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Research Article

Mirror Symmetry, Zeta Functions and Mackey Functors

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Abstract: We present two alternative and new proofs for the duality between orbifold zeta functions of Berglund-Hubsch dual invertible polynomials. We re-prove the following theorem; Assume W and W^{T} are dual invertible polynomials in $n+2$ variables. Denote by (X_W, G) and $(X_{W^T}, G^{\mathrm{T}})$ the corresponding Berglund-Hubsch dual hypersurfaces in $\mathbb{P}^{n+1}$, where G and G^{T} stands for their group of symmetries. The orbifold L-series of X_W and $X_{W^{\mathrm{T}}}$ satisfy: (*) $L_{\mathrm{orb}}(X_W, s) = L_{\mathrm{orb}}(X_{W^{\mathrm{T}}}, s)^{(-1)^n}$. We present two proofs of the above identity (*). Our methods of proof are different. The first proof uses cohomological Mackey functors on Mackey systems. The second proof is independent and uses a formula for the orbifold zeta functions. For an orbifold (X, G) we consider a Mackey system of subgroups of G and cohomological Mackey functors on this Mackey system. We investigate the relation between the above L-series of orbifolds and the Mackey functors. We show the orbifold cohomology $H^*_{\mathrm{orb}}(X, \mathbb{C})$ is an $\mathrm{End}_{\mathbb{C}G}[\bigoplus_g \mathbb{C}G/C(g)]$-module, that means; the orbifold cohomology defines a cohomological Mackey functor on the Mackey system of conjugacy classes in G. This leads one to split the zeta function according to properties of G-cohomological Mackey functors. This method allows obtaining identities on orbifold zeta functions from identities in a Grothendick group associated with subgroup quotients of G. In this context, the relation (1) is a consequence of cohomological mirror symmetry and Mackey structure. In other words, we obtain the identity (1) from Mackey type of identities in a Grothendieck group followed by a multiplicative homomorphism constructed from zeta functions of a Galois representation. The second proof uses a duality between age functions $\iota: G \to \mathbb{Z}$, and $\iota^{\mathrm{T}}: G^{\mathrm{T}} \to \mathbb{Z}$, of the dual invertible polynomials. It is known that G^{T} is the character group of G. We show that these age functions are Fourier transforms of each other with respect to a unitary representation obtained from the natural pairing between G and G^{T}. Using a formula of the orbifold zeta function in terms of the age functions, we deduce a comparison of zeta functions for the two dual invertible polynomials as given in the above.

Keywords: Orbifold Hodge structure, Chen-Ruan cohomology, zeta and L-series, Mackey system, Mackey functor, mirror symmetry, Berglund-Hubsch duality

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1. Introduction

The zeta functions of complex analytic varieties is an invariant of the variety which characterizes many (topological) properties of the ambient space. Despite its simplicity the zeta or L-series encodes substantial information about the manifold or singularities, [1–5]. In this paper, we deal with the interaction between zeta functions and the Mirror symmetry of Chen-Ruan. Mirror symmetry is a branch of mathematics that studies a duality phenomenon that appears in properties of Calabi-Yau (CY) varieties. The name "mirror" reflects the fact that CY varieties appear in mirror pairs. Mirror properties of CY Manifolds were first discovered by Physicists but later became a systematic program of research for mathematicians. Homological Mirror symmetry considers a duality between the Hodge decompositions of two mirror Calabi-Yau manifolds. This property was first discovered by M. Kontsevich. A form of this duality was studied by Berglund-Hubsch for dual polynomials (see below) or hypersurfaces concerning orbifold Hodge structure [6–10].

We study quotient manifolds of type Y/G where Y is a complex topological manifold and G is a finite group. We consider some additional structures in these spaces relevant to the stabilizers of the G-action on the points in Y. We call such a structure an orbifold structure. This structure naturally appears in the finite quotients of homogeneous hypersurfaces X in $\mathbb{P}^N$. The quotient here is by the group of nontrivial symmetries of the homogeneous polynomial W defining X. We present and will work with an example of orbifolds in Section 1.3. A major step is to define a cohomology theory for these objects, as the usual cohomology of a multisector space denoted by ΣX. This space is a disjoint union of the manifolds $Y^g/C(g)$, $g \in G$, where Y^g is the points fixed by g in Y and $C(g)$ is the conjugacy class of g in G. The resulting cohomology is called orbifold cohomology or Chen-Ruan (CR) cohomology. The orbifold cohomology has a special Hodge structure and in the étale settings is a representation of the absolute Galois group of the number field where W is defined over. In this context, one defines a new sort of zeta function associated with this representation, called orbifold zeta function. Berglund-Hubsch duality is a way to assign to a special homogeneous polynomial W (with the same number of terms as variables) another polynomial W^{T} of the same form. Then one may study the effect of this assignment on the associated orbifold zeta function.

We introduce some of the terminologies that we use in the course of presenting our main result. We first introduce orbifold cohomology and Hodge structure for orbifolds X.

1.1 *Orbifold Hodge structure*

Assume $X = [Y/G]$ is an orbifold where G is a finite group. By this, roughly speaking we mean that we consider the quotient space Y/G together with the inertia (stabilizer) structure at each point of Y ([1, 8, 11–13], see also Appendix A for a detailed discussion). In [7], Ruan introduced a new cohomology theory of orbifolds by defining

$$H^k_{\text{orb}}(X) := H^k(\Sigma X) = \bigoplus_{g \in T} H^{k-2\iota(g)}(X_{(g)}), \qquad X_{(g)} = Y^g/C(g), \tag{1}$$

where $\Sigma X := \coprod_g X_{(g)}$ is the multisector space, and T denotes a set of representatives for the conjugacy classes in G. The superfix by g, means the elements fixed by g and $C(g)$ is the conjugacy class of $g \in G$. The function $\iota \colon G \to \mathbb{Q}$, $(g \mapsto \iota_g)$ is the age function that appears in the grading of $H^k_{\text{orb}}(X)$ (see Appendix A). In our case, the age function takes values in $\mathbb{Z}$. The orbifold cohomology satisfies a Hodge structure;

$$H^k_{\text{orb}}(X) = \bigoplus_{p+q=k} \bigoplus_{g} H^{p-\iota_g,\, q-\iota_g}_{\text{orb}}(X_{(g)}) \tag{2}$$

The above Hodge structure is called orbifold Hodge structure, see [6, 8–11, 14]. One can also define the Chen-Ruan orbifold cohomology in the étale setting for the orbifold X (see [1]), in a similar way. The definition is as follows, (see [1])

$$H^k_{et,\,\mathrm{orb}}(\overline{X},\,\mathbb{Q}_l) := H^k_{et}((\Sigma\overline{X},\,\mathbb{Q}_l) = \bigoplus_{a+2b=k} H^a_{et}(\mathrm{age}^{-1}(b),\,\mathbb{Q}_l) \tag{3}$$

Here $\Sigma X := \coprod_g Y^g/C(g)$ is the inertia scheme, [1, 11].

1.2 *Orbifold zeta and L-series*

The étale cohomology is naturally a Galois module, that is a module over $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We have a Galois representation

$$\rho\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}(H^d_{et,\,\mathrm{orb}}(\bar{X},\,\mathbb{Q}_l)) \tag{4}$$

The Frobenius substitutions

$$\mathrm{Frob}_{\mathfrak{p}} = (\mathfrak{p},\,F/\mathbb{Q})\colon b \mapsto b^p \pmod{\mathfrak{p}} \tag{5}$$

of $\mathfrak{p}$ for F finite and Galois over $\mathbb{Q}$ are topological generators of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The induced action on the étale cohomology groups is denoted by the same symbol $\mathrm{Frob}_{\mathfrak{p}}$ (agrees with the inverse of Frobenius called geometric Frobenius) defines the representation (4). In the orbifold case, one needs to modify the representation (4) as (see [1] prop. 1.1);

$$F_{\mathfrak{p},\,\mathrm{orb}}\colon H^*_{\mathrm{orb}}(\overline{X_{et}},\,\mathbb{Q}_l) \to H^*_{\mathrm{orb}}(\overline{X_{et}},\,\mathbb{Q}_l), \qquad \alpha \mapsto q^{-\iota(\alpha)}\mathrm{Frob}_{\mathfrak{p}}(\alpha) \tag{6}$$

Convention: We consider a convention on the determinant and trace function on graded vector spaces or complexes. For a linear map $F\colon V \to V$ on a $\mathbb{Z}$-graded vector space we write $V = \oplus_i V_i$, $F = \oplus F_i$ and we denote $\det(F|V) = \prod_i \det(F_i|V_i)^{(-1)^{i+1}}$, and $\mathrm{Tr}(F|V) = \sum_i (-1)^i \mathrm{Tr}(F_i|V_i)$.

The orbifold zeta function of X is defined as follows.

