On the Non-Uniqueness of the Sets Computing a Partially Symmetric Rank at Most Three

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Abstract: We describe all partially symmetric tensors which have rank two in more than one way and gives many examples, perhaps all, for rank three partially symmetric tensors.
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1. Introduction

Let $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a multiprojective space. For all $(d_1, \dots, d_k) \in (\mathbb{N} \setminus \{0\})^k$ let $\nu_{(d_1, \dots, d_k)} : Y \to \mathbb{P}^r$, $r = -1 + \prod_{i=1}^{k} \binom{n_i+d_i}{n_i}$, denote the Segre-Veronese embedding of Y, i.e. the embedding of Y by the complete linear system $|\mathcal{O}_Y(d_1, \dots, d_k)|$. The case $k = 1$ is just the Veronese embedding of the projective space $\mathbb{P}^{n_1}$. The case $d_i = 1$ for all i corresponds to the Segre embedding of Y. Set $X := \nu_{d_1,\dots,d_k}(Y)$. We recall that the elements of $H^0(\mathcal{O}_Y(1, \dots, 1))^\vee$ correspond to the partially symmetric tensors of format $\binom{n_1+d_1}{n_1} \times \cdots \times \binom{n_k+d_k}{n_k}$ and hence the elements of $\mathbb{P}^r$ correspond to partially symmetric tensors of that format, up to a non-zero multiplicative constant. Fix $q \in \mathbb{P}^r$. The X-rank $r_X(q)$ of q (or the partially symmetric rank of any non-zero tensor with q as its equivalence class) is the minimal cardinality of a finite set $S \subset Y$ such that $q \in \langle \nu_{d_1,\dots,d_k}(S) \rangle$, where $\langle \ \rangle$ denote the linear span. The solution set $\mathcal{S}(X, q)$ is the set of all $S \subset Y$ such that $\#S = r_X(q)$ and $q \in \langle \nu_{d_1,\dots,d_k}(S) \rangle$. Obviously $\mathcal{S}(X, q) = \emptyset$. We recall that $q \in \mathbb{P}^r$ is said to be a concise tensor or a concise partially symmetric tensor if there is no multiprojective space $Y' \subsetneq Y$ such that $q \in \langle \nu_{d_1,\dots,d_k}(Y') \rangle$.

In this note we prove the following result.

Theorem 1.1 Fix an integer $k \geq 1$ and positive integers n_i, d_i, $1 \leq i \leq k$, such that $(d_1, \dots, d_k) \neq (1, \dots, 1)$. There is a concise tensor q with $r_X(q) = 2$ and $\#\mathcal{S}(X, q) \neq 1$ if and only if either $k = 1$, $n_1 = 1$, $d_1 = 2$ and $\mathcal{S}(X, q)$ is $\mathbb{P}^1$ minus two points or $k = 2$, $n_1 = n_2 = 1$, $(d_1, d_2) \in \{(2,1), (1, 2)\}$ and q, $\mathcal{S}(X, q)$ are as in Example 3.2.

We discuss several examples with $r_X(q) = 3$ and $\#\mathcal{S}(X, q) > 1$ and we wonder if they are the only ones. In particular we described all cases with $\#\mathcal{S}(X, q) > 1$ when $r_X(q) = 3$ and $q \in \tau(X)$ (Proposition 4.1). We always assume $d_i \geq 2$ for at least one integer i, because the case of the Segre variety is done in [4].

Question 1.2 Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. What is the maximal integer $\alpha_X > 0$ (resp. α'_X) such that for each set $A \subset X$ with $\#A \leq \alpha_X - 1$ and $A \in \mathcal{S}(X, q)$ for some $q \in \mathbb{P}^r$, we have $r_X(q') = \#A + 1$ for a general $o \in X$ and a general $q' \in \langle \{o, q\} \rangle$ (resp. all $q' \in \langle \{o, q'\} \rangle \setminus \{o, q'\}$)?

