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Research Article

Extended Newton-Traub Method of Order Six

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Abstract: In various scientific and engineering fields, many applications can be simplified as the task of solving equations or systems of equations within a carefully chosen abstract space. However, analytical solutions for such problems are often difficult or even impossible to find. As a result, iterative methods are widely employed to obtain the desired solutions. This article focuses on introducing a highly efficient three-step iterative method that exhibits sixth order convergence. The analysis presented here thoroughly explores the local and semi-local convergence properties, taking into account the continuity conditions imposed on the operators present in the method. The innovative methodology described in this article is not limited to specific methods but can be extended to a broader range of approaches that involve the utilization of inverse operations on linear operators or matrices.

***Keywords*:** non-linear equations, Fréchet derivative, convergence, Banach space

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1. Introduction

In the field of applied science and technology, numerous challenges can be tackled by formulating them as non-linear equations of the form:

$$G(x) = 0 \tag{1}$$

In this context, G: $\Theta \subset \Omega_1 \rightarrow \Omega_2$ denotes a differentiable function according to the Fréchet sense, where Ω_1 and Ω_2 represent complete normed linear spaces, and Θ corresponds to a non-empty, open, and convex set.

Finding closed-form solutions for these nonlinear equations is typically challenging. Therefore, iterative methods are commonly employed to seek their solutions. Among these methods, Newton's method [1–5] is widely used due to its quadratic convergence and is a popular choice for solving equation (1). In recent times, substantial advancements have been achieved in the realms of science and mathematics, resulting in the identification and utilization of multiple sophisticated iterative approaches for resolving nonlinear equations [3, 6–9]. However, these methods often suffer from the drawback of requiring the computation of second and higher-order derivatives, which limits their practicality in real-world

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applications. The computational cost associated with evaluating G'' in each iteration makes classical cubic convergent schemes less suitable. Notably, many of these methods rely on Taylor expansions, which necessitate derivatives of higher order not found in the method itself.

Evaluating the local and semi-local characteristics of iterative techniques offers valuable insights into convergence traits, error limits, and the area of uniqueness for solutions [10–14]. Numerous research endeavors have concentrated on exploring the local and semi-local convergence of effective iterative approaches, yielding noteworthy outcomes like convergence radii, error approximations, and the broadened applicability of these methods [15–19]. These findings are particularly valuable as they shed light on the intricacies involved in selecting appropriate initial points for the iterative process.

In this article, we introduce and examine a specific method that consists of three sequential steps. The primary aim of this investigation is to establish convergence theorems for the method, further developing the foundation laid in a prior study [9]. The three-step method is defined for $x_0 \in \Theta$ and each $i = 0, 1, 2, \cdots$ by

$$\begin{aligned} y_i &= x_i - G'(x_i)^{-1}G(x_i) \\ z_i &= x_i - \Big[\frac{3}{2}I - \frac{1}{2}G'(x_i)^{-1}G'(y_i)\Big]G'(x_i)^{-1}G(x_i) \\ x_{i+1} &= z_i - \Big[\frac{7}{2}I + \Big(-4I + \frac{3}{2}G'(x_i)^{-1}G'(y_i)\Big)G'(x_i)^{-1}G'(y_i)\Big]G'(x_i)^{-1}G(z_i). \end{aligned} \tag{2}$$

This method uses two operator evaluations, two derivatives and only one inversion of $G'(x_i)$ per iteration. The local convergence order is shown to be six in the special case when $\Omega_1 = \mathbb{R}^j$ for j a natural number [9]. The article also includes the construction of the method and a favorable comparison to other approaches that utilize similar information. Nevertheless, the convergence assumptions necessitate the existence of at least the seventh derivative. This limits the method's applicability to solving equations where $G^{(2)}$ exists, although the method may still converge even if $G^{(2)}, \cdots, G^{(7)}$ do not exist. Other problems with the Taylor series approach have been reported in earlier chapters. To exemplify the concept, let's consider a motivating example where G is defined on the interval $\Theta = [-0.5, 1.5]$ as follows:

$$G(x) = \begin{cases} \frac{1}{3}x^3\ln(x) + 8x^5 - 8x^4, & \text{if } \; x \neq 0 \\ 0, & \text{if } \; x = 0 \end{cases} \tag{3}$$