Definition 1 [1, 2] The orbifold cohomological zeta function is given by

$$L_{\mathrm{orb}}(X,\,t) := \det\left(1 - F_{\mathrm{orb}}.t|_{H^*_{\mathrm{orb}}(\overline{X_{et}},\,\mathbb{Q}_l)}\right) = \exp\left(\sum_{r=1}^{\infty} \mathrm{Tr}\left(F^r_{\mathrm{orb}}|_{H^*_{\mathrm{orb}}(\overline{X_{et}},\,\mathbb{Q}_l)}\right)\frac{t^r}{r}\right) \tag{7}$$

In this case one computes (cf. [1] section 6),

$$\mathrm{Tr}(F_{\mathrm{orb}}|H^*_{\mathrm{orb}}) = \sum_g \mathrm{Tr}(F_{\mathrm{orb}}|_{X_g}|H^*) = \sum_g q^{-\iota_g} \sum_a \frac{q^{-\dim(a)}}{\sharp\mathrm{Aut}(a)} \tag{8}$$

where $\mathrm{Aut}(a)$ is the automorphism group the sector corresponding to a. Plugging (8) in equation (7) then the formula (7) becomes

$$L_{\text{orb}}(X, t) = \exp\left(\sum_{r=1}^{\infty}\sum_{\xi}\frac{(q^r)^{-\text{age}(\xi)-\dim(\xi)}}{\sharp\text{Aut}(\xi)}\cdot\frac{t^r}{r}\right) \tag{9}$$

We refer to [1] definition 6.1 for details of the computation, see also [3]. The zeta function (7) is (also) associated with the representation of the Galois group of the number field K over which the variety X is defined, see Appendix B. In this text, we deal with the orbifold zeta functions of some special hypersurfaces, and $K = \mathbb{Q}$.

1.3 *Berglund-Hubsch dual invertible polynomials*

We work in the weighted projective space $\mathbb{P}(q_1, \ldots, q_{n+2})$ where $q_i = w_i/d$ have common denominators and we have $gcd(w_1, \ldots, w_{n+2}) = 1$. In [15] Bergland and Hubsch look at the orbifolds X defined by weighted homogeneous polynomials with the same number of variables as monomials

$$W = \sum_{i=1}^{n+2} x_1^{m_{i,1}} \ldots x_{n+2}^{m_{i,n+2}}, \qquad W(\lambda^{q_1}x_1, \ldots, \lambda^{q_{n+2}}x_{n+2}) = \lambda W(x_1, .., x_{n+2}) \tag{10}$$

We assume $W: \mathbb{C}^{n+2} \to \mathbb{C}$ has isolated singularity at the origin. The group of non-trivial automorphism of W is

$$G = \text{Aut}(W)/J_W, \qquad J_W = \langle \text{diag}(\exp(2\pi i q_1), \ldots, \exp(2\pi i q_{n+2}))\rangle \tag{11}$$

In the notation used at the beginning of this section. The (Chen-Ruan) orbifold cohomology of the hypersurface $X_W = \{W = 0\}$ inside the weighted projective space $\mathbb{P}(w_1, \ldots, w_{n+2})$ is defined by

$$H^k_{CR}([X_W/G], \mathbb{C}) := H^k_{orb}([X_W/G], \mathbb{C}) \tag{12}$$

To each such polynomial one may associate an integer matrix $A = (m_{ij})$ that encode the exponent of x_j in the i-th monomial. In the mirror symmetry setup we always assume that the matrix $A = (m_{ij})$ is invertible. One can make a duality by transposition of these exponents with $(i, j) \mapsto (j, i)$ obtaining another polynomial W^{T}. Mirror symmetry studies the properties of the orbifolds Y/G and $Y^{\text{T}}/G^{\text{T}}$ via the Bergland-Hubsch duality, [2, 7, 12, 16–20]. The mirror map establishes an isomorphism between the Hodge pieces of orbifold cohomologies of the corresponding orbifolds in different levels in the form

$$H^{p,q}_{CR}([X_W/G], \mathbb{C}) = H^{\dim -p,q}_{CR}([X_{W^{\text{T}}}/G^{\text{T}}], \mathbb{C}) \tag{13}$$

On account of the correspondence (13), one may investigate possible relations between the orbifold zeta functions of W and W^{T}.

1.4 *Contribution of the text*

We compare the orbifold zeta functions of the varieties $W = 0$ and $W^{\text{T}} = 0$. We reprove the following theorem by a systematic application of Mackey functors.

Theorem 1 The orbifold L-series of X_W and X_{W^T} satisfy

$$L_{\text{orb}}(X_W, s) = L_{\text{orb}}(X_{W^T}, s)^{(-1)^n} \tag{14}$$

In [2] a proof of this theorem has already been presented. We give two new proofs of the above theorem which provide a mirror symmetric insights. Our proofs also connects several contexts in mathematics, namely Mirror symmetry, MacKey Functors, and zeta functions.

Our result appears as the proof of the Theorem 4. We also provide several lemmas and propositions on the way to obtain the above result. They appear as Lemma 2, Definition 3 and Propositions 1 and 2. The results appear in Section 3.

1.5 *Method of the proof*

We prove the identity (14) in two ways different from that in [2]. Our method in the first proof employs MacKey functors defined for MacKey systems, [21–23]. We consider a Mackey system $(\mathscr{C}, \mathscr{O})$ of subgroups of G in (11) and cohomological a Mackey functor defined by the Chen-Ruan orbifold cohomology. As a result, the Chen-Ruan cohomology defines a Mackey functor $X \mapsto H^*_{\text{orb}}(X, \mathbb{C})$ on the Mackey system $(G, \mathscr{C} = \{C(g) | \, g \in G\})$ of subgroups $C(g)$ of conjugacy classes in the group G of non-trivial symmetries of the polynomial W. That is the assignment $C(g) \mapsto H^{n-2\iota_g}_{\text{orb}}(X_{(g)})$ makes $H^*_{\text{orb}}(X, \mathbb{C})$ an $\text{End}_{\mathbb{C}G}[\bigoplus_g \mathbb{C}G/C(g)]$-module. This leads us to split the zeta function according to some properties of G-modules as cohomological Mackey functors. The method is theoretical and gives insights beyond Mirror symmetry.

The second proof uses some formulas of the orbifold zeta functions as quotient stacks, [1]. The formula concerns the age function of the associated orbifold. In fact we apply mirror symmetry to the context of orbifold quotient spaces. A main ingredient of the proof is an inversion relation between the age functions of two dual invertible polynomials. Specifically these age functions are Fourier transform of each other with respect to the unitary representation induced by a natural pairing.

1.6 *Related works*

There is already a proof of the Theorem 4 in [2]. We provide an alternative proof of this Theorem based on the MacKey systems and functors [21]. Some related facts and conjectures about the classical zeta functions of mirror CY-manifolds are provided in [16–19, 24, 25] and the references therein.

1.7 *Organization of the text*

In Section 2 we introduce Mackey systems of subgroups of a given group and their Mackey functors as systematic operations on this collection of objects. Section 3 contains our main result namely Theorem 4. We reprove a duality between the orbifold L-series of Berglund-Hubsch invertible polynomials using an application of Mackey functors to zeta functions. In Appendix A we introduce basic definitions and properties of orbifolds. We define the Chen-Ruan orbifold cohomology and explain its orbifold Hodge structure. In continuation, we also explain the étale orbifold cohomology and describe a similar structure on the étale cohomology of orbifolds. In Appendix B we provide the definitions related to zeta and L-series, both in the usual and orbifold case.

2. Cohomological Mackey functors

In this section, we introduce Mackey systems of groups and their Mackey functors following [21], see also [22, 23, 26, 27]. We wish to use this terminology for the orbifold cohomology and their zeta functions in the next section. Let G be a group. Mackey functors on Mackey systems are powerful tools appearing in many branches of mathematics. We first define these systems.

Definition 2 [21] A Mackey system $(\mathscr{C}, \mathscr{O})$ for G is the data

(S1) $\mathscr{C}$ is a set of subgroups of G closed under conjugation and finite intersection. In this case for H a subgroup of G we set $\mathscr{C}(H) = \{ U \subset H | \, U \in \mathscr{C} \}$.

(S2) A family $\mathscr{O}(H) \subset \mathscr{C}(H)$ for $H \in \mathscr{O}$ such that

- $[H\colon U]$ is finite for $U \in \mathscr{O}(H)$.
- $\mathscr{O}(U) \subset \mathscr{O}(H)$ for $U \in \mathscr{O}(H)$
- $\mathscr{O}(gHg^{-1}) = g\mathscr{O}(H)g^{-1}$
- $U \cap V \in \mathscr{V}$ for $V \in \mathscr{C}(H)$

Assume k is a commutative ring with 1. Let $(G, \mathscr{C}, \mathscr{O})$ be a Mackey system. A k-Mackey functor M on this system (also called G-functor) into the category of k-modules is the data

(F1) A family of k-modules $M(H)$.

(F2) A family of k-linear maps $c_H^g\colon M(H) \to M({}^gH)$ for each $H \in \mathscr{C}$ and $g \in G$ namely conjugation.

(F3) A family of k-linear maps $res_I^H\colon M(H) \to M(I)$ for each $I \leq H$ in $\mathscr{C}$, called restriction maps.

(F4) A family of k-linear maps $ind_I^H\colon M(I) \to M(H)$ for each $I \leq H$ with $H \in \mathscr{C}$ and $I \in \mathscr{O}(H)$ called induction or transfer maps. These maps are all supposed to be transitive on subgroups,

(F5) The restriction and induction commute with conjugation and satisfy

$$res_I^H \circ ind_J^H = \sum_{h \in I \backslash H / J} ind_{{}^hJ \cap I} \circ res_{{}^hJ \cap I} \circ c_J^h \tag{15}$$

called Mackey formula or Mackey decomposition.

