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Research Article

Highly Dispersive Optical Solitons with Quadratic-Cubic Nonlinear form of Self-Phase Modulation by Sardar Sub-Equation Approach

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Abstract: The highly dispersive optical solitons with a quadratic-cubic form of self-phase modulation structure are derived. The governing model was reduced to an ordinary differential equation by the traveling wave hypothesis. Subsequently, the Sardar sub-equation method and its modified version are used to locate the soliton solutions. A full spectrum of optical solitons is thus obtained.

***Keywords*:** traveling waves, sardar sub-equation, quadratic-cubic

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1. Introduction

The concept of highly dispersive (HD) optical solitons emerged about half a decade ago [1–5]. This involves the nonlinear Schrödinger's equation with inter-modal dispersion (IMD), chromatic dispersion (CD), third-order dispersion (3OD), fourth-order dispersion (4OD), fifth-order dispersion (5OD), and sixth-order dispersion (6OD). When the CD levels become critically low and there is an imminent risk of depletion, the implementation of additional distinct effects becomes necessary. These effects serve as a strategic response to counterbalance the diminishing levels of CD. Essentially, they are introduced to offset the adverse effects or limitations caused by the low count of CD. By deploying these supplementary measures, the aim is to maintain functionality or achieve desired outcomes despite the scarcity of CD resources. In essence, these additional effects act as a form of intervention to address the challenges posed by the impending depletion of CD [1–5]. The negative effects of the inclusion of such higher-order dispersion terms are ignored. One of the issues is the drastic slowdown of optical solitons, while the other is the immense radiative effect that the model will experience. By ignoring these effects, the study will focus on the soliton dynamics of the model.

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The self-phase modulation (SPM) structure is considered to be one of the non-Kerr laws. This work addresses the model with a quadratic-cubic form of SPM. This extension to the usual Kerr law of SPM includes the quadratic term as an addendum. Earlier this year, this form of the nonlinear Schrödinger's equation was examined using the Lie symmetry approach, and a few soliton solutions were established [6, 7]. The current paper implements the Sardar sub-equation and Sardar modified sub-equation approaches to obtain a complete spectrum of optical solitons for the model. First, the model is reduced to an ordinary differential equation, and then both approaches are successfully applied to recover the results.

The details are presented in the remainder of the paper. Section 2 outlines the governing model, providing the foundational framework for understanding the system's dynamics. Section 3 explores traveling wave solutions within this framework, analyzing wave propagation characteristics. Section 4 introduces and applies the Sardar Sub-Equation method to derive analytical solutions for the governing model. Finally, Section 5 concludes the study by summarizing findings, discussing implications, and suggesting future research directions.

2. Governing model

The HD-NLSE, featuring quadratic-cubic (QC) nonlinearity, is discussed in [8], as presented below

$$i\,\Phi_t + i\,a_1\Phi_x + a_2\Phi_{xx} + i\,a_3\Phi_{xxx} + a_4\Phi_{xxxx} + ia_5\Phi_{xxxxx} + a_6\Phi_{xxxxxx} + \left(b_1\,|\Phi| + b_2\,|\Phi|^2\right)\Phi = 0. \tag{1}$$

Here, the dependent variable $\Phi = \Phi(x,\,t)$ in the dimensionless form of Equation (1) is derived from the soliton profile and represents a complex-valued function, where x and t are the independent variables representing spatial and temporal coordinates, respectively. Subsequently, $i = \sqrt{-1}$ serves as the coefficient for the linear temporal evolution of the pulses. Also, b_j for $j = 1,\,2$ represents the quadratic and cubic coefficients of the SPM effect. Finally, the IMD, CD, 3OD, 4OD, 5OD, and 6OD are sequentially represented by the coefficients of a_j for $j = 1,\,\ldots,\,6$.

3. Travelling wave solution

The solution for Eq. (1) is provided in [9–14], as indicated below

$$\Phi(x,\,t) = u(\xi)\,e^{i\theta(x,\,t)}. \tag{2}$$

Here, $\xi = x - \gamma t$ characterizes the wave variable, with representing the soliton speed. Furthermore, the phase component of the soliton is given by $\theta(x,\,t) = -kx + \omega t + \theta_0$, where k signifies the soliton frequency, ω denotes the wavenumber, and θ_0 stands for the phase constant. Lastly, $u(\xi)$ denotes the amplitude component of the soliton. By utilizing Eq. (2) and its derivatives:

$$i\Phi_t = \left[-i\gamma u' - \omega u\right] e^{i\theta(x,\,t)}, \tag{3}$$

$$\Phi_x = \left[u' - kiu\right] e^{i\theta(x,\,t)}, \tag{4}$$

$$\Phi_{xx} = \left[u'' - 2iku' - k^2 u\right] e^{i\theta(x,\,t)}, \tag{5}$$

$$\Phi_{xxx} = \left[u^{(3)} - 3iku'' - 3k^2u' + ik^3u\right] e^{i\theta(x,\, t)}, \tag{6}$$

$$\Phi_{xxxx} = \left[u^{(4)} - 4iku^{(3)} - 6k^2u'' + 4ik^3u' + k^4u\right] e^{i\theta(x,\, t)}, \tag{7}$$

$$\Phi_{xxxxx} = \left[u^{(5)} - 5iku^{(4)} - 10k^2u^{(3)} + 10ik^3u'' + 5k^4u' - ik^5u\right] e^{i\theta(x,\, t)}, \tag{8}$$

$$\Phi_{xxxxxx} = \left[u^{(6)} - 6iku^{(5)} - 15k^2u^{(4)} + 20ik^3u^{(3)} + 15k^4u'' - 6ik^5u' - k^6u\right] e^{i\theta(x,\, t)}. \tag{9}$$

Eq. (1) becomes:

$$\begin{aligned}
&\left[-i\gamma u' - \omega u\right] + a_1\left[iu' + ku\right] + a_2\left[u'' - 2iku' - k^2u\right] \\
&+a_3\left[iu^{(3)} + 3ku'' - i3k^2u' - k^3u\right] + a_4\left[u^{(4)} - 4iku^{(3)} - 6k^2u'' + 4ik^3u' + k^4u\right] \\
&+a_5\left[iu^{(5)} + 5ku^{(4)} - i10k^2u^{(3)} - 10k^3u'' + i5k^4u' + k^5u\right] \\
&+a_6\left[u^{(6)} - 6iku^{(5)} - 15k^2u^{(4)} + 20ik^3u^{(3)} + 15k^4u'' - 6ik^5u' - k^6u\right] + \left(b_1|\Phi| + b_2|\Phi|^2\right)\Phi = 0.
\end{aligned} \tag{10}$$

Eq. (10) can be decomposed into real and imaginary parts, which are respectively expressed as

$$\begin{aligned}
&\left(-\omega + a_1k - a_2k^2 - a_3k^3 + a_4k^4 + a_5k^5 - a_6k^6\right)u + \left(a_2 + 3a_3k - 6a_4k^2 - 10a_5k^3 + 15a_6k^4\right)u'' \\
&+\left(a_4 + 5a_5k - 15a_6k^2\right)u^{(4)} + a_6u^{(6)} + (b_1 + b_2)u^3 = 0,
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
&\left(-\gamma + a_1 - 2ka_2 - 3a_3k^2 + 4a_4k^3 + 5a_5k^4 - 6a_6k^5\right)u' \\
&+\left(a_3 - 4a_4k - 10a_5k^2 + 20a_6k^3\right)u^{(3)} + (a_5 - 6a_6k)u^{(5)} = 0.
\end{aligned} \tag{12}$$

In accordance with Eq. (12), we arrive at

$$\gamma = a_1 - 2ka_2 - 3a_3k^2 + 4a_4k^3 + 5a_5k^4 - 6a_6k^5, \tag{13}$$

whenever

$$a_3 = \left(4a_4k + 10a_5k^2 - 20a_6k^3\right), \tag{14}$$

and

$$a_5 = 6a_6k. \tag{15}$$

Accordingly, Eq. (11) can be represented as

$$\begin{aligned} &\left(-\omega + a_1k - a_2k^2 - 3a_4k^4 - 35a_6k^6\right)u + \left(a_2 + 6a_4k^2 + 75a_6k^4\right)\, u'' \\ &+ \left(a_4 + 15a_6k^2\right)u^{(4)} + a_6u^{(6)} + \left(b_1 + b_2\right)u^3 = 0. \end{aligned} \tag{16}$$

4. Sardar sub-equation method (SSEM)

The SSEM offers a significant advantage in its ability to produce diverse soliton solutions, ranging from dark, bright, and singular forms to more complex configurations such as mixed dark-bright, dark-singular, bright-singular, and mixed singular solitons. Moreover, it facilitates the derivation of rational, periodic, trigonometric, and other solution types. In this approach, we solve Eq. (16) by assuming that the solution conforms to the form described in [15–21]:

$$u(\xi) = \sum_{n=0}^{\mathrm{N}} \lambda_n \Psi^n(\xi), \;\; \lambda_{\mathrm{N}} \neq 0. \tag{17}$$

Here, the constants λ_n (where $n = 0, 1, \ldots, N$)) are to be calculated subsequently. The integer N is determined using the homogeneous balance method, ensuring a balance between the nonlinear term and the highest-order derivative in Eq. (16). Additionally, the function $\Psi^n(\xi)$ in Eq. (17) must fulfill the following equation:

$$\Psi'(\xi) = \sqrt{\eta_2\Psi(\xi)^4 + \eta_1\Psi(\xi)^2 + \eta_0}, \tag{18}$$

where η_l (with $l = 0, 1, 2$) represents constants.