Obviously the integer α_X in Question 1.2 is at most the generic X-rank $r_{X,\text{gen}}$ of $\mathbb{P}^r$, i.e. the minimal integer t such that $\sigma_t(X) = \mathbb{P}^r$, where $\sigma_t(X)$ denote the t-secant variety of X[12-13]. In very special cases $r_{X,\text{gen}} = \alpha_X$. For instance this is true if X is a rational normal curve by Sylvester's theorem[9,12-13].

We work over an algebraically closed field of characteristic 0.

2. Notation and preliminary remarks

Let $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, $k \geq 1$, $n_i \geq 1$ for all i, be any multiprojective space. For any i $\in (1, \dots, k\}$ let $\pi_i : Y \to \mathbb{P}^{n_i}$ denote the projection onto the i-th factor of Y. If $k \geq 2$ set $Y_i := \prod_{h \neq i} \mathbb{P}^{n_h}$ and let $\eta_i : Y \to Y_i$ the morphisms which forget the i-th coordinate of any $p = (p_1, \dots, p_k) \in Y$. Let ε_i (resp. $\hat{\varepsilon}_i$ denote the element $(a_1, \dots, a_k) \in \mathbb{N}^k$ with $a_h = 1$ for all $h \neq i$ and $a_i = 0$ (resp $a_i = 0$ and $a_h = 1$ for all $h \neq i$). Thus $\hat{\varepsilon}_i + \varepsilon = (1, \dots, 1)$.

Remark 2.1 Fix $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and $(d_1, \dots, d_k) \in (\mathbb{N} \setminus \{0\})^k$. Let $\nu_{d_1,\dots,d_k} : Y \to \mathbb{P}^r$, $r = -1 + \prod_{i=1}^{k} \binom{n_i+d_i}{n_i}$, be the Segre-

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Veronese embedding of Y with multidegree (d_1, …, d_k). Let $Y' \subseteq$ Y be a multiprojective subspace. Obviously $\nu_{d_1,\ldots,d_k|Y'}$ is the Segre-Veronese embedding of Y' with multidegree (d_1, …, d_k). Fix $q \in \langle \nu_{d_1,\ldots,d_k}(Y')\rangle$. It is known that $r_{\nu_{d_1,\ldots,d_k}(Y)}(q) = r_{\nu_{d_1,\ldots,d_k}(Y')}(q)$ and that $\mathcal{S}(\nu_{d_1,\ldots,d_k}(Y), q) = \mathcal{S}(\nu_{d_1,\ldots,d_k}(Y'), q)$. We will call Autarky or concision this property. For any finite set $A \subset Y$ the multiprojective space $\prod_{i=1}^{k}\langle \pi_i(A)\rangle$ is the minimal multipro jective subspace of Y containing A. Thus Autarky means that for any $q' \in \mathbb{P}^r$ knowing one solution $S' \in \mathcal{S}(\nu_{d_1,\ldots,d_k}(Y), q')$ we reconstruct the minimal multiprojective space Y' $\subseteq Y$ such that $q' \in \langle \nu_{d_1,\ldots,d_k}(Y')\rangle$. Note that Y' is uniquely determined by q'.

Let $D \subset Y$ be an effective divisor. For any line bundle $\mathcal{L}$ on Y and any finite set $S \subset Y$ there is an exact sequence

$$0 \to \mathcal{I}_{S\setminus S\cap D} \otimes \mathcal{L}(-D) \to \mathcal{I}_S \otimes \mathcal{L} \to \mathcal{L}_{S\cap D, D} \to 0 \qquad (1)$$

3. Proof of theorem 1.1

In this section we prove Theorem 1.1. By Autarky we have $n_i = 1$ for all i.

Remark 3.1 Assume $k = 1$ and hence $Y = \mathbb{P}^1$. Fix $q \in \mathbb{P}^r$, $r = d_1$, with $r_X(q) = 2$. The case $d_1 \geq 3$ is excluded by Sylvester's theorem[9,12] (1.36,1.40). Now assume $d_1 = 2$ and hence $r = 2$ and $X \subset \mathbb{P}^2$ and $r_X(q) = 2$ if and only if $q \in \mathbb{P}^r \setminus X$. The constructible set $\mathcal{S}(X, q)$ is isomorphic to the pencil of lines $L \subset \mathbb{P}^2$ containing q, minus the tangent lines to X containing q. Since we are not in characteristic 2, there are exactly 2 lines passing through q and tangent to X.