We can observe that the solution $\mu^* = 1 \in \Theta$ and the third derivative is expressed as:

$$G'''(x) = \frac{11}{3} - 192x + 480x^2 + 2\ln(x).$$

It is obvious that G''' is unbounded on Θ. Thus, utilising the findings in [9], convergence isn't always guaranteed. To enhance the method's applicability, we introduce local convergence by utilizing w-continuity on G', focusing solely on the operators within the method, and within the broader context of a Banach space. Moreover the more interesting case of semi-local convergence is presented here, but not given in [9], using majorizing sequences.

The subsequent sections of this paper are structured as follows: Section 2 focuses on the investigation of local convergence properties of the method (2). In Section 3, we introduce and investigate majorizing sequences for the purpose of conducting the Semi-Local Convergence Analysis of (2). Section 4 includes numerical applications that utilize the convergence results derived in the previous sections. Ultimately, Section 5 wraps up this paper with concluding remarks.

2. Local convergence analysis

In this article, we use the symbol $\mathscr{L}(\Omega_1, \Omega_2)$ to denote the space of bounded linear operators from Ω_1 into Ω_2. Moreover, the set $U(x, a)$ is the open ball with center $x \in \Omega$ and of radius $a > 0$. Furthermore, the set $U[x_0, a]$ is the closure of $U(x_0, a)$.

The conditions below are applied for $P = [0, \infty)$:

(A_1) A solution $\mu^* \in \Theta$ exists for the equation $G(x) = 0$.

(A_2) A continuous and non-decreasing function w_0: $P \to P$ exists, satisfying the equation $w_0(t) - 1 = 0$, with the smallest positive solution τ_0. Let $P_0 = [0, \tau_0)$.

(A_3) There exists an operator $L \in \mathscr{L}(\Omega_1, \Omega_2)$ such that $L^{-1} \in \mathscr{L}(\Omega_2, \Omega_1)$ and for each $x \in \Theta$, $\|L^{-1}(G'(x) - L)\| \leq w_0(\|x - \mu^*\|)$. Set $\Theta_0 = U(\mu^*, \tau_0) \cap \Theta$.

(A_4) A continuous and non-decreasing function w: $P_0 \to P$ exists such that the equations $m_1(t) - 1 = 0$ and $m_2(t) - 1 = 0$ are satisfied, where the functions m_1 and m_2 are defined on P_0 for

$$\overline{w}(t) = \begin{cases} w\Big((1 + m_1(t))t\Big) \\ \text{or} \\ w_0(t) + w_0(m_1(t)t) \end{cases}$$

by

$$m_1(t) = \frac{\int_0^1 w((1-\alpha)t)d\alpha}{1 - w_0(t)},$$

$$m_2(t) = \frac{\int_0^1 w((1-\alpha)t)d\alpha}{1 - w_0(t)} + \frac{1}{2}\frac{\overline{w}(t)\Big(1 + \int_0^1 w_0(\alpha t)d\alpha\Big)}{\Big(1 - w_0(t)\Big)^2}$$

have smallest solutions $r_1, r_2 \in P_0 - \{0\}$, respectively. We shall choose the smallest of the two versions of the function $\overline{w}(t)$ in the numerical examples to obtain the largest radius of convergence.

(A_5) There exists smallest solution $\tau \in P_0 - \{0\}$ of the equation $w_0(m_1(t)t) - 1 = 0$. Set $P_1 = [0, \tau)$.