Accordingly, based on the values of the parameters η_l, Eq. (18) exhibits various known solutions, listed as follows [22–27]:

Case 1 When $\eta_0 = 0$, $\eta_1 > 0$, and $\eta_2 \neq 0$, soliton solutions are obtained:

$$\Psi_1^{\pm}(\xi) = \pm\sqrt{-pq\eta_1/\eta_2}\,\mathrm{sech}_{pq}\left(\sqrt{\eta_1}\xi\right), \;\; \eta_2 < 0, \tag{19}$$

and

$$\Psi_2^{\pm}(\xi) = \pm\sqrt{pq\eta_1/\eta_2}\,\mathrm{csch}_{pq}(\sqrt{\eta_1}\xi), \;\; \eta_2 > 0, \tag{20}$$

where

$$\begin{aligned} \mathrm{sech}_{pq}(\sqrt{\eta_1}\xi) &= \frac{2}{p\, e^{\sqrt{\eta_1}\xi} + q\, e^{-\sqrt{\eta_1}\xi}}, \\ \mathrm{csch}_{pq}(\sqrt{\eta_1}\xi) &= \frac{2}{p\, e^{\sqrt{\eta_1}\xi} - q\, e^{-\sqrt{\eta_1}\xi}}. \end{aligned} \tag{21}$$

Case 2 If $\eta_0 = \dfrac{1}{4}\dfrac{\eta_1^2}{\eta_2}$, $\eta_2 > 0$, and $\eta_1 < 0$, soliton solutions are derived:

$$\Psi_3^{\pm}(\xi) = \pm\sqrt{-\eta_1/2\eta_2}\,\tanh_{pq}\left(\sqrt{-\frac{\eta_1}{2}}\xi\right), \tag{22}$$

$$\Psi_4^{\pm}(\xi) = \pm\sqrt{-\eta_1/2\eta_2}\,\coth_{pq}\left(\sqrt{-\frac{\eta_1}{2}}\xi\right), \tag{23}$$

$$\Psi_5^{\pm}(\xi) = \pm\sqrt{-\eta_1/2\eta_2}\left(\tanh_{pq}\left(\sqrt{-2\eta_1}\xi\right) \pm i\sqrt{pq}\,\mathrm{sech}_{pq}\left(\sqrt{-2\eta_1}\xi\right)\right), \tag{24}$$

$$\Psi_6^{\pm}(\xi) = \pm\sqrt{-\eta_1/2\eta_2}\left(\coth_{pq}\left(\sqrt{-2\eta_1}\xi\right) \pm \sqrt{pq}\,\mathrm{csch}_{pq}\left(\sqrt{-2\eta_1}\xi\right)\right), \tag{25}$$

and

$$\Psi_7^{\pm}(\xi) = \pm\frac{1}{2}\sqrt{-\eta_1/2\eta_2}\left(\tanh_{pq}\left(\sqrt{-\frac{\eta_1}{8}}\xi\right) \pm \coth_{pq}\left(\sqrt{\frac{-\eta_1}{8}}\xi\right)\right), \tag{26}$$

where

$$\begin{aligned} \tanh_{pq}(\sqrt{\eta_1}\xi) &= \frac{p\, e^{\sqrt{\eta_1}\xi} - qe^{-\sqrt{\eta_1}\xi}}{pe^{\sqrt{\eta_1}\xi} + qe^{-\sqrt{\eta_1}\xi}}, \\ \coth_{pq}(\sqrt{\eta_1}\xi) &= \frac{pe^{\sqrt{\eta_1}\xi} + qe^{-\sqrt{\eta_1}\xi}}{pe^{\sqrt{\eta_1}\xi} - qe^{-\sqrt{\eta_1}\xi}}. \end{aligned} \tag{27}$$

4.1 *Application of the modified Sardar sub-equation method*

Our analysis commenced with the application of the homogeneous balance method principle [28–33], balancing the nonlinear term $u^{(6)}$ with the linear term u^3 in Eq. (16). This yielded the equation $N+6=3N$, from which we derived $N=3$. Consequently, Eq. (17) takes the form:

$$u(\xi)=\left(\lambda_0+\lambda_1\Psi+\lambda_2\Psi^2+\lambda_3\Psi^3\right), \tag{28}$$

$$u'(\xi)=\left(\lambda_1+2\lambda_2\Psi+3\lambda_3\Psi^2\right)\sqrt{(\eta_2\Psi^4+\eta_1\Psi^2+\eta_0)}, \tag{29}$$

$$u''=\begin{pmatrix} 12\lambda_3\eta_2\Psi^5+6\lambda_2\eta_2\Psi^4+(2\lambda_1\eta_2+9\lambda_3\eta_1)\Psi^3 \\ +4\lambda_2\eta_1\Psi^2+(\lambda_1\eta_1+6\lambda_3\eta_0)\Psi+2\lambda_2\eta_0 \end{pmatrix}, \tag{30}$$

$$u^{(3)}=\begin{pmatrix} 60\,\lambda_3\eta_2\Psi^4+24\lambda_2\eta_2\Psi^3+3(2\lambda_1\eta_2+9\lambda_3\eta_1)\Psi^2 \\ +8\lambda_2\eta_1\Psi+(\lambda_1\eta_1+6\lambda_3\eta_0) \end{pmatrix} \times\sqrt{(\eta_2\Psi^4+\eta_1\Psi^2+\eta_0)}, \tag{31}$$

$$u^{(4)}=\begin{pmatrix} 360\,\lambda_3{\eta_2}^2\Psi^7+120\lambda_2{\eta_2}^2\Psi^6+6\eta_2(4\lambda_1\eta_2+68\lambda_3\eta_1)\Psi^5 \\ +120\lambda_2\eta_1\eta_2\Psi^4+\left(252\lambda_3\eta_0\eta_2+20\lambda_1\eta_1\eta_2+81\lambda_3{\eta_1}^2\right)\Psi^3 \\ +(16\lambda_2\eta_1\eta_1+72\lambda_2\eta_0\eta_2)\Psi^2 \\ +\left(\lambda_1{\eta_1}^2+60\,\lambda_3\eta_0\eta_1+12\eta_0\lambda_1\eta_2\right)\Psi+8\lambda_2\eta_0\eta_1 \end{pmatrix}, \tag{32}$$

$$u^{(5)}=\begin{pmatrix} 2520\,\lambda_3{\eta_2}^2\Psi^6+720\lambda_2{\eta_2}^2\Psi^5+30\eta_2(4\lambda_1\eta_2+68\lambda_3\eta_1)\Psi^4 \\ +480\lambda_2\eta_1\eta_2\Psi^3+3\left(252\lambda_3\eta_0\eta_2+20\lambda_1\eta_1\eta_2+81\lambda_3{\eta_1}^2\right)\Psi^2 \\ +2(16\lambda_2\eta_1\eta_1+72\lambda_2\eta_0\eta_2)\Psi+\left(\lambda_1{\eta_1}^2+60\,\lambda_3\eta_0\eta_1++12\eta_0\lambda_1\eta_2\right) \end{pmatrix} \times\sqrt{(\eta_2\Psi^4+\eta_1\Psi^2+\eta_0)}, \tag{33}$$

and

$$u^{(6)}=\left\{\begin{matrix} 20160\,\lambda_3{\eta_2}^3\Psi^9+5040\lambda_2{\eta_2}^3\Psi^8+180{\eta_2}^2(4\lambda_1\eta_2+152\eta_1)\Psi^7 \\ +2520\,\lambda_3\eta_1{\eta_2}^2\Psi^7+6720\lambda_2\eta_1{\eta_2}^2\Psi^6 \end{matrix}\right.$$

