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Two-Wavelet Multipliers and Applications

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Abstract: This paper delves into the Dunkl-Bessel operator on $\mathbb{R}_+^{d+1}$ and its corresponding harmonic analysis. A generalized form of Heisenberg-type uncertainty inequality is established. Schatten-von Neumann properties for the two-wavelet multiplier within the Dunkl-Bessel theory framework are elucidated. Additionally, the trace formula for a two-wavelet Dunkl-Bessel multiplier is proven as a bounded linear operator in the trace class from $L^2_{k,\beta}(\mathbb{R}_+^{d+1})$ into $L^2_{k,\beta}(\mathbb{R}_+^{d+1})$. Furthermore, subject to appropriate conditions, the $L^p_{k,\beta}$ boundedness and compactness of these Dunkl-Bessel two-wavelet multipliers are proven, applicable to $L^p_{k,\beta}(\mathbb{R}_+^{d+1})$, $1 \le p \le \infty$. Finally, using a class of concentration operators for the Dunkl-Bessel two-wavelet, we show that the eigenfunctions of the Dunkl-Bessel two-wavelet are maximally concentrated in the time-frequency domain. Leveraging this result, we derive approximation inequalities for functions that exhibit significant concentration within specific regions of the time-frequency plane.

***Keywords*:** Dunkl-Bessel transform, Dunkl-Bessel two-wavelet multipliers, uncertainty principle

MSC: 81S30, 42C40, 42B10, 47G10

1. Introduction

The term "localization operators" was first introduced by Daubechies in 1988 [1], utilized as a mathematical tool to localize signals on the time-frequency plane. These operators are also referred to as Toeplitz operators or short-time Fourier transform multipliers. Operators that localize in both time and frequency domains hold significance across various applications in optics and signal analysis, serving as a mathematical framework for defining function restrictions to specific regions in the time-frequency plane.

Extensive research on localization operators has been conducted, notably by Slepian and Pollak in a series of excellent papers [2], with further contributions from Landau and Pollak [3, 4], as well as Slepian [5, 6]. Consequently, wavelet multiplier operators can be considered as a variant of localization operators. Subsequently, the theory of wavelet multipliers has been initiated by He and Wong in [7], developed in the paper [8] by Du and Wong, and detailed in the book [9] by Wong. Next, this subject has been extended for the generalized Fourier transforms (see [10–15] and others).

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In classical scenarios, quantitative uncertainty principles, which are special inequalities, provide insights into the relationship between a function and its Fourier transform. These principles draw parallels to the classical Heisenberg inequality, which has played a significant role in advancing quantum physics. Numerous authors have investigated quantitative uncertainty principles for various Fourier transforms, including examples from literature [16–23]).

The Dunkl-Bessel transform is an important tool which has the scope of and potential for applications in many areas of the mathematical sciences. Very recently, many authors have been investigating the behavior of the Dunkl-Bessel transform to several problems already studied for the Fourier transform; for instance, men value theorem [24], uncertainty principles [25, 26], Dunkl-Bessel Gabor transform [27], Sobolev spaces of exponential type [28], Dunk-Bessel wavelet transform [29], time-frequency analysis [30] and so on. We mention that the Dunkl-Bessel transform generalize the usual Fourier transform, the Weinstein transform [17, 31–33] and the multi-variables Bessel transform [34], which give an impact for any subject studied in the Dunkl-Bessel setting.

The first focus of this paper is to extend the study of uncertainty principles for the Dunkl-Bessel transform by comparing different measures of localization. Specifically, we concentrate on uncertainty principles where concentration is assessed in terms of dispersion or the smallest support. Furthermore, we aim to expose and study the two-wavelet multipliers in the setting of the Dunkl-Bessel transform.

The second aim of this paper is to prove results on the L^p-boundedness and the L^p-compactness of the two-wavelet multipliers associated with the Dunkl-Bessel transform.

The third aim is to construct and study an example of generalized two-wavelet multipliers. Indeed, we prove that the generalized two-wavelet multiplier is unitary equivalent to a scalar multiple of the generalized Landau-Pollak-Slepian Operator.

The fourth aim is to give some applications on the generalized two-wavelet multipliers. In fact, in the first application we use the ε-localization measure introduced in [35] to state a new uncertainty inequality involving the generalized two-wavelet multiplier. More precisely, we present a proof of an uncertainty principle of Donoho-Stark type which involve in a new way generalized two-wavelet multipliers, the concept of ε-concentration and the standard deviation of L^2 functions. We show how our results improve the classical Donoho-Stark estimate.

The second application, on the fundamental example constructed, is the study of some spectral problems. In particular, we prove that a signal which is almost time and almost bandlimited can be approximated by its projection on the span of the first eigenfunctions of the phase space restriction operator (special case of the generalized Landau-Pollak-Slepian operator), corresponding to the largest eigenvalues which are close to one.

We recall that, the time-limited functions and bandlimited functions are basic tools of signal and image processing. Unfortunately, the simplest form of the uncertainty principle tells us that a signal cannot be simultaneously time and bandlimited. This leads to the investigation of the set of almost time and almost bandlimited functions, which was initially carried by Landau, Pollak [4, 36] and then by Donoho, Stark [37].

The structure of this paper is as follow:

In Section 2, we recall the main results about the harmonic analysis associated with the Dunkl-Bessel theory. In Section 3, we obtain new uncertainty inequalities by means of local uncertainty principles for functions either in $L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ or in $L^1_{k,\beta}(\mathbb{R}^{d+1}_+) \cap L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$. In Section 4, we introduce and we study the two-wavelet multipliers in the setting of the Dunkl-Bessel transform. More precisely, the Schatten-von Neumann properties of these two-wavelet multipliers are established, and for trace class generalized two-wavelet multipliers, the traces and the trace class norm inequalities are presented. Next, we investigate the L^p-boundedness and compactness of these two-wavelet multipliers, when suitable conditions on the symbols and two admissible wavelets are satisfied. In the last section, firstly we introduce the generalized Landau-Pollak-Slepian operator. We give the link between this operator and the generalized two-wavelet multipliers. As applications we prove the Donoho-Stark uncertainty principle for the Dunkl-Bessel transform, next we study some spectral problems associated for the generalized Landau-Pollak-Slepian operator. More precisely, we use the compositions of time and bandlimiting operators and consider the eigenvalue problem associated with these operators. The resulting operators yield an orthonormal set of eigenfunctions (well-known as prolate spheroidal functions in the Euclidean Analysis) which satisfy some optimality in concentration in a region in the time-frequency domain. We prove a characterization of functions that are approximately time and bandlimited in the region of interest, and we obtain approximation inequalities for such functions using a finite linear combination of eigenfunctions.

2. Preliminaries

In this section we recall some basic results in the Dunkl theory, harmonic analysis associated with the Dunkl-Bessel Laplace operator and Schatten-von Neumann classes. Main references are [24, 38–40].

2.1 *The Dunkl operators*

Let $\mathbb{R}^d$, be the Euclidean space equipped with a scalar product $\langle , \rangle$ and let $||x|| = \sqrt{\langle x, x\rangle}$. For α in $\mathbb{R}^d\backslash\{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α, i.e. for $x \in \mathbb{R}^d$,

$$\sigma_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{||\alpha||^2}\alpha.$$

A finite set $R \subset \mathbb{R}^d\backslash\{0\}$ is called a root system if $R \cap \mathbb{R}\ \alpha = \{\alpha, -\alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. For a given root system R reflections σ_α, $\alpha \in R$, generate a finite group $W \subset O(d-1)$, called the reflection group associated with R. We fix a $\beta \in \mathbb{R}^d\backslash \bigcup_{\alpha\in R} H_\alpha$ and define a positive root system $R_+ = \Big\{\alpha \in R\colon\ \langle \alpha, \beta\rangle > 0\Big\}$. We normalize each $\alpha \in R_+$ as $\langle \alpha, \alpha\rangle = 2$. A function $k\colon R \longrightarrow \mathbb{C}$ on R is called a multiplicity function if it is invariant under the action of W. We introduce the index γ as

$$\gamma = \gamma(k) = \sum_{\alpha\in R_+} k(\alpha).$$

Throughout this paper, we will assume that $k(\alpha) \geq 0$ for all $\alpha \in R$. We denote by ω_k the weight function on $\mathbb{R}^d$ given by

$$\omega_k(x) = \prod_{\alpha\in R_+} |\langle \alpha, x\rangle|^{2k(\alpha)},$$

which is invariant and homogeneous of degree 2γ.

The Dunkl operators T_j, $j = 1, 2, \ldots, d$, on $\mathbb{R}^d$ associated with the positive root system R_+ and the multiplicity function k are given by

$$T_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha\in R_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle}, \quad f \in C^1(\mathbb{R}^d).$$

We define the Dunkl-Laplace operator $\triangle_k$ on $\mathbb{R}^d$ for f of class C^2 on $\mathbb{R}^d$ by

$$\triangle_k f(x) = \sum_{j=1}^{d} T_j^2 f(x) = \triangle f(x) + 2\sum_{\alpha\in R^+} k(\alpha)\Big(\frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle^2}\Big),$$

where $\triangle$ and ∇ are the usual Euclidean Laplacian and nabla operators on $\mathbb{R}^d$ respectively. Then for each $y \in \mathbb{R}^d$, the system

$$T_j u(x, y) = y_j u(x, y), \quad j = 1, \ldots, d,$$

$$u(0, y) = 1$$

admits a unique analytic solution $K(x, y)$, $x \in \mathbb{R}^d$, called the Dunkl kernel.

2.2 *Harmonic analysis associated with the Dunkl-Bessel Laplace operator*

In this subsection we collect some notations and results on the Dunkl-Bessel kernel, the Dunkl-Bessel transform, and the Dunkl-Bessel convolution. (cf. [24]).

In the following we denote by

$$\mathbb{R}^{d+1}_{+} = \mathbb{R}^d \times [0, \infty).$$

$$x = (x_1, \ldots, x_d, x_{d+1}) = (x', x_{d+1}) \in \mathbb{R}^{d+1}_{+}.$$

χ_U	the characteristic function of the measurable subset U.
$C_*(\mathbb{R}^{d+1})$	the space of continuous functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
$C^p_*(\mathbb{R}^{d+1})$	the space of functions of class C^p on $\mathbb{R}^{d+1}$, even with respect to the last variable.
$\mathscr{E}_*(\mathbb{R}^{d+1})$	the space of C^∞-functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
$\mathscr{S}_*(\mathbb{R}^{d+1})$	the Schwartz space of rapidly decreasing functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
$D_*(\mathbb{R}^{d+1})$	the space of C^∞-functions on $\mathbb{R}^{d+1}$ which are of compact support, even with respect to the last variable.

We consider the Dunkl-Bessel Laplace operator $\triangle_{k,\beta}$ defined by $\forall x = (x', x_{d+1}) \in \mathbb{R}^d \times [0, \infty)$,

$$\triangle_{k,\beta} f(x) = \triangle_{k,x'} f(x', x_{d+1}) + \mathscr{L}_{\beta, x_{d+1}} f(x', x_{d+1}), \; f \in C^2_*(\mathbb{R}^{d+1}), \tag{1}$$

where $\triangle_k$ is the Dunkl-Laplace operator on $\mathbb{R}^d$, and $\mathscr{L}_\beta$ the Bessel operator on $[0, \infty)$ given by

$$\mathscr{L}_\beta = \frac{d^2}{dx^2_{d+1}} + \frac{2\beta+1}{x_{d+1}} \frac{d}{dx_{d+1}}, \quad \beta > -\frac{1}{2}.$$

The Dunkl-Bessel kernel Λ is given by

$$\Lambda_{k,\beta}(x, z) = K(ix', z') j_\beta(x_{d+1} z_{d+1}), \; (x, z) \in \mathbb{R}^{d+1}_{+} \times \mathbb{C}^{d+1}, \tag{2}$$

where $K(ix', z')$ is the Dunkl kernel and $j_\beta(x_{d+1} z_{d+1})$ is the normalized Bessel function. The Dunkl-Bessel kernel satisfies the following properties:

i) For all $z, t \in \mathbb{C}^{d+1}$, we have

$$\Lambda_{k,\beta}(z,t)=\Lambda_{k,\beta}(t,z);\quad \Lambda_{k,\beta}(z,0)=1 \quad \text{and} \quad \Lambda_{k,\beta}(\lambda z,t)=\Lambda_{k,\beta}(z,\lambda t),\quad \text{for all} \quad \lambda\in\mathbb{C}. \tag{3}$$

ii) For all $\nu\in\mathbb{N}^{d+1}$, $x\in\mathbb{R}_+^{d+1}$ and $z\in\mathbb{C}^{d+1}$, we have

$$|D_z^\nu\Lambda_{k,\beta}(x,z)|\leq ||x||^{|\nu|}\exp(||x||\,||Imz||), \tag{4}$$

where $D_z^\nu=\frac{\partial^\nu}{\partial z_1^{\nu_1}...\partial z_{d+1}^{\nu_{d+1}}}$ and $|\nu|=\nu_1+...+\nu_{d+1}$. In particular

$$|\Lambda_{k,\beta}(x,y)|\leq 1,\quad \text{for all} \quad x,y\in\mathbb{R}_+^{d+1}. \tag{5}$$

We denote by $L_{k,\beta}^p(\mathbb{R}_+^{d+1})$ the space of measurable functions on $\mathbb{R}_+^{d+1}$ such that

$$||f||_{L_{k,\beta}^p}=\left(\int_{\mathbb{R}_+^{d+1}}|f(x)|^p d\mu_{k,\beta}(x)\right)^{\frac{1}{p}}<\infty,\quad \text{if} \quad 1\leq p<\infty,$$

$$||f||_{L_{k,\beta}^\infty}=\operatorname*{ess\ sup}_{x\in\mathbb{R}_+^{d+1}}|f(x)|<\infty,$$

where $d\mu_{k,\beta}$ is the measure on $\mathbb{R}_+^{d+1}$ given by

$$d\mu_{k,\beta}(x',x_{d+1})=\frac{1}{m_{k,\beta}}\omega_k(x')x_{d+1}^{2\beta+1}dx'dx_{d+1}.$$

Here

$$m_{k,\beta}=\int_{\mathbb{R}^{d+1}}e^{-\frac{||x||^2}{2}}\omega_k(x')x_{d+1}^{2\beta+1}dx'dx_{d+1}. \tag{6}$$

For $p=2$, we provide this space with the scalar product

$$\langle f,g\rangle_{L_{k,\beta}^2}:=\int_{\mathbb{R}_+^{d+1}}f(x)\overline{g(x)}d\mu_{k,\beta}(x).$$

The Dunkl-Bessel transform is given for f in $L_{k,\beta}^1(\mathbb{R}_+^{d+1})$ by

$$\mathscr{F}_{k,\beta}(f)(y)=\int_{\mathbb{R}_+^{d+1}}f(x)\Lambda_{k,\beta}(-x,y)d\mu_{k,\beta}(x),\quad \text{for all} \quad y=(y',y_{d+1})\in\mathbb{R}_+^{d+1}. \tag{7}$$

i) For f in $L_{k,\beta}^1(\mathbb{R}_+^{d+1})$,

$$||\mathscr{F}_{k,\beta}(f)||_{L^\infty_{k,\beta}} \leq ||f||_{L^1_{k,\beta}}. \tag{8}$$

ii) For all f in $L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$, if $\mathscr{F}_{k,\beta}(f)$ belongs to $L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$, then

$$f(y) = \int_{\mathbb{R}^{d+1}_+} \mathscr{F}_{k,\beta}(f)(x)\Lambda_{k,\beta}(x,y)d\mu_{k,\beta}(x). \quad a.e. \tag{9}$$

iii) The Dunkl-Bessel transform $\mathscr{F}_{k,\beta}$ provides a natural generalization of the usual Fourier transform, to which it reduces in the case $k=0$ and $\beta = -\frac{1}{2}$, and if $f(x) = F(||x||)$ is a radial function on $\mathbb{R}^{d+1}$, then

$$\forall y \in \mathbb{R}^{d+1}, \quad \mathscr{F}_{k,\beta}(f)(y) = \mathscr{F}_\beta^{\gamma+\beta+\frac{d}{2}}(F)(||y||), \tag{10}$$

where the transform $\mathscr{F}_\beta^{\gamma+\beta+\frac{d}{2}}$ is the Bessel transform given by

$$\forall \lambda \geq 0, \ \mathscr{F}_\beta^{\gamma+\beta+\frac{d}{2}} g(\lambda) = \frac{1}{2^{\gamma+\beta+\frac{d}{2}}\Gamma(\gamma+\beta+\frac{d+2}{2})} \int_0^\infty g(r) j_{\gamma+\beta+\frac{d}{2}}(\lambda r) r^{2\gamma+2\beta+d+1} dr. \tag{11}$$

iv) For $f \in \mathscr{S}_*(\mathbb{R}^{d+1})$, if we define

$$\overline{\mathscr{F}_{k,\beta}}(f)(y) = \mathscr{F}_{k,\beta}(f)(-y),$$

then

$$\mathscr{F}_{k,\beta}\overline{\mathscr{F}_{k,\beta}} = \overline{\mathscr{F}_{k,\beta}}\mathscr{F}_{k,\beta} = Id.$$

v) Let $\lambda > 0$. The dilation operator $\mathscr{D}_\lambda$, is defined by

$$\mathscr{D}_\lambda f(x) = \frac{1}{\lambda^{A^d_{\gamma,\beta}}} f(\frac{x}{\lambda}),$$

where $A^d_{\gamma,\beta} = \gamma + \beta + 1 + \frac{d}{2}$. This operator satisfies

$$\mathscr{F}_{k,\beta}\mathscr{D}_\lambda = \mathscr{D}_{\frac{1}{\lambda}}\mathscr{F}_{k,\beta}. \tag{12}$$

Proposition 1 i) Plancherel's formula. The Dunkl-Bessel transform $\mathscr{F}_{k,\beta}$ is a topological isomorphism from $\mathscr{S}_*(\mathbb{R}^{d+1})$ onto itself and for all f in $\mathscr{S}_*(\mathbb{R}^{d+1})$,

$$\int_{\mathbb{R}_+^{d+1}} |f(x)|^2 d\mu_{k,\beta}(x) = \int_{\mathbb{R}_+^{d+1}} |\mathscr{F}_{k,\beta}(f)(\xi)|^2 d\mu_{k,\beta}(\xi). \tag{13}$$

ii) In particular, The Dunkl-Bessel transform $f \to \mathscr{F}_{k,\beta}(f)$ can be uniquely extended to an isometric isomorphism on $L^2_{k,\beta}(\mathbb{R}_+^{d+1})$.