Example 3.2 Take $n_1 = n_2 = 1$ and $(d_1, d_2) \in \{(1, 2), (2, 1)\}$. Just to fix the notation we assume $d_1 = 2$ and $d_2 = 1$. We have $\sigma_2(X) = \mathbb{P}^5$. It is well-known that in this case X is an OADP, i.e. $\#\mathcal{S}(X, q) = 1$ for a general $q \in \mathbb{P}^r$ [8,10,17]; in the terminology of [8, Proposition 2.3] X is the scroll $S(2, 2)$). We will prove that all $q \in \mathbb{P}^5 \setminus X$ have $r_X(q) = 2$, that $\#\mathcal{S}(X, q) = 1$ if $q \in \mathbb{P}^5 \setminus \tau(X)$ and that there are two types of $q \in \tau(X) \setminus X$, one with $\dim \mathcal{S}(X, q) = 1$ and one with $\dim \mathcal{S}(Y, q) = 3$.

Take $L \in |\mathcal{O}_Y(1, 0)|$. Note that $D := \nu_{2,1}(L)$ is a smooth conic and $\langle D\rangle \nsubseteq X$. Hence each $q \in \langle D\rangle \setminus X \cap \langle D\rangle$ has $r_X(q) = 2$ and $\mathcal{S}(X, q)$ is infinite. More precisely there is a 1-dimensional family of $\mathcal{S}(X, q)$ formed by the solutions spanning a line contained in $\langle D\rangle$.

Claim 1: Assume $q \in \langle D\rangle \setminus X \cap \langle D\rangle$. Every $A \in \mathcal{S}(X, q)$ is contained in D and $\mathcal{S}(X, q)$ is isomorphic to $\mathbb{P}^1$ minus 2 points. Moreover $A \cap A' = \emptyset$ for all $A, A' \in \mathcal{S}(X, q)$ such that $A \neq A'$.

Proof of Claim 1: The set of all $E \subset L$ such that $\#E = 2$ and $q \in \langle \nu_{(2,1)}(E)\rangle$ is isomorphic to the set of all lines $T \subset \langle \nu_{(2,1)}(D)\rangle$ containing q and not tangent to D and hence is isomorphic to $\mathbb{P}^1$ minus 2 points. Note that any two such different elements are disjoint. Fix $A \in \mathcal{S}(X, q)$. There is $B \subset D$ such that $B \in \mathcal{S}(X, q)$ and $B \cap A = \emptyset$, because there is a line $T \subset \langle \nu_{(2,1)}(D)\rangle$ containing q, not tangent to D and with $T \subset \nu_{(2,1)}(A) = \emptyset$. Set $S := A \cup B$ and assume $A \nsubseteq D$, i.e. $S \setminus S \cap D = \emptyset$. Since $S \nsubseteq D$, [2, Lemma 5.1] gives $h^1(\mathcal{I}_{S\setminus S\cap D}(1,1)) > 0$. Since $B \subset D$, $\#(S \setminus S \cap D) \leq 2$. Thus the very ampleness of $\mathcal{O}_Y(1, 1)$ gives $h^1(\mathcal{I}_{S\setminus S\cap D}(1, 1)) = 0$, a contradiction.

Note that any q as in Claim 1 is an element of $\tau(X) \setminus X$. All other elements of $\tau(X) \setminus X$ are obtained in the following way. Let $v \subset Y$ be a connected degree 2 zero-dimensional scheme contained neither in some L $\in |\mathcal{O}_Y(\varepsilon_1)|$ nor in some $R \in |\mathcal{O}_Y(\varepsilon_2)|$ (because $\nu_{2,1}(R) \subset X$ and so $\langle \nu_{2,1}(v) \subset X$ if $v \subset R$). There is a smooth $C \in |\mathcal{I}_v(1, 1)|$. Since $\dim\langle \nu_{2,1}(C)\rangle = 3$ and $\nu_{2,1}(C)$ is a rational normal curve of $\langle \nu_{2,1}(C)\rangle$, Sylvester's theorem gives $r_X(q) = 3$ and $\dim \mathcal{S}(\nu_{2,1}(C)) = 2$ for all $q \in \langle \nu_{2,1}(v)\rangle \setminus \nu_{2,1}(v_{\mathrm{red}})$. Since there are ∞^1 $C \in |\mathcal{I}_v(1,1)|$ and any two of them meet only along v (because $\mathcal{O}_Y(1, 1) \cdot \mathcal{O}_Y(1, 1) = 2$), we get $\dim \mathcal{S}(Y, q) = 3$.