(A_6) The equation $m_3(t) - 1 = 0$ has a smallest solution $r_3 \in P_1 - \{0\}$, where the function m_3 with domain P_1 is defined by

$$m_3(t) = \left[\frac{\int_0^1 w((1-\alpha)m_2(t)t)d\alpha}{1-w_0(m_2(t)t)} + \frac{1}{2} \frac{\overline{\overline{w}}(t)\left(1+\int_0^1 w_0(\alpha m_2(t)t)d\alpha\right)}{\left(1-w_0(t)\right)\left(1-w_0(m_2(t)t)\right)} \right.$$

$$\left. + \frac{1}{2} \frac{\overline{p}(t)\left(1+\int_0^1 w_0(\alpha m_2(t)t)d\alpha\right)}{\left(1-w_0(t)\right)} \right] m_2(t),$$

where

$$\overline{\overline{w}}(t) = \begin{cases} w\Big((1+m_2(t))t\Big) \\ \text{or} \\ w_0(t) + w_0(m_2(t)t) \end{cases}$$

and

$$\overline{p}(t) = \frac{1}{2}\left[3\left(\frac{\overline{w}(t)}{1-w_0(t)}\right)^2 + 14\left(\frac{\overline{w}(t)}{1-w_0(t)}\right)\right].$$

We shall choose the smallest of the two versions of the function $\overline{\overline{w}}(t)$ in the numerical examples to obtain the largest radius of convergence.

The convergence radius for the method (1) is defined as

$$r = \min\{r_j \colon j = 1, 2, 3\},$$

and

(A_7) $U[\mu^*, r] \subset D$.

Notice that $L = G'(\mu^*)$ is a possible choice. In this case μ^* is a simple solution of the equation $G(x) = 0$. However, L can be any other linear operator satisfying these conditions. Then, the solution μ^* does not have to be simple.

The motivation for the introduction of these real functions is given in a series of the following calculations starting from the first sub step of the method

$$y_i - \mu^* = x_i - \mu^* - G'(x_i)^{-1}G(x_i)$$

$$= \int_0^1 G'(x_i)^{-1}\Big(G'(\mu^* + \alpha(x_i - \mu^*)) - G'(x_i)\Big)d\alpha(x_i - \mu^*),$$

$$
\begin{aligned}
\|y_i-\mu^*\| \leq & \frac{\int_0^1 w\Big((1-\alpha)\|x_i-\mu^*\|\Big)d\alpha\|x_i-\mu^*\|}{1-w_0(\|x_i-\mu^*\|)} \\
\leq & m_1(\|x_i-\mu^*\|)\|x_i-\mu^*\| \\
\leq & \|x_i-\mu^*\| < r, \\
z_i-\mu^* = & x_i-\mu^*-G'(x_i)^{-1}G(x_i) \\
& +\frac{1}{2}G'(x_i)^{-1}\Big(G'(x_i)-G'(y_i)\Big)G'(x_i)^{-1}G(z_i), \\
\|z_i-\mu^*\| \leq & \Bigg[\frac{\int_0^1 w((1-\alpha)\|x_i-\mu^*\|)d\alpha}{1-w_0(\|x_i-\mu^*\|)} \\
& +\frac{1}{2}\frac{\overline{w}_i\Big(1+\int_0^1 w_0(\alpha\|x_i-\mu^*\|)d\alpha\Big)}{\Big(1-w_0(\|x_i-\mu^*\|)\Big)^2}\Bigg]\|x_i-\mu^*\| \\
\leq & m_2(\|x_i-\mu^*\|)\|x_i-\mu^*\| \leq \|x_i-\mu^*\|,
\end{aligned}
$$

where,

$$
\begin{aligned}
\|L^{-1}\Big(G'(x_i)-G'(y_i)\Big)\| & \leq w(\|x_i-y_i\|) \\
& \leq w(\|x_i-\mu^*\|+\|y_i-\mu^*\|)=\overline{w}_i, \\
\|L^{-1}\Big(G'(x_i)-G'(y_i)\Big)\| & \leq \|L^{-1}(G'(x_i)-L)\|+\|L^{-1}(G'(y_i)-L)\| \\
& \leq w_0(\|x_i-\mu^*\|)+w_0(\|y_i-\mu^*\|)=\overline{w}_i,
\end{aligned}
$$