iii) Parseval's formula. For all $f, g \in L^2_{k,\beta}(\mathbb{R}_+^{d+1})$, we have

$$\int_{\mathbb{R}_+^{d+1}} f(x)\overline{g(x)} d\mu_{k,\beta}(x) = \int_{\mathbb{R}_+^{d+1}} \mathscr{F}_{k,\beta}(f)(\xi)\overline{\mathscr{F}_{k,\beta}(g)(\xi)} d\mu_{k,\beta}(\xi). \tag{14}$$

By using the Dunkl-Bessel kernel, we introduce a generalized translation and a convolution structure. For a function $f \in \mathscr{S}_*(\mathbb{R}^{d+1})$ and $y \in \mathbb{R}_+^{d+1}$ the Dunkl-Bessel translation $\tau_y^{k,\beta} f$ is defined by the following relation:

$$\mathscr{F}_{k,\beta}(\tau_y^{k,\beta} f)(x) = \Lambda_{k,\beta}(x, y)\mathscr{F}_{k,\beta}(f)(x). \tag{15}$$

By using the Dunkl-Bessel translation, we define the Dunkl-Bessel convolution product $f *_{k,\beta} g$ of functions $f, g \in \mathscr{S}_*(\mathbb{R}^{d+1})$ as follows:

$$f *_{k,\beta} g(x) = \int_{\mathbb{R}_+^{d+1}} \tau_x^{k,\beta} f(-y) g(y) d\mu_{k,\beta}(y). \tag{16}$$

This convolution is commutative and associative and satisfies the following :

Proposition 2 i) For all $f, g \in \mathscr{S}_*(\mathbb{R}_+^{d+1})$, $f *_{k,\beta} g$ belongs to $\mathscr{S}_*(\mathbb{R}_+^{d+1})$ and

$$\forall y \in \mathbb{R}_+^{d+1}, \quad \mathscr{F}_{k,\beta}(f *_{k,\beta} g)(y) = \mathscr{F}_{k,\beta}(f)(y)\mathscr{F}_{k,\beta}(g)(y). \tag{17}$$

ii) Let $1 \le p, q, r \le \infty$ such that $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. If $f \in L^p_{k,\beta}(\mathbb{R}_+^{d+1})$ and $g \in L^q_{k,\beta}(\mathbb{R}_+^{d+1})$ is radial, then $f *_{k,\beta} g \in L^r_{k,\beta}(\mathbb{R}_+^{d+1})$ and

$$\left\| f *_{k,\beta} g \right\|_{L^r_{k,\beta}} \le \|f\|_{L^p_{k,\beta}} \|g\|_{L^q_{k,\beta}}. \tag{18}$$

iii) Moreover, for f, g in $L^2_{k,\beta}(\mathbb{R}_+^{d+1})$, the function $f *_{k,\beta} g$ belongs to $L^2_{k,\beta}(\mathbb{R}_+^{d+1})$ if and only if the function $\mathscr{F}_{k,\beta}(f)\mathscr{F}_{k,\beta}(g)$ belongs to $L^2_{k,\beta}(\mathbb{R}^d)$ and (17) holds.

2.3 *Schatten-von Neumann classes*

Notations. We denote by

• $l^p(\mathbb{N})$ the set of all infinite sequences of real (or complex) numbers $x := (x_j)_{j\in\mathbb{N}}$, such that

$$||x||_p := \Big(\sum_{j=1}^{\infty} |x_j|^p\Big)^{\frac{1}{p}} < \infty, \quad \text{if} \quad 1 \le p < \infty,$$

$$||x||_\infty := \sup_{j \in \mathbb{N}} |x_j| < \infty.$$

For $p = 2$, we provide this space $l^2(\mathbb{N})$ with the scalar product

$$\langle x, y \rangle_2 := \sum_{j=1}^{\infty} x_j \overline{y_j}.$$

• $B(L^2_{k,\beta}(\mathbb{R}^{d+1}_+))$ the space of bounded operators from $L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ into itself.

Definition 1 (i) The singular values $(s_n(A))_{n\in\mathbb{N}}$ of a compact operator A in $B(L^2_{k,\beta}(\mathbb{R}^{d+1}_+))$ are the eigenvalues of the positive self-adjoint operator $|A| = \sqrt{A^*A}$.

(ii) For $1 \le p < \infty$, the Schatten class S_p is the space of all compact operators whose singular values lie in $l^p(\mathbb{N})$. The space S_p is equipped with the norm

$$||A||_{S_p} := \Big(\sum_{n=1}^{\infty} (s_n(A))^p\Big)^{\frac{1}{p}}. \tag{19}$$

Remark 1 We note that the space S_2 is the space of Hilbert-Schmidt operators, and S_1 is the space of trace class operators.

Definition 2 The trace of an operator A in S_1 is defined by

$$tr(A) = \sum_{n=1}^{\infty} \langle Av_n, v_n \rangle_{L^2_{k,\beta}} \tag{20}$$

where $(v_n)_n$ is any orthonormal basis of $L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$.

Remark 2 If A is positive, then

$$tr(A) = ||A||_{S_1}. \tag{21}$$

Moreover, a compact operator A on the Hilbert space $L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ is Hilbert-Schmidt, if the positive operator A^*A is in the space of trace class S_1. Then

$$||A||^2_{HS} := ||A||^2_{S_2} = ||A^*A||_{S_1} = tr(A^*A) = \sum_{n=1}^{\infty} ||Av_n||^2_{L^2_{k,\beta}} < \infty \tag{22}$$

for any orthonormal basis $(v_n)_n$ of $L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$.

Definition 3 We define $S_\infty := B(L^2_{k,\beta}(\mathbb{R}^{d+1}_+))$, equipped with the norm,

$$||A||_{S_\infty} := \sup_{v\in L^2_{k,\beta}:\, ||v||_{L^2_{k,\beta}}=1} ||Av||_{L^2_{k,\beta}}. \tag{23}$$

3. Uncertainty principles by means of the frequency limiting operator

Let define firstly, for $\sigma \in L^\infty_{k,\beta}(\mathbb{R}^{d+1}_+)$, the linear operator $M_\sigma\colon L^2_{k,\beta}(\mathbb{R}^{d+1}_+) \to L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ by

$$M_\sigma(f) = \mathscr{F}^{-1}_{k,\beta}(\sigma \mathscr{F}_{k,\beta}(f)). \tag{24}$$

This operator is called the Dunkl-Bessel multiplier. Involving Plancherel's formula (13), we deduce that M_σ is bounded with

$$||M_\sigma||_{S_\infty} \le ||\sigma||_{L^\infty_{k,\beta}}.$$

Notice that, if σ is a characteristic function ($\sigma = \chi_A$), then we write M_A instead of M_σ. The operator $M_A\colon L^2_{k,\beta}(\mathbb{R}^{d+1}_+) \to L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$, is a self-adjoint projection, which is known as the frequency limiting operator on $L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ and has many applications in time-frequency analysis.

We would like to find non-zero functions $f \in L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$, which are time-limited on a subset $S \subset \mathbb{R}^{d+1}_+$ (i.e. $f \subset S$) and bandlimited on a subset $\Sigma \subset \mathbb{R}^{d+1}_+$ (i.e. $\mathscr{F}_{k,\beta} f \subset \Sigma$), for sets S and Σ with finite measure. Unfortunately, such functions do not exist, because if f is time and bandlimited on subsets of finite measure, then $f = 0$ (see [19]). As a result, it is natural to replace the exact support by the essential support, and to focus on functions that are essentially time and bandlimited to a bounded region like $S \times \Sigma$ in the time-frequency plane. To do this, we define the time limiting operator

$$E_S f = \chi_S f, \quad f \in L^1_{k,\beta}(\mathbb{R}^{d+1}_+) \cup L^2_{k,\beta}(\mathbb{R}^{d+1}_+).$$

We recall the following notions.

Definition 4 Let $0 \le \varepsilon < 1$ and let $S, \Sigma \subset \mathbb{R}^{d+1}_+$. Then

1. a nonzero function $f \in L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ is ε-concentrated on S if $\|E_{S^c} f\|_{L^2_{k,\beta}} \le \varepsilon \|f\|_{L^2_{k,\beta}}$,
2. a nonzero function $f \in L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$ is ε-timelimited on S if $\|E_{S^c} f\|_{L^1_{k,\beta}} \le \varepsilon \|f\|_{L^1_{k,\beta}}$,
3. a nonzero function $f \in L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ is ε-bandlimited on Σ if $\|M_{\Sigma^c} f\|_{L^2_{k,\beta}} \le \varepsilon \|f\|_{L^2_{k,\beta}}$,
4. a nonzero function $f \in L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ is ε-localized with respect to an operator

$$L\colon L^2_{k,\beta}(\mathbb{R}^{d+1}_+) \to L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$$

if

$$\|Lf - f\|_{L^2_{k,\beta}} \le \varepsilon \|f\|_{L^2_{k,\beta}}.$$

Here $A^c = \mathbb{R}_+^{d+1} \backslash A$ is the complement of A in $\mathbb{R}_+^{d+1}$. It is clear that, if f is ε-bandlimited on Σ then by Inequality (13), $\mathscr{F}_{k,\beta} f$ is ε-concentrated on Σ and we recall that, the time limiting operator, E_S: $L^2_{k,\beta}(\mathbb{R}_+^{d+1}) \to L^2_{k,\beta}(\mathbb{R}_+^{d+1})$ is a self-adjoint projection.

Let $\varepsilon_1, \varepsilon_2 \in (0, 1)$ and let S, Σ two measurable subsets of $\mathbb{R}_+^{d+1}$ such that

$$0 < \mu_{k,\beta}(S) := \int_S d\mu_{k,\beta}(t), \quad \mu_{k,\beta}(\Sigma) := \int_\Sigma d\mu_{k,\beta}(t) < \infty,$$

We denote by $L^2_{k,\beta}(\varepsilon_1, \varepsilon_2, S, \Sigma)$ the subspace of $L^2_{k,\beta}(\mathbb{R}_+^{d+1})$ consisting of functions that are ε_1-concentrated on S and ε_2-bandlimited on Σ (clearly $L^2_{k,\beta}(0, 0, S, \Sigma) = \emptyset$). We denote also by $L^1_{k,\beta} \cap L^2_{k,\beta}(\varepsilon_1, \varepsilon_2, S, \Sigma)$ the subspace of $L^1_{k,\beta}(\mathbb{R}_+^{d+1}) \cap L^2_{k,\beta}(\mathbb{R}_+^{d+1})$ consisting of functions that are ε_1-timelimited on S and ε_2-bandlimited on Σ. And if $\varepsilon_1 = \varepsilon_2$, we denote by $L^2_{k,\beta}(\varepsilon, S, \Sigma)$ instead of $L^2_{k,\beta}(\varepsilon, \varepsilon, S, \Sigma)$.

In the next subsection, we will use local uncertainty principles to obtain new uncertainty inequalities.

3.1 *Uncertainty principles on the space $L^2_{k,\beta}(\varepsilon_1, \varepsilon_2, S, \Sigma)$*

The first known result for functions in $L^2_{k,\beta}(\varepsilon_1, \varepsilon_2, S, \Sigma)$ is the following Donoho-Stark type uncertainty principle, see [19, Inequality (3.4)].

Theorem 1 Let $\varepsilon_1, \varepsilon_2 \in (0, 1)$ such that $\varepsilon_1^2 + \varepsilon_2^2 < 1$. Then if $f \in L^2_{k,\beta}(\varepsilon_1, \varepsilon_2, S, \Sigma)$ we have

$$\mu_{k,\beta}(S)\mu_{k,\beta}(\Sigma) \geq \left(1 - \sqrt{\varepsilon_1^2 + \varepsilon_2^2}\right)^2. \tag{25}$$

In the case of the usual Fourier transform,the last inequality has been proven by to Donoho and Stark [37]. Inequality (25) implies that the essential support of f and $\mathscr{F}_{k,\beta} f$ can't be too small.

Now, recall the following Faris-local uncertainty inequalities, see [19].

Theorem 2

1. If $0 < s < A^d_{\gamma,\beta}$, then there exists $C > 0$ such that for every $f \in L^2_{k,\beta}(\mathbb{R}_+^{d+1})$ and all measurable subset $\Sigma \subset \mathbb{R}_+^{d+1}$ of finite measure $0 < \mu_{k,\beta}(\Sigma) < \infty$,

$$\|M_\Sigma f\|^2_{L^2_{k,\beta}} \leq C \left(\mu_{k,\beta}(\Sigma)\right)^{\frac{s}{A^d_{\gamma,\beta}}} \||x|^s f\|^2_{L^2_{k,\beta}}. \tag{26}$$

2. If $s > A^d_{\gamma,\beta}$, then there exists a positive constant C such that for all $f \in L^2_{k,\beta}(\mathbb{R}_+^{d+1})$ and all measurable subset $\Sigma \subset \mathbb{R}_+^{d+1}$ of finite measure $0 < \mu_{k,\beta}(\Sigma) < \infty$,

$$\|M_\Sigma f\|^2_{L^2_{k,\beta}} \leq C \mu_{k,\beta}(\Sigma) \|f\|^{2 - \frac{2A^d_{\gamma,\beta}}{s}}_{L^2_{k,\beta}} \||x|^s f\|^{\frac{2A^d_{\gamma,\beta}}{s}}_{L^2_{k,\beta}}. \tag{27}$$

Next, take $s = A^d_{\gamma,\beta}$. Then, if we apply the first inequality (26) with $A^d_{\gamma,\beta}(1 - \varepsilon)$, $\varepsilon \in (0, 1)$, replacing s and then apply the following classical inequality

$$\||x|^{A^d_{\gamma,\beta}-A^d_{\gamma,\beta}\varepsilon} f\|_{L^2_{k,\beta}} \leq C\|f\|^{\varepsilon}_{L^2_{k,\beta}} \left\||x|^{A^d_{\gamma,\beta}} f\right\|^{1-\varepsilon}_{L^2_{k,\beta}}, \tag{28}$$

we obtain for all $\varepsilon \in (0, 1)$,

$$\|M_\Sigma f\|^2_{L^2_{k,\beta}} \leq C\left(\mu_{k,\beta}(\Sigma)\right)^{1-\varepsilon} \|f\|^{2\varepsilon}_{L^2_{k,\beta}} \||x|^{A^d_{\gamma,\beta}} f\|^{2-2\varepsilon}_{L^2_{k,\beta}}. \tag{29}$$

Consequently we deduce the following first corollary comparing the support of $\mathscr{F}_{k,\beta}(f)$ and the generalized time dispersion $\||x|^s f\|_{L^2_{k,\beta}}$ for function in the range of M_Σ:

$$\operatorname{Im}(M_\Sigma) = \{f \in L^2_{k,\beta}(\mathbb{R}^{d+1}_+) : \mathscr{F}_{k,\beta}(f) \subset \Sigma\}.$$

Corollary 1 Let $s > 0$. Then there exists $C > 0$ such that for every $f \in \operatorname{Im}(M_\Sigma)$,

$$\mu_{k,\beta}\left(\mathscr{F}_{k,\beta}(f)\right) \||x|^s f\|^{\frac{2A^d_{\gamma,\beta}}{s}}_{L^2_{k,\beta}} \geq C\|f\|^{\frac{2A^d_{\gamma,\beta}}{s}}_{L^2_{k,\beta}}. \tag{30}$$

Proof. Let $s > 0$ and $f \in \operatorname{Im}(M_\Sigma)$. Then $f = M_\Sigma f$, and we apply (26), (27), (29) to obtain the desired result. □

Notice that, if $\mu_{k,\beta}\left(\mathscr{F}_{k,\beta}(f)\right)$ is finite, then $\mu_{k,\beta}(f)$ is infinite, because f and $\mathscr{F}_{k,\beta}(f)$ cannot be simultaneously supported on subsets of finite measure, see [19, Corollary 3.7]. This result is known as the Benedicks-Amrein-Berthier uncertainty principle.

Moreover, we can also obtain an inequality comparing the essential support of $\mathscr{F}_{k,\beta}(f)$ and the generalized time dispersion $\||x|^s f\|_{L^2_{k,\beta}}$ for functions that are ε_2-bandlimited on Σ.

Corollary 2 Let $s > 0$.

1. If $0 < s < A^d_{\gamma,\beta}$, then there exists $C > 0$ such that for every function f that is ε_2-bandlimited on Σ,

$$\left(\mu_{k,\beta}(\Sigma)\right)^{\frac{s}{A^d_{\gamma,\beta}}} \||x|^s f\|^2_{L^2_{k,\beta}} \geq C\left(1-\varepsilon_2^2\right)\|f\|^2_{L^2_{k,\beta}}. \tag{31}$$

2. If $s > A^d_{\gamma,\beta}$, then there exists $C > 0$ such that for every function f that is ε_2-bandlimited on Σ,

$$\left(\mu_{k,\beta}(\Sigma)\right)^{\frac{s}{A^d_{\gamma,\beta}}} \||x|^s f\|^2_{L^2_{k,\beta}} \geq C\left(1-\varepsilon_2^2\right)^{\frac{s}{A^d_{\gamma,\beta}}} \|f\|^2_{L^2_{k,\beta}}. \tag{32}$$

3. For all $\varepsilon \in (0, 1)$, there exists $C > 0$ such that for every function f that is ε_2-bandlimited on Σ,

$$\mu_{k,\beta}(\Sigma)\left\||x|^{A^d_{\gamma,\beta}} f\right\|^2_{L^2_{k,\beta}} \geq C(1-\varepsilon_2^2)^{\frac{1}{1-\varepsilon}}\|f\|^2_{L^2_{k,\beta}}. \tag{33}$$

Proof. Since $f \in L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ is ε_2-bandlimited on Σ, then

$$\|M_\Sigma f\|^2_{L^2_{k,\beta}} = \|f\|^2_{L^2_{k,\beta}} - \|M_{\Sigma^c} f\|^2_{L^2_{k,\beta}} \geq (1-\varepsilon_2^2)\|f\|^2_{L^2_{k,\beta}}.$$

For the first result, we use the local inequalities (26). Analogously, for the second inequality, we use (27), and finally, for the third inequality, we use (29). □

Now, since, $\|M_\Sigma f\|_{L^2_{k,\beta}} = \left\|E_\Sigma \mathscr{F}_{k,\beta}(f)\right\|_{L^2_{k,\beta}}$, then by interchanging the roles of f and $\mathscr{F}_{k,\beta}(f)$ in Theorem 2, Corollary 1 and Corollary 2, we obtain the following results involving the time limiting operator instead of the frequency limiting operator, and the frequency dispersion instead of the time dispersion.