Claim 2: Fix $a \in \sigma_2(X) \setminus \tau(X)$. We have $r_X(a) = 2$ and $\#\mathcal{S}(X, a) = 1$.

Proof of Claim 2: Since X is smooth and $a \in \sigma_2(X) \setminus \tau(X)$, $r_X(a) = 2$. Assume that $\mathcal{S}(Y, a)$ is not a singleton and take $E, F \in \mathcal{S}(X, a)$ such that $E \neq F$. Set $G := E \cup F$. Since any two different lines either are disjoint or meets at one point and $a \in \langle \nu_{2,1}(E)\rangle \cap \langle \nu_{2,1}(F)\rangle$, we have $E \cap F = 0$. Hence $h^1(\mathcal{I}_G(1, 1)) > 0$[1](Lemma 1). Thus any $C \in |\mathcal{O}_Y(1, 1)|$ containing 3 points of G contains the fourth one. Thus there is $C \in |\mathcal{I}_G(1, 1)|$. Since $a \notin \tau(X)$, we saw that neither E nor F are contained in a ruling of Y. Thus C is smooth. By assumption a / $\tau(\nu_{2,1}(C))$. Hence $r_{\nu_{2,1}(C)}(a) = 2$ and $E, F \in \mathcal{S}(\nu_{2,1}(C), a)$, contradicting[12] (Theorem 1.40).

Proof of Theorem 1.1: Remark 3.1 describes the case $k = 1$. From now on we assume $k \geq 2$.

See Example 3.2 for the case $k = 2$, $n_1 = n_2 = 1$ and $(d_1, d_2) \in \{(1, 2), (2, 1)\}$.

(b) Assume $k = 2$, $d_1 = d_2 = 2$. Take $H \in |\mathcal{O}_Y(2, 0)|$ containing A. Either $S \subset H$ or $h^1(\mathcal{I}_{B\setminus B\cap H}(0, 2)) > 0$[2-3] (Lemma 5.1 or Lemma 2.4).

(b1) First assume $S \subset H$. Since q is concise, there is no $M \in |\mathcal{O}_Y(1, 0)|$ containing S. Since S is a finite set, we get that $H \neq 2M$ for any $M \in |\mathcal{O}_Y(1, 0)|$. Thus $H = H \cup H''$ with $H', H'' \in |\mathcal{O}_Y(1, 0)|$. With no loss of generality we may assume $\#(S \cap H') \geq \#(S \cap H'')$. By [2, Lemma 5.1] or [3, Lemma 2.4] we have $h^1(\mathcal{I}_{S\cap H''}(1, 1)) > 0$. Since $\#(S \cap H'') \leq 2$ and $\mathcal{O}_Y(1, 2)$

is very ample, we get a contradiction.

(b2) Now assume $h^1(H, \mathcal{I}_{B \setminus B \cap H}(0, 2)) > 0$. Since $h^1(\mathbb{P}^1, \mathcal{I}_Z(2)) = 0$ for any scheme $Z \subset \mathbb{P}^1$ with $\deg(Z) \le 3$, we get $\#(B \setminus B \cap H) = 2$ (i.e. $B \cap H = \emptyset$ and $\#(\pi_1(B)) = 1$. Set $M := \pi_1^{-1}(\pi_1(M)) \in |\mathcal{I}_B(1, 0)|$. Since $\mathcal{O}_Y(1, 2)$ is very ample, we have $h^1(\mathcal{I}_A(1, 1)) = 0$. Since $S \setminus S \cap M \subseteq A$, [2, Lemma 5.1] or [3, Lemma 2.4] give $S \subset M$, contradicting the assumption that q is concise.