$$
\begin{aligned}
x_{i+1}-\mu^* =& z_i-\mu^*-G'(z_i)^{-1}G(z_i)\\
&+\left(G'(z_i)^{-1}-G'(x_i)^{-1}\right)G(z_i)\\
&-\left[\frac{5}{2}I-4G'(x_i)^{-1}G'(y_i)+\frac{3}{2}\left(G'(x_i)^{-1}G'(y_i)\right)\right]^2\\
&\times G'(x_i)^{-1}G(z_i)\\
=& z_i-\mu^*-G'(z_i)^{-1}G(z_i)+G'(z_i)^{-1}\left(G'(x_i)-G'(z_i)\right)\\
&\times G'(z_i)^{-1}G(z_i)\\
&-\frac{1}{2}\Bigg[3\left(G'(x_i)^{-1}(G'(x_i)-G'(y_i))\right)^2\\
&-14G'(x_i)^{-1}(G'(x_i)-G'(y_i))\Bigg]G'(x_i)^{-1}G(z_i),
\end{aligned}
$$

$$
\begin{aligned}
\|x_{i+1}-\mu^*\| \leq& \Bigg[\frac{\int_0^1 w((1-\alpha)\|z_i-\mu^*\|)d\alpha}{1-w_0(\|z_i-\mu^*\|)}\\
&+\frac{1}{2}\frac{\overline{\overline{w}}(\|x_i-\mu^*\|)\left(1+\int_0^1 w_0(\|z_i-\mu^*\|)d\alpha\right)}{\left(1-w_0(\|x_i-\mu^*\|)\right)\left(1-w_0(\|z_i-\mu^*\|)\right)}\\
&+\frac{\overline{p}(t)\left(1+\int_0^1 w_0(\alpha\|z_i-\mu^*\|)d\alpha\right)}{\left(1-w_0(\|x_i-\mu^*\|)\right)}\Bigg]\|z_i-\mu^*\|\\
\leq& m_3(\|x_i-\mu^*\|)\|x_i-\mu^*\|\\
\leq& \|x_i-\mu^*\|,
\end{aligned}
$$

where,

$$\frac{1}{2}\Big[3\|G'(x_i)^{-1}(G'(x_i)-G'(y_i))\|^2+14\|G'(x_i)^{-1}(G'(x_i)-G'(y_i))\|\Big]$$

$$\leq\frac{1}{3}\left[3\left(\frac{\overline{w}_i}{1-w_0(\|x_i-\mu^*\|)}\right)+14\left(\frac{\overline{w}_i}{1-w_0(\|x_i-\mu^*\|)}\right)\right]$$

$$=\overline{p}(t)$$

and since

$$G(z_i)=G(z_i)-G(\mu^*)=\int_0^1 G'(\mu^*+\alpha(z_i-\mu^*))d\alpha(z_i-\mu^*),$$

$$\|L^{-1}G(z_i)\|=\|L^{-1}\int_0^1\Big(G'(\mu^*+\alpha(z_i-\mu^*))d\alpha-L+L\Big)(z_i-\mu^*)\|$$

$$\leq\Big(1+\int_0^1 w_0(\alpha\|z_i-\mu^*\|)d\alpha\Big)\|z_i-\mu^*\|.$$

Then, under the conditions $(A_1)-(A_7)$ the iterates $\{x_i\}\subset U(\mu^*,\ r)$ and $\lim_{i\to+\infty} x_i=\mu^*$. Hence, we showed by induction:

Theorem 1 Given the assumptions (A_1)-(A_7), it is established that $\{x_i\}\subset U(\mu^*,\ r)$ and converges to μ^* as i tends to positive infinity, provided that the initial value x_0 lies in the set $U(\mu^*,\ r)-\{\mu^*\}$.