Theorem 3 Let $t > 0$.

1. If $0 < t < A^d_{\gamma,\beta}$, then

(a) there exists a constant $C > 0$ such that for all $f \in L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ and all measurable subset $S \subset \mathbb{R}^{d+1}_+$ of finite measure $0 < \mu_{k,\beta}(S) < \infty$,

$$\|E_S f\|^2_{L^2_{k,\beta}} \leq C\big(\mu_{k,\beta}(S)\big)^{\frac{t}{A^d_{\gamma,\beta}}} \||x|^t \mathscr{F}_{k,\beta}(f)\|^2_{L^2_{k,\beta}}, \tag{34}$$

(b) there exists a constant $C > 0$ such that for every function f which is ε_1-concentrated on S,

$$\big(\mu_{k,\beta}(S)\big)^{\frac{t}{A^d_{\gamma,\beta}}} \||\xi|^t \mathscr{F}_{k,\beta}(f)\|^2_{L^2_{k,\beta}} \geq C\left(1-\varepsilon_1^2\right)\|f\|^2_{L^2_{k,\beta}}. \tag{35}$$

2. If $t > A^d_{\gamma,\beta}$, then

(a) there exists a constant $C > 0$ such that for all $f \in L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ and all measurable subset $S \subset \mathbb{R}^{d+1}_+$ of finite measure $0 < \mu_{k,\beta}(S) < \infty$,

$$\|E_S f\|^2_{L^2_{k,\beta}} \leq C\mu_{k,\beta}(S)\|f\|^{2-\frac{2A^d_{\gamma,\beta}}{t}}_{L^2_{k,\beta}} \||\xi|^\beta \mathscr{F}_{k,\beta}(f)\|^{\frac{2A^d_{\gamma,\beta}}{t}}_{L^2_{k,\beta}}, \tag{36}$$

(b) there exists a constant $C > 0$ such that for every function f which is ε_1-concentrated on S,

$$\big(\mu_{k,\beta}(S)\big)^{\frac{t}{A^d_{\gamma,\beta}}} \||\xi|^t \mathscr{F}_{k,\beta}(f)\|^2_{L^2_{k,\beta}} \geq C\left(1-\varepsilon_1^2\right)^{\frac{t}{A^d_{\gamma,\beta}}} \|f\|^2_{L^2_{k,\beta}}. \tag{37}$$

3. For all $\varepsilon \in (0, 1)$,

(a) there exists $C > 0$ such that for every $f \in L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ and all measurable subset $S \subset \mathbb{R}^{d+1}_+$ of finite measure $0 < \mu_{k,\beta}(S) < \infty$,

$$\|E_S f\|^2_{L^2_{k,\beta}} \leq C\left(\mu_{k,\beta}(S)\right)^{1-\varepsilon} \|f\|^{2\varepsilon}_{L^2_{k,\beta}} \||\xi|^{A^d_{\gamma,\beta}} \mathscr{F}_{k,\beta}(f)\|^{2-2\varepsilon}_{L^2_{k,\beta}}, \tag{38}$$

(b) there exists a constant $C > 0$ such that for every function f which is ε_1-concentrated on S,

$$\mu_{k,\beta}(S)\left\||\xi|^{A^d_{\gamma,\beta}}\mathscr{F}_{k,\beta}(f)\right\|^2_{L^2_{k,\beta}} \geq C\left(1-\varepsilon_1^2\right)^{\frac{1}{1-\varepsilon}}\|f\|^2_{L^2_{k,\beta}}. \tag{39}$$

4. There exists a constant $C > 0$ such that for all $f \in \operatorname{Im}(E_S) = \{L^2_{k,\beta}(\mathbb{R}^{d+1}_+) : f \subset S\}$,

$$\mu_{k,\beta}(f)\,\||\xi|^t\mathscr{F}_{k,\beta}(f)\|^{\frac{2a}{t}}_{L^2_{k,\beta}} \geq C\|f\|^{\frac{2A^d_{\gamma,\beta}}{t}}_{L^2_{k,\beta}}. \tag{40}$$

Finally we can formulate our new Heisenberg-type uncertainty inequalities for functions in $L^2_{k,\beta}(\varepsilon_1, \varepsilon_2, S, \Sigma)$, with constants that depend on $\varepsilon_1,\ \varepsilon_2,\ S$ and Σ.

Theorem 4 Let $s, t > 0$. Then for any $f \in L^2_{k,\beta}(\varepsilon_1, \varepsilon_2, S, \Sigma)$:

1. if $0 < s, t < A^d_{\gamma,\beta}$,

$$\||x|^s f\|^t_{L^2_{k,\beta}}\,\||\xi|^t\mathscr{F}_{k,\beta}(f)\|^s_{L^2_{k,\beta}} \geq C\,\frac{(1-\varepsilon_1^2)^{s/2}(1-\varepsilon_2^2)^{t/2}}{\left(\mu_{k,\beta}(S)\mu_{k,\beta}(\Sigma)\right)^{\frac{st}{2A^d_{\gamma,\beta}}}}\|f\|^{s+t}_{L^2_{k,\beta}}, \tag{41}$$

2. if $s, t > A^d_{\gamma,\beta}$,

$$\||x|^s f\|^t_{L^2_{k,\beta}}\,\||\xi|^t\mathscr{F}_{k,\beta}(f)\|^s_{L^2_{k,\beta}} \geq C\left(\frac{(1-\varepsilon_1^2)(1-\varepsilon_2^2)}{\mu_{k,\beta}(S)\mu_{k,\beta}(\Sigma)}\right)^{\frac{st}{2A^d_{\gamma,\beta}}}\|f\|^{s+t}_{L^2_{k,\beta}}, \tag{42}$$

3. for all $\varepsilon \in (0, 1)$,

$$\left\||x|^{A^d_{\gamma,\beta}}f\right\|_{L^2_{k,\beta}}\left\||\xi|^{A^d_{\gamma,\beta}}\mathscr{F}_{k,\beta}(f)\right\|_{L^2_{k,\beta}} \geq C\,\frac{\left((1-\varepsilon_1^2)(1-\varepsilon_2^2)\right)^{\frac{1}{2-2\varepsilon}}}{\sqrt{\mu_{k,\beta}(S)\mu_{k,\beta}(\Sigma)}}\|f\|^2_{L^2_{k,\beta}}. \tag{43}$$

Remark 3

1. Notice that Corollary 2 and Inequalities (35), (37) and (39) give separately a lower bounds for the measures of the time dispersion $\||x|^s f\|_{L^2_{k,\beta}}$ and the frequency dispersion $\||\xi|^t\mathscr{F}_{k,\beta}(f)\|_{L^2_{k,\beta}}$, which give more information than a lower bound of the product between them in Theorem 4.

2. On the other hand, from Corollary 2 and Inequalities (35), (37) and (39), we can obtain separately a lower bounds, that depend of the signal $f \in L^2_{k,\beta}(\varepsilon_1, \varepsilon_2, S, \Sigma)$, for the measures of $\mu_{k,\beta}(S)$ and $\mu_{k,\beta}(\Sigma)$, from which we deduce, in the spirit of [41], the following lower bounds for the product between them:

$$\mu_{k,\beta}(S)\mu_{k,\beta}(\Sigma) \geq \begin{cases} C.C_f(s, A^d_{\gamma,\beta}, t)\left((1-\varepsilon_1^2)^{\frac{1}{t}}(1-\varepsilon_2^2)^{\frac{1}{s}}\right)^{A^d_{\gamma,\beta}}, & 0 < s, t < A^d_{\gamma,\beta}, \\ C.C_f(s, A^d_{\gamma,\beta}, t)\,(1-\varepsilon_1^2)(1-\varepsilon_2^2), & s, t > A^d_{\gamma,\beta}, \\ C.C_f(A^d_{\gamma,\beta}, A^d_{\gamma,\beta}, A^d_{\gamma,\beta})\left((1-\varepsilon_1^2)(1-\varepsilon_2^2)\right)^{\frac{1}{1-\varepsilon}}, & \text{otherwise}, \end{cases} \tag{44}$$

where C is a constant that depend only on $s, A^d_{\gamma,\beta}, t, \varepsilon,$ and

$$C_f(s, A^d_{\gamma,\beta}, t) = \left(\frac{\|f\|^{s+t}_{L^2_{k,\beta}}}{\|x^s f\|^t_{L^2_{k,\beta}}\||\xi|^t \mathscr{F}_{k,\beta}(f)\|^s_{L^2_{k,\beta}}}\right)^{\frac{2A^d_{\gamma,\beta}}{st}}. \tag{45}$$

3.2 *Uncertainty principles on the space $L^1_{k,\beta} \cap L^2_{k,\beta}(\varepsilon_1, \varepsilon_2, S, \Sigma)$*

The first known result for functions in $L^1_{k,\beta} \cap L^2_{k,\beta}(\varepsilon_1, \varepsilon_2, S, \Sigma)$ is the following Donoho-Stark type uncertainty inequality, see [20, Proposition 2.6].

Theorem 5 Let $\varepsilon_1, \varepsilon_2 \in (0, 1)$. Then if $f \in L^1_{k,\beta} \cap L^2_{k,\beta}(\varepsilon_1, \varepsilon_2, S, \Sigma)$ we have

$$\mu_{k,\beta}(S) \geq \frac{\|f\|^2_{L^1_{k,\beta}}}{\|f\|^2_{L^2_{k,\beta}}}(1-\varepsilon_1)^2, \quad \mu_{k,\beta}(\Sigma) \geq \frac{\|f\|^2_{L^2_{k,\beta}}}{\|f\|^2_{L^1_{k,\beta}}}(1-\varepsilon_2^2), \tag{46}$$

and then

$$\mu_{k,\beta}(S)\mu_{k,\beta}(\Sigma) \geq (1-\varepsilon_1)^2(1-\varepsilon_2^2). \tag{47}$$

Theorem 5 is stronger then Theorem 1, in the sense that the previous theorem give a lower bound of $\mu_{k,\beta}(S)$ and $\mu_{k,\beta}(\Sigma)$ separately, which is not possible in Theorem 1.

We proceed as [20, Proposition 2.2, Proposition 2.3], we prove the following Carlson-type and Nash-type inequalities.

Theorem 6 Let $s, t > 0$. Then we have:

1. *A Carlson-type inequality:* there exists a constant $C_1 = C(s, A^d_{\gamma,\beta}) > 0$ such that for all $f \in L^1_{k,\beta}(\mathbb{R}^{d+1}_+) \cap L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$,

$$\|f\|^{1+\frac{s}{A^d_{\gamma,\beta}}}_{L^1_{k,\beta}} \leq C_1 \|f\|^{\frac{s}{A^d_{\gamma,\beta}}}_{L^2_{k,\beta}} \||x|^s f\|_{L^1_{k,\beta}}. \tag{48}$$

2. *A Nash-type inequality:* there exists a constant $C_2 = C(t, A^d_{\gamma,\beta}) > 0$ such that for all $f \in L^1_{k,\beta}(\mathbb{R}^{d+1}_+) \cap L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$,

$$\|f\|_{L^2_{k,\beta}}^{1+\frac{t}{A^d_{\gamma,\beta}}} \le C_2 \|f\|_{L^1_{k,\beta}}^{\frac{t}{A^d_{\gamma,\beta}}} \||\xi|^t \mathscr{F}_{k,\beta}(f)\|_{L^2_{k,\beta}}. \tag{49}$$

Consequently we obtain a lower bounds for the time and frequency dispersions:

$$\||x|^s f\|_{L^1_{k,\beta}} \ge C_1 \left(\frac{\|f\|_{L^1_{k,\beta}}}{\|f\|_{L^2_{k,\beta}}} \right)^{\frac{s}{A^d_{\gamma,\beta}}} \|f\|_{L^1_{k,\beta}} \quad \text{and} \quad \||\xi|^t \mathscr{F}_{k,\beta}(f)\|_{L^2_{k,\beta}} \ge C_2 \left(\frac{\|f\|_{L^2_{k,\beta}}}{\|f\|_{L^1_{k,\beta}}} \right)^{\frac{t}{A^d_{\gamma,\beta}}} \|f\|_{L^2_{k,\beta}}. \tag{50}$$

Corollary 3 Let $s, t > 0$. Then

1. there exists a constant $C_3 = C(k, \beta, t, s, d) > 0$ such that for all $f \in L^1_{k,\beta}(\mathbb{R}^{d+1}_+) \cap L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$,

$$\||x|^s f\|^t_{L^1_{k,\beta}} \||\xi|^t \mathscr{F}_{k,\beta}(f)\|^s_{L^2_{k,\beta}} \ge C_3 \|f\|^t_{L^1_{k,\beta}} \|f\|^s_{L^2_{k,\beta}}, \tag{51}$$

2. there exists a constant $C_4 = C(k, \beta, s, d) > 0$ such that for all $f \in L^1_{k,\beta}(\mathbb{R}^{d+1}_+) \cap L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ and all measurable subset of Σ of finite measure,

$$\|M_\Sigma f\|^2_{L^2_{k,\beta}} \le C_4 \mu_{k,\beta}(\Sigma) \|f\|_{L^2_{k,\beta}}^{\frac{2s}{A^d_{\gamma,\beta}+s}} \||x|^s f\|_{L^1_{k,\beta}}^{\frac{2A^d_{\gamma,\beta}}{A^d_{\gamma,\beta}+s}}, \tag{52}$$

3. there exists a constant $C_5 = C(k, \beta, t, d) > 0$ such that for all $f \in L^1_{k,\beta}(\mathbb{R}^{d+1}_+) \cap L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ and all measurable subset S of finite measure,

$$\|E_S f\|^2_{L^1_{k,\beta}} \le C_5 \mu_{k,\beta}(S) \|f\|_{L^1_{k,\beta}}^{\frac{2t}{A^d_{\gamma,\beta}+t}} \||\xi|^\beta \mathscr{F}_{k,\beta}(f)\|_{L^2_{k,\beta}}^{\frac{2A^d_{\gamma,\beta}}{A^d_{\gamma,\beta}+t}}, \tag{53}$$

4. there exists a constant $C_6 = C(k, \beta, s, d) > 0$ such that for all $f \in L^1_{k,\beta}(\mathbb{R}^{d+1}_+) \cap L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ with $\mathscr{F}_{k,\beta}(f) \subset \Sigma$,

$$\mu_{k,\beta}(\mathscr{F}_{k,\beta}(f)) \||x|^s f\|_{L^1_{k,\beta}}^{\frac{2A^d_{\gamma,\beta}}{A^d_{\gamma,\beta}+s}} \ge C_6 \|f\|_{L^2_{k,\beta}}^{\frac{2A^d_{\gamma,\beta}}{A^d_{\gamma,\beta}+s}}, \tag{54}$$

5. there exists a constant $C_7 = C(k, \beta, t, d) > 0$ such that for all $f \in L^1_{k,\beta}(\mathbb{R}^{d+1}_+) \cap L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ with $f \subset S$,

$$\mu_{k,\beta}(f) \||\xi|^t \mathscr{F}_{k,\beta}(f)\|_{L^2_{k,\beta}}^{\frac{2A^d_{\gamma,\beta}}{A^d_{\gamma,\beta}+t}} \ge C_7 \|f\|_{L^1_{k,\beta}}^{\frac{2A^d_{\gamma,\beta}}{A^d_{\gamma,\beta}+t}}. \tag{55}$$

Proof. The first inequality follows by combining the Carlson's inequality (48) and the Nash's inequality (49). Next by (13) and (8),

$$\|M_\Sigma f\|^2_{L^2_{k,\beta}} = \left\|\chi_\Sigma \mathscr{F}_{k,\beta}(f)\right\|^2_{L^2_{k,\beta}} \le \mu_{k,\beta}(\Sigma)\left\|\mathscr{F}_{k,\beta}(f)\right\|^2_\infty \le \mu_{k,\beta}(\Sigma)\|f\|^2_{L^1_{k,\beta}},$$

and by the Carlson's inequality (48) we obtain (52). Now by the Cauchy-Schwartz's inequality we have,

$$\|E_S f\|^2_{L^1_{k,\beta}} \le \mu_{k,\beta}(S)\|f\|^2_{L^2_{k,\beta}},$$

and by the Nash type inequality (49) we deduce (53). Finally (54) follows directly from (52) by taking $\Sigma = \mathscr{F}_{k,\beta}(f)$ and if we take $S = f$ in (53) we obtain (55). $\square$

Remark 4 Clearly, the inequalities in (50) imply also that there exist a positive constant $\mathscr{C}$, for all $f \in L^1_{k,\beta}(\mathbb{R}^{d+1}_+) \cap L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$,

$$\||x|^s f\|^{A^d_{\gamma,\beta}+t}_{L^1_{k,\beta}} \||\xi|^t \mathscr{F}_{k,\beta}(f)\|^{A^d_{\gamma,\beta}+s}_{L^2_{k,\beta}} \ge \mathscr{C}\|f\|^{A^d_{\gamma,\beta}+s}_{L^1_{k,\beta}} \|f\|^{A^d_{\gamma,\beta}+t}_{L^2_{k,\beta}}. \tag{56}$$

Corollary 4 Let $s, t > 0$. Then

1. there exists a constant $C > 0$ such that for every function f, which is ε_1-timelimited on S,

$$\left(\mu_{k,\beta}(S)\right)^{\frac{A^d_{\gamma,\beta}+t}{2A^d_{\gamma,\beta}}} \||\xi|^t \mathscr{F}_{k,\beta}(f)\|_{L^2_{k,\beta}} \ge C(1-\varepsilon_1)^{\frac{A^d_{\gamma,\beta}+t}{A^d_{\gamma,\beta}}} \|f\|_{L^1_{k,\beta}}, \tag{57}$$

2. there exists a constant $C > 0$ such that for every function f, which is ε_2-bandlimited on Σ,

$$\left(\mu_{k,\beta}(\Sigma)\right)^{\frac{A^d_{\gamma,\beta}+s}{2A^d_{\gamma,\beta}}} \||x|^s f\|_{L^1_{k,\beta}} \ge C\left(1-\varepsilon_2^2\right)^{\frac{A^d_{\gamma,\beta}+s}{2A^d_{\gamma,\beta}}} \|f\|_{L^2_{k,\beta}}, \tag{58}$$

3. there exists a constant C such that for all $f \in L^1_{k,\beta} \cap L^2_{k,\beta}(\varepsilon_1, \varepsilon_2, S, \Sigma)$,

$$\||x|^s f\|^{A^d_{\gamma,\beta}+t}_{L^1_{k,\beta}} \||\xi|^t \mathscr{F}_{k,\beta}(f)\|^{A^d_{\gamma,\beta}+s}_{L^2_{k,\beta}} \ge C\left(\frac{(1-\varepsilon_1)^2(1-\varepsilon_2^2)}{\mu_{k,\beta}(S)\mu_{k,\beta}(\Sigma)}\right)^{\frac{(A^d_{\gamma,\beta}+s)(A^d_{\gamma,\beta}+t)}{2A^d_{\gamma,\beta}}} \|f\|^{A^d_{\gamma,\beta}+s}_{L^1_{k,\beta}} \|f\|^{A^d_{\gamma,\beta}+t}_{L^2_{k,\beta}}. \tag{59}$$

Proof. If f is ε_1-timelimited, then

$$\|E_S f\|_{L^1_{k,\beta}} \ge \|f\|_{L^1_{k,\beta}} - \|E_{S^c} f\|_{L^1_{k,\beta}} \ge (1-\varepsilon_1)\|f\|_{L^1_{k,\beta}},$$

and if f is ε_2-bandelimited, then

$$\|M_\Sigma f\|^2_{L^2_{k,\beta}} = \|f\|^2_{L^2_{k,\beta}} - \|M_{\Sigma^c} f\|^2_{L^2_{k,\beta}} \geq (1-\varepsilon_2^2)\|f\|^2_{L^2_{k,\beta}}.$$

Hence the desired result follows from (52) and (53). □

Remark 5 Let $s, t > 0$ and let $f \in L^1_{k,\beta}(\mathbb{R}^{d+1}_+) \cap L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$.