(c) Assume $k = 2$, $d_1 = 3$ and $d_2 = 1$. Fix $H \in |\mathcal{O}_Y(2, 0)|$ containing A. By [2-3] either $S \subset H$ or $h^1(\mathcal{I}_{B \setminus B \cap H}(0, 1)) > 0$.

(c1) Assume $S \subset H$. Since q is concise, there is no $M \in |\mathcal{O}_Y(1,0)|$ containing S. Since S is a finite set, we get that $H \ne 2M$ with $M \in |\mathcal{O}_Y(1, 0)|$. Thus $H = H' \cup H''$ with $H', H'' \in |\mathcal{O}_Y(1,0)|$ and $H' \ne H''$. With no loss of generality we may assume $\#(S \cap H') > \#(S \cap H'')$. By [2, Lemma 5.1] or [3, Lemma 2.4] we have $h^1(\mathcal{I}_{S \cap H''}(2, 1)) > 0$. Since $\#(S \cap H'') \le 2$ and $\mathcal{O}_Y(1, 2)$ is very ample, we get a contradiction.

(c2) Assume $h^1(\mathcal{I}_{B \setminus B \cap M}(0, 1)) > 0$. Since $\mathcal{O}_Y(0, 1)$ is spanned, we get $B \cap M = \emptyset$ and $\#(\pi_1(B)) = 1$, contradicting concision.

(d) As in steps (b) and (c) we exclude all other cases with $k = 2$. Among the cases with $k > 2$ we immediately see that it is sufficient to exclude the case $k = 3$, $d_1 = 2$ and $d_2 = d_3 = 1$. Assume $k = 3$, $d_1 = 2$ and $d_2 = d_3 = 1$. Fix $H \in |\mathcal{O}_Y(1, 0, 0)|$ containing at least one point of S. By Autarky we have $S \nsubseteq H$ and hence $h^1(\mathcal{I}_{S \setminus S \cap H}(1, 1, 1)) > 0$. Since $\#(S \setminus S \cap H) \le 3$ and $\nu_{1,1,1}(Y)$ is cut out by quadrics, we get $\#(S \setminus S \cap H) = 3$ (i.e. $\#(S \cap H) = 1$) and the existence of $i \in (1, 2, 3\}$ such that $\#(\pi_1(S \setminus S \cap H)) = 1$. Take $M := \pi_1^{-1}(\pi_1(S \setminus S \cap H)) \in |\mathcal{O}_Y(\varepsilon_i)|$. Since $\mathcal{O}_Y(2, 1, 1)(-\varepsilon_i)$ is spanned and $\#(S \cap H) = 1$, [2, Lemma 5.1] or [3, Lemma 2.4] gives a contradiction.

4. $r_X(q) = 3$, $q \in t(X)$

In this section we prove the following result.

Proposition 4.1 Fix $q \in \tau(X)$ with $2 \le r_X(q) \le 3$, $k \ge 2$. Then $Y = \mathbb{P}^1 \times \mathbb{P}^1$, $(d_1, d_2) \in \{(2, 1), (1, 2)\}$ and q is as in Example 3.2.

By section 3 we may assume $r_X(q) = 3$. Fix $A \in \mathcal{S}(X, q)$ and a degree 2 connected scheme $v \subset Y$ such that $q \in \langle \nu_{d_1,\dots,d_k}(v)\rangle$. Set $\{o\} := v_{\text{red}}$. Since $\deg(v) = 2$ and Y is minimal among the multiprojective spaces containing v, we have $n_i = 1$ for all i. With no loss of generality we may assume $d_1 \ge d_2 \ge \cdots \ge d_k > 0$. By assumption $k \ge 2$ and $d_1 \ge 2$. Set $E := v \cup A$. We have $h^1(\mathcal{I}_E(d_1, ..., d_k)) > 0$[1] (Lemma 1).