(F6) A Mackey functor is called cohomological if

$$ind_I^H \circ res_I^H = [H\colon I].id_{M(H)}, \;\; H \in \mathscr{C}, \;\; I \in \mathscr{O}(H) \tag{16}$$

A basic example of Mackey systems appears for Galois groups of field extensions in number theory. An interesting example of Mackey functor is the Galois cohomology, where the appropriate axioms correspond to well-known properties of Galois cohomology.

Mackey functors can be presented in several different ways. We briefly mention two of them to give some sense of their behavior.

A Mackey functor can alternatively be defined as a bi-functor $M\colon D(G) \to Ab$ from the category of G-sets and G-maps to the category of abelian groups and group homomorphisms. By a bifunctor, we mean pair of functors $(M_*, M^*)\colon G$-sets $\to Ab$ where these two functors are the same on objects $M_*(S) = M^*(S), \;\; S \in D(G)$ but the first M_* is covariant and the second M^* is contravariant, and they satisfy certain compatibility conditions, op. cit (see [21] definition 2.6). One shows that the category of Mackey functors $\mathbf{Mack}_k(G)$ is equivalent to the category of bifunctors with certain compatibilities, ([21] theorem 2.7), see also [22, 23].

Mackey functors are also related to permutation G-modules. Let $\mathbf{Per}(kG)$ be the category of permutation kG-modules, that are kG-modules that have a finite basis that is permuted by the action of G. The following theorem explains this relation.

Theorem 2 (Yoshida) [21, 26, 27] If G is finite, there is an equivalence of categories

$$\mathscr{Y}\colon \mathbf{Mack}_k(G) \xrightarrow{\cong} \mathrm{Funct}_k(\mathbf{Per}(kG), \mathbf{Mod}(k)) \tag{17}$$

such that $\mathscr{Y}(M)(k[G/H]) \cong M(H)$, where $\mathrm{Funct}_k(\mathbf{Per}(kG), \mathbf{Mod}(k))$ denotes the category of k-functors from $\mathbf{Per}(kG)$ to $\mathbf{Mod}(k)$ (category of k-modules).

Let $\mathscr{E} := \mathrm{End}_{kG}(\bigoplus_{H \leq G} k[G/H])$. The Yoshida theorem says that the category of cohomological Mackey functors (with compatible natural transformations) is equivalent to $\mathscr{E}$-modules. The following theorem is one of the main results of [21].

Theorem 3 [21] Let $(\mathscr{C}, \mathscr{O})$ be a Mackey system on G. Assume that k is a field with $char(k) = 0$. Let M be a cohomological k-Mackey functor for this Mackey system. Suppose $H \in \mathscr{C}$ and $I \trianglelefteq H$ is a normal subgroup such that H/I is not cyclic. Then one has

$$\bigoplus_{I \leq H_0 < \ldots < H_n = H, \quad n \text{ odd}} M(H_0)^{|H_0/I|} \cong \bigoplus_{I \leq H_0 < \ldots < H_n = H, \quad n \text{ even}} M(H_0)^{|H_0/I|} \tag{18}$$

of k-modules.

First applying the theorem to G/I shows that I can be taken to be identity. Second is that the theorem also holds in characteristics $p > 0$, however, one needs to assume H/I does not have a normal p-subgroup with a cyclic quotient. Third, by defining

$$\mu(I, H) := \sum_{I \leq H_0 < \ldots < H_n = H} (-1)^n \tag{19}$$

then, the relation (19) defines an identity in a Grothendieck group. Theorem 3 has a vast of applications in different areas of mathematics such as number theory, or representation theory, and also the theory of zeta and L-series. We present just the following simple example to understand the Theorem 3 better, (see [21] for various examples).

Example 1 [21] For example if $G = S_3$ the symmetric group of 3 elements, then we obtain

$$M(1) \oplus M(G)^2 \cong M(C_3) \oplus M(C_2) \tag{20}$$

where C_2, C_3 are subgroups of order 2 and 3. For $G = A_4$ the alternative group of four elements we obtain two possible identities,

$$\begin{aligned} &M(1) \oplus M(G)^3 \cong M(C_3)^3 \oplus M(V_4) \\ &M(1) \oplus M(V)^4 \cong M(C_2)^3 \end{aligned} \tag{21}$$

where C_i denotes the subgroups of order i and V_4 the subgroup of order 4.

An example of a Mackey system appears for G being the Galois group of number fields or p-adic fields. It is a straightforward checkup that the system of the subgroups satisfies the axioms of a Mackey system. We want to see some application of the Mackey functors to zeta and L-series. We need the following lemma that shows the behavior of L-series under Ind_H^G functor.

Notation: Below we use a convention between zeta functions and L series as $L(s, \chi, \rho) := \zeta(s, \chi \otimes \rho)$ for a representation ρ and character χ of the absolute Galois group G_K, (see Appendix B for more details).

Lemma 1 ([21] Theorem 6.9, and proposition 6.10) Assume $\rho: G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(\mathbb{Q}_l)$ is an l-adic representation of the absolute Galois group of $\mathbb{Q}$. Assume H is a subgroup of finite index in G. Then

$$L(s, \mathrm{Ind}_H^G(\chi), \rho) = L(s, \chi, \rho|_H), \qquad \forall \chi \colon H \to \mathbb{Q}_l^* \tag{22}$$

where $L(s, \mathrm{Ind}_H^G \chi, \rho) := \zeta(s, \chi \otimes \rho)$.

We are now ready to see an instance application to zeta functions.

Example 2 ([21] section 6 and Theorem 1.8) Let G be a Galois group and consider the Grothendieck group $K_0(G)$ on the base elements $[G/H]$ where $H \leq G$ runs through subgroups of finite index in G. Let ρ be a representation of G. One defines a map

$$\zeta \colon K_0(G) \to \mathscr{M}, \qquad [G/H] \mapsto \zeta(s, \rho|_H) \tag{23}$$

This map has been studied in ([21] sec. 6). Using the Lemma 1 with Theorem 3 by applying the map ζ in (23) to identities as (18) one obtains the following computation,

$$\begin{aligned} \prod_{G_L \leq U \leq G} \zeta(s, \rho)^{|U/G_L|\mu(U, G)} &= \prod_{G_L \leq U \leq G} L(s, \mathrm{Ind}_U^G(1), \rho)^{|U/G_L|\mu(U, G)} \\ &= L(s, \sum_{G_L \leq U \leq G} |U/G_L|\mu(U, G)\mathrm{Ind}_U^G(1), \rho) \\ &= L(s, 0, \rho) = 1 \end{aligned} \tag{24}$$

where U runs through open subgroups of G with U/G_L finite. We provide a more systematic application in the next section.

3. Orbifold zeta functions and mirror symmetry

Mirror symmetry compares the Chen-Ruan orbifold cohomology associated with each pair of dual polynomials as

$$W = \sum_{i=1}^{n+2} x_1^{m_{i,1}} \ldots x_{n+2}^{m_{i,n+2}} \qquad \leftrightarrows \qquad W^{\mathrm{T}} = \sum_{i=1}^{n+2} x_1^{m_{1,i}} \ldots x_{n+2}^{m_{n+2,i}} \tag{25}$$

with the same number of monomials as variables. Here we have chosen the number of variables to be $n+2$ so that the dimension of the variety becomes n in the projective space. The (Chen-Ruan) orbifold cohomology of the hypersurface $X_W = \{W = 0\}$ inside the weighted projective space $\mathbb{P}(w_1, \ldots, w_{n+2})$ is defined by

$$H^k_{\mathrm{CR}}([X_W/G], \mathbb{C}) := H^k_{\mathrm{orb}}([X_W/G], \mathbb{C}) \tag{26}$$

The Chen-Ruan cohomology of X_W has a polarized Hodge structure, see Appendix A. As we mentioned in the introduction one assigns the matrix $A = (m_{ij})$ to the polynomial W which we assume is an invertible $(n+2) \times (n+2)$-

matrix. The polynomial W^{T} is obtained by transposing the matrix A. We usually denote the inverse by $A^{-1} = (m^{ij})$. Then to each column $[m^{1j}, ..., m^{(n+2)j}]^{\mathrm{T}}$ one can associate the diagonal matrix

$$\rho_j = \mathrm{diag}\ [\exp(2\pi i m^{ij}), ..., \exp(2\pi i m^{(n+2)j})] \tag{27}$$

We set $G = \mathrm{Aut}(W)/J_W$ where

$$\mathrm{Aut}(W) = \{\alpha = \mathrm{diag}\ [\alpha_1, ..., \alpha_{n+2}] \mid \alpha^* W = W\} = \langle \rho_1, ..., \rho_{n+2} \rangle \tag{28}$$

as the group of nontrivial symmetries of the polynomial W. This construction can be repeatedly done starting from $A_{W^{\mathrm{T}}}$. Then one obtains a group G^{T} defined by

$$G^{\mathrm{T}} = \left\langle \rho_1^{\mathrm{T}}, ..., \rho_{n+2}^{\mathrm{T}} \right\rangle / J_{W^{\mathrm{T}}} \tag{29}$$

where ρ_i^{T} corresponds to the i-th column of $A_{W^{\mathrm{T}}}^{-1} = (A^{\mathrm{T}})^{-1}$ and $J_{W^{\mathrm{T}}}$ is defined similarly. The group G^{T} can be identified with the group of characters of G. The (Homological) mirror map is an isomorphism between the orbifold Hodge pieces of the dual pairs (W, W^{T}),

$$H_{\mathrm{CR}}^{p,q}([X_W/G], \mathbb{C}) = H_{\mathrm{CR}}^{n-p,q}([X_{W^{\mathrm{T}}}/G^{\mathrm{T}}], \mathbb{C}) \tag{30}$$

Both of the orbifold cohomologies $H_{CR}^k([X/G], \mathbb{C})$ and its Hodge pieces $H_{CR}^{p,q}([X_W/G])$ are modules over $\mathscr{E}_0 = \mathrm{End}_{\mathbb{C}G}(\bigoplus_{(g)} \mathbb{C}G/C(g))$. Therefore by Yoshida theorem 2 they define Mackey functors on the set of subgroups of the form $\mathscr{C}(G) = \{C(g) \mid g \in G\}$. The family $\mathscr{O}$ in the definition of the Mackey system reduces to the identity subgroup only. The proof of the following theorem is our main result.

