^{3}\beta_{1}+n^{5}\beta_{2}-24n^{3}c_{1}\sigma_{4}a_{2}^{2}+6n^{5}\beta_{4}\,a_{2}^{2}=0, \end{array}\\
&\left(Y^{6}+\frac{1}{Y^{6}}\right): \mu^{2}c_{1}\sigma_{1}[190n^{3}+1131n^{2}-601n+96]-n^{3}\beta_{1}+40n^{3}c_{1}\sigma_{4}a_{2}^{2}=0,\\
&\left(Y^{4}+\frac{1}{Y^{4}}\right): \begin{array}{l} \mu^{4}c_{1}\sigma_{1}[180n^{3}-2632n^{2}+1512n+272]-32\mu^{2}n^{3}\beta_{1} \\ +4n^{5}\beta_{2}-132\mu^{2}n^{3}c_{1}\sigma_{4}a_{2}^{2}+15n^{5}\beta_{4}a_{2}^{2}=0, \end{array}
\end{aligned} \tag{48}$$

$$\left(Y^2+\frac{1}{Y^2}\right)\mu^2 c_1\sigma_1[-1024n^3+928n^2-608n+128]+48n^3\beta_1+160n^3c_1\sigma_4 a_2^2=0,$$

$$\left(Y^0+\frac{1}{Y^0}\right): \begin{array}{l} 10\mu^4 c_1\sigma_1[-301n^3+458n^2-218n+28]-48n^3\mu^2\beta_1 \\ +6n^5\beta_2+20n^5\beta_4 a_2^2-184n^3\mu^2 c_1\sigma_4 a_2^2=0. \end{array}$$

Solving the set of algebraic equations (48) yields:

$$\mu=\frac{n}{2}\sqrt{\frac{-\left(a+6\,k^2c_1\sigma_1\right)n}{2\left(332+3n-2333n^2-2220n^3\right)c_{\,1}\sigma_1}}. \tag{49}$$

Family I

$$a_2=b_2=\mp\frac{2\mu^2}{n}\sqrt{-\frac{2c_1\sigma_1[49n^3+128n^2+38n+2]}{n\left(4\mu^2nc_1\sigma_4+n^2\beta_4\right)}}. \tag{50}$$

Accordingly, a dark-singular straddled soliton solution shapes up as

$$\Phi_5(x,\ t)=\left[\mp\frac{2\mu^2}{n}\sqrt{-\frac{2c_1\sigma_1[49n^3+128n^2+38n+2]}{n\left(4\mu^2nc_1\sigma_4+n^2\beta_4\right)}}\left\{\begin{array}{c} tanh^2\left(\mu\left(x-\gamma t\right)\right) \\ +coth^2\left(\mu\left(x-\gamma t\right)\right) \end{array}\right\}\right]^{\frac{1}{n}} \tag{51}$$

$$\times\exp[i\left(-\kappa x+\omega t+\theta_0\right)].$$

Family II

$$a_2=b_2=\mp\frac{2\mu}{n}\sqrt{\frac{2[-56n^3-62\,n^2+2n+8]}{n}}. \tag{52}$$

Consequently, a dark-singular straddled soliton solution turns out to be

$$\Phi_6(x,\ t)=\left[\mp\frac{2\mu}{n}\sqrt{\frac{2[-56n^3-62n^2+2n+8]}{n}}\left\{\begin{array}{c} tanh^2\left(\mu\left(x-\gamma t\right)\right) \\ +coth^2\left(\mu\left(x-\gamma t\right)\right) \end{array}\right\}\right]^{\frac{1}{n}} \tag{53}$$

$$\times\exp[i\left(-\kappa x+\omega t+\theta_0\right)].$$

Family III

$$a_2 = b_2 = \mp\frac{2\mu^2}{n}\sqrt{\frac{c_1\sigma_1\left[-83n^3-106n^2+26n+4\right]+4\mu^2n^3\beta_1+n^5\beta_2}{3n\left(4c_1\sigma_4-n^3\beta_4\right)}}. \tag{54}$$