$$+\begin{bmatrix} 6\eta_2\left(2772\lambda_3\eta_0\eta_2+20\lambda_1\eta_1\eta_2+81\lambda_3{\eta_1}^2\right)+30\eta_1\eta_2(4\lambda_1\eta_2+68\lambda_3\eta_1) \\ +6\eta_2\left(252\,\lambda_3\eta_0\eta_2+20\lambda_1\eta_1\eta_2+81\lambda_3{\eta_1}^2\right)+120\eta_1\eta_2(4\lambda_1\eta_2+68\lambda_3\eta_1) \end{bmatrix}\Psi^5$$

$$
\begin{aligned}
&+\left[2\eta_2\left(736\lambda_2\eta_1{}^2+72\lambda_2\eta_0\eta_2\right)+4\eta_2\left(16\lambda_2\eta_1\eta_1+72\lambda_2\eta_0\eta_2\right)+480\lambda_2\eta_1{}^2\eta_2+3600\lambda_2\eta_0\eta_2{}^2\right]\Psi^4\\
&+\left[\begin{array}{c}2\eta_2\left(\lambda_1\eta_1{}^2+60\,\lambda_3\eta_0\eta_1++12\eta_0\lambda_1\eta_2\right)+6\eta_1\left(252\lambda_3\eta_0\eta_2+20\lambda_1\eta_1\eta_2+81\lambda_3\eta_1{}^2\right)\\+3\eta_1\left(252\lambda_3\eta_0\eta_2+20\lambda_1\eta_1\eta_2+81\lambda_3\eta_1{}^2\right)+120\eta_0\eta_2\left(4\lambda_1\eta_2+68\lambda_3\eta_1\right)\end{array}\right]\Psi^3\\
&+\left[2\eta_1\left(16\lambda_2\eta_1\eta_1+72\lambda_2\eta_0\eta_2\right)+1440\lambda_2\eta_0\eta_1\eta_2+2\left(16\lambda_2\eta_1{}^3+72\lambda_2\eta_0\eta_1{}^2\eta_2\right)\right]\Psi^2\\
&+\left[\eta_1\left(\lambda_1\eta_1{}^2+60\,\lambda_3\eta_0\eta_1+12\eta_0\lambda_1\eta_2\right)+6\eta_0\left(252\lambda_3\eta_0\eta_2+20\lambda_1\eta_1\eta_2+81\lambda_3\eta_1{}^2\right)\right]\Psi\\
&+2\eta_0\left(16\lambda_2\eta_1{}^2+72\lambda_2\eta_0\eta_2\right)\}.
\end{aligned}
\tag{34}
$$

By substituting equations (30), (32), and (34) into equation (16) and considering equation (18), we derive:

$$
\begin{aligned}
&\left(-\omega\,+a_1k\,-a_2k^2\,-3a_4k^4-35\,a_6k^6\right)\left(\lambda_0+\lambda_1\Psi+\lambda_2\Psi^2+\lambda_3\Psi^3\right)\\
&+\left(a_2+6a_4k^2+75\,a_6k^4\right)\left(\begin{array}{c}12\lambda_3\eta_2\Psi^5+6\lambda_2\eta_2\Psi^4+\left(2\lambda_1\eta_2+9\lambda_3\eta_1\right)\Psi^3\\+4\lambda_2\eta_1\Psi^2+\left(\lambda_1\eta_1+6\lambda_3\eta_0\right)\Psi+2\lambda_2\eta_0\end{array}\right)\\
&+\left(a_4+15a_6k^2\right)\Big(360\lambda_3\eta_2{}^2\Psi^7+120\lambda_2\eta_2{}^2\Psi^6+6\eta_2\left(4\lambda_1\eta_2+68\lambda_3\eta_1\right)\Psi^5\\
&+120\lambda_2\eta_1\eta_2\Psi^4+\left(252\lambda_3\eta_0\eta_2+20\lambda_1\eta_1\eta_2+81\lambda_3\eta_1{}^2\right)\,\Psi^3+\left(16\lambda_2\eta_1\eta_1+72\lambda_2\eta_0\eta_2\right)\Psi^2\\
&+\,\left(\lambda_1\eta_1{}^2+60\,\lambda_3\eta_0\eta_1+12\eta_0\lambda_1\eta_2\right)\Psi+8\lambda_2\eta_0\eta_1\Big)\\
&+a_6\left\{\begin{array}{c}20160\lambda_3\eta_2{}^3\Psi^9+5040\lambda_2\eta_2{}^3\Psi^8+180\eta_2{}^2\left(4\lambda_1\eta_2+152\eta_1\right)\Psi^7\\+2520\,\lambda_3\eta_1\eta_2{}^2\Psi^7+6720\lambda_2\eta_1\eta_2{}^2\Psi^6\end{array}\right.\\
&+\left[\begin{array}{c}6\eta_2\left(2772\lambda_3\eta_0\eta_2+20\lambda_1\eta_1\eta_2+81\lambda_3\eta_1{}^2\right)+30\eta_1\eta_2\left(4\lambda_1\eta_2+68\lambda_3\eta_1\right)\\+6\eta_2\left(252\,\lambda_3\eta_0\eta_2+20\lambda_1\eta_1\eta_2+81\lambda_3\eta_1{}^2\right)+120\eta_1\eta_2\left(4\lambda_1\eta_2+68\lambda_3\eta_1\right)\end{array}\right]\Psi^5\\
&+\left[\begin{array}{c}2\eta_2\left(736\lambda_2\eta_1{}^2+72\lambda_2\eta_0\eta_2\right)+4\eta_2\left(16\lambda_2\eta_1\eta_1+72\lambda_2\eta_0\eta_2\right)\\+480\lambda_2\eta_1{}^2\eta_2+3600\lambda_2\eta_0\eta_2{}^2\end{array}\right]\Psi^4\\
&+\left[\begin{array}{c}2\eta_2\left(\lambda_1\eta_1{}^2+60\,\lambda_3\eta_0\eta_1++12\eta_0\lambda_1\eta_2\right)+6\eta_1\left(252\lambda_3\eta_0\eta_2+20\lambda_1\eta_1\eta_2+81\lambda_3\eta_1{}^2\right)\\+3\eta_1\left(252\lambda_3\eta_0\eta_2+20\lambda_1\eta_1\eta_2+81\lambda_3\eta_1{}^2\right)+120\eta_0\eta_2\left(4\lambda_1\eta_2+68\lambda_3\eta_1\right)\end{array}\right]\Psi^3
\end{aligned}
$$

$$
\begin{aligned}
&+\left[2\eta_1\left(16\lambda_2\eta_1\eta_1+72\lambda_2\eta_0\eta_2\right)+1440\lambda_2\eta_0\eta_1\eta_2+2\left(16\lambda_2{\eta_1}^3+72\lambda_2\eta_0{\eta_1}^2\eta_2\right)\right]\Psi^2\\
&+\eta_1\left(\lambda_1{\eta_1}^2+60\,\lambda_3\eta_0\eta_1+12\eta_0\lambda_1\eta_2\right)+6\eta_0\left(252\lambda_3\eta_0\eta_2+20\lambda_1\eta_1\eta_2+81\lambda_3{\eta_1}^2\right)\Psi\\
&+2\eta_0\left(16\lambda_2{\eta_1}^2+72\lambda_2\eta_0\eta_2\right)\}+(b_1+b_2)\left({\lambda_0}^3+3{\lambda_0}^2\,\lambda_1\Psi+3\left({\lambda_0}^2\lambda_2+\lambda_0{\lambda_1}^2\right)\Psi^2\right.\\
&+\left({\lambda_1}^3+3{\lambda_0}^2\lambda_3+6\lambda_0\lambda_1\lambda_2\right)\Psi^3+\left(3{\lambda_1}^2\lambda_2+3{\lambda_2}^2\lambda_0+6\lambda_0\lambda_1\lambda_3\right)\Psi^4\\
&+\left(3{\lambda_1}^2\lambda_3+3{\lambda_2}^2\lambda_1+6\lambda_0\lambda_2\lambda_3\right)\Psi^5+\left(2{\lambda_3}^2\lambda_0+{\lambda_2}^3+6\lambda_1\lambda_2\lambda_3\right)\Psi^6\\
&+3\left({\lambda_3}^2\lambda_1+{\lambda_2}^2\lambda_3\right)\Psi^7+3{\lambda_3}^2\lambda_2\Psi^8+{\lambda_3}^3\Psi^9=0.
\end{aligned}
\tag{35}
$$

Collecting and setting the coefficients of the independent functions $\Psi^j(\xi)$ to zero, we derive the following set of algebraic equations for each case:

Case I $\eta_0=0,\ \lambda_0=0,\ \lambda_1=0.$

Hence, Eq. (35) simplifies to the following equation:

$$
\begin{aligned}
&\left(-\omega\ +a_1k\ -a_2k^2\ -3a_4k^4-35\ a_6k^6\right)\left(\lambda_2\Psi^2+\lambda_3\Psi^3\right)\\
&+\left(a_2+6a_4k^2+75\ a_6k^4\right)\left(12\lambda_3\eta_2\Psi^5+6\lambda_2\eta_2\Psi^4+9\lambda_3\eta_1\Psi^3+4\lambda_2\eta_1\Psi^2\right)\\
&+\left(a_4+15a_6k^2\right)\left(\begin{array}{c}360\ \lambda_3{\eta_2}^2\Psi^7+120\lambda_2{\eta_2}^2\Psi^6+408\lambda_3\eta_1\eta_2\Psi^5\\+120\lambda_2\eta_1\eta_2\Psi^4+81\lambda_3{\eta_1}^2\ \Psi^3+16\lambda_2{\eta_1}^2\Psi^2\end{array}\right)\\
&+a_6\left\{\begin{array}{c}20160\lambda_3{\eta_2}^3\Psi^9+5040\ \lambda_2{\eta_2}^3\Psi^8+29880\lambda_3\eta_1{\eta_2}^2\Psi^7+6720\lambda_2\eta_1{\eta_2}^2\Psi^6\\+11172\ \eta_2\lambda_3{\eta_1}^2\ \Psi^5+2016\lambda_2{\eta_1}^2\eta_2\ \Psi^4+729\lambda_3{\eta_1}^3\Psi^3+64\lambda_2{\eta_1}^3\Psi^2\end{array}\right\}\\
&+(b_1+b_2)\left({\lambda_2}^3\Psi^6+3{\lambda_2}^2\lambda_3\Psi^7+3{\lambda_3}^2\lambda_2\Psi^8+{\lambda_3}^3\Psi^9\right)=0.
\end{aligned}
\tag{36}
$$

As a result, this leads to the derivation of the following set of algebraic equations for Ψ^j, where j ranges from 2 to 9:

$$\Psi^9: 20160a_6{\eta_2}^3+(b_1+b_2)\ {\lambda_3}^2=0,$$

$$\Psi^8: 1680a_6{\eta_2}^3+(b_1+b_2)\,{\lambda_3}^2=0,$$

$$\Psi^7: 120\left(a_4+15a_6k^2+83\ \eta_1a_6\right){\eta_2}^2+(b_1+b_2)\,{\lambda_2}^2=0,$$

$$\Psi^6: 120\left(\left(a_4+15a_6k^2\right)+56\eta_1 a_6\right)\eta_2{}^2+\left(b_1+b_2\right)\lambda_2{}^2=0,$$

$$\Psi^5: \left(a_2+6a_4k^2+75a_6k^4\right)+34\left(a_4+15a_6k^2\right)\eta_1+931a_6\eta_1{}^2=0,$$

$$\Psi^4: \left(a_2+6a_4k^2+75a_6k^4\right)+20\left(a_4+15a_6k^2\right)\eta_1+336a_6\eta_1{}^2=0,$$

$$\begin{aligned}\Psi^3: &\left(-\omega+a_1k-a_2k^2-3a_4k^4-35a_6k^6\right)+9\left(a_2+6a_4k^2+75a_6k^4\right)\eta_1\\ &+81\left(a_4+15a_6k^2\right)\eta_1{}^2+729a_6\eta_1{}^3=0,\\ \Psi^2: &\left(-\omega+a_1k-a_2k^2-3a_4k^4-35a_6k^6\right)+4\left(a_2+6a_4k^2+75a_6k^4\right)\eta_1\\ &+16\left(a_4+15a_6k^2\right)\eta_1{}^2+64a_6\eta_1{}^3=0.\end{aligned} \tag{37}$$

The solution to the set of algebraic equations (37) yields:

$$\omega=k\left(a_1-a_2k-3a_4k^3-35a_6k^5\right). \tag{38}$$

Family 1

$$\lambda_2=\mp 2\sqrt{-\frac{30\left(a_4+15a_6k^2+83\eta_1a_6\right)}{\left(b_1+b_2\right)}}\eta_2,\ \lambda_3=\mp 8\sqrt{-\frac{315a_6\eta_2}{\left(b_1+b_2\right)}}\eta_2,\ \eta_1=-\frac{266\left(a_2+6a_4k^2+75a_6k^4\right)}{2527\left(a_4+15a_6k^2\right)}.$$

Consequently, bright and singular soliton solutions emerge as follows:

$$\begin{aligned}\Phi_{1,\,a}(x,\ t)=&\left(-pq\frac{\eta_1}{\eta_2}\right)\left[\lambda_2\text{sech}^2{}_{pq}\left(\sqrt{\eta_1}\left(x-\gamma t\right)\right)+\lambda_3\sqrt{\left(-pq\frac{\eta_1}{\eta_2}\right)}\text{sech}^3{}_{pq}\left(\sqrt{\eta_1}\left(x-\gamma t\right)\right)\right]\\ &\times exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right],\ \ \eta_2<0,\end{aligned} \tag{39}$$

and

$$\begin{aligned}\Phi_{1,\,b}(x,\ t)=&\left(pq\frac{\eta_1}{\eta_2}\right)\left[\lambda_2 csch^2{}_{pq}\left(\sqrt{\eta_1}\left(x-\gamma t\right)\right)+\lambda_3\sqrt{\left(pq\frac{\eta_1}{\eta_2}\right)}csch^3{}_{pq}\left(\sqrt{\eta_1}\left(x-\gamma t\right)\right)\right]\\ &\times exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right],\ \ \eta_2>0,\end{aligned} \tag{40}$$

respectively.

Family 2

$$\lambda_2 = \mp 2\sqrt{-\frac{30\left(a_4+15a_6k^2+83\eta_1a_6\right)}{(b_1+b_2)}}\eta_2,$$

$$\lambda_3 = \mp 4\sqrt{-\frac{105a_6\eta_2}{(b_1+b_2)}}\eta_2,$$

$$\eta_1 = -\frac{266\left(a_2+6a_4k^2+75a_6k^4\right)}{2527\left(a_4+15a_6k^2\right)}$$

Thus, bright and singular soliton solutions are revealed as

$$\begin{aligned}\Phi_{2,\,a}(x,\,t) = &\left(-pq\frac{\eta_1}{\eta_2}\right)\left[\lambda_2\text{sech}^2{}_{pq}\left(\sqrt{\eta_1}\left(x-\gamma t\right)\right)+\lambda_3\sqrt{\left(-pq\frac{\eta_1}{\eta_2}\right)}\text{sech}^3{}_{pq}\left(\sqrt{\eta_1}\left(x-\gamma t\right)\right)\right]\\ &\times exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right],\;\; \eta_2<0,\end{aligned} \tag{41}$$

and

$$\begin{aligned}\Phi_{2,\,b}(x,\,t) = &\left(pq\frac{\eta_1}{\eta_2}\right)\left[\lambda_2\text{csch}^2{}_{pq}\left(\sqrt{\eta_1}\left(x-\gamma t\right)\right)+\lambda_3\sqrt{\left(pq\frac{\eta_1}{\eta_2}\right)}\text{csch}^3{}_{pq}\left(\sqrt{\eta_1}\left(x-\gamma t\right)\right)\right]\\ &\times exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right],\;\; \eta_2>0,\end{aligned} \tag{42}$$

respectively.

Family 3

$$\lambda_2 = \mp 2\sqrt{\frac{30\left(a_4+15a_6k^2+56\eta_1a_6\right)}{(b_1+b_2)}}\eta_2,$$

$$\lambda_3 = \mp 8\sqrt{-\frac{315a_6\eta_2}{(b_1+b_2)}}\eta_2,$$

$$\eta_1 = -\frac{266\left(a_2+6a_4k^2+75a_6k^4\right)}{2527\left(a_4+15a_6k^2\right)}.$$

As a result, bright and singular soliton solutions take form as

$$\Phi_{3,\,a}(x,\,t)=\left(-pq\frac{\eta_1}{\eta_2}\right)\left[\lambda_2\text{sech}^2{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma t)\right)+\lambda_3\sqrt{\left(-pq\frac{\eta_1}{\eta_2}\right)}\text{sech}^3{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma\, t)\right)\right]$$
$$\times\, exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right],\ \ \eta_2<0, \tag{43}$$

and

$$\Phi_{3,\,b}(x,\,t)=\left(pq\frac{\eta_1}{\eta_2}\right)\left[\lambda_2 csch^2{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma t)\right)+\lambda_3\sqrt{\left(pq\frac{\eta_1}{\eta_2}\right)}csch^3{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma t)\right)\right]$$
$$\times\, exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right],\ \ \eta_2>0, \tag{44}$$

respectively.