Theorem 4 The orbifold L-series of X_W and $X_{W^{\mathrm{T}}}$ satisfy

$$L_{\mathrm{orb}}(X_W, s) = L_{\mathrm{orb}}(X_{W^{\mathrm{T}}}, s)^{(-1)^n} \tag{31}$$

The relation (31) is referred to as a duality between orbifold zeta functions. However, this expression is conceptual, referring to the dual invertible polynomials (hypersurfaces) in the sense explained above. We reprove the Theorem 4. This is our Main result. Our method of proof gives a deeper mathematical understanding of the relation (31). We first prove the following lemma.

Lemma 2 Let (X, G) be an orbifold. Consider the Mackey system $\mathscr{C}(X) = (G, \mathscr{C} = \{C(g) \mid g \in G\})$ in the sense defined in Section 2. Then, the functors $M(g)$: $X_{(g)} \mapsto H^{*-2\iota_g}(X_{(g)}, \mathbb{C})^{C(g)}$, $g \in G$ define a Mackey functor on $\mathscr{C}(X)$.

Proof. The conditions F(1)-F(6) are formal consequences of the way the orbifold cohomology is defined as the direct sum of the cohomologies $H^{*-2\iota_g}(X_{(g)}, \mathbb{C})^{C(g)}$ and are based on the fact how the conjugacy classes in a finite abelian group are structured, see also the discussion in the beginning of Appendix A. By Yoshida Theorem 2 $H_{\mathrm{orb}}^*(X)$ becomes an $\mathrm{End}_{\mathbb{C}G}[\bigoplus_g \mathbb{C}G/C(g)]$-module, where g acts through the component $X_{(g)} = X^g/C(g)$. The corresponding cohomological functor is M: $\mathscr{C} \to \mathrm{Vect}_{/\mathbb{C}}$, $C(g) \mapsto H^{*-2\iota_g}(X_{(g)})^{C(g)}$, where $\mathrm{Vect}_{/\mathbb{C}}$ is the category of vector spaces over $\mathbb{C}$. □

The following definition is an orbifold analog of a similar map for ordinary zeta functions, defined in [21] section 6.

Definition 3 Let (X, G) be an orbifold. Consider the representation $\rho\colon G_L \to \mathrm{GL}(H^k_{\mathrm{orb},\, et}(\overline{X}, \mathbb{Q}_l))$. Let $K_0^{\rho}(G)$ be the Grothendieck group with base elements $[G/C(g)]$, where $g \in G$. Define the homomorphism

$$L^{\rho}_{\mathrm{orb}}\colon K_0^{\rho}(G) \to \mathscr{M}(\mathbb{C}), \qquad [G/C(g)] \mapsto L_{\mathrm{orb}}(s, \rho|_{C(g)}) \tag{32}$$

where $\mathscr{M}(\mathbb{C})$ is the meromorphic functions on $\mathbb{C}$. Here $L_{\mathrm{orb}}(s, \rho|_{C(g)})$ is defined as Appendix B, item 5.

As a consequence, we can prove the following.

Proposition 1 Let (X, G) be an orbifold. Then, we have the following formula

$$L_{\mathrm{orb}}(X, s) = \prod_g L_{\mathrm{orb}}(s, \rho_{C(g)})^{|G/C(g)|} \tag{33}$$

Proof. According to Lemma 2, the functors $M(g)\colon X_{(g)} \mapsto H^{*-2\iota_g}(X_{(g)}, \mathbb{C})^{C(g)},\ g \in G$ define a Mackey functor on $\mathscr{C}(X)$. By the theorem 3 we have

$$M(G)^{|G/G|} \cong \bigoplus_g M(C(g))^{|G/C(g)|} \tag{34}$$

Applying the map L^{ρ}_{orb} described in 3 to the both sides of (34), we get (33), see also Example 2 or [21] section 6 for a similar argument. □

Proposition 1 allows to obtain identities between orbifold zeta functions from identities in the Grothendieck group $K_0^{\rho}(G)$. We use this property to give our first proof of Theorem 4.

Proof. (First Proof of Theorem 4) From Proposition 1 we also get

$$L_{\mathrm{orb}}(X^{\mathrm{T}}, s) = \prod_{g'} L_{\mathrm{orb}}(s, \rho_{C(g')})^{|G^{\mathrm{T}}/C(g')|}.$$

On the other hand the subgroups $C(g') \leq G$ are dual to the subgroups $C(g') \leq G^{\mathrm{T}}$. Because G^{T} is the group of characters of the group G, this means that the subgroups $C(g')$ are the kernels of the maps $G^{\mathrm{T}} \to C(g)^{\mathrm{T}}$ associated to the inclusion $C(g) \hookrightarrow G$. In this way we have $|G/C(g)| = |G^{\mathrm{T}}/C(g')|$ for the choice of g and g' as explained. The factors $L_{\mathrm{orb}}(s, \rho_{C(g)})$ and $L_{\mathrm{orb}}(s, \rho_{C(g')})$ are the zeta functions of the sectors $X_{(g)}$ and $X^{\mathrm{T}}_{(g')}$. In case that X and X^{T} are CY-varieties where one has the cohomological mirror isomorphism $H^{p,\,q}_{\mathrm{orb}}(X, \mathbb{C}) = H^{n-p,\,q}_{\mathrm{orb}}(X^{\mathrm{T}}, \mathbb{C})$, we obtain the same representation of the absolute Galois group on these isomorphic pieces. Thus the Hodge decomposition 5 the multiplicative factors in the $\det\left(1 - F_{\mathrm{orb}}.t|_{H^*_{\mathrm{orb}}(\overline{X_{et}},\, \mathbb{Q}_l)}\right)$ and $\det\left(1 - F_{\mathrm{orb}}.t|_{H^*_{\mathrm{orb}}(\overline{X^{\mathrm{T}}_{et}},\, \mathbb{Q}_l)}\right)$ just (may!) have different exponents $\pm 1 = (-1)^n$. This proves the equation (31). □

The above proof shows that the duality relation mentioned in Theorem 4, is mainly based Mackey identities in the Grothendieck group $K_0^{\rho}(X_W)$.

We also provide another proof of Theorem 4, by a more direct method. The following proposition explains the relation between the age functions of two dual invertible polynomials. We will use this fact in the proof of Theorem 4.

Proposition 2 Let $A = (m_{ij})$ be the matrix defined by W. Consider the pairing

$$\underline{A} = \langle ., . \rangle \colon G \times G^{\mathrm{T}} \to \mathbb{C}^*, \qquad \langle g, g' \rangle = \exp[2\pi i\, \alpha.A.\beta^{\mathrm{T}}]$$

$$g = \mathrm{diag}[2\pi i \alpha_1, \ldots, 2\pi i \alpha_{n+2}], \qquad g' = \mathrm{diag}[2\pi i \beta_1, \ldots, 2\pi i \beta_{n+2}], \tag{35}$$

$$alpha = (\alpha_1, \ldots, \alpha_{n+2}), \qquad \beta = (\beta_1, \ldots, \beta_{n+2}).$$

If $\iota \colon G \to \mathbb{Q},\ g \mapsto \iota_g$ is the age function of the polynomial W. Then the age function of W^{T} namely, $\iota^{\mathrm{T}} \colon G^{\mathrm{T}} \to \mathbb{Q},\ g' \mapsto \iota^{\mathrm{T}}_{g'}$ is given by the Fourier transform of ι (denoted $\hat{\iota}$), i.e., $\iota^{\mathrm{T}} = \hat{\iota}$.

Proof. (sketch) The group G^{T} is the group of characters of G, i.e., $G^{\mathrm{T}} = \mathrm{Hom}(G, \mathbb{C}^*)$ cf. [2, 20, 28]. If $g = \mathrm{diag}[\exp(2\pi i a_1), \ldots, \exp(2\pi i a_{n+2})]$, then $\iota(g) = a_1 + \cdots + a_{n+2}$. The maps $\langle g, . \rangle$ and $\langle ., g' \rangle$ define 1-dimensional unitary representations of G^{T} and G respectively. In this sense, we regard the elements of G as a (unique) unitary representation of G' and vice versa. Let us for simplicity by abuse call g the exponential of α in equation (35). By ([28] page 5) the elements of G^{T} are generated by the (some-not all) exponentials of the rows of the matrix A^{-1} (see the condition in [28]), and elements of G are between exponentials of the columns of A^{-1}. The age functions just calculate the sum of the entries in these rows and columns. In this way, the Fourier transform with respect to $\underline{A}$ should correspond to the operation of taking the transpose of A in this process. This explains the claim of this proposition. □

Remark 1 From the proof of Proposition 2 it is easy to see that $\sum_g \iota_g = \sum_{g'} \iota_{g'}$. This formula also follows from the basic character theory of finite groups by Proposition 2.