As a result, a dark-singular straddled soliton solution becomes

$$\Phi_7(x,\,t) = \left[\begin{array}{l}\mp\frac{2\mu^2}{n}\sqrt{\frac{c_1\sigma_1\left[-83n^3-106n^2+26n+4\right]+4\mu^2n^3\beta_1+n^5\beta_2}{3n\left(4c_1\sigma_4-n^3\beta_4\right)}}\\ \times\left\{tanh^2\left(\mu\left(x-\gamma t\right)\right)+coth^2\left(\mu\left(x-\gamma t\right)\right)\right\}\end{array}\right]^{\frac{1}{n}} \tag{55}$$

$$\times\exp[i(-\kappa x+\omega t+\theta_0)].$$

Family IV

$$a_2 = b_2 = \mp\frac{1}{2n}\sqrt{\frac{n^3\beta_1-\mu^2c_1\sigma_1\left[190n^3+1131n^2-601n+96\right]}{10nc_1\sigma_4}}. \tag{56}$$

Thus, our outcome includes a dark-singular straddled soliton solution

$$\Phi_8(x,\,t) = \left[\begin{array}{l}\mp\frac{1}{2n}\sqrt{\frac{n^3\beta_1-\mu^2c_1\sigma_1\left[190n^3+1131n^2-601n+96\right]}{10nc_1\sigma_4}}\\ \times\left\{tanh^2\left(\mu\left(x-\gamma t\right)\right)+coth^2\left(\mu\left(x-\gamma t\right)\right)\right\}\end{array}\right]^{\frac{1}{n}} \tag{57}$$

$$\times\exp[i(-\kappa x+\omega t+\theta_0)$$

Family V

$$a_2 = b_2 = \mp\frac{1}{n}\sqrt{\frac{\mu^4c_1\sigma_1\left[180n^3-2632n^2+1512n+272\right]-32\mu^2n^3\beta_1+4n^5\beta_2}{n\left(132\mu^2c_1\sigma_4-15n^2\beta_4\right)}}. \tag{58}$$

Therefore, a dark-singular straddled soliton solution is acquired:

$$\Phi_9(x,\ t) = \begin{bmatrix} \mp\frac{1}{n}\sqrt{\frac{\mu^4 c_1\sigma_1\left[180n^3-2632n^2+1512n+272\right]-32\mu^2n^3\beta_1+4n^5\beta_2}{n\left(132\mu^2c_1\sigma_4-15n^2\beta_4\right)}} \\ \times\left\{tanh^2(\mu(x-\gamma t))+\coth^2(\mu(x-\gamma t))\right\} \end{bmatrix}^{\frac{1}{n}} \tag{59}$$

$$\times\exp[i(-\kappa x+\omega t+\theta_0)].$$

Family VI

$$a_2 = b_2 = \mp\frac{1}{4n}\sqrt{\frac{\mu^2c_1\sigma_1[-1024n^3+928n^2-608n+128]+48n^3\beta_1}{-10nc_1\sigma_4}}. \tag{60}$$

Accordingly, a dark-singular straddled soliton solution is obtained from the analysis, as follows

$$\Phi_{10}(x,\ t) = \begin{bmatrix} \mp\frac{1}{4n}\sqrt{\frac{\mu^2c_1\sigma_1[-1024n^3\ +928n^2-608n+128]+48n^3\beta_1}{-10nc_1\sigma_4}} \\ \times\left\{tanh^2(\mu(x-\gamma t))+coth^2(\mu(x-\gamma t))\right\} \end{bmatrix}^{\frac{1}{n}} \tag{61}$$

$$\times\exp[i(-\kappa x+\omega t+\theta_0)].$$

Family VII

$$a_2 = b_2 = \mp\frac{1}{2n}\sqrt{\frac{10\mu^4c_1\sigma_1[-301n^3+458n^2-218n+28]-48n^3\mu^2\beta_1+6n^5\beta_2}{n\left(46\mu^2c_1\sigma_4-5n^2\beta_4\right)}}. \tag{62}$$