Family 4

$$\lambda_2=\mp 2\sqrt{\frac{30\left(a_4+15a_6k^2+56\eta_1 a_6\right)}{(b_1+b_2)}}\eta_2,$$

$$\lambda_3=\mp 4\sqrt{-\frac{105\,a_6\eta_2}{(b_1+b_2)}}\eta_2,$$

$$\eta_1=-\frac{266\left(a_2+6a_4k^2+75\,a_6k^4\right)}{2527\left(a_4+15a_6k^2\right)}.$$

Accordingly, bright and singular soliton solutions shape up as

$$\Phi_{4,\,a}(x,\,t)=\left(-pq\frac{\eta_1}{\eta_2}\right)\left[\lambda_2\text{sech}^2{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma t)\right)+\lambda_3\sqrt{\left(-pq\frac{\eta_1}{\eta_2}\right)}\text{sech}^3{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma t)\right)\right]$$
$$\times\, exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right],\ \ \eta_2<0, \tag{45}$$

and

$$\Phi_{4,\,b}(x,\,t)=\left(pq\frac{\eta_1}{\eta_2}\right)\left[\lambda_2\text{csch}^2{}_{pq}\left(\sqrt{-\eta_1}\,(x-\gamma t)\right)+\lambda_3\sqrt{\left(pq\frac{\eta_1}{\eta_2}\right)}\text{csch}^3{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma t)\right)\right]$$
$$\times\, exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right],\ \ \eta_2>0, \tag{46}$$

respectively.

Family 5

$$\lambda_2 = \mp 2\sqrt{-\frac{30\left(a_4+15a_6k^2+83\eta_1a_6\right)}{\left(b_1+b_2\right)}}\eta_2,$$

$$\lambda_3 = \mp 8\sqrt{-\frac{315a_6\eta_2}{\left(b_1+b_2\right)}}\eta_2,$$

$$\eta_1 = -\frac{595\left(a_2+6a_4k^2+75a_6k^4\right)}{7196\left(a_4+15a_6k^2\right)}.$$

Consequently, bright and singular soliton solutions turn out to be

$$\Phi_{5,\,a}(x,\,t) = \left(-pq\frac{\eta_1}{\eta_2}\right)\left[\lambda_2\text{sech}^2{}_{pq}\left(\sqrt{\eta_1}\left(x-\gamma t\right)\right)+\lambda_3\sqrt{\left(-pq\frac{\eta_1}{\eta_2}\right)}\text{sech}^3{}_{pq}\left(\sqrt{\eta_1}\left(x-\gamma t\right)\right)\right] \tag{47}$$

$$\times\, exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right],\quad \eta_2<0,$$

and

$$\Phi_{5,\,b}(x,\,t) = \left(pq\frac{\eta_1}{\eta_2}\right)\left[\lambda_2\text{csch}^2{}_{pq}\left(\sqrt{\eta_1}\left(x-\gamma t\right)\right)+\lambda_3\sqrt{\left(pq\frac{\eta_1}{\eta_2}\right)}\text{csch}^3{}_{pq}\left(\sqrt{\eta_1}\left(x-\gamma t\right)\right)\right] \tag{48}$$

$$\times\, exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right],\quad \eta_2>0,$$

respectively.

Family 6

$$\lambda_2 = \mp 2\sqrt{-\frac{30\left(a_4+15a_6k^2+83\eta_1a_6\right)}{\left(b_1+b_2\right)}}\eta_2,$$

$$\lambda_3 = \mp\, 4\sqrt{-\,\frac{105\,a_6\,\eta_2}{\left(b_1+b_2\right)}}\,\eta_2,$$

$$\eta_1 = -\frac{595\left(a_2+6a_4k^2+75\,a_6k^4\right)}{7196\,\left(a_4+15a_6k^2\right)}.$$

As a result, bright and singular soliton solutions arise as

$$
\Phi_{6,\,a}(x,\;t)=\left(-pq\frac{\eta_1}{\eta_2}\right)\left[\lambda_2\mathrm{sech}^2{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma\,t)\right)+\lambda_3\sqrt{\left(-pq\frac{\eta_1}{\eta_2}\right)}\mathrm{sech}^3{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma\,t)\right)\right]
$$

$$
\times\, exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right],\;\; \eta_2<0, \tag{49}
$$

and

$$
\Phi_{6,\,b}(x,\;t)=\left(pq\frac{\eta_1}{\eta_2}\right)\left[\lambda_2\mathrm{csch}^2{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma\,t)\right)+\lambda_3\sqrt{\left(pq\frac{\eta_1}{\eta_2}\right)}\mathrm{csch}^3{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma\,t)\right)\right]
$$

$$
\times\, exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right],\;\; \eta_2>0, \tag{50}
$$

respectively.

Family 7

$$
\lambda_2=\mp 2\sqrt{\frac{30\;\left(a_4+15a_6k^2+56\;\eta_1 a_6\right)}{(b_1+b_2)}}\eta_2,
$$

$$
\lambda_3=\mp\, 8\sqrt{-\;\frac{315\;a_6\;\eta_2}{(b_1+b_2)}}\;\;\eta_2,
$$

$$
\eta_1=-\frac{595\left(a_2+6a_4k^2+75\;a_6k^4\right)}{7196\;\left(a_4+15a_6k^2\right)}.
$$

Consequently, bright and singular soliton solutions emerge as follows:

$$
\Phi_{7,\,a}(x,\;t)=\left(-pq\frac{\eta_1}{\eta_2}\right)\left[\lambda_2\mathrm{sech}^2{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma\,t)\right)+\lambda_3\sqrt{\left(-pq\frac{\eta_1}{\eta_2}\right)}\mathrm{sech}^3{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma\,t)\right)\right]
$$

$$
\times\, exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right],\;\; \eta_2<0, \tag{51}
$$

and

$$
\Phi_{7,\,b}(x,\;t)=\left(pq\frac{\eta_1}{\eta_2}\right)\left[\lambda_2\mathrm{csch}^2{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma\,t)\right)+\lambda_3\sqrt{\left(pq\frac{\eta_1}{\eta_2}\right)}\mathrm{csch}^3{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma\,t)\right)\right]
$$

$$
\times\, exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right],\;\; \eta_2>0, \tag{52}
$$

respectively.

Family 8

$$\lambda_2 = \mp 2\sqrt{\frac{30\ (a_4 + 15a_6k^2 + 56\ \eta_1 a_6)}{(b_1 + b_2)}}\eta_2,$$

$$\lambda_3 = \mp 4\sqrt{-\ \frac{105\ a_6\ \eta_2}{(b_1 + b_2)}}\ \eta_2,$$

$$\eta_1 = -\frac{595\left(a_2 + 6a_4k^2 + 75\ a_6k^4\right)}{7196\ (a_4 + 15a_6k^2)}.$$

Thus, bright and singular soliton solutions are revealed as

$$\Phi_{8,\ a}(x,\ t) = \left(-pq\frac{\eta_1}{\eta_2}\right)\left[\lambda_2 \mathrm{sech}^2{}_{pq}\left(\sqrt{\eta_1}\,(x - \gamma\, t)\right) + \lambda_3\sqrt{\left(-pq\frac{\eta_1}{\eta_2}\right)}\mathrm{sech}^3{}_{pq}\left(\sqrt{\eta_1}\,(x - \gamma\, t)\right)\right]$$
$$\times\, exp\left[i\left(-\kappa x + \omega t + \theta_0\right)\right],\ \ \eta_2 < 0, \tag{53}$$

and

$$\Phi_{8,\ b}(x,\ t) = \left(pq\frac{\eta_1}{\eta_2}\right)\left[\lambda_2 \mathrm{csch}^2{}_{pq}\left(\sqrt{\eta_1}\,(x - \gamma\, t)\right) + \lambda_3\sqrt{\left(pq\frac{\eta_1}{\eta_2}\right)}\mathrm{csch}^3{}_{pq}\left(\sqrt{\eta_1}\,(x - \gamma\, t)\right)\right]$$
$$\times\, exp\left[i\left(-\kappa x + \omega t + \theta_0\right)\right],\ \ \eta_2 > 0, \tag{54}$$

respectively.