1. If f is ε_1-timelimited on S, then

$$\mu_{k,\beta}(S) \geq C\left(\frac{\|f\|_{L^1_{k,\beta}}}{\||\xi|^t \mathscr{F}_{k,\beta}(f)\|_{L^2_{k,\beta}}}\right)^{\frac{2A^d_{\gamma,\beta}}{A^d_{\gamma,\beta}+t}} (1-\varepsilon_1)^2. \tag{60}$$

2. If f is ε_2-bandlimited on Σ, then

$$\mu_{k,\beta}(\Sigma) \geq C\left(\frac{\|f\|_{L^2_{k,\beta}}}{\||x|^s f\|_{L^1_{k,\beta}}}\right)^{\frac{2A^d_{\gamma,\beta}}{A^d_{\gamma,\beta}+s}} (1-\varepsilon_2^2). \tag{61}$$

3. If $f \in L^1_{k,\beta} \cap L^2_{k,\beta}(\varepsilon_1, \varepsilon_2, S, \Sigma)$, then

$$\mu_{k,\beta}(S)\mu_{k,\beta}(\Sigma) \geq C.\tilde{C}_f(A^d_{\gamma,\beta}, s, t)(1-\varepsilon_1)^2(1-\varepsilon_2^2), \tag{62}$$

where

$$\tilde{C}_f(A^d_{\gamma,\beta}, s, t) = \left(\frac{\|f\|^{A^d_{\gamma,\beta}+s}_{L^1_{k,\beta}} \|f\|^{A^d_{\gamma,\beta}+t}_{L^2_{k,\beta}}}{\||x|^s f\|^{A^d_{\gamma,\beta}+t}_{L^1_{k,\beta}} \||\xi|^t \mathscr{F}_{k,\beta}(f)\|^{A^d_{\gamma,\beta}+s}_{L^2_{k,\beta}}}\right)^{\frac{2A^d_{\gamma,\beta}}{(A^d_{\gamma,\beta}+t)(A^d_{\gamma,\beta}+s)}}. \tag{63}$$

4. Dunkl-Bessel two-wavelet multipliers

Our motivation in this section comes mainly from results established in [11, 42, 43]

Definition 5 Let u, v, σ be measurable functions on $\mathbb{R}^{d+1}_+$, we define the Dunkl-Bessel two-wavelet multiplier operator noted by $\mathscr{P}_{u,v}(\sigma)$, on $L^p_{k,\beta}(\mathbb{R}^{d+1}_+)$, $1 \leq p \leq \infty$, by

$$\mathscr{P}_{u,v}(\sigma)(f)(y) = \int_{\mathbb{R}^{d+1}_+} \sigma(\xi)\mathscr{F}_{k,\beta}(uf)(\xi)\Lambda_{k,\beta}(y,\xi)\overline{v(y)}d\mu_{k,\beta}(\xi), \ y \in \mathbb{R}^{d+1}_+. \tag{64}$$

In the case when $\sigma = \chi_A$ is the characteristic function of the subset $A \subset \mathbb{R}_+^{d+1}$, then we write $\mathscr{P}_{u,v}(\sigma)$ as $\mathscr{P}_{u,v}(A)$, if $u \neq v$ and by $\mathscr{P}_u(A)$ if $u = v$.

Often, it is more convenient to interpret the definition of $\mathscr{P}_{u,v}(\sigma)$ in a weak sense, that is, for f in $L^p_{k,\beta}(\mathbb{R}_+^{d+1})$, $p \in [1, \infty]$, and g in $L^{p'}_{k,\beta}(\mathbb{R}_+^{d+1})$,

$$\langle \mathscr{P}_{u,v}(\sigma)(f), g\rangle_{L^2_{k,\beta}} = \int_{\mathbb{R}_+^{d+1}} \sigma(\xi)\mathscr{F}_{k,\beta}(uf)(\xi)\overline{\mathscr{F}_{k,\beta}(vg)(\xi)}d\mu_{k,\beta}(\xi) \tag{65}$$

Proposition 3 Let $p \in [1, \infty)$. The adjoint of linear operator

$$\mathscr{P}_{u,v}(\sigma)\colon L^p_{k,\beta}(\mathbb{R}_+^{d+1}) \to L^p_{k,\beta}(\mathbb{R}_+^{d+1})$$

is $\mathscr{P}_{v,u}(\overline{\sigma})\colon L^{p'}_{k,\beta}(\mathbb{R}_+^{d+1}) \to L^{p'}_{k,\beta}(\mathbb{R}_+^{d+1})$.

Proof. For all f in $L^p_{k,\beta}(\mathbb{R}_+^{d+1})$ and g in $L^{p'}_{k,\beta}(\mathbb{R}_+^{d+1})$ it follows immediately from (64)

$$\begin{aligned}\langle \mathscr{P}_{u,v}(\sigma)(f), g\rangle_{L^2_{k,\beta}} &= \int_{\mathbb{R}_+^{d+1}} \sigma(\xi)\mathscr{F}_{k,\beta}(uf)(\xi)\overline{\mathscr{F}_{k,\beta}(vg)(\xi)}d\mu_{k,\beta}(\xi)\\ &= \overline{\int_{\mathbb{R}_+^{d+1}} \overline{\sigma(\xi)\mathscr{F}_{k,\beta}(uf)(\xi)}\mathscr{F}_{k,\beta}(vg)(\xi)d\mu_{k,\beta}(\xi)}\\ &= \overline{\langle \mathscr{P}_{v,u}(\overline{\sigma})(g), f\rangle_{L^2_{k,\beta}}} = \langle f, \mathscr{P}_{v,u}(\overline{\sigma})(g)\rangle_{L^2_{k,\beta}}.\end{aligned}$$

Thus we get

$$\mathscr{P}^*_{u,v}(\sigma) = \mathscr{P}_{v,u}(\overline{\sigma}). \tag{66}$$

□

Proposition 4 Let $\sigma \in L^1_{k,\beta}(\mathbb{R}_+^{d+1}) \cup L^\infty_{k,\beta}(\mathbb{R}_+^{d+1})$ and let $u, v \in L^2_{k,\beta}(\mathbb{R}_+^{d+1}) \cap L^\infty_{k,\beta}(\mathbb{R}_+^{d+1})$. Then

$$\langle \mathscr{P}_{u,v}(\sigma)(f), g\rangle_{L^2_{k,\beta}} = \langle \bar{v} M_\sigma(uf), g\rangle_{L^2_{k,\beta}}. \tag{67}$$

Proof. For all f, g in $L^2_{k,\beta}(\mathbb{R}_+^{d+1})$, it follows by the use of the relations (65) and (24) and Parseval's formula (14) that

$$\langle \mathscr{P}_{u,v}(\sigma)(f), g\rangle_{L^2_{k,\beta}} = \int_{\mathbb{R}_+^{d+1}} M_\sigma(uf)(x)\overline{(vg)(x)}d\mu_{k,\beta}(x) = \langle \bar{v} M_\sigma(uf), g\rangle_{L^2_{k,\beta}}.$$

So, the prove is achieved. □

4.1 *Boundedness for $\mathscr{P}_{u,v}(\sigma)$ on S_∞*

In this subsection, using interpolation theorem we will prove the boundedness of the operators $\mathscr{P}_{u,v}(\sigma)$ for $\sigma \in L^p_{k,\beta}(\mathbb{R}^{d+1}_+)$, $1 \le p \le \infty$ on S_∞.

In sequel, in this subsection, u and v will be any functions in $L^2_{k,\beta}(\mathbb{R}^{d+1}_+) \cap L^\infty_{k,\beta}(\mathbb{R}^{d+1}_+)$ such that

$$\|u\|_{L^2_{k,\beta}} = \|v\|_{L^2_{k,\beta}} = 1.$$

Proposition 5 Let σ be in $L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$, then, $\mathscr{P}_{u,v}(\sigma)$ is in S_∞ and

$$||\mathscr{P}_{u,v}(\sigma)||_{S_\infty} \leqslant \|\sigma\|_{L^1_{k,\beta}}. \tag{68}$$

Proof. From (64), it's easy to see, for every functions f and g in $L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$, that

$$|\langle \mathscr{P}_{u,v}(\sigma)(f), g\rangle_{L^2_{k,\beta}}| \leqslant \|\mathscr{F}_{k,\beta}(uf)\|_{L^\infty_{k,\beta}} \|\mathscr{F}_{k,\beta}(vg)\|_{L^\infty_{k,\beta}} \|\sigma\|_{L^1_{k,\beta}}.$$

On the other hand, from (5) and the Cauchy-Schwarz's inequality, we get

$$\|\mathscr{F}_{k,\beta}(uf)\|_{L^\infty_{k,\beta}} \le \|u\|_{L^2_{k,\beta}} \|f\|_{L^2_{k,\beta}}, \quad \|\mathscr{F}_{k,\beta}(vg)\|_{L^\infty_{k,\beta}} \le \|v\|_{L^2_{k,\beta}} \|g\|_{L^2_{k,\beta}}.$$

Therefore, since $\|u\|_{L^2_{k,\beta}} = \|v\|_{L^2_{k,\beta}} = 1$, we obtain

$$|\langle \mathscr{P}_{u,v}(\sigma)(f), g\rangle_{L^2_{k,\beta}}| \leqslant \|f\|_{L^2_{k,\beta}} \|g\|_{L^2_{k,\beta}} \|\sigma\|_{L^1_{k,\beta}}.$$

Using (23), we derive the result. □

Proposition 6 Let σ be in $L^\infty_{k,\beta}(\mathbb{R}^{d+1}_+)$, then the operator $\mathscr{P}_{u,v}(\sigma)$ is in S_∞ and we have

$$||\mathscr{P}_{u,v}(\sigma)||_{S_\infty} \leqslant \|u\|_{L^\infty_{k,\beta}} \|v\|_{L^\infty_{k,\beta}} \|\sigma\|_{L^\infty_{k,\beta}}.$$

Proof. Using Cauchy-Schwarz's inequality, we infer

$$|\langle \mathscr{P}_{u,v}(\sigma)(f), g\rangle_{L^2_{k,\beta}}| \leqslant \|\sigma\|_{L^\infty_{k,\beta}} \|\mathscr{F}_{k,\beta}(uf)\|_{L^2_{k,\beta}} \|\mathscr{F}_{k,\beta}(vg)\|_{L^2_{k,\beta}}.$$

Involving Plancherel's formula (13), we derive that

$$|\langle \mathscr{P}_{u,v}(\sigma)(f), g\rangle_{L^2_{k,\beta}}| \leqslant \|u\|_{L^\infty_{k,\beta}} \|v\|_{L^\infty_{k,\beta}} \|\sigma\|_{L^\infty_{k,\beta}} \|f\|_{L^2_{k,\beta}} \|g\|_{L^2_{k,\beta}}.$$

From this and (23), we obtain the result. □

Theorem 7 Let σ be in $L^p_{k,\beta}(\mathbb{R}^{d+1}_+)$, $1 \leq p \leq \infty$. Then there exists a unique bounded linear operator $\mathscr{P}_{u,v}(\sigma)$: $L^2_{k,\beta}(\mathbb{R}^{d+1}_+) \to L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$, such that

$$||\mathscr{P}_{u,v}(\sigma)||_{S_\infty} \leqslant (\|u\|_{L^\infty_{k,\beta}}\|v\|_{L^\infty_{k,\beta}})^{\frac{p-1}{p}}\|\sigma\|_{L^p_{k,\beta}}.$$

Proof. Let f be in $L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$. We consider the following operator

$$\mathscr{T}: L^1_{k,\beta}(\mathbb{R}^{d+1}_+)\bigcap L^\infty_{k,\beta}(\mathbb{R}^{d+1}_+) \quad \to \quad L^2_{k,\beta}(\mathbb{R}^{d+1}_+),$$

given by

$$\mathscr{T}(\sigma) := \mathscr{P}_{u,v}(\sigma)(f).$$

Then by Proposition 5 and Proposition 6

$$||\mathscr{T}(\sigma)||_{L^2_{k,\beta}} \leqslant ||f||_{L^2_{k,\beta}}\|\sigma\|_{L^1_{k,\beta}} \tag{69}$$

and

$$||\mathscr{T}(\sigma)||_{L^2_{k,\beta}} \leq \|u\|_{L^\infty_{k,\beta}}\|v\|_{L^\infty_{k,\beta}}||f||_{L^2_{k,\beta}}\|\sigma\|_{L^\infty_{k,\beta}}. \tag{70}$$

Thus, by (69), (70) and the Riesz-Thorin interpolation's theorem (see [[44], Theorem 2] we see also [[9], Theorem 2.11]). We obtain the following result

$$||\mathscr{P}_{u,v}(\sigma)(f)||_{L^2_{k,\beta}} = ||\mathscr{T}(\sigma)||_{L^2_{k,\beta}} \leq (\|u\|_{L^\infty_{k,\beta}}\|v\|_{L^\infty_{k,\beta}})^{\frac{p-1}{p}}||f||_{L^2_{k,\beta}}\|\sigma\|_{L^p_{k,\beta}}. \tag{71}$$

Since (71) is true for arbitrary functions f in $L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$, then we obtain the desired result. □

4.2 *Shatten class properties*

In this subsection, u and v will be any functions in $L^2_{k,\beta}(\mathbb{R}^{d+1}_+) \cap L^\infty_{k,\beta}(\mathbb{R}^{d+1}_+)$ such that

$$\|u\|_{L^2_{k,\beta}} = \|v\|_{L^2_{k,\beta}} = 1.$$

Let us begin with the following theorem

Theorem 8 Let σ be in $L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$, then $\mathscr{P}_{u,v}(\sigma)$ is an Hilbert Schmidt operator and

$$\|\mathscr{P}_{u,\,v}(\sigma)\|_{S_2} \leqslant \|\sigma\|_{L^1_{k,\,\beta}}.$$

Proof. Let $\{\phi_j,\ j = 1,\ 2...\}$ be an orthonormal basis for $L^2_{k,\,\beta}(\mathbb{R}^{d+1}_+)$. Then by (65), Fubini's theorem and Parseval's identity (13), we obtain

$$\begin{aligned}
&\sum_{j=1}^{\infty} ||\mathscr{P}_{u,\,v}(\sigma)(\phi_j)||^2_{L^2_{k,\,\beta}} \\
&= \sum_{j=1}^{\infty} \langle \mathscr{P}_{u,\,v}(\sigma)(\phi_j),\ \mathscr{P}_{u,\,v}(\sigma)(\phi_j)\rangle_{L^2_{k,\,\beta}} \\
&= \sum_{j=1}^{\infty} \int_{\mathbb{R}^{d+1}_+} \sigma(\xi)\langle \phi_j,\ \bar{u}\Lambda_{k,\,\beta}(\xi,\ .)\rangle_{L^2_{k,\,\beta}} \overline{\langle \mathscr{P}_{u,\,v}(\sigma)(\phi_j),\ \bar{v}\Lambda_{k,\,\beta}(\xi,\ .)\rangle}_{L^2_{k,\,\beta}} d\mu_{k,\,\beta}(\xi) \\
&= \int_{\mathbb{R}^{d+1}_+} \sigma(\xi) \sum_{j=1}^{\infty} \langle \mathscr{P}^*_{u,\,v}(\sigma)(\bar{v}\Lambda_{k,\,\beta}(\xi,\ .)),\ \phi_j\rangle_{L^2_{k,\,\beta}} \langle \phi_j,\ \bar{u}\Lambda_{k,\,\beta}(\xi,\ .)\rangle_{L^2_{k,\,\beta}} d\mu_{k,\,\beta}(\xi) \\
&= \int_{\mathbb{R}^{d+1}_+} \sigma(\xi)\langle \mathscr{P}^*_{u,\,v}(\sigma)(\bar{v}\Lambda_{k,\,\beta}(\xi,\ .)),\ \bar{u}\Lambda_{k,\,\beta}(\xi,\ .)\rangle_{L^2_{k,\,\beta}} d\mu_{k,\,\beta}(\xi).
\end{aligned}$$

Therefore from Proposition 5, the relation (5), we derive

$$\sum_{j=1}^{\infty} ||\mathscr{P}_{u,\,v}(\sigma)(\phi_j)||^2_{L^2_{k,\,\beta}} \leq \int_{\mathbb{R}^{d+1}_+} |\sigma(\xi)|\ ||\mathscr{P}^*_{u,\,v}(\sigma)||_{S_\infty} d\mu_{k,\,\beta}(\xi) \leq \|\sigma\|^2_{L^1_{k,\,\beta}} < \infty. \tag{72}$$

So, by (72) and the Proposition 2.8 in the book [9], by Wong,

$$\mathscr{P}_{u,\,v}(\sigma)\colon L^2_{k,\,\beta}(\mathbb{R}^{d+1}_+) \to L^2_{k,\,\beta}(\mathbb{R}^{d+1}_+)$$

is in the Hilbert-Schmidt class S_2 and hence compact. □

Proposition 7 Let σ be a symbol in $L^p_{k,\,\beta}(\mathbb{R}^{d+1}_+),\ 1 \leqslant p < \infty$. Then the operator $\mathscr{P}_{u,\,v}(\sigma)$ is compact.