Proof. (Second Proof of Theorem 4) We have by definition the formula

$$L_{\mathrm{orb}}(X_W, t) \colon = \det\left(1 - F_{\mathrm{orb}}.t|_{H^*_{\mathrm{orb}}(\overline{X_{W,\,et}},\, \mathbb{Q}_l)}\right) = \exp\left(\sum_{r=1}^{\infty} \mathrm{Tr}(F^r_{\mathrm{orb}}|_{H^*_{\mathrm{orb}}(\overline{X_{W,\,et}},\, \mathbb{Q}_l)}) \frac{t^r}{r}\right) \tag{36}$$

and similar for $X_{W^{\mathrm{T}}}$. By the equation (72) in Appendix II, or [1] def. 6.1, we have

$$\mathrm{Tr}(F_{\mathrm{orb}}|H^*_{\mathrm{CR}}) = \sum_g \mathrm{Tr}(F_{\mathrm{orb}}|_{X_g}|H^*) = \sum_g q^{-\iota_g} \sum_a \frac{q^{-\dim(a)}}{\sharp \mathrm{Aut}(a)} \tag{37}$$

Using the formula (37) we obtain

$$L_{\mathrm{orb}}(X_W, t) = \exp\left(\sum_{r=1}^{\infty} \sum_{\xi \in [X_W(\mathbb{F}_q)]} \frac{(q^r)^{-\mathrm{age}(\xi) - \dim(\xi)}}{\sharp \mathrm{Aut}(\xi)} . \frac{t^r}{r}\right) \tag{38}$$

Expanding the exponential function we are led to compare terms of the form

$$\exp\left(\sum_{\xi \in [X(F_q)]} \frac{(q^r)^{-\mathrm{age}(\xi) - \dim(\xi)}}{\sharp \mathrm{Aut}(\xi)}\right) = \exp\left(\mathrm{Tr}(F^r_{\mathrm{orb}}|_{H^*_{\mathrm{CR}}})\right) \tag{39}$$

The identity (39) holds for any r and can be compared with (37). It follows that the expressions $\frac{q^{-\mathrm{age}(\xi)-\dim(\xi)}}{\sharp\mathrm{Aut}(\xi)}$ are eigenvalues of $F_{\mathrm{orb}}|H^*_{\mathrm{CR}}$. By the well-known formula $\exp\mathrm{Tr}(M) = \det e^M$ in a computation of the determinant of $F_{\mathrm{orb}}|H^i_{\mathrm{CR}}$ for some i we have a product of these expressions. Thus, we have to compare the sums $\sum_\xi -\mathrm{age}(\xi) - \dim(\xi)$ for the dual invertible polynomials W and W^{T}. So we are led to compare the two sums $\sum_g \iota(g)$ and $\sum_{g'} \iota^{\mathrm{T}}(g')$. By Proposition 2 ι^{T} is the Fourier transform of ι with respect to the pairing $\underline{A}$. The two sums $\sum_g \iota(g)$ and $\sum_{g'} \iota^{\mathrm{T}}(g')$ are equal by basic character theory for finite abelian groups.

We summarize the above. It follows that the only difference that affects between the orbifold zeta function formula $L_{\mathrm{orb}}(X_W,\ t)\colon = \det\left(1 - F_{\mathrm{orb}}.t|_{H^*_{\mathrm{orb}}(\overline{X_{W,\,et}},\,\mathbb{Q}_l)}\right)$ for two dual invertible polynomials is on the power degrees in $\det\left(1 - F_{\mathrm{orb}}.t|_{H^i_{\mathrm{orb}}(\overline{X_{W,\,et}},\,\mathbb{Q}_l)}\right)^{(-1)^{i+1}}$. The only thing that remains, is to look at how the different factors in (75) or (33) are corresponded via the Mirror isomorphism. According to the homological mirror symmetry, (30) these exponents differ by $(-1)^n$. This proves the theorem. $\square$

Remark 2 The relation in Theorem 4 is proved in [2] by another method. Our approach was somewhat different and mainly on the power of Mackey functors in Mirror symmetry. The relation

$$\chi_{\mathrm{orb}}(X_W,\, G) = (-1)^{\dim}\chi_{\mathrm{orb}}(X_{W^{\mathrm{T}}},\, G) \tag{40}$$

also holds for the orbifold Euler characteristic defined analogously, see [2].

Remark 3 [29] Given a smooth projective variety X, denote by $D^b(X)$ the derived category of coherent sheaves on X. For two projective varieties X, Y the Fourier-Mukai transform with kernel $P \in D^b(X \times Y)$ is

$$F\colon D^b(X) \to D^b(Y), \qquad F = (p_2)_*(p_1^*(-)), \tag{41}$$

where $X \xleftarrow{p_1} X \times Y \xrightarrow{p_2} Y$ are projections. There is a well-known map

$$v\colon D^b(X) \to H^*(X,\mathbb{C}), \qquad v(E) = \mathrm{ch}(E)\sqrt{\mathrm{td}(X)} \tag{42}$$

called Mukai vector which is a covariant functorial isomorphism. In Mirror symmetry, the Mirror isomorphism between two pairs X and Y (X_W and $X_{W^{\mathrm{T}}}$ in our case) can also be explained by the Fourier-Mukai transform between their derived categories. Then one can deduce that

$$\mathrm{Tr}(\mathrm{Frob}|H^{even}(X)) = \mathrm{Tr}(\mathrm{Frob}|H^{even}(Y))$$

$$\mathrm{Tr}(\mathrm{Frob}|H^{odd}(X)) = \mathrm{Tr}(\mathrm{Frob}|H^{odd}(Y))$$

however one can not establish the equality between their zeta functions from these relations, except in low dimensions or when most of the cohomologies may vanish.

Remark 4 [30] Suppose G is as in the setting of this section. Set $\mathbb{X} = \oplus_g \mathrm{Ind}_{C(g)}^G 1$. Then one defines the Hecke algebra by $H_{(g)} = \mathrm{End}_{\mathbb{C}[G]}(\mathrm{Ind}_{C(g)}^G 1)$ and $H = \left(\bigoplus_g H_{(g)}\right)^{\mathrm{op}}$. Then there exists natural adjoint functors

$$\mathbf{Mod}(H) \xrightarrow{A} \mathbf{Mod}_{\mathbb{C}}(G) \qquad M \longmapsto \mathbb{X} \otimes_H M \tag{43}$$

$$\mathbf{Mod}(H) \xleftarrow{\mathfrak{h}} \mathbf{Mod}_{\mathbb{C}}(G) \qquad \mathrm{Hom}_G(\mathbb{X}, V) \longleftarrow V \tag{44}$$

This correspondence reduces the representation theory of the group G to module theory over H. The associated correspondence can also be made at the level of derived Hecke algebras as

$$\mathbf{Mod}(H^\bullet) \xrightarrow{A} \mathbf{Mod}_{\mathbb{C}}(G) \qquad M^\bullet \longmapsto I^\bullet \otimes_{H^\bullet} M^\bullet \tag{45}$$

$$\mathbf{Mod}(H^\bullet) \xleftarrow{\mathfrak{h}} \mathbf{Mod}_{\mathbb{C}}(G) \qquad \mathrm{Hom}_G(\mathbb{X}, V^\bullet) \longleftarrow V^\bullet \tag{46}$$

where $\mathbb{X} \to I^\bullet$ is an injective resolution. The $\mathbb{C}[G]$-module $H^*_{\mathrm{orb}}(X)$ can be regarded as an H-module in the derived sense. This reflects the fact that the theory of L-series is an assignment over complexes and their cohomologies, see [31]. The interesting fact is to study the structure of the Ext-algebra

$$\mathrm{Ext}^*_{\mathbb{C}G}(\mathbb{X}) = \bigoplus_g H^*(C(g), \mathbb{X}) \tag{47}$$

induced by Frobenius reciprocity. It is a fundamental fact in the theory of Hecke algebras that the multiplication in $H^\bullet$ is induced by the (opposite of) Yoneda product on Ext^*. The Ext-algebra has the structure of an A_∞-algebra, i.e. a differential graded algebra with higher multiplications. If the field $\mathbb{C}$ is replaced by a field with $char = p > 0$ this formalism breaks down unless when all the subgroups $C(g)$ are pro-p groups and p-torsion free. In this case the two functors A and $\mathfrak{h}$ are quasi-inverse to each other by a theorem of P. Schneider, loc. cit. .

Remark 5 (Congruence mirror symmetry-D. Wan) [24] An interesting question is if for two mirror pair of Calabi-Yau varieties X and Y one has $\sharp X(\mathbb{F}_q) = \sharp Y(\mathbb{F}_q)$. This question has been positively answered in [24] for strong mirror pairs (X_λ, Y_λ) in mirror families of CY-varieties parametrized by λ, as $\sharp X_\lambda(\mathbb{F}_q) = \sharp Y_\lambda(\mathbb{F}_q)$.

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Conflict of interest

The authors declare no competing financial interest.