Henceforth, the solution turns out to be a dark-singular straddled soliton solution

$$\Phi_{11}(x,\ t) = \begin{bmatrix} \mp\frac{1}{2n}\sqrt{\frac{10\mu^4c_1\sigma_1[-301n^3+458n^2-218n+28]-48n^3\mu^2\ \beta_1+6n^5\beta_2}{n\left(46\mu^2c_1\sigma_4-5n^2\beta_4\right)}} \\ \times\left\{tanh^2(\mu(x-\gamma t))+coth^2(\mu(x-\gamma t))\right\} \end{bmatrix}^{\frac{1}{n}} \tag{63}$$

$$\times\exp[i(-\kappa x+\omega t+\theta_0)].$$

5. Results and discussion

Utilizing the Sardar Sub-Equation Method (SSEM) and the Tanh-Coth method technique, we have successfully derived exact analytical solutions for the concatenation model with power law nonlinearity, as described in Equation (1). We transform the concatenation model with power law nonlinearity into a system of real and imaginary equations, described in Equations (4) and (5). Subsequently, we explore the analytical solutions of this system using the Sardar Sub-Equation Method (SSEM) and Tanh-Coth method technique. Special assumptions are employed in extracting the soliton solutions, such as the bright, dark and singular soliton solution. Bright solitons are fascinating phenomena in nonlinear optics, offering stable, localized wave packets that propagate through dispersive media without dispersion-induced spreading. These solitons, governed by the nonlinear Schrödinger equation (NLSE), exhibit distinctive features influenced by the nonlinearity of the medium.

In Figure 1, we explore the behavior of bright soliton solutions under the influence of power law nonlinearity, as described by Eq. (39). This equation characterizes the evolution of the soliton profile, $\Phi(x, t)$, under various power law exponents (n), highlighting the impact of different nonlinear regimes. Figure 1(a) presents a surface plot illustrating the bright soliton solution at $t = 0$ with a power law exponent of $t = 1$, corresponding to the Kerr law nonlinearity. The surface plot vividly portrays the stable, bell-shaped profile of the soliton, indicative of its self-trapping nature and resilience against dispersion. Figure 1(b) delves deeper into the behavior of the bright soliton solution by presenting 2D plots for various power law exponents ($n = 1$, 1.5, 2, 2.5, 3, 3.5, and 4). As n deviates from unity, we observe significant alterations in the soliton's profile. For $n = 1$, the soliton profile broadens, reflecting the weakening of nonlinear effects and increased susceptibility to dispersion. Conversely, as n surpasses unity, the soliton's profile narrows and intensifies, indicative of enhanced self-focusing and stronger nonlinear interactions. This transition highlights the crucial role of the power law exponent in shaping the dynamics of optical solitons. By setting the time variable $t = 0$, we focus on the initial state of the bright soliton solution, capturing its intrinsic properties before any temporal evolution occurs. This instantaneous snapshot allows us to analyze the soliton's behavior at the onset of propagation, providing valuable insights into its stability and nonlinear characteristics.

6. Conclusion

In the current paper, the entire discriminant approach was employed to study the concatenation model with the power-law of SPM. For each of the power-law parameter 'n' values, the integration schemes, namely the Sardar Sub-Equation Method (SSEM) and the Tanh-Coth method, derived bright, dark, and singular soliton solutions. By utilizing parameter constraints, all of these soliton existence requirements are provided. The model holds a promising future, and the results are truly encouraging. Furthermore, methods other than the Sardar Sub-Equation Method (SSEM) and Tanh-Coth method technique will also be explored for related quiescent optical solitons and gap solitons. Additionally, it is possible to examine the model with various waveguide types, such as optical metamaterials [31–35].

In addition to the current findings, future research could explore several promising directions. Firstly, investigating the derived soliton solutions under various perturbations would provide valuable insights into their practical applicability. Secondly, extending the analysis to include higher-order nonlinearities or non-local effects could yield a more comprehensive understanding of soliton dynamics in complex optical systems. Furthermore, exploring the interaction of solitons with external potentials or defects within the medium could lead to the discovery of novel phenomena and potential applications in nonlinear optics. Lastly, considering multi-dimensional soliton solutions and their propagation characteristics in different geometries could open up new avenues for exploring nonlinear wave phenomena in diverse physical systems.