Proof. Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L^1_{k,\,\beta}(\mathbb{R}^{d+1}_+) \bigcap L^\infty_{k,\,\beta}(\mathbb{R}^{d+1}_+)$ such that $\sigma_n \to \sigma$ in $L^p_{k,\,\beta}(\mathbb{R}^{d+1}_+)$ as $n \to \infty$. Then by Theorem 7 we get:

$$||\mathscr{P}_{u,\,v}(\sigma_n) - \mathscr{P}_{u,\,v}(\sigma)||_{S_\infty} \leq (\|u\|_{L^\infty_{k,\,\beta}} \|v\|_{L^\infty_{k,\,\beta}})^{\frac{p-1}{p}} ||\sigma_n - \sigma||_{L^p_{k,\,\beta}}.$$

Therefore $\mathscr{P}_{u,\,v}(\sigma_n) \to \mathscr{P}_{u,\,v}(\sigma)$ in S_∞ as $n \to \infty$. Now, since by Theorem 8, the operators are $\mathscr{P}_{u,\,v}(\sigma_n)$ in S_2 and hence compact, and since the set of compact operators is closed subspace of S_∞ it follows that $\mathscr{P}_{u,\,v}(\sigma)$ is also compact.

□

Theorem 9 Let σ be in $L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$. Then,
1. $\mathscr{P}_{u,v}(\sigma)$: $L^2_{k,\beta}(\mathbb{R}^{d+1}_+) \to L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ is trace class and we have

$$\frac{2}{\|u\|^2_{L^\infty_{k,\beta}} + \|v\|^2_{L^\infty_{k,\beta}}} \|\widetilde{\sigma}\|_{L^1_{k,\beta}} \leqslant \|\mathscr{P}_{u,v}(\sigma)\|_{S_1} \leqslant \|\sigma\|_{L^1_{k,\beta}}, \tag{73}$$

where $\widetilde{\sigma}$ is given by

$$\widetilde{\sigma}(\xi) = \langle \mathscr{P}_{u,v}(\sigma)\Lambda_{k,\beta}(\xi,.)u,\ \Lambda_{k,\beta}(\xi,.)v\rangle_{L^2_{k,\beta}}, \quad \xi \in \mathbb{R}^{d+1}_+.$$

2. We have the following trace formula

$$tr(\mathscr{P}_{u,v}(\sigma)) = \int_{\mathbb{R}^{d+1}_+} \sigma(\xi)\langle \bar{v}\Lambda_{k,\beta}(\xi,.),\ \bar{u}\Lambda_{k,\beta}(\xi,.)\rangle_{L^2_{k,\beta}} d\mu_{k,\beta}(\xi). \tag{74}$$

Proof. 1. Since σ is in $L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$, by Theorem 8, $\mathscr{P}_{u,v}(\sigma)$ is in S_2. Using [9, Theorem 2.2], there exists an orthonormal basis $\{\phi_j,\ j = 1, 2...\}$ for the orthogonal complement of the kernel of the operator $\mathscr{P}_{u,v}(\sigma)$, consisting of eigenvectors of $|\mathscr{P}_{u,v}(\sigma)|$ and $\{\psi_j,\ j = 1, 2...\}$ an orthonormal set in $L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$, such that

$$\mathscr{P}_{u,v}(\sigma)(f) = \sum_{j=1}^{\infty} s_j \langle f, \phi_j\rangle_{L^2_{k,\beta}} \psi_j, \tag{75}$$

where s_j, $j = 1, 2...$ are the positive singular values of $\mathscr{P}_{u,v}(\sigma)$ corresponding to ϕ_j. Then, we get

$$\|\mathscr{P}_{u,v}(\sigma)\|_{S_1} = \sum_{j=1}^{\infty} s_j = \sum_{j=1}^{\infty} \langle \mathscr{P}_{u,v}(\sigma)(\phi_j),\ \psi_j\rangle_{L^2_{k,\beta}}.$$

Thus, by Fubini's theorem, Parseval's identity, Bessel's inequality, Cauchy-Schwarz's inequality, (5), and $||u||_{L^2_{k,\beta}} = ||u||_{L^2_{k,\beta}} = 1$, we get

$$\begin{aligned}
\|\mathscr{P}_{u,v}(\sigma)\|_{S_1} &= \sum_{j=1}^{\infty} \langle \mathscr{P}_{u,v}(\sigma)(\phi_j),\ \psi_j\rangle_{L^2_{k,\beta}} \\
&= \sum_{j=1}^{\infty} \int_{\mathbb{R}^{d+1}_+} \sigma(\xi)\mathscr{F}_{k,\beta}(u\phi_j)(\xi)\overline{\mathscr{F}_{k,\beta}(v\psi_j)(\xi)} d\mu_{k,\beta}(\xi) \\
&= \int_{\mathbb{R}^{d+1}_+} \sigma(\xi) \sum_{j=1}^{\infty} \langle \phi_j, \bar{u}\Lambda_{k,\beta}(\xi,.)\rangle_{L^2_{k,\beta}} \langle \bar{v}\Lambda_{k,\beta}(\xi,.),\ \psi_j\rangle_{L^2_{k,\beta}} d\mu_{k,\beta}(\xi)
\end{aligned}$$

$$\leq \int_{\mathbb{R}_+^{d+1}} |\sigma(\xi)| \Big(\sum_{j=1}^{\infty} |\langle \phi_j, \bar{u}\Lambda_{k,\beta}(\xi, .)\rangle_{L^2_{k,\beta}}| \Big)^{\frac{1}{2}} \Big(\sum_{j=1}^{\infty} |\langle \bar{v}\Lambda_{k,\beta}(\xi, .), \psi_j\rangle_{L^2_{k,\beta}}| \Big)^{\frac{1}{2}} d\mu_{k,\beta}(\xi)$$

$$\leq \int_{\mathbb{R}_+^{d+1}} |\sigma(\xi)| ||\bar{u}\Lambda_{k,\beta}(\xi, .)||_{L^2_{k,\beta}} ||\bar{v}\Lambda_{k,\beta}(\xi, .)||_{L^2_{k,\beta}} d\mu_{k,\beta}(\xi)$$

$$\leqslant \|\sigma\|_{L^1_{k,\beta}}.$$

Thus

$$\|\mathscr{P}_{u,v}(\sigma)\|_{S_1} \leqslant \|\sigma\|_{L^1_{k,\beta}}.$$

We now prove that $\mathscr{P}_{u,v}(\sigma)$ satisfies the first member of (73). It is easy to see that $\widetilde{\sigma}$ belongs to $L^1_{k,\beta}(\mathbb{R}_+^{d+1})$, and using formula (75), we get

$$\begin{aligned}
|\widetilde{\sigma}(\xi)| &= \Big|\langle \mathscr{P}_{u,v}(\sigma)(\Lambda_{k,\beta}(\xi, .)u), \Lambda_{k,\beta}(\xi, .)v\rangle_{L^2_{k,\beta}}\Big| \\
&= \Big| \sum_{j=1}^{\infty} s_j \langle \Lambda_{k,\beta}(\xi, .)u, \phi_j\rangle_{L^2_{k\beta}} \langle \psi_j, \Lambda_{k,\beta}(\xi, .)v\rangle_{L^2_{k,\beta}} \Big| \\
&\leqslant \frac{1}{2} \sum_{j=1}^{\infty} s_j \Big(\Big|\langle \Lambda_{k,\beta}(\xi, .)u, \phi_j\rangle_{L^2_{k,\beta}}\Big|^2 + \Big|\langle \Lambda_{k,\beta}(\xi, .)v, \psi_j\rangle_{L^2_{k,\beta}}\Big|^2 \Big).
\end{aligned}$$

Then, using Plancherel's formula given by relation (13) and Fubini's theorem, we obtain

$$\begin{aligned}
\int_{\mathbb{R}_+^{d+1}} |\widetilde{\sigma}(\xi) d\mu_{k,\beta}(\xi) \quad &\leq \tfrac{1}{2} \sum_{j=1}^{\infty} s_j \Big(\int_{\mathbb{R}_+^{d+1}} |\langle \Lambda_{k,\beta}(\xi, .)u, \phi_j\rangle_{L^2_{k,\beta}}|^2 d\mu_{k,\beta}(\xi) \\
&\quad + \int_{\mathbb{R}_+^{d+1}} |\langle \Lambda_{k,\beta}(\xi, .)v, \psi_j\rangle_{L^2_{k,\beta}}|^2 d\mu_{k,\beta}(\xi) \Big).
\end{aligned}$$

Thus

$$\int_{\mathbb{R}_+^{d+1}} |\widetilde{\sigma}(\xi)| d\mu_{k,\beta}(\xi) \leq \frac{\|u\|^2_{L^\infty_{k,\beta}} + \|v\|^2_{L^\infty_{k,\beta}}}{2} \sum_{j=1}^{\infty} s_j = \frac{\|u\|^2_{L^\infty_{k,\beta}} + \|v\|^2_{L^\infty_{k,\beta}}}{2} \|\mathscr{P}_{u,v}(\sigma)\|_{S_1},$$

this allows to conclude.

2. Let $\{\phi_j,\ j=1,2...\}$ be an orthonormal basis for $L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$. From the previous assertion, the Dunkl-Bessel two-wavelet multiplier $\mathscr{P}_{u,v}(\sigma)$ belongs to S_1, then by the definition of the trace given by the relation (20), Fubini's theorem and Parseval's identity, we have

$$\begin{aligned} tr(\mathscr{P}_{u,v}(\sigma)) &= \sum_{j=1}^{\infty}\langle \mathscr{P}_{u,v}(\sigma)(\phi_j),\ \phi_j\rangle_{L^2_{k,\beta}} \\ &= \sum_{j=1}^{\infty}\int_{\mathbb{R}^{d+1}_+}\sigma(\xi)\langle \phi_j,\ \Lambda_{k,\beta}(\xi,.)\overline{u}\rangle_{L^2_{k,\beta}}\overline{\langle \phi_j,\ \Lambda_{k,\beta}(\xi,.)\overline{v}\rangle}_{L^2_{k,\beta}}d\mu_{k,\beta}(\xi) \\ &= \int_{\mathbb{R}^{d+1}_+}\sigma(\xi)\sum_{j=1}^{\infty}\langle \phi_j,\ \Lambda_{k,\beta}(\xi,.)\overline{u}\rangle_{L^2_{k,\beta}}\langle \Lambda_{k,\beta}(\xi,.)\overline{v},\ \phi_j\rangle_{L^2_{k,\beta}}d\mu_{k,\beta}(\xi) \\ &= \int_{\mathbb{R}^{d+1}_+}\sigma(\xi)\langle \Lambda_{k,\beta}(\xi,.)\overline{v},\ \Lambda_{k,\beta}(\xi,.)\overline{u}\rangle_{L^2_{k,\beta}}d\mu_{k,\beta}(\xi). \end{aligned}$$

Thus thee proof is complete. □

Involving Theorem 7, relation (73) and by interpolation argument (See [9, Theorem 2.10 and Theorem 2.11]), we deduce the following result.

Corollary 5 Let σ be in $L^p_{k,\beta}(\mathbb{R}^{d+1}_+)$, $1\leqslant p\leqslant\infty$. Then, the Dunkl-Bessel two-wavelet multiplier $\mathscr{P}_{u,v}(\sigma)\colon L^2_{k,\beta}(\mathbb{R}^{d+1}_+)\longrightarrow L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ is in S_p and we have

$$\|\mathscr{P}_{u,v}(\sigma)\|_{S_p}\leqslant(\|u\|_{L^\infty_{k,\beta}}\|v\|_{L^\infty_{k,\beta}})^{\frac{p-1}{p}}\|\sigma\|_{L^p_{k,\beta}}.$$

Remark 6 If $u=v$ and if σ is a real valued and nonnegative function in $L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$ then

$$\mathscr{P}_{u,v}(\sigma)\colon L^2_{k,\beta}(\mathbb{R}^{d+1}_+)\to L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$$

is a positive operator. Moreover, using (21) and relation (74), we obtain

$$||\mathscr{P}_{u,v}(\sigma)||_{S_1}=\int_{\mathbb{R}^{d+1}_+}\sigma(\xi)||\Lambda_{k,\beta}(\xi,.)u||^2_{L^2_{k,\beta}}d\mu_{k,\beta}(\xi). \tag{76}$$

The trace of products of Dunkl-Bessel two-wavelet multipliers is given in the following result.

Corollary 6 Let σ_1 and σ_2 be any real-valued and non-negative functions in $L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$. We assume that $u=v$ and u is a function in $L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ such that $||u||_{L^2_{k,\beta}}=1$. Then, the Dunkl-Bessel two-wavelet multipliers $\mathscr{P}_{u,v}(\sigma_1)$, $\mathscr{P}_{u,v}(\sigma_2)$ are positive trace class operators and

$$\left|\left|\Big(\mathscr{P}_{u,v}(\sigma_1)\,\mathscr{P}_{u,v}(\sigma_2)\Big)^n\right|\right|_{S_1}\quad\leq\quad\left|\left|\mathscr{P}_{u,v}(\sigma_1)\right|\right|^n_{S_1}\left|\left|\mathscr{P}_{u,v}(\sigma_2)\right|\right|^n_{S_1},$$

for all natural numbers n.

Proof. By Theorem 1 in the paper [45] by Liu we know that if A and B are in the trace class S_1 and are positive operators, then

$$\forall n \in \mathbb{N}, \quad tr(AB)^n \leq \Big(tr(A)\Big)^n \Big(tr(B)\Big)^n.$$

So, if we take $A = \mathscr{P}_{u,v}(\sigma_1)$, $B = \mathscr{P}_{u,v}(\sigma_2)$ and we invoke the previous remark, the proof is complete. □

4.3 L^p Boundedness of $P_{u,v}(\sigma)$

The aim of this subsection is to give a sufficient conditions on the symbols σ and the functions u and v, for which $\mathscr{P}_{u,v}(\sigma)$: $L^p_{k,\beta}(\mathbb{R}^{d+1}_+) \to L^p_{k,\beta}(\mathbb{R}^{d+1}_+)$, $1 \leq p \leq \infty$ be bounded.

Let us start with the following propositions.

Proposition 8 Let σ be in $L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$, $u \in L^\infty_{k,\beta}(\mathbb{R}^{d+1}_+)$ and $v \in L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$, then the Dunkl-Bessel two-wavelet multiplier $\mathscr{P}_{u,v}(\sigma)$: $L^1_{k,\beta}(\mathbb{R}^{d+1}_+) \longrightarrow L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$ is a bounded linear operator and we have

$$||\mathscr{P}_{u,v}(\sigma)||_{B(L^1_{k,\beta}(\mathbb{R}^{d+1}_+))} \leqslant \|u\|_{L^\infty_{k,\beta}} \|v\|_{L^1_{k,\beta}} \|\sigma\|_{L^1_{k,\beta}}.$$

Proof. For every function f in $L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$, we have

$$||\mathscr{P}_{u,v}(\sigma)(f)||_{L^1_{k,\beta}} \leqslant \int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+} |\sigma(\xi)||\mathscr{F}_{k,\beta}(uf)(\xi)||\Lambda_{k,\beta}(\xi, y)v(y)|d\mu_{k,\beta}(\xi)d\mu_{k,\beta}(y),$$

Involving the relations (8) and (5), we derive

$$||\mathscr{P}_{u,v}(\sigma)(f)||_{L^1_{k,\beta}} \leqslant \|f\|_{L^1_{k,\beta}} ||u||_{L^\infty_{k,\beta}} ||v||_{L^1_{k,\beta}} \|\sigma\|_{L^1_{k,\beta}},$$

then we obtain the desire result. □

Therefore we have the following result.

Proposition 9 Let σ be in $L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$ and let $u \in L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$, $v \in L^\infty_{k,\beta}(\mathbb{R}^{d+1}_+)$, then the Dunkl-Bessel two-wavelet multiplier

$$\mathscr{P}_{u,v}(\sigma): L^\infty_{k,\beta}(\mathbb{R}^{d+1}_+) \longrightarrow L^\infty_{k,\beta}(\mathbb{R}^{d+1}_+)$$

is a bounded linear operator such that

$$||\mathscr{P}_{u,v}(\sigma)||_{B(L^\infty_{k,\beta})} \leqslant \|u\|_{L^1_{k,\beta}} \|v\|_{L^\infty_{k,\beta}} \|\sigma\|_{L^1_{k,\beta}}.$$

Proof. Since the adjoint of $\mathscr{P}_{v,u}(\overline{\sigma})$: $L^1_{k,\beta}(\mathbb{R}^{d+1}_+) \to L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$ is

$$\mathscr{P}_{u,v}(\sigma)\colon L^{\infty}_{k,\beta}(\mathbb{R}^{d+1}_{+}) \to L^{\infty}_{k,\beta}(\mathbb{R}^{d+1}_{+}),$$

then by the Proposition 8 we obtain

$$||\mathscr{P}_{u,v}(\sigma)||_{B(L^{\infty}_{k,\beta})} = ||\mathscr{P}_{v,u}(\overline{\sigma})||_{B(L^{1}_{k,\beta})} \leqslant \|u\|_{L^{1}_{k,\beta}}\|v\|_{L^{\infty}_{k,\beta}}\|\sigma\|_{L^{1}_{k,\beta}}.$$

This completes the proof □

Using an interpolation of Propositions 8 and 9, we obtain the following result.