Family 9

$$\lambda_2 = \mp 2\sqrt{-\frac{30\,(a_4 + 15a_6k^2 + 83\ \eta_1 a_6)}{(b_1 + b_2)}}\ \eta_2,$$

$$\lambda_3 = \mp 8\sqrt{-\ \frac{315\ a_6\ \eta_2}{(b_1 + b_2)}}\ \ \eta_2,$$

$$\eta_1 = -\frac{203\left(a_2 + 6a_4k^2 + 75\ a_6k^4\right)}{1708\ (a_4 + 15a_6k^2)}.$$

As a result, bright and singular soliton solutions take form as

$$\Phi_{9,\,a}(x,\,t)=\left(-pq\frac{\eta_1}{\eta_2}\right)\left[\lambda_2\mathrm{sech}^2{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma\,t)\right)+\lambda_3\sqrt{\left(-pq\frac{\eta_1}{\eta_2}\right)}\mathrm{sech}^3{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma\,t)\right)\right]$$
$$\times\, exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right],\ \ \eta_2<0, \tag{55}$$

and

$$\Phi_{9,\,b}(x,\,t)=\left(pq\frac{\eta_1}{\eta_2}\right)\left[\lambda_2\mathrm{csch}^2{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma\,t)\right)+\lambda_3\sqrt{\left(pq\frac{\eta_1}{\eta_2}\right)}\mathrm{csch}^3{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma\,t)\right)\right]$$
$$\times\, exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right],\ \ \eta_2>0, \tag{56}$$

respectively.

Family 10

$$\lambda_2=\mp 2\sqrt{-\frac{30\left(a_4+15a_6k^2+83\,\eta_1 a_6\right)}{(b_1+b_2)}}\,\eta_2,$$

$$\lambda_3=\mp\,4\sqrt{-\,\frac{105\,a_6\,\eta_2}{(b_1+b_2)}}\,\eta_2,$$

$$\eta_1=-\frac{203\left(a_2+6a_4k^2+75\,a_6k^4\right)}{1708\,\left(a_4+15a_6k^2\right)}.$$

Accordingly, bright and singular soliton solutions shape up as

$$\Phi_{10,\,a}(x,\,t)=\left(-pq\frac{\eta_1}{\eta_2}\right)\left[\lambda_2\mathrm{sech}^2{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma\,t)\right)+\lambda_3\sqrt{\left(-pq\frac{\eta_1}{\eta_2}\right)}\mathrm{sech}^3{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma\,t)\right)\right]$$
$$\times\, exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right],\ \ \eta_2<0, \tag{57}$$

and

$$\Phi_{10,\,b}(x,\,t)=\left(pq\frac{\eta_1}{\eta_2}\right)\left[\lambda_2\mathrm{csch}^2{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma\,t)\right)+\lambda_3\sqrt{\left(pq\frac{\eta_1}{\eta_2}\right)}\mathrm{csch}^3{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma\,t)\right)\right]$$
$$\times\, exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right],\ \ \eta_2>0, \tag{58}$$

respectively.

Family 11

$$\lambda_2 = \mp 2\sqrt{\frac{30\ (a_4+15a_6k^2+56\ \eta_1 a_6)}{(b_1+b_2)}}\eta_2,$$

$$\lambda_3 = \mp\, 8\sqrt{-\ \frac{315\ a_6\ \eta_2}{(b_1+b_2)}}\ \eta_2,$$

$$\eta_1 = -\frac{203\left(a_2+6a_4k^2+75\ a_6k^4\right)}{1708\ (a_4+15a_6k^2)}.$$

Consequently, bright and singular soliton solutions turn out to be

$$\Phi_{11,\ a}(x,\ t) = \left(-pq\frac{\eta_1}{\eta_2}\right)\left[\lambda_2 \mathrm{sech}^2{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma t)\right)+\lambda_3\sqrt{\left(-pq\frac{\eta_1}{\eta_2}\right)}\mathrm{sech}^3{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma t)\right)\right]$$
$$\times\, exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right],\ \ \eta_2<0, \tag{59}$$

and

$$\Phi_{11,\ b}(x,\ t) = \left(pq\frac{\eta_1}{\eta_2}\right)\left[\lambda_2 \mathrm{csch}^2{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma t)\right)+\lambda_3\sqrt{\left(pq\frac{\eta_1}{\eta_2}\right)}\mathrm{csch}^3{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma t)\right)\right]$$
$$\times\, exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right],\ \ \eta_2>0, \tag{60}$$

respectively.

Family 12

$$\lambda_2 = \mp 2\sqrt{\frac{30\ (a_4+15a_6k^2+56\ \eta_1 a_6)}{(b_1+b_2)}}\eta_2,$$

$$\lambda_3 = \mp\, 4\sqrt{-\ \frac{105\ a_6\ \eta_2}{(b_1+b_2)}}\ \eta_2,$$

$$\eta_1 = -\frac{203\left(a_2+6a_4k^2+75\ a_6k^4\right)}{1708\ (a_4+15a_6k^2)}.$$

As a result, bright and singular soliton solutions arise as

$$
\Phi_{12,\,a}(x,\,t) = \left(-pq\frac{\eta_1}{\eta_2}\right)\left[\lambda_2 \mathrm{sech}^2{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma t)\right) + \lambda_3\sqrt{\left(-pq\frac{\eta_1}{\eta_2}\right)}\mathrm{sech}^3{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma t)\right)\right]
$$
$$
\times\, exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right],\ \ \eta_2<0, \tag{61}
$$

and

$$
\Phi_{12,\,b}(x,\,t) = \left(pq\frac{\eta_1}{\eta_2}\right)\left[\lambda_2 \mathrm{csch}^2{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma t)\right) + \lambda_3\sqrt{\left(pq\frac{\eta_1}{\eta_2}\right)}\mathrm{csch}^3{}_{pq}\left(\sqrt{\eta_1}\,(x-\gamma t)\right)\right]
$$
$$
\times\, exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right],\ \ \eta_2>0, \tag{62}
$$

respectively.

Case 2 $\eta_0 = \frac{1}{4}\frac{\eta_1^2}{\eta_2},\ \lambda_0 = 0,\ \lambda_1 = 0,\ \lambda_2 = 0,\ \eta_2 > 0.$

Eq. (35) is simplified to the following equation:

$$
\begin{aligned}
&\left(-\omega + a_1 k - a_2 k^2 - 3a_4 k^4 - 35\,a_6 k^6\right)\left(\lambda_3 \Psi^3\right) + \left(a_2 + 6a_4 k^2 + 75\,a_6 k^4\right)\left(12\lambda_3\eta_2\Psi^5\right. \\
&\left.+ \left(9\lambda_3\eta_1\right)\Psi^3 + \left(6\lambda_3\eta_0\right)\Psi\right) + \left(a_4 + 15a_6 k^2\right)\begin{pmatrix} 360\,\lambda_3\eta_2{}^2\Psi^7 + 6\eta_2\left(68\lambda_3\eta_1\right)\Psi^5 \\ +\left(252\lambda_3\eta_0\eta_2 + 81\lambda_3\eta_1{}^2\right)\Psi^3 + \left(60\,\lambda_3\eta_0\eta_1\right)\Psi \end{pmatrix} \\
&+\lambda_3 a_6\left\{20160\,\eta_2{}^3\Psi^9 + 180\eta_2{}^2\left(152\eta_1\right)\Psi^7 + 2520\,\eta_1\eta_2{}^2\Psi^7\right. \\
&+\left[6\eta_2\left(2772\eta_0\eta_2 + 81\eta_1{}^2\right) + 30\eta_1\eta_2\left(68\eta_1\right) + 6\eta_2\left(252\,\eta_0\eta_2 + 81\eta_1{}^2\right) + 120\eta_1\eta_2\left(68\eta_1\right)\right]\Psi^5 \\
&+\left[2\eta_2\left(60\,\eta_0\eta_1\right) + 6\eta_1\left(252\eta_0\eta_2 + 81\eta_1{}^2\right) + 3\eta_1\left(252\eta_0\eta_2 + 81\eta_1{}^2\right) + 120\eta_0\eta_2\left(68\eta_1\right)\right]\Psi^3 \\
&\left.+\left[\eta_1\left(60\,\eta_0\eta_1\right) + 6\eta_0\left(252\eta_0\eta_2 + 81\eta_1{}^2\right)\right]\Psi\right\} + \left(b_1 + b_2\right)\left(\lambda_3{}^3\Psi^9\right) = 0.
\end{aligned} \tag{63}
$$