We now present a finding that confirms the sole existence of the solution within the framework of local convergence.

Proposition 1 Let's assume that there is a solution, denoted as ξ, in the neighborhood $U(\mu^*,\ \tau_3)$, where $\tau_3>0$, to the equation $G(x)=0$.

Furthermore, assume the condition in (A_3) is satisfied within the ball $U(\mu^*,\ \tau_3)$, and there exists a larger radius $\tau_4\geq\tau_3$ such that

$$\int_0^1 w_0(\alpha\tau_4)d\alpha<1. \tag{4}$$

Let $U_1=\Theta\cap U[\mu^*,\ \tau_4]$. Therefore, μ^* is the only solution to the equation $G(x)=0$ within the set U_1.

Proof. Define the linear operator $\mathscr{W}=\int_0^1 G'(\mu^*+\alpha(\xi-\mu^*))d\alpha$. Utilizing condition in (A_3) and (4), we can deduce the following:

$$\|G'(\mu^*)^{-1}(\mathscr{W}-G'(\mu^*))\| \le \int_0^1 w_0(\alpha\|\xi-\mu^*\|)d\alpha$$

$$\le \int_0^1 w_0(\alpha\tau_4)d\alpha$$

$$< 1.$$

Therefore, $\mathscr{W}^{-1} \in \mathscr{L}(\Omega_2, \Omega_1)$, and based on the approximation

$$\xi-\mu^* = \mathscr{W}^{-1}(G(\xi)-G(\mu^*)) = \mathscr{W}^{-1}(0) = 0, \tag{5}$$

we conclude that $\xi = \mu^*$. □

3. Semi-local convergence

The conditions are:

(H_1) There is a continuous and non-decreasing function v_0: $P \to P$ that meets the equation $v_0(t)-1=0$, and the smallest positive solution is denoted as s.

(H_2) There exists an operator $L \in \mathscr{L}(\Omega_1, \Omega_2)$ such that $L^{-1} \in \mathscr{L}(\Omega_2, \Omega_1)$ and each $x \in \Theta$, $\|L^{-1}(G'(x)-L)\| \le v_0(\|x-x_0\|)$ for some x_0. It follows by (H_1) that $v_0(0) < 1$, thus $G'(x_0)^{-1} \in \mathscr{L}(\Omega_2, \Omega_1)$ and the iterate y_0 is well defined by the first sub step of the method (2). Set $\|G'(x_0)^{-1}G(x_0)\| \le b_0$ and $\Theta_2 = U(x_0, s) \cap \Theta$.

(H_3) There exists a continuous and non-decreasing function v with values in P such that for each x, $y \in \Theta_2$, $\|L^{-1}(G'(y)-G'(x))\| \le v(\|y-x\|)$. Define the scalar sequence $\{a_i\}$ for $a_0 = 0$ and $b_0 \in [0, s)$ such that

$$\bar{v}(i) = \begin{cases} v(b_i-a_i) \\ \text{or} \\ v_0(a_i)+v_0(b_i), \end{cases}$$

$$c_i = b_i + \left(1+\frac{1}{2}\frac{\bar{v}_i}{1-v_0(a_i)}\right)(b_i-a_i),$$

$$q_i = \left(1+\int_0^1 v_0(a_i+\alpha(c_i-a_i))d\alpha\right)(c_i-a_i)+(1+v_0(a_i))(b_i-a_i),$$

$$a_{i+1} = c_i + \left[1 + 3\left(\frac{\bar{v}_i}{1 - v_0(a_i)}\right)^2 + 14\left(\frac{\bar{v}_i}{1 - v_0(a_i)}\right)\right]\frac{q_i}{1 - v_0(a_i)},$$

$$\delta_{i+1} = \int_0^1 v((1-\alpha)(a_{i+1} - a_i))d\alpha(a_{i+1} - a_i) + (1 + v_0(a_i))(a_{i+1} - b_i),$$

and

$$b_{i+1} = a_{i+1} + \frac{\delta_{i+1}}{1 - v_0(a_{i+1})},$$

where $\bar{v}(i)$ is chosen to be the smaller of the two versions of it.