Theorem 10 Let u and v be functions in $L^{1}_{k,\beta}(\mathbb{R}^{d+1}_{+}) \cap L^{\infty}_{k,\beta}(\mathbb{R}^{d+1}_{+})$. Then for all σ in $L^{1}_{k,\beta}(\mathbb{R}^{d+1}_{+})$, there exists a unique bounded linear operator $\mathscr{P}_{u,v}(\sigma)\colon L^{p}_{k,\beta}(\mathbb{R}^{d+1}_{+}) \longrightarrow L^{p}_{k,\beta}(\mathbb{R}^{d+1}_{+})$, $1 \le p \le \infty$, such that

$$||\mathscr{P}_{u,v}(\sigma)||_{B(L^{p}_{k,\beta})} \leqslant \|u\|^{\frac{1}{p'}}_{L^{1}_{k,\beta}}\|v\|^{\frac{1}{p}}_{L^{1}_{k,\beta}}\|u\|^{\frac{1}{p}}_{L^{\infty}_{k,\beta}}\|v\|^{\frac{1}{p'}}_{L^{\infty}_{k,\beta}}\|\sigma\|_{L^{1}_{k,\beta}}.$$

We can give another version of the $L^{p}_{k,\beta}$-boundedness. Firstly we generalize and we improve Proposition 9.

Proposition 10 Let σ be in $L^{1}_{k,\beta}(\mathbb{R}^{d+1}_{+})$, $v \in L^{p}_{k,\beta}(\mathbb{R}^{d+1}_{+})$ and $u \in L^{p'}_{k,\beta}(\mathbb{R}^{d+1}_{+})$, for $1 < p \le \infty$, then the Dunkl-Bessel two-wavelet multiplier $\mathscr{P}_{u,v}(\sigma)\colon L^{p}_{k,\beta}(\mathbb{R}^{d+1}_{+}) \longrightarrow L^{p}_{k,\beta}(\mathbb{R}^{d+1}_{+})$ is a bounded linear operator, and we have

$$||\mathscr{P}_{u,v}(\sigma)||_{B(L^{p}_{k,\beta})} \leqslant \|u\|_{L^{p'}_{k,\beta}}\|v\|_{L^{p}_{k,\beta}}\|\sigma\|_{L^{1}_{k,\beta}}.$$

Proof. For any $f \in L^{p}_{k,\beta}(\mathbb{R}^{d+1}_{+})$, consider the linear functional

$$\begin{aligned} \mathscr{I}_f\colon L^{p'}_{k,\beta}(\mathbb{R}^{d+1}_{+}) &\to \mathbb{C} \\ g &\mapsto \langle g, \mathscr{P}_{u,v}(\sigma)(f)\rangle_{L^{2}_{k,\beta}}. \end{aligned}$$

From the relation (65)

$$\begin{aligned} |\langle \mathscr{P}_{u,v}(\sigma)(f), g\rangle_{L^{2}_{k,\beta}}| &\leqslant \int_{\mathbb{R}^{d+1}_{+}} |\sigma(\xi)||\mathscr{F}_{k,\beta}(uf)(\xi)||\mathscr{F}_{k,\beta}(vg)(\xi)|d\mu_{k,\beta}(\xi) \\ &\leqslant \|\sigma\|_{L^{1}_{k,\beta}}\|\mathscr{F}_{k,\beta}(uf)\|_{L^{\infty}_{k,\beta}}\|\mathscr{F}_{k,\beta}(vg)\|_{L^{\infty}_{k,\beta}}. \end{aligned}$$

Using the relation (7), (5) and Hölder's inequality, we get

$$|\langle \mathscr{P}_{u,v}(\sigma)(f), g\rangle_{L^{2}_{k,\beta}}| \leqslant \|\sigma\|_{L^{1}_{k,\beta}}\|u\|_{L^{p'}_{k,\beta}}\|v\|_{L^{p}_{k,\beta}}\|f\|_{L^{p}_{k,\beta}}\|g\|_{L^{p'}_{k,\beta}}.$$

Thus, the operator $\mathscr{I}_f$ is a continuous linear functional on $L^{p'}_{k,\beta}(\mathbb{R}^{d+1}_+)$, and the operator norm

$$||\mathscr{I}_f||_{B(L^{p'}_{k,\beta})} \leqslant \|u\|_{L^{p'}_{k,\beta}} \|v\|_{L^p_{k,\beta}} \|f\|_{L^p_{k,\beta}} \|\sigma\|_{L^1_{k,\beta}}.$$

As $\mathscr{I}_f(g) = \langle g, \mathscr{P}_{u,v}(\sigma)(f)\rangle_{L^2_{k,\beta}}$, by the Riesz representation theorem, we have

$$||\mathscr{P}_{u,v}(\sigma)(f)||_{B(L^p_{k,\beta})} = ||\mathscr{I}_f||_{B(L^{p'}_{k,\beta})} \leqslant \|u\|_{L^{p'}_{k,\beta}} \|v\|_{L^p_{k,\beta}} \|f\|_{L^p_{k,\beta}} \|\sigma\|_{L^1_{k,\beta}},$$

which establishes the proposition. □

Combining Proposition 8 and Proposition 10, we have the following theorem.

Theorem 11 Let σ be in $L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$, $v \in L^p_{k,\beta}(\mathbb{R}^{d+1}_+)$ and $u \in L^{p'}_{k,\beta}(\mathbb{R}^{d+1}_+)$, for $1 \le p \le \infty$, then the Dunkl-Bessel two-wavelet multiplier $\mathscr{P}_{u,v}(\sigma)$: $L^p_{k,\beta}(\mathbb{R}^{d+1}_+) \longrightarrow L^p_{k,\beta}(\mathbb{R}^{d+1}_+)$ is a bounded linear operator, and we have

$$||\mathscr{P}_{u,v}(\sigma)||_{B(L^p_{k,\beta})} \leqslant \|u\|_{L^{p'}_{k,\beta}} \|v\|_{L^p_{k,\beta}} \|\sigma\|_{L^1_{k,\beta}}.$$

With a Schur technique, we can obtain an $L^p_{k,\beta}$-boundedness result as in the Theorem 10, but the estimate for the norm $||\mathscr{P}_{u,v}(\sigma)||_{B(L^p_{k,\beta})}$ is cruder.

Theorem 12 Let σ be in $L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$, u and v in $L^1_{k,\beta}(\mathbb{R}^{d+1}_+) \bigcap L^\infty_{k,\beta}(\mathbb{R}^{d+1}_+)$. Then there exists a unique bounded linear operator $\mathscr{P}_{u,v}(\sigma)$: $L^p_{k,\beta}(\mathbb{R}^{d+1}_+) \longrightarrow L^p_{k,\beta}(\mathbb{R}^{d+1}_+)$, $1 \le p \le \infty$ such that

$$||\mathscr{P}_{u,v}(\sigma)||_{B(L^p_{k,\beta})} \leqslant \max(\|u\|_{L^1_{k,\beta}} \|v\|_{L^\infty_{k,\beta}}, \|u\|_{L^\infty_{k,\beta}} \|v\|_{L^1_{k,\beta}}) \|\sigma\|_{L^1_{k,\beta}}.$$

Proof. Let $\mathscr{N}$ be the function defined on $\mathbb{R}^{d+1}_+ \times \mathbb{R}^{d+1}_+$ by

$$\mathscr{N}(y, z) = \int_{\mathbb{R}^{d+1}_+} \sigma(\xi)\Lambda_{k,\beta}(\xi, y)\overline{v(y)}\Lambda_{k,\beta}(-\xi, z)u(z)d\mu_{k,\beta}(\xi). \tag{77}$$

We have

$$\mathscr{P}_{u,v}(\sigma)(f)(y) = \int_{\mathbb{R}^{d+1}_+} \mathscr{N}(y, z)f(z)d\mu_{k,\beta}(z).$$

By simple calculations, it is easy to see that

$$\int_{\mathbb{R}^{d+1}_+} |\mathscr{N}(y, z)|d\mu_{k,\beta}(y) \le \|u\|_{L^\infty_{k,\beta}} \|v\|_{L^1_{k,\beta}} \|\sigma\|_{L^1_{k,\beta}}, \ z \in \mathbb{R}^{d+1}_+,$$

and

$$\int_{\mathbb{R}_+^{d+1}} |\mathscr{N}(y,z)| d\mu_{k,\beta}(z) \leq \|u\|_{L^1_{k,\beta}} \|v\|_{L^\infty_{k,\beta}} \|\sigma\|_{L^1_{k,\beta}}, \ y \in \mathbb{R}_+^{d+1}.$$

Thus by Schur Lemma (cf. [46]), we can conclude that

$$\mathscr{P}_{u,v}(\sigma) \colon L^p_{k,\beta}(\mathbb{R}_+^{d+1}) \longrightarrow L^p_{k,\beta}(\mathbb{R}_+^{d+1})$$

is a bounded linear operator for $1 \leq p \leq \infty$, and we have

$$||\mathscr{P}_{u,v}(\sigma)||_{B(L^p_{k,\beta})} \leqslant \max(\|u\|_{L^1_{k,\beta}} \|v\|_{L^\infty_{k,\beta}}, \|u\|_{L^\infty_{k,\beta}} \|v\|_{L^1_{k,\beta}}) \|\sigma\|_{L^1_{k,\beta}}.$$

□

Remark 7 The previous Theorem tells us that the unique bounded linear operator on $L^p_{k,\beta}(\mathbb{R}_+^{d+1})$, $1 \leq p \leq \infty$, obtained by interpolation in Theorem 10 is in fact the integral operator on $L^p_{k,\beta}(\mathbb{R}_+^{d+1})$ with kernel $\mathscr{N}$ given by (77).

We can now state and prove the main result in this subsection.

Theorem 13 Let σ be in $L^r_{k,\beta}(\mathbb{R}_+^{d+1})$, $r \in [1, 2]$, and $u, v \in L^1_{k,\beta}(\mathbb{R}_+^{d+1}) \bigcap L^\infty_{k,\beta}(\mathbb{R}_+^{d+1})$. Then there exists a unique bounded linear operator $\mathscr{P}_{u,v}(\sigma) \colon L^p_{k,\beta}(\mathbb{R}_+^{d+1}) \longrightarrow L^p_{k,\beta}(\mathbb{R}_+^{d+1})$ for all $p \in [r, r']$, and we have

$$||\mathscr{P}_{u,v}(\sigma)||_{B(L^p_{k,\beta})} \leqslant C_1^t C_2^{1-t} \|\sigma\|_{L^p_{k,\beta}} \|u\|_{L^{p'}_{k,\beta}} \|v\|_{L^p_{k,\beta}}, \tag{78}$$

where

$$C_1 = \left(\|u\|_{L^\infty_{k,\beta}} \|v\|_{L^1_{k,\beta}}\right)^{\frac{2}{r}-1} \left(\|u\|_{L^\infty_{k,\beta}} \|v\|_{L^\infty_{k,\beta}}\right)^{\frac{1}{r'}},$$

$$C_2 = \left(\|u\|_{L^1_{k,\beta}} \|v\|_{L^\infty_{k,\beta}}\right)^{\frac{2}{r}-1} \left(\|u\|_{L^\infty_{k,\beta}} \|v\|_{L^\infty_{k,\beta}}\right)^{\frac{1}{r'}},$$

and

$$\frac{t}{r} + \frac{1-t}{r'} = \frac{1}{p}.$$

Proof. Consider the linear functional

$$\begin{aligned} \mathscr{I} \colon \left(L^1_{k,\beta}(\mathbb{R}_+^{d+1}) \cap L^2_{k,\beta}(\mathbb{R}_+^{d+1})\right) \times \left(L^1_{k,\beta}(\mathbb{R}_+^{d+1}) \cap L^2_{k,\beta}(\mathbb{R}_+^{d+1})\right) &\to L^1_{k,\beta}(\mathbb{R}_+^{d+1}) \cap L^2_{k,\beta}(\mathbb{R}_+^{d+1}) \\ (\sigma, f) &\mapsto \mathscr{P}_{u,v}(\sigma)(f). \end{aligned}$$

Then by Proposition 8 and Theorem 7

$$||\mathscr{I}(\sigma, f)||_{L^1_{k,\beta}} \leqslant \|u\|_{L^\infty_{k,\beta}}\|v\|_{L^1_{k,\beta}}||f||_{L^1_{k,\beta}}\|\sigma\|_{L^1_{k,\beta}} \tag{79}$$

and

$$||\mathscr{I}(\sigma, f)||_{L^2_{k,\beta}} \leq \sqrt{\|u\|_{L^\infty_{k,\beta}}\|v\|_{L^\infty_{k,\beta}}}||f||_{L^2_{k,\beta}}\|\sigma\|_{L^2_{k,\beta}}. \tag{80}$$

Therefore, by (79), (80) and the the multi-linear interpolation theory, see Section 10.1 in [47] for reference, we get a unique bounded linear operator

$$\mathscr{I}(\sigma, f)\colon L^r_{k,\beta}(\mathbb{R}^{d+1}_+) \times L^r_{k,\beta}(\mathbb{R}^{d+1}_+) \to L^r_{k,\beta}(\mathbb{R}^{d+1}_+)$$

such that

$$||\mathscr{I}(\sigma, f)||_{L^r_{k,\beta}} \leq C_1||f||_{L^r_{k,\beta}}\|\sigma\|_{L^r_{k,\beta}}, \tag{81}$$

where

$$C_1 = \left(\|u\|_{L^\infty_{k,\beta}}\|v\|_{L^1_{k,\beta}}\right)^\theta \left(\|u\|_{L^\infty_{k,\beta}}\|v\|_{L^\infty_{k,\beta}}\right)^{\frac{1-\theta}{2}}$$

and

$$\frac{\theta}{1} + \frac{1-\theta}{2} = \frac{1}{r}.$$

By the definition of $\mathscr{I}$, we have

$$||\mathscr{P}_{u,v}(\sigma)||_{B(L^r_{k,\beta})} \leqslant C_1\|\sigma\|_{L^r_{k,\beta}}.$$

As the adjoint of $\mathscr{P}_{u,v}(\sigma)$ is $\mathscr{P}_{v,u}(\overline{\sigma})$, so $\mathscr{P}_{u,v}(\sigma)$ is a bounded linear map on $L^{r'}_{k,\beta}(\mathbb{R}^{d+1}_+)$ with its operator norm

$$||\mathscr{P}_{u,v}(\sigma)||_{B(L^{r'}_{k,\beta})} = ||\mathscr{P}_{v,u}(\overline{\sigma})||_{B(L^r_{k,\beta})} \leq C_2\|\sigma\|_{L^r_{k,\beta}}, \tag{82}$$

where

$$C_2 = \left(\|u\|_{L^1_{k,\beta}}\|v\|_{L^\infty_{k,\beta}}\right)^\theta \left(\|u\|_{L^\infty_{k,\beta}}\|v\|_{L^\infty_{k,\beta}}\right)^{\frac{1-\theta}{2}}.$$

Using an interpolation of (81) and (82), we have that, for any $p \in [r, r']$,

$$||\mathscr{P}_{u,v}(\sigma)||_{B(L^p_{k,\beta})} \leqslant C_1^t C_2^{1-t} \|\sigma\|_{L^p_{k,\beta}} \|u\|_{L^{p'}_{k,\beta}} \|v\|_{L^p_{k,\beta}},$$

with

$$\frac{t}{r} + \frac{1-t}{r'} = \frac{1}{p}.$$

□

4.4 *Compactness of $\mathscr{P}_{u,v}(\sigma)$*

In this section we will give sufficient conditions on σ, u, v so that the bounded operator Dunkl-Bessel two-wavelet multiplier $\mathscr{P}_{u,v}(\sigma)$: $L^p_{k,\beta}(\mathbb{R}^{d+1}_+) \longrightarrow L^p_{k,\beta}(\mathbb{R}^{d+1}_+)$ is compact. Our first result is the following proposition.

Proposition 11 Under the same hypothesis of Theorem 10, the Dunkl-Bessel two-wavelet multiplier $\mathscr{P}_{u,v}(\sigma)$: $L^1_{k,\beta}(\mathbb{R}^{d+1}_+) \longrightarrow L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$ is compact.

Proof. Let $(f_n)_{n\in\mathbb{N}} \in L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$ such that $f_n \rightharpoonup 0$ weakly in $L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$ as $n \to \infty$. It is enough to prove that

$$\lim_{n\to\infty} ||\mathscr{P}_{u,v}(\sigma)(f_n)||_{L^1_{k,\beta}} = 0.$$

We have

$$||\mathscr{P}_{u,v}(\sigma)(f_n)||_{L^1_{k,\beta}} \leqslant \int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+} |\sigma(\xi)||\langle f_n, \Lambda_{k,\beta}(\xi, .)u\rangle_{L^2_{k,\beta}}||\Lambda_{k,\beta}(\xi, y)v(y)|d\mu_{k,\beta}(\xi)d\mu_{k,\beta}(y). \tag{83}$$

Now as $f_n \rightharpoonup 0$ weakly in $L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$ as $n \to \infty$, then on the one hand

$$\forall \xi, y \in \mathbb{R}^{d+1}_+, \quad \lim_{n\to\infty} |\sigma(\xi)||\langle f_n, \Lambda_{k,\beta}(\xi, .)u\rangle_{L^2_k}||\Lambda_{k,\beta}(\xi, y)v(y)| = 0. \tag{84}$$

Moreover, as $f_n \rightharpoonup 0$ weakly in $L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$ as $n \to \infty$, then there exists a positive constant C such that $||f_n||_{L^1_{k,\beta}} \le C$. Hence by simple calculations we get

$$\forall \xi, y \in \mathbb{R}^{d+1}_+, \quad |\sigma(\xi)||\langle f_n, \Lambda_{k,\beta}(\xi, .)u\rangle_{L^2_{k,\beta}}||\Lambda_{k,\beta}(\xi, y)v(y)| \le C|\sigma(\xi)|\,||u||_{L^\infty_{k,\beta}}|v(y)|. \tag{85}$$

Therefore, by Fubini's theorem and relation (5), we have

$$\int_{\mathbb{R}_+^{d+1}}\int_{\mathbb{R}_+^{d+1}}|\sigma(\xi)||\langle f_n, \Lambda_{k,\beta}(\xi, .)u\rangle_{L^2_{k,\beta}}||\Lambda_{k,\beta}(\xi, y)v(y)|d\mu_{k,\beta}(\xi)d\mu_{k,\beta}(y)$$

$$\leq C||u||_{L^\infty_{k,\beta}}\int_{\mathbb{R}_+^{d+1}}|\sigma(\xi)|\int_{\mathbb{R}_+^{d+1}}|v(y)|d\mu_{k,\beta}(y)d\mu_{k,\beta}(\xi)$$

$$\leq C||u||_{L^\infty_{k,\beta}}||v||_{L^1_{k,\beta}}\|\sigma\|_{L^1_{k,\beta}} < \infty. \tag{86}$$

Thus from the Lebesgue dominated convergence's theorem and the relations (83), (84), (85), (86) we deduce that

$$\lim_{n\to\infty}||\mathscr{P}_{u,v}(\sigma)(f_n)||_{L^1_{k,\beta}} = 0$$

and the proof is complete. □

Consequently we have the following three results for compactness of the Dunkl-Bessel two-wavelet multiplier operators.