Confilict of interest

The authors claim that there is no conflict of interest.

References

[1] Ankiewicz A, Wang Y, Wabnitz S, Akhmediev N. Extended nonlinear Schrödinger equation with higher-order odd and even terms and its rogue wave solutions. *Physical Review E*. 2014; 89(1): 012907.

[2] Ankiewicz A, Akhmediev N. Higher-order integrable evolution equation and its soliton solutions. *Physics Letters A*. 2014; 378(4): 358-361.

[3] Arnous AH, Biswas A, Kara AH, Yıldırım Y, Moraru L, Iticescu C, et al. Optical solitons and conservation laws for the concatenation model with spatio-temporal dispersion (internet traffic regulation). *Journal of the European Optical Society-Rapid Publications*. 2023; 19(2): 35.

[4] Jawad A, Biswas A. Solutions of resonant nonlinear Schrödinger's equation with exotic non-Kerr law nonlinearities. *Al-Rafidain Journal of Engineering Sciences*. 2024; 2(1): 43-50.

[5] Arnous AH, Biswas A, Kara AH, Yıldırım Y, Moraru L, Iticescu C, et al. Optical solitons and conservation laws for the concatenation model: power–law nonlinearity. *Ain Shams Engineering Journal*. 2024; 15(2): 102381.

[6] Biswas A, Vega-Guzman JM, Yildirim Y, Moshokoa SP, Aphane M, Alghamdi AA. Optical solitons for the concatenation model with power-law nonlinearity: undetermined coefficients. *Ukrainian Journal of Physical Optics*. 2023; 24(3).

[7] Tang L, Biswas A, Yıldırım Y, Alghamdi AA. Bifurcation analysis and optical solitons for the concatenation model. *Physics Letters A*. 2023; 480: 128943.

[8] Jawad AJM, Abu-AlShaeer MJ. Highly dispersive optical solitons with cubic law and cubic-quinticseptic law nonlinearities by two methods. *Al-Rafidain Journal of Engineering Sciences*. 2023; 1(1): 1-8.

[9] Triki H, Sun Y, Zhou Q, Biswas A, Yıldırım Y, Alshehri HM. Dark solitary pulses and moving fronts in an optical medium with the higher-order dispersive and nonlinear effects. *Chaos, Solitons & Fractals*. 2022; 164: 112622.

[10] Yıldırım Y, Biswas A, Moraru L, Alghamdi AA. Quiescent optical solitons for the concatenation model with nonlinear chromatic dispersion. *Mathematics*. 2023; 11(7): 1709.

[11] Tang L, Biswas A, Yildirim Y, Asiri A. Bifurcation analysis and chaotic behavior of the concatenation model with power-law nonlinearity. *Contemporary Mathematics*. 2023; 4(4): 1014-1025.

[12] Rezazadeh H, Inc M, Baleanu D. New solitary wave solutions for variants of (3+1)-dimensional Wazwaz-Benjamin-Bona-Mahony equations. *Frontiers in Physics*. 2020; 8: 332.

[13] Muhammad T, Hamoud AA, Emadifar H, Hamasalh FK, Azizi H, Khademi M. Traveling wave solutions to the Boussinesq equation via Sardar sub-equation technique. *AIMS Mathematics*. 2022; 7: 11134-11149.

[14] Jawad A, Biswas A, Yildirim Y, Alghamdi AA. Dispersive optical solitons with Schrödinger-Hirota equation by a couple of integration schemes. *Journal of Optoelectronics and Advanced Materials*. 2023; 25(3-4): 203-209.

[15] Wang S. Novel soliton solutions of CNLSEs with Hirota bilinear method. *Journal of Optics*. 2023; 52(3): 1602-1607.

[16] Kopçasız B, Yaşar E. The investigation of unique optical soliton solutions for dual-mode nonlinear Schrödinger's equation with new mechanisms. *Journal of Optics*. 2023; 52(3): 1513-1527.