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Appendix A. Orbifold Hodge structures

We briefly provide some preliminaries on orbifolds and orbifold cohomologies following [8, 11], see also [6, 9, 10, 14]. We start with the definition of an orbifold and orbifold bundles. An orbifold is a Housdorff second countable topological space X with an atlas $\mathfrak{A} = \{ (V_p, G_p, \pi_p) \mid p \in X \}$ called uniformizing system, where V_p is an n-dimensional manifold, G_p is a finite group acting on V_p in a smooth way, and π_p: $V_p \to X$ inducing a local homeomorphism $\tilde{\pi}_p$: $V_p/G_p \longrightarrow X$, $p \in V_p \subset X$. The group G_p is called the isotropy group of p, or the local group at p. An orbifold vector bundle or an orbibundle of rank k is given by a surjective map pr: $E \longrightarrow X$, with uniformizing systems (V, G, π) and $(V \times \mathbb{R}^k, G, \hat{\pi})$ for X and E respectively, such that the action of G on V is given by $g.(x, v) = (g.x, \rho(x, g)v)$, where ρ: $V \times G \to \mathrm{Aut}(\mathbb{R}^k)$ is smooth map satisfying $\rho(g.x, h) \circ \rho(x, g) = \rho(x, h \circ g)$. Finally we require that $\tilde{\mathrm{pr}}$: $V \times \mathbb{R}^k \to V$ satisfies $\pi \circ \tilde{\mathrm{pr}} = \mathrm{pr} \circ \tilde{\pi}$. Maps between orbifolds and orbibundles are defined to be compatible with the orbifold structures.

Define the multi-sector $\Sigma_k X$ as the set of pairs $(p, (\underline{g}))$ where $\underline{g} = (g_1, \ldots, g_k)$ is the conjugacy class of elements $(g_1, \ldots, g_k)$ in G_p. $\Sigma_k X$ can be locally seen as $V_p^{(\underline{g})}/C(\underline{g})$ where $V_p^{(\underline{g})} = V_p^{g_1} \cap \ldots \cap V_p^{g_k}$ and $C(\underline{g}) = C(g_1) \cap \ldots \cap C(g_k)$. The notation V_p^g stands for the fixed point of the action of g on V_p and $C(g)$ for the centralizer of g in G_p. We have the following properties;

1. There exists the decomposition

$$\Sigma_k X = \bigsqcup_{(\underline{g}) \in T_k} X_{(\underline{g})}, \qquad X_{(\underline{g})} = \{ (p, (\underline{g}') \mid (\underline{g}') \in (\underline{g}) \} \tag{48}$$

where T_k is the set of equivalence classes obtained from maps $G_q \to G_p$ for $q \in U_p = \pi(V_p)$. The $X_{(g)}$ for $g \neq 1$ is called twisted sector and $X_{(1)}$ the untwisted one.

2. For $l \leq k$ there are evaluation maps

$$e_{i_1, \ldots, i_l}: \Sigma_k X \to \Sigma_l X, \qquad e_{i_1, \ldots, i_l}(x, (g_1, \ldots, g_k)) = (x, (g_{i_1}, \ldots, g_{i_l})) \tag{49}$$

3. We have an involution

$$I: \Sigma_k X \to \Sigma_k X, \qquad (x, (\underline{g})) \to (x, (\underline{g}^{-1})) \tag{50}$$

4. The case we are interested in is a global quotient $X = Y/G$ where G is a finite group. In this case

$$\Sigma X = \bigsqcup_{(g)} Y^g/C(g), \qquad X_{(g)} = Y^g/C(g) \tag{51}$$

We assume the orbifold X has an almost complex manifold with a complex structure given by J as a smooth section of $\mathrm{End}(TX)$ such that $J^2 = -1$. For each $p \in X$ the almost complex structure gives a representation ρ_p: $G_p \to \mathrm{GL}_n(\mathbb{C})$ where $n = \dim X$ that can be diagonalized as $\mathrm{diag}\left(e^{2\pi i \frac{m_{1,g}}{m_g}}, \ldots, e^{2\pi i \frac{m_{n,g}}{m_g}}\right)$ where m_g is the order of $g \in G_p$ and $0 \leq m_{i,g} < m_g$.

5. There exists a locally constant function

$$\iota: \Sigma X \to \mathbb{Q}, \qquad \iota(p, (g)) = \sum_{j=1}^{n} m_{j,g}/m_g \tag{52}$$

called the degree shifting number. It is integer-valued iff $\rho_p(g) \in \mathrm{SL}_n(\mathbb{C})$. In this case, X is called an SL-orbifold.

The orbifold cohomology groups of X are defined as

$$H^d_{\mathrm{orb}}(X, \mathbb{C}) := H^d(\Sigma X, \mathbb{C}) = \bigoplus_{(g)\in T_1} H^{d-2\iota_{(g)}}(Y^g, \mathbb{C})^{C(g)} \tag{53}$$

where the superfix means the fixed points. Applying this argument to Dolbeault cohomologies we obtain

$$H^{p,q}_{\mathrm{orb}}(X) = \bigoplus_{(g)} H^{p-\iota_{(g)}, q-\iota_{(g)}}(X_{(g)}) \tag{54}$$

Suppose $\omega \in H^{1,1}(X, \mathbb{R})$ is a Kähler class and let L_ω: $H^*_{\mathrm{orb}}(X) \to H^*_{\mathrm{orb}}(X)$ be the wedge operator with the Kähler class (the Lefschetz operator). Because

$$\dim X_{(g)} = \dim X - i_{(g)} - i_{(g^{-1})} \tag{55}$$

then L^p_ω pairs H^{n-p}_{orb} with H^{n+p}_{orb} subject to the condition that

$$\iota_{(g)} = \iota_{(g^{-1})} \qquad (*) \tag{56}$$

In this case L^p_ω: $H^{n-p}_{\mathrm{orb}} \to H^{n+p}_{\mathrm{orb}}$ is an isomorphism. The primitive orbifold classes are defined by

$$(H_{\mathrm{orb}})_0(X, \mathbb{C}) \colon \ker(L^{n-p+1}_\omega \colon H^p_{\mathrm{orb}} \to H^{2n-p+2}_{\mathrm{orb}}) \tag{57}$$

The following theorem explains the Hodge structure on the orbifold cohomology, based on the notations we introduced.

Theorem 5 [8] Let X be a projective *Sl*-orbifold satisfying condition (*). Then for each *k*,

$$H^k_{\mathrm{orb}}(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_{\mathrm{orb}}(X), \qquad H^{q,p}_{\mathrm{orb}}(X) = \overline{H^{p,q}_{\mathrm{orb}}(X)} \tag{58}$$

is a Hodge structure (HS) of weight *k*. The primitive cohomology also inherits a HS of weight *k* in a natural way from this decomposition. This Hodge structure is polarized by the form

$$Q_{(g)}(a, b) = (-1)^{(k(k-1)/2)+\iota_{(g)}} \int_{X_{(g)}} a \wedge b, \qquad a \in H^{k-2\iota_{(g)}} \tag{59}$$

The Theorem states that the orbifold intersection form splits as $\bigotimes Q_g$ according to the sector decomposition $\coprod_g X_g$ and

$$Q_g(H^{a-\iota_g,\, b-\iota_g}(X_g),\, H^{c-\iota_g,\, d-\iota_g}(X_g)) = 0, \qquad \text{unless } a+c=n \tag{60}$$

which implies $Q(F^p, F^{n-p+1}) = 0$. The two filtrations

$$W_l = \bigoplus_{k \geq 2n-l} H^k_{\text{orb}}, \qquad F^p = \bigoplus_{a} \bigoplus_{b \leq n-p} H^{a,\, b}_{\text{orb}} \tag{61}$$

define a polarized MHS on H^*_{orb} polarized by L_w. The Lefschetz operator L_w is an infinitesimal isometry for Q_g;

$$Q_g(L_w\alpha,\, \beta) + Q_g(\alpha,\, L_w\beta) = 0 \tag{62}$$

and the HS of weight $n+l$ induced by F on $\ker(L_w^{l+1}\colon Gr^W_{n+l} \to Gr^W_{n-l-2})$ is polarized by $Q(.,\, L^l_w.)$.

A1 *The étale setting*

[1] Let X be an orbifold as above, and define over the number field L. Then define similarly

$$H^k_{et,\, \text{orb}}(\overline{X},\, \mathbb{Q}_l)\colon = H^k_{et}((\Sigma\overline{X},\, \mathbb{Q}_l) = \bigoplus_{a+2b=k} H^a_{et}(\text{age}^{-1}(b),\, \mathbb{Q}_l) \tag{63}$$

for $0 \leq k \leq 2n$, where $\Sigma\overline{X}$ and the age function are defined similarly. The conjugation property in (58) will fail in the étale setting but one still has $\dim H^{q,\, p}_{et,\, orb}(\Sigma X) = \dim H^{p,\, q}_{et,\, orb}(\Sigma X)$. The groups $H^k_{et,\, \text{orb}}(\overline{X},\, \mathbb{Q}_l)$ are $\mathbb{Q}_l$-vector spaces endowed with continuous action $\rho\colon G_L \to \text{GL}(H^k_{\text{orb},\, et}(\overline{X},\, \mathbb{Q}_l))$. Most of the properties and definitions in the étale setting are similar to the case over the complex numbers, see [1] for details.