Theorem 14 Under the hypothesis of Theorem 10, the bounded linear operator

$$\mathscr{P}_{u,v}(\sigma)\colon L^p_{k,\beta}(\mathbb{R}_+^{d+1}) \longrightarrow L^p_{k,\beta}(\mathbb{R}_+^{d+1})$$

is compact for $1 \leq p \leq \infty$.

Proof. From the previous proposition, we only need to show that the conclusion holds for $p = \infty$. In fact, the operator $\mathscr{P}_{u,v}(\sigma)\colon L^\infty_{k,\beta}(\mathbb{R}_+^{d+1}) \longrightarrow L^\infty_{k,\beta}(\mathbb{R}_+^{d+1})$ is the adjoint of the operator

$$\mathscr{P}_{v,u}(\overline{\sigma})\colon L^1_{k,\beta}(\mathbb{R}_+^{d+1}) \longrightarrow L^1_{k,\beta}(\mathbb{R}_+^{d+1}),$$

which is compact by the previous Proposition. Thus by the duality property,

$$\mathscr{P}_{u,v}(\sigma)\colon L^\infty_{k,\beta}(\mathbb{R}_+^{d+1}) \longrightarrow L^\infty_{k,\beta}(\mathbb{R}_+^{d+1})$$

is compact. Finally, by an interpolation of the compactness on $L^1_{k,\beta}(\mathbb{R}_+^{d+1})$ and on $L^\infty_{k,\beta}(\mathbb{R}_+^{d+1})$ such as the one given on pages 202 and 203 of the book [48] by Bennett and Sharpley, the proof is complete. □

The following result is an analogue of Theorem 13 for compact operators.

Theorem 15 Under the hypotheses of Theorem 13, the bounded linear operator

$$\mathscr{P}_{u,v}(\sigma)\colon L^p_{k,\beta}(\mathbb{R}_+^{d+1}) \longrightarrow L^p_{k,\beta}(\mathbb{R}_+^{d+1})$$

is compact for all $p \in [r, r']$.

Proof. The result is an immediate consequence of an interpolation of Corollary 5 and Proposition 11. See again pages 202 and 203 of the book [48] by Bennett and Sharpley for the interpolation used. □

Using similar ideas as above we can prove the following.

Theorem 16 Under the hypothesis of Theorem 11, the bounded linear operator

$$\mathscr{P}_{u,v}(\sigma)\colon L^p_{k,\beta}(\mathbb{R}^{d+1}_+) \longrightarrow L^p_{k,\beta}(\mathbb{R}^{d+1}_+)$$

is compact for $1 \leq p \leq \infty$.

5. The generalized Landau-Pollak-Slepian Operator

Let $U \subset \mathbb{R}^{d+1}$, be a measurable subset. As above, we define $\mu_{k,\beta}(U)$ by

$$\mu_{k,\beta}(U) \colon = \int_U d\mu_{k,\beta}(t).$$

5.1 *Traces formula*

Let R and R_1 and R_2 be positive numbers. We define the linear operators

$$Q_R\colon L^2_{k,\beta}(\mathbb{R}^{d+1}_+) \longrightarrow L^2_{k,\beta}(\mathbb{R}^{d+1}_+),$$

$$P_{R_1}\colon L^2_{k,\beta}(\mathbb{R}^{d+1}_+) \longrightarrow L^2_{k,\beta}(\mathbb{R}^{d+1}_+),$$

$$P_{R_2}\colon L^2_{k,\beta}(\mathbb{R}^{d+1}_+) \longrightarrow L^2_{k,\beta}(\mathbb{R}^{d+1}_+),$$

as

$$Q_R f \colon = E_{B(0,R)} = \chi_{B(0,R)} f, \;\; P_{R_i} f \colon = M_{B(0,R_i)} = (\mathscr{F}_{k,\beta})^{-1}(\chi_{B(0,R_i)} \mathscr{F}_{k,\beta}(f)), \;\; i = 1, 2.$$

We adapt the proof of Proposition 20.1 in the book [9] by Wong, we prove the following.

Proposition 12 The linear operators $Q_R\colon L^2_{k,\beta}(\mathbb{R}^{d+1}_+) \longrightarrow L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$, $P_{R_1}\colon L^2_{k,\beta}(\mathbb{R}^{d+1}_+) \longrightarrow L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ and $P_{R_2}\colon L^2_{k,\beta}(\mathbb{R}^{d+1}_+) \longrightarrow L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$, are self-adjoint projections.

The bounded linear operator $P_{R_2} Q_R P_{R_1}\colon L^2_{k,\beta}(\mathbb{R}^{d+1}_+) \longrightarrow L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$, it is called the generalized Landau-Pollak-Slepian operator. We can show that this operator is a Dunkl-Bessel two-wavelet multiplier.

Theorem 17 Let u and v be the functions on $\mathbb{R}^{d+1}_+$ defined by

$$u = \frac{1}{\sqrt{\mu_{k,\beta}(B(0,R_1))}} \chi_{B(0,R_1)}, \quad v = \frac{1}{\sqrt{\mu_{k,\beta}(B(0,R_2))}} \chi_{B(0,R_2)}.$$

Then the generalized Landau-Pollak-Slepian operator $P_{R_2}Q_RP_{R_1}\colon L^2_{k,\beta}(\mathbb{R}^{d+1}_+)\longrightarrow L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ is unitary equivalent to a scalar multiple of the Dunkl-Bessel two-wavelet multiplier

$$\mathscr{P}_{u,v}(\chi_{B(0,R)})\colon L^2_{k,\beta}(\mathbb{R}^{d+1}_+)\longrightarrow L^2_{k,\beta}(\mathbb{R}^{d+1}_+).$$

In fact

$$P_{R_2}Q_RP_{R_1}=C_{k,\beta}(R_1,R_2)(\mathscr{F}_{k,\beta})^{-1}(\mathscr{P}_{u,v}(\chi_{B(0,R)}))\mathscr{F}_{k,\beta}, \tag{87}$$

where

$$C_{k,\beta}(R_1,R_2):=\sqrt{\mu_{k,\beta}(B(0,R_1)\mu_{k,\beta}(B(0,R_2))}.$$

Proof. It is easy to see that u and v belong to $L^2_{k,\beta}(\mathbb{R}^{d+1}_+)\cap L^\infty_{k,\beta}(\mathbb{R}^{d+1}_+)$ and

$$||u||_{L^2_{k,\beta}}=||v||_{L^2_{k,\beta}}=1.$$

On the other hand we have

$$\langle\mathscr{P}_{u,v}(\chi_{B(0,R)})(f),g\rangle_{L^2_{k,\beta}}=\int_{\mathbb{R}^{d+1}_+}M_{\chi_{B(0,R)}}(uf)(\xi)\overline{(vg)(\xi)}d\mu_{k,\beta}(\xi).$$

By simple calculations we find

$$\begin{aligned}
\langle\mathscr{P}_{u,v}(\chi_{B(0,R)})(f),g\rangle_{L^2_{k,\beta}(\mathbb{R}^{d+1}_+)} &= \frac{1}{C_{k,\beta}(R_1,R_2)}\int_{\mathbb{R}^{d+1}_+}\chi_{B(0,R)}(\xi)P_{R_1}(\mathscr{F}^{-1}_{k,\beta}(f))(\xi)\overline{P_{R_2}(\mathscr{F}^{-1}_{k,\beta}(g))(\xi)}d\mu_{k,\beta}(\xi)\\
&= \frac{1}{C_{k,\beta}(R_1,R_2)}\int_{B(0,R)}P_{R_1}(\mathscr{F}^{-1}_{k,\beta}(f))(\xi)\overline{P_{R_2}(\mathscr{F}^{-1}_{k,\beta}(g))(\xi)}d\mu_{k,\beta}(\xi)\\
&= \frac{1}{C_{k,\beta}(R_1,R_2)}\int_{\mathbb{R}^{d+1}_+}Q_RP_{R_1}(\mathscr{F}^{-1}_{k,\beta}(f))(\xi)\overline{P_{R_2}(\mathscr{F}^{-1}_{k,\beta}(g))(\xi)}d\mu_{k,\beta}(\xi)\\
&= \frac{1}{C_{k,\beta}(R_1,R_2)}\langle Q_RP_{R_1}(\mathscr{F}^{-1}_{k,\beta}(f)),P_{R_2}(\mathscr{F}^{-1}_{k,\beta}(g)\rangle_{L^2_{k,\beta}}\\
&= \frac{1}{C_{k,\beta}(R_1,R_2)}\langle P_{R_2}Q_RP_{R_1}(\mathscr{F}^{-1}_{k,\beta}(f)),(\mathscr{F}^{-1}_{k,\beta}(g)\rangle_{L^2_{k,\beta}}
\end{aligned}$$

$$= \frac{1}{C_{k,\beta}(R_1, R_2)} \langle \mathscr{F}_{k,\beta} P_{R_2} Q_R P_{R_1}(\mathscr{F}_{k,\beta}^{-1}(f)), g \rangle_{L^2_{k,\beta}}$$

for all f, g in $\mathscr{S}_*(\mathbb{R}^{d+1})$ and hence the proof is complete. □

The next result gives a formula for the trace of the generalized Landau-Pollak-Slepian operator $P_{R_2} Q_R P_{R_1} : L^2_{k,\beta}(\mathbb{R}^{d+1}_+) \longrightarrow L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$.

Corollary 7 We have

$$tr(P_{R_2} Q_R P_{R_1}) = C_{k,\beta}(R_1, R_2) \int_{B(0,R)} \int_{B(0,\min(R_1,R_2))} |\Lambda_{k,\beta}(\xi, y)|^2 d\mu_{k,\beta}(y) d\mu_{k,\beta}(\xi).$$

Proof. The result is an immediate consequence of Theorem 17 and relation (74). □

Remark 8 (i) The analogues of the previous results were studied for the classical wavelet multipliers by Catană (cf. [49]).

(ii) Let $S, \Sigma_1, \Sigma_2 \subset \mathbb{R}^{d+1}_+$ be a measurable subsets with $0 < \mu_{k,\beta}(\Sigma_i), \mu_{k,\beta}(S) < \infty$, $i = 1, 2$. Using similar ideas used in Theorem 17, we prove that

$$M_{\Sigma_2} E_S M_{\Sigma_1} = C_{k,\beta}(\Sigma_1, \Sigma_2)(\mathscr{F}_{k,\beta})^{-1}(\mathscr{P}_{u,v}(\chi_S))\mathscr{F}_{k,\beta}, \tag{88}$$

where

$$E_S h = \chi_S h, \; M_{\Sigma_i} h = (\mathscr{F}_{k,\beta})^{-1}(\chi_{\Sigma_i} \mathscr{F}_{k,\beta}(h)), \; i = 1, 2,$$

$$u = \frac{1}{\sqrt{\mu_{k,\beta}(\Sigma_1)}} \chi_{\Sigma_1}, \quad v = \frac{1}{\sqrt{\mu_{k,\beta}(\Sigma_2)}} \chi_{\Sigma_2}$$

and

$$C_{k,\beta}(\Sigma_1, \Sigma_2) := \sqrt{\mu_{k,\beta}(\Sigma_1)\mu_{k,\beta}(\Sigma_2)}.$$

5.2 *Donoho-Stark type uncertainty principle*

In this subsection we will assume that u and v satisfy $\|u\|_{L^\infty_{k,\beta}} \|v\|_{L^\infty_{k,\beta}} = 1$.

Now let $\sigma_1 = \chi_S$ and $\sigma_2 = \chi_\Sigma$ and let $L_1 = \mathscr{P}_{u,v}(\sigma_1)$ and $L_2 = \mathscr{P}_{u,v}(\sigma_2)$.

The main of this subsection is to prove the following Donoho-Stark type uncertainty principle.

Theorem 18 Let $\varepsilon_1, \varepsilon_2 \in (0, 1)$ such that $\varepsilon_1 + \varepsilon_2 < 1$. If $f \in L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ is ε_1-localized with respect to L_1 and ε_2-localized with respect to L_2 then,

$$\mu_{k,\beta}(S)\mu_{k,\beta}(\Sigma) \geq (1 - \varepsilon_1 - \varepsilon_2). \tag{89}$$

Proof. From Proposition 6,

$$\begin{aligned}\|f-L_2L_1f\|_{L^2_{k,\beta}} &\leq \|f-L_2f\|_{L^2_{k,\beta}}+\|L_2f-L_2L_1f\|_{L^2_{k,\beta}}\\ &\leq \|L_2f-f\|_{L^2_{k,\beta}}+\|L_2\|_{S_\infty}\|L_1f-f\|_{L^2_{k,\beta}}\\ &\leq (\varepsilon_2+\varepsilon_1)\|f\|_{L^2_{k,\beta}}.\end{aligned}$$

Therefore

$$\begin{aligned}\|L_2L_1f\|_{L^2_{k,\beta}} &\geq \|f\|_{L^2_{k,\beta}}-\|f-L_2L_1f\|_{L^2_{k,\beta}}\\ &\geq (1-\varepsilon_1-\varepsilon_2)\|f\|_{L^2_{k,\beta}}.\end{aligned}$$

Thus from Proposition 5 it follows that

$$\begin{aligned}1-\varepsilon_1-\varepsilon_2 &\leq \|L_2L_1\|_{S_\infty}\\ &\leq \|L_1\|_{S_\infty}\|L_2\|_{S_\infty}\\ &\leq \mu_{k,\beta}(S)\mu_{k,\beta}(\Sigma).\end{aligned}$$

This proves the desired result. □

We proceed as above theorem we obtain the following result.

Corollary 8 If $f\in L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ is an eigenfunction of L_1 and L_2 corresponding to the same eigenvalue 1, then

$$\mu_{k,\beta}(S)\mu_{k,\beta}(\Sigma)\geq 1. \tag{90}$$

Proof. Notice that, when $\varepsilon_1=\varepsilon_2=0$ we have in this case $S=f$, $\Sigma=\mathscr{F}_{k,\beta}(f)$ and we proceed as above theorem we obtain the result. □

Remark 9 (1) As a first result, we can remark that the essential supports S and Σ cannot be too small.

(2) The result involves the couple (L_2f, L_1f) and the rectangle $\Sigma\times S$ analogously to the Donoho-Stark UP which involves the couple $(f, \mathscr{F}_{k,\beta}(f))$ and the same rectangle.

(3) The estimate

$$\mu_{k,\beta}(S)\mu_{k,\beta}\Sigma)\geq 1-\varepsilon_1-\varepsilon_2$$

is stronger then the classical Donoho-Stark estimate

$$\mu_{k,\beta}(S)\mu_{k,\beta}(\Sigma) \geq (1-\varepsilon_1-\varepsilon_2)^2.$$

5.3 Approximation inequalities

In this subsection, we prove that the Dunkl-Bessel wavelet multiplier is unitary equivalent to a scalar multiple of the phase space restriction operator $L_{S,\Sigma} = E_S M_\Sigma E_S$ on $L^2_{k,\beta}\mathbb{R}^{d+1}_+$ arising from the Landau theory in signal analysis ([4]). For this we define the phase space restriction operator by

$$L_{S,\Sigma} = E_S M_\Sigma E_S = (M_\Sigma E_S)^* M_\Sigma E_S.$$

And in the case when $\sigma = \chi_A$ is the characteristic function of the subset $A \subset \mathbb{R}^{d+1}_+$, then we write $\mathscr{P}_{u,v}(\sigma)$ as $\mathscr{P}_{u,v}(A)$ if $u \neq v$ and $\mathscr{P}_u(A)$ if $u = v$.

The operator $M_\Sigma E_S$ is Hilbert-Schmidt, and since the pair (S, Σ) is strongly annihilating, then we have

$$\|L_{S,\Sigma}\|_{S_\infty} = \|E_S M_\Sigma\|^2_{S_\infty} = \|M_\Sigma E_S\|^2_{S_\infty} < 1. \tag{91}$$

Moreover, the operator $L_{S,\Sigma}$ is self-adjoint, positive and from (22) it is compact and even trace class with

$$\|L_{S,\Sigma}\|_{S_1} = \|M_\Sigma E_S\|^2_{S_2}, \tag{92}$$

The compact operator $L_{S,\Sigma}$: $L^2_{k,\beta}(\mathbb{R}^{d+1}_+) \to L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ is self-adjoint and then can be diagonalized as

$$L_{S,\Sigma} f = \sum_{n=1}^{\infty} \lambda_n \langle f, \varphi_n\rangle_{L^2_{k,\beta}} \varphi_n, \tag{93}$$

where $\{\lambda_n = \lambda_n(S, \Sigma)\}_{n=1}^{\infty}$ are the positive eigenvalues arranged in a non-increasing manner

$$\lambda_n \leq \cdots \leq \lambda_1 < 1, \tag{94}$$

and $\{\varphi_n = \varphi_n(S, \Sigma)\}_{n=1}^{\infty}$ is the corresponding orthonormal set of eigenfunctions. In particular

$$\|L_{S,\Sigma}\|_{S_\infty} = \lambda_1, \tag{95}$$

where λ_1 is the first eigenvalue corresponding to the first eigenfunction φ_1 of the compact operator $L_{S,\Sigma}$. This eigenfunction realizes the maximum of concentration on the set $S \times \Sigma$. On the other hand, since φ_n is an eigenfunction of $L_{S,\Sigma}$ with eigenvalue λ_n, then

$$\|L_{S,\Sigma}\varphi_n-\varphi_n\|_{L^2_{k,\beta}}=\langle\varphi_n-L_{S,\Sigma}\varphi_n,\,\varphi_n\rangle_{L^2_{k,\beta}}=1-\lambda_n, \tag{96}$$

and

$$\begin{aligned}\|L_{S,\Sigma}(L_{S,\Sigma}\varphi_n)-L_{S,\Sigma}\varphi_n\|_{L^2_{k,\beta}} &= \lambda_n^{-1}\langle L_{S,\Sigma}\varphi_n-L_{S,\Sigma}(L_{S,\Sigma}\varphi_n),\,L_{S,\Sigma}\varphi_n\rangle_{L^2_{k,\beta}}\\ &= \lambda_n(1-\lambda_n)=(1-\lambda_n)\|L_{S,\Sigma}\varphi_n\|_{L^2_{k,\beta}}.\end{aligned} \tag{97}$$

Thus, for all n, the functions φ_n and $L_{S,\Sigma}\varphi_n$ are $(1-\lambda_n)$-localized with respect to $L_{S,\Sigma}$. More generally, we have the following comparisons of the measures of localization.