[17] Tang L. Bifurcations and optical solitons for the coupled nonlinear Schrödinger equation in optical fiber Bragg gratings. *Journal of Optics*. 2023; 52(3): 1388-1398.

[18] Thi TN, Van LC. Supercontinuum generation based on suspended core fiber infiltrated with butanol. *Journal of Optics*. 2023; 52(4): 2296-2305.

[19] Li Z, Zhu E. Optical soliton solutions of stochastic Schrödinger-Hirota equation in birefringent fibers with spatiotemporal dispersion and parabolic law nonlinearity. *Journal of Optics*. 2023; 53: 1302-1308.

[20] Han T, Li Z, Li C, Zhao L. Bifurcations, stationary optical solitons and exact solutions for complex Ginzburg-Landau equation with nonlinear chromatic dispersion in non-Kerr law media. *Journal of Optics*. 2023; 52(2): 831-844.

[21] Tang L. Phase portraits and multiple optical solitons perturbation in optical fibers with the nonlinear Fokas-Lenells equation. *Journal of Optics*. 2023; 52(4): 2214-2223.

[22] Nandy S, Lakshminarayanan V. Adomian decomposition of scalar and coupled nonlinear Schrödinger equations and dark and bright solitary wave solutions. *Journal of Optics*. 2015; 44: 397-404.

[23] Chen W, Shen M, Kong Q, Wang Q. The interaction of dark solitons with competing nonlocal cubic nonlinearities. *Journal of Optics*. 2015; 44: 271-280.

[24] Xu SL, Petrović N, Belić MR. Two-dimensional dark solitons in diffusive nonlocal nonlinear media. *Journal of Optics*. 2015; 44: 172-177.

[25] Dowluru RK, Bhima PR. Influences of third-order dispersion on linear birefringent optical soliton transmission systems. *Journal of Optics*. 2011; 40: 132-142.

[26] Singh M, Sharma AK, Kaler R. Investigations on optical timing jitter in dispersion managed higher order soliton system. *Journal of Optics*. 2011; 40: 1-7.

[27] Janyani V. Formation and propagation-dynamics of primary and secondary soliton-like pulses in bulk nonlinear media. *Journal of Optics*. 2008; 37: 1-8.
[28] Hasegawa A. Application of optical solitons for information transfer in fibers-A tutorial review. *Journal of Optics*. 2004; 33(3): 145-156.
[29] Mahalingam A, Uthayakumar A, Anandhi P. Dispersion and nonlinearity managed multisoliton propagation in an erbium doped inhomogeneous fiber with gain/loss. *Journal of Optics*. 2013; 42: 182-188.
[30] AlQahtani SA, Alngar ME, Shohib R, Alawwad AM. Enhancing the performance and efficiency of optical communications through soliton solutions in birefringent fibers. *Journal of Optics*. 2024; 2024: 1-11.
[31] Abdelwahed H, Alsarhana A, El-Shewy E, Abdelrahman MA. Characteristics of new stochastic solitonic solutions for the chiral type of nonlinear Schrödinger equation. *Fractal and Fractional*. 2023; 7(6): 461.
[32] Ghafoor U, Abbas M, Akram T, El-Shewy EK, Abdelrahman MA, Abdo NF. An efficient cubic B-spline technique for solving the time fractional coupled viscous burgers equation. *Fractal and Fractional*. 2024; 8(2): 93.
[33] Abdelwahed H, Alsarhan A, El-Shewy E, Abdelrahman MA. Novel explosive and super fractional nonlinear Schrödinger structures. *Journal of Mathematics*. 2023; 2023: 9996773.
[34] Rabie WB, Ahmed HM, Darwish A, Hussein HH. Construction of new solitons and other wave solutions for a concatenation model using modified extended tanh-function method. *Alexandria Engineering Journal*. 2023; 74: 445-451.
[35] Alhojilan Y, Ahmed HM, Rabie WB. Stochastic solitons in birefringent fibers for Biswas-Arshed equation with multiplicative white noise via Itô calculus by modified extended mapping method. *Symmetry*. 2023; 15(1): 207.