Appendix B. Zeta functions

The materials in this Appendix are classical and quite well-known and can be found in various texts on zeta functions, [1–5, 20, 21]. We can define various zeta functions for the projective variety X.

1. **Artin local zeta function:** Assume X is a non-singular projective algebraic variety defined over a finite field $\mathbb{F}_q$ with $q = p^r$ elements and let $\overline{X} := X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$. The Artin (local) zeta function of X is defined by

$$Z(X, t) = \det\left(1 - \mathrm{Frob}^*|_{H^*(\overline{X}, \mathbb{Q}_l)}\right) = \exp\left(\sum_{n=1}^{\infty} \frac{\sharp X(\mathbb{F}_{q^n})}{n} t^n\right) \tag{64}$$

where $l \neq p$ is a prime, and the notation $\sharp X(\mathbb{F}_{q^n})$ is the number of rational points of X over $\mathbb{F}_{q^n}$. The identity (64) is based on the Lefschetz fixed point theorem; "if $f: X \to X$ is a continuous endomorphism of quasi-projective variety X, then $\Gamma_f.\Delta = \sum(-1)^i \mathrm{Tr}(f^*|_{H^i(X, \mathbb{Q}_l)})$ where Γ is the diagonal in $X \times X$. It follows that $\Gamma_{\mathrm{Frob}^r}.\Delta = |X(\mathbb{F}_{q^r})|$."

2. **Hasse global zeta function:** If X is defined over a number field K, the Hasse zeta function of X is

$$\zeta(X, s) = \prod_{\mathfrak{p} \text{ good}} Z_{\mathfrak{p}}\left(X(\mathfrak{p}), N\mathfrak{p}^{-s}\right) \tag{65}$$

where $Z_{\mathfrak{p}}(X(\mathfrak{p}), N\mathfrak{p}^{-s})$ is defined via (64) and $N\mathfrak{p} \in \mathbb{Z}$ is the norm of the ideal $\mathfrak{p}$. The product is over primes $\mathfrak{p}$ such that the reduction of X is a smooth variety (good primes). When $K = \mathbb{Q}$, then $\mathfrak{p} = (p)$, $N(\mathfrak{p}) = p$ and the local factors in the product (65) are defined by the formula (64), with $q = p$.

3. **Zeta function of Galois representations:** One also associates the zeta functions with the Galois representations of $G_K = \mathrm{Gal}(\overline{K}/K)$. If $\rho: G_K = \mathrm{Gal}(\overline{K}/K) \to \mathrm{Aut}_K(V)$ is a representation of the absolute Galois group of K, then the zeta function of ρ is

$$\zeta(\rho, s) = \prod_{\mathfrak{p} \neq \infty} \det\left(1 - \mathrm{Frob}_{\mathfrak{p}} N\mathfrak{p}^{-s}|_{V^{I_{\mathfrak{p}}}}\right)^{(-1)} \tag{66}$$

If $D_{\mathfrak{p}}$ is the decomposition group at $\mathfrak{p}$, then $I_{\mathfrak{p}} \trianglelefteq D_{\mathfrak{p}}$ is the inertia subgroup which fit into the short exact sequence

$$0 \to I_{\mathfrak{p}} \to D_{\mathfrak{p}} \to \mathrm{Gal}\left(\overline{k(\mathfrak{p})}/k(\mathfrak{p})\right) \to 0 \tag{67}$$

When $\mathfrak{p}$ is good, then $I_{\mathfrak{p}} = 1$ (identity subgroup). The zeta function in (65) is in fact the zeta function of the natural representation of G_K on the cohomology $H^*(\overline{X}, \mathbb{Q}_p)$.

4. ***L*-series of projective varieties:** If X is a non-singular projective variety defined over the number field K, then the L-series of X is defined by

$$L(X, s) = \prod_i \prod_{\mathfrak{p} \neq \infty} \det\left(1 - \mathrm{Frob}_{\mathfrak{p}} N\mathfrak{p}^{-s}|_{H^i_{et}(\overline{X}, \mathbb{Q}_p)^{I_{\mathfrak{p}}}}\right)^{(-1)^{i+1}} \tag{68}$$

We use the notation of L-series, where $L(s, \chi, \rho) = \zeta(s, \chi \otimes \rho)$ for a character χ of G_K. In this sense, we identify the zeta and L-series, in the text. Zeta and L series can also be associated to Galois representations. In this form, one regards $H^i_{et}(X, \mathbb{Q}_l)$ as Galois representations, and the definition is the same as the previous item.

5. **Orbifold L-series:** The zeta and L-series can also be defined for orbifolds or orbifold cohomologies. As before the arithmetic Frobenius acts on the orbifold cohomology. However, unlike the usual case, this action is no longer a ring homomorphism on $H^*_{\text{orb}}(\overline{X}, \mathbb{Q}_p)$. To make the action on the orbifold cohomology a ring homomorphism one needs to modify it as,

$$A \longmapsto q^{-\iota_{g(A)}} \text{Frob}_{\mathfrak{p}}(A) \tag{69}$$

We sometimes denote $\iota_{g(A)}$ simply by ι_A when $A \in H^*_{\text{orb}}(\overline{X}, \mathbb{Q}_p)$. Following [1] or [3], it is natural to define orbifold Frobenius morphism as

$$F_{\mathfrak{p}, orb}: H^*_{\text{orb}}(\overline{X_{et}}, \mathbb{Q}_l) \to H^*_{\text{orb}}(\overline{X_{et}}, \mathbb{Q}_l), \qquad \alpha \mapsto q^{-\iota(\alpha)} \text{Frob}_{\mathfrak{p}}(\alpha) \tag{70}$$

where the second $\text{Frob}_{\mathfrak{p}}$ is the usual one. In [1] it is explained that $F_{\mathfrak{p}, orb}$ is a ring homomorphism. The orbifold L-series of the orbifold X is defined as

$$L_{\text{orb}}(X, t) := \det\left(1 - F_{\text{orb}}.t|_{H^*_{\text{orb}}(\overline{X_{et}}, \mathbb{Q}_l)}\right) = \exp\left(\sum_{r=1}^{\infty} \text{Tr}\left(F^r_{\text{orb}}|_{H^*_{\text{orb}}(\overline{X_{et}}, \mathbb{Q}_l)}\right) \frac{t^r}{r}\right) \tag{71}$$

One computes (cf. [1] section 6),

$$\text{Tr}(F_{\text{orb}}|H^*_{\text{CR}}) = \sum_g \text{Tr}(F_{\text{orb}}|_{X_g}|H^*) = \sum_g q^{-\iota_g} \sum_a \frac{q^{-\dim(a)}}{\sharp\text{Aut}(a)} \tag{72}$$

where $\text{Aut}(a)$ is the automorphism group the sector corresponding to a. Plugging (72) in equation (71) then the formula (71) becomes

$$L_{\text{orb}}(X, t) = \exp\left(\sum_{r=1}^{\infty} \sum_{\xi \in [X(F_q)]} \frac{(q^r)^{-\text{age}(\xi) - \dim(\xi)}}{\sharp\text{Aut}(\xi)} . \frac{t^r}{r}\right) \tag{73}$$

We refer to [1] definition 6.1 for details of the computation, see also [3]. The formula (73) allows us to compare L-series according to the age functions. We wish to do this for a mirror pair polynomials of special forms, according to some inversion formula between their age function, (see Section 3).

6. **Zeta functions of Deligne-Mumford stacks:** [3, 32]. Here again the definition is based on the Lefschetz trace formula. In this case, the trace formula reads $\text{Tr}F^q|_{H(X_{et}, \mathbb{Q}_l)} = \sharp X(\mathbb{F}_q)$. For technical reasons we reformulate this formula in terms of the arithmetic Frobenius ϕ_q On $H(X_{et}, \mathbb{Q}_l)$ that acts as the inverse of F_q. For ϕ_q the trace formula reads $q^{\dim(X)}\text{Tr}\phi_q|_{H(X_{et})} = \sharp X(\mathbb{F}_q)$. This follows from Poincaré duality.

Algebraic stacks relate to algebraic varieties in the same way groupoids relate to sets. A groupoid is a category all of whose morphisms are isomorphisms. A set X is considered as a groupoid denoted also X, by taking for objects the elements of the sets and morphisms only the identity morphisms. A group G is considered as a groupoid denoted BG with one object whose automorphism group is G. A G-set X is considered as a groupoid denoted X_G or $[X/G]$ by taking as objects the elements of X and for the set of morphisms from x to y the transporter the elements of G that take x to y through the action. For a groupoid $\mathfrak{X}$ we define

$$\sharp\mathfrak{X} = \sum_{\xi\in[\mathfrak{X}]} \frac{1}{\mathrm{Aut}\xi}.$$

where the sum is taken over the set of isomorphism classes of $\mathfrak{X}$ and for an isomorphism class $\mathrm{Aut}(\xi)$ is the automorphism group of any representative. If Aut happens to be infinite we set $\frac{1}{\mathrm{Aut}} = 0$.