We reach the following set of algebraic equations for the corresponding Ψ^j, where j extends from 1 to 9:

$$
\begin{aligned}
&\Psi^9 : 20160\, a_6\eta_2{}^3 + (b_1+b_2)\ \ \lambda_3{}^2 = 0,\\
&\Psi^7 : \left(a_4 + 15a_6k^2\right) + 83\, a_6\eta_1 = 0,\\
&\Psi^5 : \left(a_2 + 6a_4k^2 + 75\, a_6k^4\right) + 34\left(a_4 + 15a_6k^2\right)\ \ \eta_1 + a_6\left[1512\ \eta_0\eta_2 + 931\ \eta_1{}^2\right] = 0,\\
&\Psi^3 : \left(-\omega\ + a_1k\ - a_2k^2\ - 3a_4k^4 - 35\, a_6k^6\right) + \left(a_2 + 6a_4k^2 + 75\, a_6k^4\right)\ 9\ \eta_1\\
&\qquad + \left(a_4 + 15a_6k^2\right)\ \left(252\eta_0\eta_2 + 81\eta_1{}^2\right) + a_6\left(10548\ \eta_2\eta_0\eta_1 + 729\ \eta_1{}^3\right) = 0,\\
&\Psi : \left(a_2 + 6a_4k^2 + 75\, a_6k^4\right) + \left(a_4 + 15a_6k^2\right)\ 10\ \eta_1 + a_6\left[91\ \eta_1{}^2 + 252\eta_0\eta_2\right] = 0.
\end{aligned}
\tag{64}
$$

The set of algebraic equations (64) is solved to give:

$$
\omega = k\ \left(\ a_1\ - a_2k\ - 3a_4k^3 - 35\, a_6k^5\right). \tag{65}
$$

Family 1

$$
\eta_0 = \frac{1}{4}\frac{\eta_1^2}{\eta_2},\ \lambda_3 = \sqrt{-\frac{20160\, a_6\eta_2}{(b_1+b_2)}}\ \eta_2,\ \omega = a_1k\ - a_2k^2\ - 3a_4k^4 - 35\, a_6k^6,\ \eta_1 = -\frac{\left(a_4 + 15a_6k^2\right)}{83\, a_6}.
$$

Also, setting ${}^{\eta_1}\!/_{2\eta_2} < 0$ and $\eta_2 > 0$, we acquire the following soliton solutions:
Dark soliton solution is represented by

$$
\Phi_{13,\,a}(x,\,t) = \sqrt{\left(-{}^{\eta_1}\!/_{2\eta_2}\right)^3}\lambda_3\ \tanh^3{}_{pq}\left(\sqrt{-\frac{\eta_1}{2}}\,(x-\gamma t)\right) exp\left[i\left(-\kappa x + \omega t + \theta_0\right)\right]. \tag{66}
$$

Singular soliton solution is expressed by

$$
\Phi_{13,\,b}(x,\,t) = \sqrt{\left(-{}^{\eta_1}\!/_{2\eta_2}\right)^3}\lambda_3\ \coth^3{}_{pq}\left(\sqrt{-\frac{\eta_1}{2}}\,(x-\gamma t)\right) exp\left[i\left(-\kappa x + \omega t + \theta_0\right)\right]. \tag{67}
$$

Complexion solutions are presented as:

$$
\Phi_{13,\,c}(x,\,t) = \ \sqrt{\left(-{}^{\eta_1}\!/_{2\eta_2}\right)^3}\lambda_3\ \left[\begin{array}{c}\tanh_{pq}\left(\sqrt{-2\eta_1}\,(x-\gamma t)\right)\\ \pm i\ \sqrt{pq}\mathrm{sech}_{pq}\left(\sqrt{-2\eta_1}\,(x-\gamma t)\right)\end{array}\right]^3 \times exp\left[i\left(-\kappa x + \omega t + \theta_0\right)\right], \tag{68}
$$

and

$$\Phi_{13,\,d}(x,\,t)=\sqrt{\left(-{}^{\eta_1}\!/_{2\eta_2}\right)^3}\lambda_3\left[\begin{array}{c}\coth_{pq}\left(\sqrt{-2\eta_1}\,(x-\gamma t)\right)\\ \pm i\,\sqrt{pq}\mathrm{csch}_{pq}\left(\sqrt{-2\eta_1}\,(x-\gamma t)\right)\end{array}\right]^3\times exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right].\tag{69}$$

Straddled dark-singular soliton solution is formulated as

$$\Phi_{13,\,e}(x,\,t)=\frac{\lambda_3}{8}\sqrt{\left(-{}^{\eta_1}\!/_{2\eta_2}\right)^3}\left[\tanh_{pq}\left(\sqrt{-\frac{\eta_1}{8}}\xi\right)\pm\coth_{pq}\left(\sqrt{\frac{-\eta_1}{8}}\xi\right)\right]^3$$
$$\times exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right],\;\;\eta_2>0.\tag{70}$$

Family 2

$$\eta_0=\frac{1}{4}\frac{\eta_1^2}{\eta_2},\;\lambda_3=\sqrt{-\frac{20160\,a_6\eta_2}{(b_1+b_2)}}\,\eta_2,\;\omega=a_1k\;-a_2k^2\;-3a_4k^4-35\,a_6k^6,\;\eta_1=-\frac{55}{319}\frac{\left(a_2+6a_4k^2+75\,a_6k^4\right)}{\left(a_4+15a_6k^2\right)}.$$

Additionally, when ${}^{\eta_1}\!/_{2\eta_2}<0$ and $\eta_2>0$, the following soliton solutions are obtained:
Dark soliton solution is depicted as

$$\Phi_{14,\,a}(x,\,t)=\sqrt{\left(-{}^{\eta_1}\!/_{2\eta_2}\right)^3}\lambda_3\,\tanh^3{}_{pq}\left(\sqrt{-\frac{\eta_1}{2}}\,(x-\gamma t)\right)exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right].\tag{71}$$

Singular soliton solution is defined as

$$\Phi_{14,\,b}(x,\,t)=\sqrt{\left(-{}^{\eta_1}\!/_{2\eta_2}\right)^3}\lambda_3\,\coth^3{}_{pq}\left(\sqrt{-\frac{\eta_1}{2}}\,(x-\gamma t)\right)exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right].\tag{72}$$

Complexion solutions are provided by

$$\Phi_{14,\,c}(x,\,t)=\sqrt{\left(-{}^{\eta_1}\!/_{2\eta_2}\right)^3}\lambda_3\left[\begin{array}{c}\tanh_{pq}\left(\sqrt{-2\eta_1}\,(x-\gamma t)\right)\\ \pm i\,\sqrt{pq}\mathrm{sech}_{pq}\left(\sqrt{-2\eta_1}\,(x-\gamma t)\right)\end{array}\right]^3\times exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right],\tag{73}$$

and

$$\Phi_{14,\,d}(x,\,t)=\sqrt{\left(-{}^{\eta_1}\!/_{2\eta_2}\right)^3}\lambda_3\left[\begin{array}{c}\coth_{pq}\left(\sqrt{-2\eta_1}\,(x-\gamma t)\right)\\ \pm i\,\sqrt{pq}\mathrm{csch}_{pq}\left(\sqrt{-2\eta_1}\,(x-\gamma t)\right)\end{array}\right]^3\times exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right].\tag{74}$$

Straddled dark-singular soliton solution is characterized by

$$\Phi_{14,\,e}(x,\,t)=\frac{\lambda_3}{8}\sqrt{\left(-\eta_1/2\eta_2\right)^3}\left[\tanh_{pq}\left(\sqrt{-\frac{\eta_1}{8}}\xi\right)\pm\coth_{pq}\left(\sqrt{\frac{-\eta_1}{8}}\xi\right)\right]^3\times exp\left[i\left(-\kappa x+\omega t+\theta_0\right)\right]. \tag{75}$$

Figures 1 and 2 depict surface, contour, and 2D plots of the bright and dark soliton solutions described by Eqs. (39) and (66), respectively. In Figure 1, the chosen parameters include: $p=1$, $q=1$, $\eta_2=-1$, $\gamma=1$, $k=1$, $a_2=1$, $a_4=1$, $a_6=1$, $b_1=1$, and $b_2=1$. Meanwhile, Figure 2 addresses the parameters $p=1$, $q=1$, $a_4=1$, $a_6=1$, $k=1$, $\eta_2=1$, $b_1=-1$, $b_2=-1$, and $\gamma=1$.