There exists $s_0 \in [0, s)$ such that for each $i = 0, 1, 2, \ldots$

$$v_0(a_i) < 1 \quad \text{and} \quad a_i \leq s_0.$$

It follows that $0 \leq a_i \leq b_i \leq c_i \leq s^\star \leq s_0,$ where $s^\star = \lim_{i \to +\infty} a_i$.

and

(H_4) $U[x_0, s^\star] \subset \Theta$.

The motivations for these conditions are:

$$z_i - y_i = G'(x_i)^{-1}G(x_i) - \frac{1}{2}G'(x_i)^{-1}(G'(x_i) - G'(y_i))G'(x_i)^{-1}G(x_i)$$

$$\|z_i - y_i\| \leq \left(1 + \frac{1}{2}\frac{\bar{v}(i)}{1 - v_0(a_i)}\right)\|y_i - x_i\| \leq c_i - b_i,$$

$$x_{i+1} - z_i = -[I + 3(G'(x_i)^{-1}(G'(x_i) - G'(y_i)))^2$$

$$+ 14G'(x_i)^{-1}(G'(x_i) - G'(y_i))]G'(x_i)^{-1}G(z_i),$$

$$\|x_{i+1} - z\| \leq \left[1 + 3\left(\frac{\bar{v}_i}{1 - v_0(a_i)}\right)^2 + 14\left(\frac{\bar{v}_i}{1 - v_0(a_i)}\right)\right]\frac{q_i}{1 - v_0(a_i)},$$

since from

$$
\begin{aligned}
G(z_i) =& G(z_i) - G(x_i) + G(x_i) \\
=& \int_0^1 G'(x_i + \alpha(z_i - x_i))d\alpha(z_i - x_i) - G'(x_i)(y_i - x_i), \\
\|L^{-1}G(z_i)\| \leq& \Big(1 + \int_0^1 v_0(\|x_i - x_0\| + \alpha\|z_i - x_i\|)d\alpha\Big)\|z_i - x_i\| \\
& + (1 + v_0(\|x_i - x_0\|))\|y_i - x_i\| = \overline{q}_i \leq q_i, \\
G(x_{i+1}) =& G(x_{i+1}) - G(x_i) + G(x_i) \\
=& G(x_{i+1}) - G(x_i) - G'(x_i)(x_{i+1} - x_i) \\
& + G'(x_i)(x_{i+1} - y_i) \\
=& \int_0^1 \Big[G'(x_i + \alpha(x_{i+1} - x_i))d\alpha - G'(x_i)\Big] \\
& \times (x_{i+1} - x_i) + G'(x_i)(x_{i+1} - y_i), \\
\|L^{-1}G(x_{i+1})\| \leq& \int_0^1 v(\alpha\|x_{i+1} - x_i\|)d\alpha\|x_{i+1} - x_i\| \\
& + \Big(1 + v_0(\|x_i - x_0\|)\Big)\|x_{i+1} - y_i\| \\
=& \overline{\delta}_{i+1} \leq \delta_{i+1}, \\
\|y_{i+1} - x_{i+1}\| \leq& \|G'(x_{i+1})^{-1}L\|\|L^{-1}G(x_{i+1})\| \\
\leq& \frac{\overline{\delta}_{i+1}}{1 - v_0(\|x_{i+1} - x_0\|)} \leq \frac{\delta_{i+1}}{1 - v_0(a_{i+1})} = b_{i+1} - a_{i+1}.
\end{aligned}
$$