Proposition 13 Let $\varepsilon,\,\varepsilon_1,\,\varepsilon_2\in(0,\,1)$.

1. If $f\in L^2_{k,\beta}(\varepsilon_1,\,\varepsilon_2,\,S,\,\Sigma)$, then f is $(\varepsilon_1+\varepsilon_2)$-localized with respect to $M_\Sigma E_S$ and $(2\varepsilon_1+\varepsilon_2)$-localized with respect to $L_{S,\Sigma}$.

2. If $f\in L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ is ε-localized with respect to $L_{S,\Sigma}$, then

$$\langle f-L_{S,\Sigma}f,\,f\rangle_{L^2_{k,\beta}}\leq(\varepsilon^2+\varepsilon)\|f\|^2_{L^2_{k,\beta}}. \tag{98}$$

3. If $f\in L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ satisfies

$$\langle f-L_{S,\Sigma}f,\,f\rangle_{L^2_{k,\beta}}\leq\varepsilon\|f\|^2_{L^2_{k,\beta}}, \tag{99}$$

then f is $\sqrt{\varepsilon}$-localized with respect to $L_{S,\Sigma}$.

4. If $f\in L^2_{k,\beta}(\varepsilon_1,\,\varepsilon_2,\,S,\,\Sigma)$, then

$$\langle f-L_{S,\Sigma}f,\,f\rangle_{L^2_{k,\beta}}<(2\varepsilon_1+\varepsilon_2)\|f\|^2_{L^2_{k,\beta}}. \tag{100}$$

Proof. Recall that $\|E_S\|_{S_\infty}=\|M_\Sigma\|_{S_\infty}=1$. First we have

$$\begin{aligned}\|M_\Sigma E_Sf-f\|_{L^2_{k,\beta}} &\leq \|M_\Sigma f-f\|_{L^2_{k,\beta}}+\|M_\Sigma E_Sf-M_\Sigma f\|_{L^2_{k,\beta}}\\ &\leq \|M_{\Sigma^c}f\|_{L^2_{k,\beta}}+\|M_\Sigma\|_{S_\infty}\|E_{S^c}f\|_{L^2_{k,\beta}}\\ &\leq (\varepsilon_1+\varepsilon_2)\|f\|_{L^2_{k,\beta}}.\end{aligned}$$

Moreover,

$$
\begin{aligned}
\|L_{S,\Sigma}f - f\|_{L^2_{k,\beta}} &\leq \|E_S M_\Sigma E_S f - E_S f\|_{L^2_{k,\beta}} + \|E_S f - f\|_{L^2_{k,\beta}} \\
&\leq \|E_S\|_{S_\infty} \|M_\Sigma E_S f - f\|_{L^2_{k,\beta}} + \|E_S f - f\|_{L^2_{k,\beta}} \\
&\leq (2\varepsilon_1 + \varepsilon_2)\|f\|_{L^2_{k,\beta}}.
\end{aligned}
$$

Now since

$$
\begin{aligned}
2\langle f - L_{S,\Sigma}f, f\rangle_{L^2_{k,\beta}} &= \|L_{S,\Sigma}f - f\|^2_{L^2_{k,\beta}} + \|f\|^2_{L^2_{k,\beta}} - \|L_{S,\Sigma}f\|^2_{L^2_{k,\beta}} \\
&\leq \|L_{S,\Sigma}f - f\|^2_{L^2_{k,\beta}} + \left(\|L_{S,\Sigma}f - f\|_{L^2_{k,\beta}} + \|L_{S,\Sigma}f\|_{L^2_{k,\beta}}\right)^2 - \|L_{S,\Sigma}f\|^2_{L^2_{k,\beta}} \\
&= 2\|L_{S,\Sigma}f - f\|^2_{L^2_{k,\beta}} + 2\|L_{S,\Sigma}f - f\|_{L^2_{k,\beta}} \|L_{S,\Sigma}f\|_{L^2_{k,\beta}},
\end{aligned}
$$

and since $\|L_{S,\Sigma}\|_{S_\infty} \leq 1$, then

$$
\langle f - L_{S,\Sigma}f, f\rangle_{L^2_{k,\beta}} \leq \|L_{S,\Sigma}f - f\|^2_{L^2_{k,\beta}} + \|L_{S,\Sigma}f - f\|_{L^2_{k,\beta}} \|f\|_{L^2_{k,\beta}} \leq (\varepsilon^2 + \varepsilon)\|f\|^2_{L^2_{k,\beta}}, \tag{101}
$$

and the second result follows.

On the other hand, since

$$
\left\langle (L_{S,\Sigma})^2 f, f \right\rangle_{L^2_{k,\beta}} \leq \langle L_{S,\Sigma}f, f\rangle_{L^2_{k,\beta}}, \tag{102}
$$

and since $L_{S,\Sigma}$ is self-adjoint, then

$$
\|L_{S,\Sigma}f - f\|^2_{L^2_{k,\beta}} = \left\langle (I - L_{S,\Sigma})^2 f, f \right\rangle_{L^2_{k,\beta}} \leq \langle (I - L_{S,\Sigma})f, f\rangle_{L^2_{k,\beta}} \leq \varepsilon \|f\|^2_{L^2_{k,\beta}}. \tag{103}
$$

Finally, since

$$
\langle f - L_{S,\Sigma}f, f\rangle_{L^2_{k,\beta}} = \langle E_{S^c} f, f\rangle_{L^2_{k,\beta}} + \langle E_S f, M_{\Sigma^c} f\rangle_{L^2_{k,\beta}} + \langle M_\Sigma E_S f, E_{S^c} f\rangle_{L^2_{k,\beta}},
$$

then we obtain the last result. □

The estimate (99) is equivalent to

$$\langle L_{S,\Sigma} f, f \rangle_{L^2_{k,\beta}} \geq (1-\varepsilon)\|f\|^2_{L^2_{k,\beta}}, \tag{104}$$

and we denote by $L^2_{k,\beta}(\varepsilon, S, \Sigma)$ the subspace of $L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ consisting of functions $f \in L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ satisfying (104). Hence from (96) and (97) we have,

$$\forall n \geq 1, \qquad \varphi_n,\ L_{S,\Sigma}\varphi_n \in L^2_{k,\beta}(1-\lambda_n, S, \Sigma). \tag{105}$$

Moreover from Proposition 13, if $f \in L^2_{k,\beta}(\varepsilon_1, \varepsilon_2, S, \Sigma)$, then $f \in L^2_{k,\beta}(2\varepsilon_1 + \varepsilon_2, S, \Sigma)$, and if f is ε-localized with respect to $L_{S,\Sigma}$, then $f \in L^2_{k,\beta}(2\varepsilon, S, \Sigma)$. Therefore we are interested to study the following optimization problem

$$\text{Maximize} \qquad \langle L_{S,\Sigma} f, f \rangle_{L^2_{k,\beta}}, \qquad \|f\|_{L^2_{k,\beta}} = 1, \tag{106}$$

which aims to look for orthonormal functions in $L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$, which are approximately time and band-limited to a bounded region like $S \times \Sigma$. It follows that the number of eigenfunctions of $L_{S,\Sigma}$ whose eigenvalues are very close to one, are an optimal solutions to the problem (106), since if φ_n is an eigenfunction of $L_{S,\Sigma}$ with eigenvalue $\lambda_n \geq (1-\varepsilon)$, we have from the spectral representation,

$$\langle L_{S,\Sigma}\varphi_n, \varphi_n \rangle_{L^2_{k,\beta}} = \lambda_n \geq (1-\varepsilon). \tag{107}$$

We denote by $n(\varepsilon, S, \Sigma)$ for the number of eigenvalues λ_n of $L_{S,\Sigma}$ which are close to one, in the sense that

$$\lambda_1 \geq \cdots \geq \lambda_{n(\varepsilon,S,\Sigma)} \geq 1-\varepsilon > \lambda_{1+n(\varepsilon,S,\Sigma)} \geq \cdots, \tag{108}$$

and we denote by $V_{n(\varepsilon,S,\Sigma)} = \operatorname{span}\{\varphi_n\}_{n=1}^{n(\varepsilon,S,\Sigma)}$ the span of the first eigenfunctions of $L_{S,\Sigma}$ corresponding to the largest eigenvalues $\{\lambda_n\}_{n=1}^{n(\varepsilon,S,\Sigma)}$. Therefore, by (107) and (105), each eigenfunction φ_n and its resulting function $L_{S,\Sigma}\varphi_n$ are in $L^2_{k,\beta}(\varepsilon, S, \Sigma)$, if and only if $1 \leq n \leq n(\varepsilon, S, \Sigma)$. Now, if $f \in V_{n(\varepsilon,S,\Sigma)}$, then

$$\sum_{n=1}^{n(\varepsilon,S,\Sigma)} \lambda_n \left| \langle f, \varphi_n \rangle_{L^2_{k,\beta}} \right|^2 \geq \lambda_{n(\varepsilon,S,\Sigma)} \sum_{n=1}^{n(\varepsilon,S,\Sigma)} \left| \langle f, \varphi_n \rangle_{L^2_{k,\beta}} \right|^2 \geq (1-\varepsilon)\|f\|^2_{L^2_{k,\beta}}.$$

Thus $V_{n(\varepsilon,S,\Sigma)}$ determines the subspace of $L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ with maximum dimension that is in $L^2_{k,\beta}(\varepsilon, S, \Sigma)$. Based on the paper [50], we obtain the following theorem that characterizes functions that are in $L^2_{k,\beta}(\varepsilon, S, \Sigma)$.

Theorem 19 Let $f_{\ker}$ denote the orthogonal projection of f onto the kernel $\operatorname{Ker}(L_{S,\Sigma})$ of $L_{S,\Sigma}$. Then a function f is in $L^2_{k,\beta}(\varepsilon, S, \Sigma)$ if and only if,

$$\sum_{n=1}^{n(\varepsilon, S, \Sigma)} (\lambda_n + \varepsilon - 1) \left| \langle f, \varphi_n \rangle_{L^2_{k, \beta}} \right|^2 \geq (1-\varepsilon) \| f_{\text{ker}} \|^2_{L^2_{k, \beta}} + \sum_{n=1+n(\varepsilon, S, \Sigma)}^{\infty} (1 - \varepsilon - \lambda_n) \left| \langle f, \varphi_n \rangle_{L^2_{k, \beta}} \right|^2.$$

Proof. The eigenfunctions $\{\varphi_n^{\Sigma}\}_{n=1}^{\infty}$ form an orthonormal subset in $L^2_{k, \beta}(\mathbb{R}^{d+1}_+)$, possibly incomplete if $\text{Ker}(L^{\psi}_{\Sigma}) \neq \{0\}$; hence, we can write

$$f = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle_{L^2_{k, \beta}} \varphi_n + f_{\text{ker}}, \tag{109}$$

where $f_{\text{ker}} \in \text{Ker}(L_{S, \Sigma})$. Then

$$\langle L_{S, \Sigma} f, f \rangle_{L^2_{k, \beta}} = \sum_{n=1}^{\infty} \lambda_n \left| \langle f, \varphi_n \rangle_{L^2_{k, \beta}} \right|^2. \tag{110}$$

So the function f is in $L^2_{k, \beta}(\varepsilon, S, \Sigma)$ if and only if

$$\sum_{n=1}^{\infty} \lambda_n \left| \langle f, \varphi_n \rangle_{L^2_{k, \beta}} \right|^2 \geq (1-\varepsilon) \left(\| f_{\text{ker}} \|^2_{L^2_{k, \beta}} + \sum_{n=1}^{\infty} \left| \langle f, \varphi_n \rangle_{L^2_{k, \beta}} \right|^2 \right), \tag{111}$$

and the conclusion follows. □

While a function f that is in $L^2_{k, \beta}(\varepsilon, S, \Sigma)$ does not necessarily lies in some subspace $V_N = \text{span}\{\varphi_n\}_{n=1}^{N}$, it can be approximated using a finite number of such eigenfunctions. Let $\varepsilon_0 \in (0, 1)$ be a fixed real number and let $\mathscr{P}$ the orthogonal projection onto the subspace $V_{n(\varepsilon_0, S, \Sigma)}$.

Theorem 20 Let f be a function in $L^2_{k, \beta}(\varepsilon, S, \Sigma)$. Then

$$\left\| f - \sum_{n=1}^{n(\varepsilon_0, S, \Sigma)} \langle f, \varphi_n \rangle_{\mu} \varphi_n \right\|_{L^2_{k, \beta}} \leq \sqrt{\frac{\varepsilon}{\varepsilon_0}} \, \| f \|_{L^2_{k, \beta}}. \tag{112}$$

Proof. By an easy adaptation of the proof of Proposition 3.3 in [50], we can conclude that

$$\| \mathscr{P} f \|^2_{L^2_{k, \beta}} \geq (1 - \varepsilon/\varepsilon_0) \| f \|^2_{L^2_{k, \beta}}. \tag{113}$$

It then follows,

$$\| f \|^2_{L^2_{k, \beta}} = \| \mathscr{P} f + (f - \mathscr{P} f) \|^2_{L^2_{k, \beta}} = \| \mathscr{P} f \|^2_{L^2_{k, \beta}} + \| f - \mathscr{P} f \|^2_{L^2_{k, \beta}}.$$

Thus

$$\|f-\mathscr{P}f\|^2_{L^2_{k,\beta}}=\|f\|^2_{L^2_{k,\beta}}-\|\mathscr{P}f\|^2_{L^2_{k,\beta}}\le\|f\|^2_{L^2_{k,\beta}}-(1-\varepsilon/\varepsilon_0)\|f\|^2_{L^2_{k,\beta}}=\varepsilon/\varepsilon_0\|f\|^2_{L^2_{k,\beta}}.$$

This completes the proof of the theorem. □

Consequently and from Proposition 13, we immediately deduce the following approximation results.

Corollary 9 Let $\varepsilon, \varepsilon_1, \varepsilon_2 \in (0, 1)$.

1. If $f \in L^2_{k,\beta}(\varepsilon_1, \varepsilon_2, S, \Sigma)$, then

$$\left\| f - \sum_{n=1}^{n(\varepsilon_0, S, \Sigma)} \langle f, \varphi_n\rangle_\mu \varphi_n \right\|_{L^2_{k,\beta}} \le \sqrt{\frac{2\varepsilon_1+\varepsilon_2}{\varepsilon_0}}\, \|f\|_{L^2_{k,\beta}}. \tag{114}$$

2. If $f \in L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ is ε-localized with respect to $L_{S,\Sigma}$, then

$$\left\| f - \sum_{n=1}^{n(\varepsilon_0, S, \Sigma)} \langle f, \varphi_n\rangle_{L^2_{k,\beta}} \varphi_n \right\|_{L^2_{k,\beta}} \le \sqrt{\frac{2\varepsilon}{\varepsilon_0}}\, \|f\|_{L^2_{k,\beta}}. \tag{115}$$

Adapting the method used in [7], we will show that the phase space restriction operator $L_{S,\Sigma}$ can be viewed as a Dunkl-Bessel wavelet multiplier, and then we will deduce a trace formula.

Theorem 21 Let $u = v$ be the function on $\mathbb{R}^{d+1}_+$ defined by $u = \frac{1}{\sqrt{\mu_{k,\beta}(S)}}\chi_S$ and let $\sigma = \chi_\Sigma$. Then

$$L_{S,\Sigma} = \mu_{k,\beta}(S)\,\mathscr{P}_u(\Sigma). \tag{116}$$

Proof. Clearly, the function u belongs to $L^2_{k,\beta}(\mathbb{R}^{d+1}_+)\cap L^\infty_{k,\beta}(\mathbb{R}^{d+1}_+)$, with $\|u\|_{L^2_{k,\beta}} = 1$. Then, since E_S is self-adjoint and by Parseval's equality (14), we have for all $f, g \in L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$,

$$\begin{aligned}
\langle L_{S,\Sigma}f, g\rangle_{L^2_{k,\beta}} &= \langle M_\Sigma E_S f, \chi_S g\rangle_{L^2_{k,\beta}} \\
&= \sqrt{\mu_{k,\beta}(S)}\langle M_\Sigma E_S f, \phi g\rangle_{L^2_{k,\beta}} \\
&= \sqrt{\mu_{k,\beta}(S)}\big\langle \mathscr{F}_{k,\beta} M_\Sigma E_S f, \mathscr{F}_{k,\beta}(ug)\big\rangle_{L^2_{k,\beta}} \\
&= \sqrt{\mu_{k,\beta}(S)}\big\langle \chi_\Sigma \mathscr{F}_{k,\beta}\chi_S f, \mathscr{F}_{k,\beta}(ug)\big\rangle_{L^2_{k,\beta}} \\
&= \mu_{k,\beta}(S)\big\langle \sigma\mathscr{F}_{k,\beta}(uf), \mathscr{F}_{k,\beta}(ug)\big\rangle_{L^2_{k,\beta}}
\end{aligned}$$

$$= \mu_{k,\beta}(S)\langle \mathscr{P}_u(\Sigma), g\rangle_{L^2_{k,\beta}}.$$

This completes the proof. □

From relation (74) and Theorem 21, we deduce the following trace formula.

Corollary 10 The phase space operator $L_{S,\Sigma}$ is trace class with

$$(L_{S,\Sigma}) = \mu_{k,\beta}(S)\,(\mathscr{P}_u(\Sigma)) = \int_S\int_\Sigma |\Lambda_{k,\beta}(x,\xi)|^2 \,\mathrm{d}\mu_{k,\beta}(x)d\mu_{k,\beta}(\xi). \tag{117}$$

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Conflict of interest

The authors declare no competing financial interest.

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