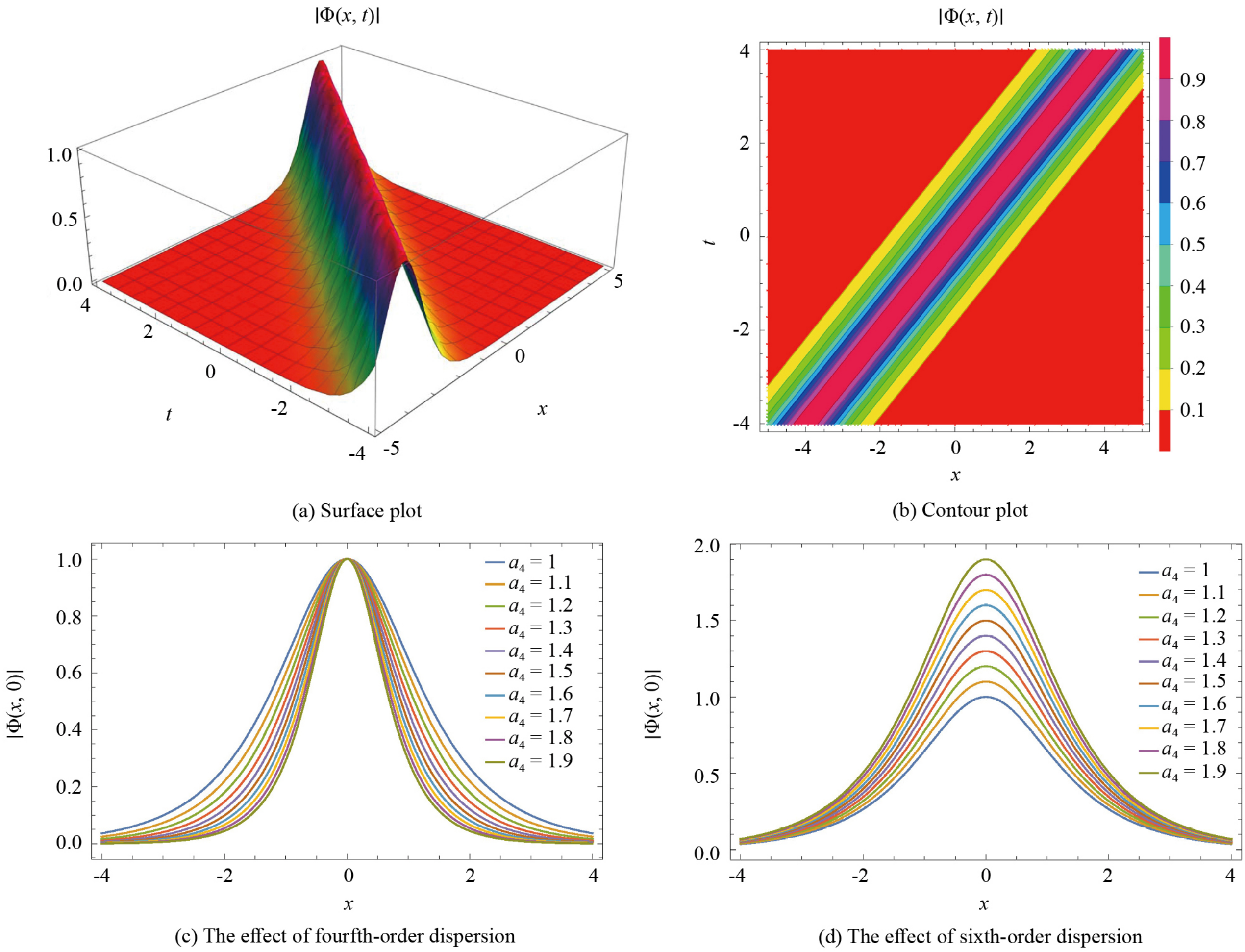
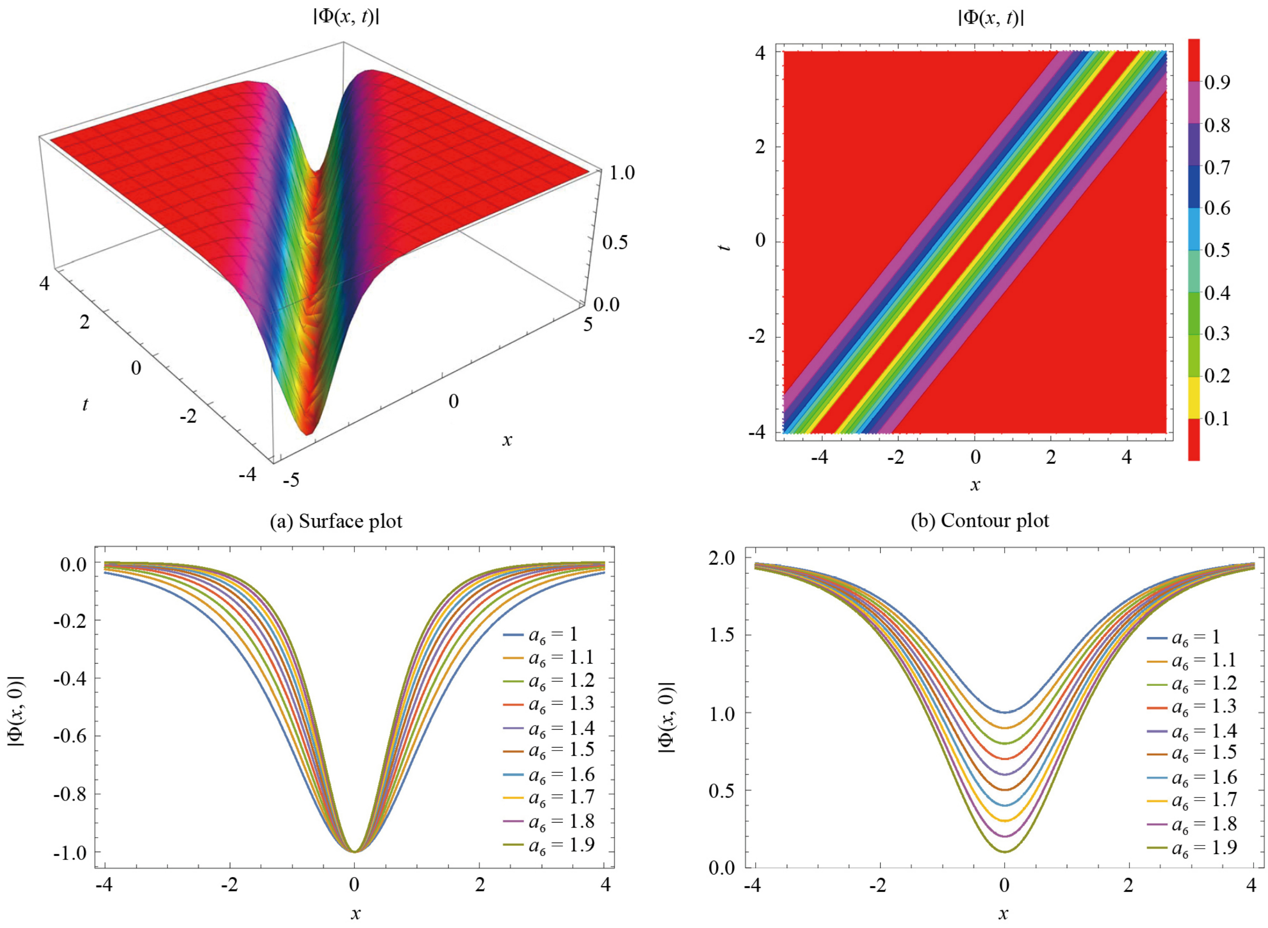


Figure 1. Profile of a bright soliton solution (39)



(a) Surface plot

(b) Contour plot

(c) The effect of fourfth-order dispersion

(d) The effect of sixth-order dispersion

Figure 2. Profile of a bright soliton solution (66)

5. Conclusion

This paper has comprehensively explored a wide range of optical solitons within the highly dispersive NLSE, which was characterized by quadratic–cubic form of SPM. The radiative effects stemming from these dispersion terms were deliberately disregarded to maintain focus on the derivation of soliton solutions. This study builds upon prior research conducted through Lie symmetry analysis, which also yielded soliton solutions. In the current work, the Sardar equation and its modified version were employed to recover a complete spectrum of single solitons for the model.

The findings of this study are intriguing and lay the groundwork for future advancements. Subsequently, the model will undergo examination with a generalized form of quadratic-cubic nonlinearity. Additionally, perturbation terms will be integrated to provide a more comprehensive understanding of soliton transmission under such dispersive effects. Once these results are synthesized and aligned with existing research, they will be disseminated for wider dissemination [29–34].

Confilict of iterest

The authors claim that there is no conflict of interest.

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