The iterates $\{x_i\}$, $\{y_i\}$, $\{z_i\} \in U(x_0, s^\star)$. Indeed, we have the estimates

$$\|y_0 - x_0\| \leq b_0 - a_0 < s^\star$$

$$\|z_i - x_0\| \leq \|z_i - y_i\| + \|y_i - x_0\| \leq c_i - b_i + b_i - a_0 = c_i < s^\star$$

$$\|x_{i+1}-x_0\| \le \|x_{i+1}-z_i\|+\|z_i-x_0\| \le a_{i+1}-c_i+c_i-a_0=a_{i+1}<s^\star,$$

$$\|y_{i+1}-x_0\| \le \|y_{i+1}-x_i\|+\|x_{i+1}-x_0\| \le b_{i+1}-a_{i+1}+a_{i+1}-a_0=b_{i+1}<s^\star.$$

Then, there exists $\mu^* \in U[x_0,\, s^\star]$ solving the equation $G(x)=0$ and satisfying the error estimate

$$\|\mu^*-x_i\| \le s^\star - a_i.$$

Hence, we arrive at the following result:

Theorem 2 Subject to the conditions (H_1)-(H_4), the sequence $\{x_i\}$ converges towards a solution $\mu^* \in U[x_0,\, s^\star]$ of the equation $G(x)=0$.

We establish the uniqueness of the solution domain in the following proposition.

Proposition 2 Assume the following conditions:

(i) There is a solution, denoted as $\bar{x}^*$, to the equation $G(x)=0$ within the neighborhood $U(x_0,\, R_1)$ for some $R_1>0$.

(ii) Condition (H_3) is satisfied within the neighborhood $U(x_0,\, R_1)$.

(iii)There is a value of R_2 that is greater than R_1 such that

$$\int_0^1 v_0((1-\alpha)R_1+\alpha R_2)d\alpha<1.$$

Set $U_4=\Theta \cap U[x_0,\, R_2]$.

In this case, the sole point in the domain U_4 that fulfills the equation $G(x)=0$ is $\bar{x}^*$.

Proof. Let us assume that there exists $x^{'} \in U_4$ such that $G(x)=0$. Conditions (ii) and (iii) allow us to obtain the following inequality:

$$\begin{aligned}\|G'(x_0)^{-1}(\mathscr{P}-G'(x_0))\| &\le \int_0^1 v_0((1-\alpha)\|\bar{x}^*-x_0\|+\alpha\|x^{'}-x_0\|)d\alpha\\ &\le \int_0^1 v_0((1-\alpha)R_1+\alpha R_2)d\alpha\\ &<1,\end{aligned}$$

where $\mathscr{P}=\int_0^1 G'(\bar{x}^*+\alpha(x^{'}-\bar{x}^*))d\alpha$. Hence, we conclude that $x^{'}=\bar{x}^*$. □

Remark 1 (i) In condition (H_4), the limit point $s^\star$ can be replaced by s.

(ii) Under all the assumptions (H_1)-(H_4), let $\bar{x}^*=\mu^*$ and $R_1=s^\star$ in Proposition 2.

4. Numerical examples

All computations in this section are performed in Mathematica programming package version 11.3.0.0 with 600 digits. These computations were carried out on an Intel(R) Core(TM) i5-8250U CPU @ 1.60 GHz 1.80 GHz with 8

GB of RAM, running on Windows 11 Home version 22H2. To stop the iterative process, we have used the criterion: $error = |x_N - x_{N-1}| < \varepsilon$, where $\varepsilon = 10^{-50}$ and N represents the number of iterations required for convergence.