In case $\mathfrak{X}$ is a set $\sharp X$ is just the number of elements of X. If $\mathfrak{X} = BG$ for a group G we have $\sharp BG = \frac{1}{\sharp G}$. If $\mathfrak{X} = X_G$ for a G set X we have, $\sharp\mathfrak{X} = \frac{\sharp X}{\sharp G}$, by the orbit formula. When $\mathfrak{X} = [X/G]$ and essentially of finite type over $\mathbb{F}_q$ we have the following

$$\frac{\sharp X(\mathbb{F}_q)}{\sharp G(\mathbb{F}_q)} = \sum_{\eta\in X(\mathbb{F}_q)/G(\mathbb{F}_q)} \frac{1}{\sharp\mathrm{Stab}(\eta)} = \sum_{\xi\in[\mathfrak{X}(F_q)]} \frac{1}{\sharp\mathrm{Aut}(\xi)} \tag{74}$$

If $\mathfrak{X}$ is a Deligne-Mumford (DM) stack of finite type over $\mathbb{F}_q$ one can define the zeta or L-series of the DM-stack $\mathfrak{X}$ as

$$L(\mathfrak{X},\, t) := \exp\left(\sum_{r=1}^{\infty} |\mathfrak{X}(\mathbb{F}_{q^n})|\frac{t^r}{r}\right) = \prod_{i=0}^{2\dim\mathfrak{X}} \det\left(1 - q^{\dim\mathfrak{X}} tF_q|_{H^i(\overline{\mathfrak{X}_{et}},\, \mathbb{Q}_l)}\right)^{(-1)^{i+1}} \tag{75}$$

where we have

$$|\mathfrak{X}(\mathbb{F}_q)| = q^{\dim\mathfrak{X}}\mathrm{Tr}(F_q)|_{H^*_{sm}(\overline{\mathfrak{X}_{et}},\, \mathbb{Q}_l)} = \sum_{\xi\in[\mathfrak{X}(F_q)]} \frac{1}{\sharp\mathrm{Aut}(\xi)} \tag{76}$$

Remark 6 (Gamma factors) [5] If $V = \bigoplus V^{p,\, q}$ is a Hodge decomposition over $\mathbb{C}$, then one defines $h^{p,\, q} = \dim V^{p,\, q}$. Then the Gamma factor attached to V is

$$\Gamma_V(s) = \prod_{p,\, q} \Gamma_{\mathbb{C}}(s - \inf(p,\, q))^{h^{p,\, q}} \tag{77}$$

If $V = \oplus_{p,\, q} V^{p,\, q}$ is an $\mathbb{R}$-Hodge structure, that is there exists an involution σ such that pairs $\sigma(V^{p,\, q}) = V^{q,\, p}$. If a factor $V^{n,\, n}$ appears in the Hodge decomposition, then the automorphism σ induces a decomposition $V^{n,\, n} = V^{n,\, +} \oplus V^{n,\, -}$, as the ± 1-eigenspaces of σ. We put $h(n,\, +) = \dim V^{n,\, +}$, $h(n,\, -) = \dim V^{n,\, -}$ and $h(n,\, n) = h(n,\, +) + h(n,\, -)$. Then the Gamma factor attached to V is

$$\Gamma_V(s) = \prod_n \Gamma_{\mathbb{R}}(s-n)^{h^{n,\, +}} \prod_n \Gamma_{\mathbb{R}}(s-n+1)^{h^{n,\, -}} \prod_{p<q} \Gamma_{\mathbb{C}}(s-p)^{h^{p,\, q}} \tag{78}$$

If X is a non-singular projective variety defined over a global field K we set $A = N(\mathfrak{f}).D^{B_m}$, $D = |d_{K/\mathbb{Q}}|$, $B_m = \dim H^m(X,\, \mathbb{C})$ where $d_{K/\mathbb{Q}}$ is the discriminant and $\mathfrak{f}$ the conductor pf $K/\mathbb{Q}$. Then one defines

$$\xi(s) = A^{s/2}\zeta(s) \prod_{v \in \Sigma_K^\infty} \Gamma_v(s) \tag{79}$$

It satisfies the functional equation

$$\xi(s) = w\,\xi(m+1-s), \qquad w = \pm 1 \tag{80}$$

The important consequence of the functional equation for zeta functions is the meromorphic continuation of $\xi(s)$ to the whole complex plane.

Remark 7 [29] In general, the functional equation for the zeta function of nonsingular varieties is a consequence of Poincaré duality $H^{2\dim - r}(X, \mathbb{Q}_l) \times H^r(X, \mathbb{Q}_l) \to \mathbb{Q}_l$. Using the (projection) formula

$$F_*(x).x' = x.F^*(x'), \qquad x \in H^{2\dim - r},\ x' \in H^r \tag{81}$$

one deduces that the eigenvalues of F^* acting on $H^{2\dim - r}$ are the same as eigenvalues of F_* acting on H^r. On the other hand the equation $F_* \circ F^* = q^d$ tells that if the eigenvalues of F^* are $\alpha_1,\ldots,\alpha_s$ then the eigenvalues of F_* must be $q^d/\alpha_1, \ldots, q^d/\alpha_s$. This implies the functional equation. The proof for stacks or orbifolds is the same. This argument can also be done via the Lefschetz operator L_w. The operator $L_w^{n-i}\colon H^i_{et,*}(X)(i\text{-}n) \xrightarrow{\cong} H^{2n-i}_{et,*}(X)$ commutes with the Frobenius map on cohomology. Therefore the aforementioned result follows from Hard Lefschetz and the fact that twisting the étale cohomology by (l) affects the eigenvalues of the Frobenius multiplied by $1/q^l$.

Remark 8 (Independence of l) [31] The coefficients of the expansion of the rational function

$$L_{\text{orb}}(X, t) = \prod P_i(t)^{(-1)^{i+1}}, \qquad P_i = \det(1 - F_{\text{orb}}t; H^i(X, \mathbb{Q}_l)) \tag{82}$$

are rational numbers and are independent of l. It is a famous conjecture that this is also true for each $P_i(t)$. It is known that the roots of $P_i(t)$ are Weil q-numbers, i.e. all their (Galois) conjugates have the same weight, which is a rational number.

Remark 9 [4] For special varieties, one may get explicit formulas for the action of the Frobenius, $F_{\mathfrak{p}}$. Consider the variety X defined by the Fermat equation,

$$X\colon x_0^d + x_1^d + \ldots + x_{n+1}^d = 0 \tag{83}$$

of degree d, where we regard as a variety of dimension n over the field $K = \mathbb{Q}(\sqrt[d]{1})$. Then $G = \bigoplus_{i=0}^{n+1} \mu_d/(\text{diagonal})$. Then the cohomology $H^n(X)$ decomposes as $H^n(X) = \bigoplus_g H^n(X)_g$, where $H^n(X)_g = \{v \mid \zeta.v = \zeta^g v,\ g \in G\}$. Then according to [4] each $H^n(X)_g$ is either one dimensional or 0. The action of the geometric Frobenius $F_{\mathfrak{p}}$ is given by the following Gauss sum.

$$F_{\mathfrak{p}}.v = J(\varepsilon_1, \ldots, \varepsilon_{n+1}) = (-1)^n \sum_{x \in \mathbb{P}^n(\mathbb{F}_q)} \prod_i \varepsilon_i(x_i), \qquad \varepsilon_i\colon \mathbb{F}_q^\times \to \mu_d \tag{84}$$

Remark 10 The fundamental fact about zeta functions is given by the Weil conjectures proved by P. Deligne. They are as follows.

(1) $Z(X, t)$ is a rational function of t and can be written

$$Z(X, t) = \frac{P_1(t)...P_{2n-1}(t)}{P_0(t)...P_{2n}(t)}, \qquad n = \dim X, \ P_i(0) = 1 \tag{85}$$

where $P_i(t)$ is a polynomial of degree β_i the i-th betti number of X. Moreover $P_0(t) = 1 - t$ and $P_{2n}(t) = 1 - q^n.t$.

(2) One has a functional equation

$$Z(X, \frac{1}{q^n.t}) = \pm q^{n\chi/2}\, t^{\chi} Z(X, t), \qquad \chi = \sum_{i=0}^{2n} (-1)^i \beta_i \tag{86}$$

(3) The polynomials $P_i(t)$ above have integral coefficients, if $P_i(t) = \prod_i (1 - w_\alpha.t)$ the complex numbers w_α have absolute value $q^{1/2}$.

Weil conjectures are also the base to define the weight filtration in Hodge-Tate decompositions as the analog of mixed Hodge structure in p-adic Hodge theory.

Remark 11 [33] Langlands program studies the deformation theory of representations ρ: $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}(H_{et}^*)$ by the existence of certain continuous map Def: $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{T}(\mathscr{H})/I$ where $\mathbb{T}(\mathscr{H})$ is a specific Hecke algebra (arisen from the cohomology of the Shimura variety parametrizing Hodge structures of H_{et}^*), with generator T_1, ..., T_n. In this context the characteristic polynomial of Def(Frob) is

$$X^n - T_1 X^{n-1} + ... + (-1)^j q^{j(j-1)} T_j + ... + (-1)^n q^{n(n-1)} T_n \tag{87}$$

It follows that the eigenvector $\phi \in \mathscr{H}$ of Hecke operators correspond bijectively to representations ρ and the characteristic polynomial of the Frobenius is

$$X^n - a_1 X^{n-1} + ... + (-1)^j q^{j(j-1)} a_j + ... + (-1)^n q^{n(n-1)} a_n \tag{88}$$

where a_j are the eigenvalues of the Hecke operators $T_i(\phi) = a_i.\phi, \ 1 \leq i \leq n, \ a_i \in \overline{\mathbb{Q}}_p$.