Example 1 Consider a system of differential equations given by:

$$G_1'(x_1) = e^{x_1}, \quad G_2'(x_2) = (e-1)x_2 + 1, \quad G_3'(x_3) = 1$$

with the initial conditions $G_1(0) = G_2(0) = G_3(0) = 0$. Let $G = (G_1, G_2, G_3)$ and $\Omega_1 = \Omega_2 = \mathbb{R}^3$ and $\Theta = U[0, 1]$. The point $\mu^* = (0, 0, 0)^T$ is a solution of the system. Define the function G on Θ for any vector $x = (x_1, x_2, x_3)^T$ as

$$G(x) = (e^{x_1} - 1, \frac{e-1}{2}x_2^2 + x_2, x_3)^T.$$

The derivative matrix of G is given by:

$$G'(x) = \begin{bmatrix} e^{x_1} & 0 & 0 \\ 0 & (e-1)x_2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and we observe that $G'(\mu^*) = 1$. In order to verify the local convergence criteria, we need to satisfy conditions $(A_1) - (A_7)$. By selecting $w_0(t) = (e-1)t$, $w(t) = e^{\frac{1}{e-1}}t$, $\tau_0 = 0.581977$, and $\Theta_0 = \Theta \cap U(\mu^*, \tau_0)$, we can meet these conditions. The radii for method (2) is provided in Table 1.

Example 2 Let $\Omega_1 = \Omega_2 = \Theta = \mathbb{R}$. Consider the function G defined on Θ as $G(x) = \sin(x)$. The derivative of G is given by $G'(x) = \cos(x)$. The fixed point is $\mu^* = 0$. In order to verify conditions $(A_1) - (A_7)$, we select $w_0(t) = w(t) = t$, $\tau_0 = 1$, and $\Theta_0 = \Theta \cap U(\mu^*, \tau_0)$. The radii of convergence can be found in Table 1.

Table 1. Estimates for Examples (1) and (2)

Radii	r_1	r_2	r_3	$r = min\{r_i\}$
Example (1)	0.382692	0.212902	0.140439	0.140439
Example (2)	0.666667	0.368726	0.24305	0.24305

Example 3 We consider a system of two equations:

$$x_1 + e^{x_2} - \cos x_2 = 0$$

$$3x_1 - \sin x_1 - x_2 = 0$$

with an initial value $x_0 = \{-1, 1\}^T$, aiming to find the solution $\mu^* = \{0, 0\}^T$. The Table 2 provides error estimates for the solution. After performing 5 iterations, we can observe that the system of equations converges to the solution μ^*.

Example 4 In this example, we examine a system of twenty nonlinear equations:

$$x_i - \cos\left(2x_i - \sum_{j\neq i,\, j=1}^{20} x_j\right) = 0, \quad 1 \leq i \leq 20.$$

We start with the initial approximation $x_0 = \{-0.89, -0.89, \ldots, -0.89\}^T$ to find the solution:

$$\mu^* = \{-0.8970747863329200, -0.8970747863329200, \ldots, -0.8970747863329200\}^T.$$

The Table 2 provides error estimates for the solution. After analyzing the system of equations, we observe that it converges to μ^* within a maximum of 4 iterations.

Table 2. Estimates for Examples (3) and (4)

Errors	$\|x_1-\bar{x}\|$	$\|x_2-\bar{x}\|$	$\|x_3-\bar{x}\|$	$\|x_4-\bar{x}\|$
Example (3)	0.7263140974005179	0.3073578993301358	$1.500290193984357*10^{-2}$	$1.200874212027866*10^{-6}$
Example (4)	$5.888571011502531*10^{-13}$	$3.2769450822864*10^{-14}$	$2.979040983896728*10^{-15}$	$4.965068306494546*10^{-16}$

5. Conclusion

We have presented a novel technique that enables the analysis of both local and semi-local convergence properties of high-order methods without relying on additional derivatives beyond those inherent to the method itself. Unlike previous approaches that assumed the existence of high-order derivatives, our technique eliminates this requirement and provides previously unavailable error bounds and uniqueness results. Importantly, our technique is highly versatile and applicable to a wide range of higher-order methods, including single step and multi-step methods [3–5, 20, 21]

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Conflict of interest

The authors declare no competing financial interest.

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