(a) First assume $d_1 \ge 3$. Take $T_i \in |\mathcal{O}_Y(\varepsilon_i)|$, $1 \le i \le 3$, such that $T_1 \cup T_2 \cup T_3 \supset A$ and call $T \subseteq T_1 \cup T_2 \cup T_3$ containing A. We have $v \nsubseteq T$, because T is reduced and $\deg(\pi_1(v)) = 2$. Thus $h^1(\mathcal{I}_v(0, d_2, \dots, d_k)) > 0$[2-3] (Lemma 5.1 or Lemma 2.4), contradicting the assumptions $k \ge 2$ and $\deg(\pi_2(v)) \ge 2$.

(b) By step (a) from now on we assume $d_1 = 2$. Take $T \in |\mathcal{O}_Y(2\varepsilon_i)|$ containing v. Note that $T = 2K$ with $\{K\} = |\mathcal{I}_o(\varepsilon_i)|$. Either $h^1(\mathcal{I}_{A \setminus A \cap T}(0, d_2, \dots, d_k)) > 0$ or $A \subset T$ and hence $A \subset K$. The latter is impossible, because K is a proper multiprojective subspace of Y. Thus $h^1(\mathcal{I}_{A \setminus A \cap T}(0, d_2, \dots, d_k)) > 0$. Since Y is the minimal multiprojective space containing A by Autarky, we have $\#(\pi_1(A)) > 1$ for all i. Since $h^1(\mathcal{I}_{A \setminus A \cap T}(0, d_2, \dots, d_k)) > 0$, there are $a, b \in A \setminus A \cap T$ such that $\pi_i(a) = \pi_i(b)$ for all $i > 1$. Write $A = \{a, b, c\}$.

(b1) Assume $k \ge 3$. Take $\{M\} := |\mathcal{I}_a(\varepsilon_2)|$ and $M' := |\mathcal{I}_c(\varepsilon_3))|$. Note that $A \subset M \cup M'$. Since $h^1(\mathcal{I}_v(2\varepsilon_1)) = 0$, we get $v \subset M \cup M'$, i.e. $\pi_2(a) = \pi_2(o)$ and $\pi_3(c) = \pi_3(o)$. Using $|\mathcal{I}_c(\varepsilon_2))|$ and $|\mathcal{I}_a(\varepsilon_3)|$ we get $\pi_2(c) = \pi_2(o)$. Thus $\#(\pi_2(A))= 1$, contradicting the minimality of Y.

(b2) By $k = 2$. If $d_2 = 1$ q is as in Example 3.2. Assume $d_2 = 2$. Using $T' \in |\mathcal{O}_Y(2\varepsilon_2)|$ instead of T as in the first part of step (b) we get the existence of $a', b' \in A$ such that $a' \ne b'$, $\pi_1(a') = \pi_1(b')$ and $\pi_2(c') = \pi_2(o)$, where $\{c'\} := A \setminus (a', b'\}$. Since $\#(A) = 3$ and $(a', b'\} = \{a, b\}$ we may assume $a' = a$ and $b' = c$. Thus $c' = b$. Thus $\pi_2(b) = \pi_2(o) = \pi_2(c)$. Write $\{H\} := |\mathcal{I}_o(\varepsilon_2)|$. We have $A \nsubseteq H$ by the minimality of Y. Since $\text{Res}_H(E) \subseteq \{a, o\}$, we have $h^1(\mathcal{I}_{\text{Res}_H(E)}(2, 1)) = 0$, contradicting [2-3]) and concluding the proof of Proposition 4.1.

5. Other examples with $r_X(q) = 3$

Remark 5.1 Take $k = 2$, $n_1 = n_2 = d_2 = 1$ and $d_1 = 2$. Thus $r = 5$. The case $r_X(q) = 2$ is done in Example 3.2. Since X is not the Veronese surface, we have $\sigma_2(X) = \mathbb{P}^5$. Thus all $q \in \mathbb{P}^5$ with $r_X(q) = 3$ are contained in $\tau(X) \setminus X$. These case is described in Example 3.2.

Remark 5.2 Take $k = 2$, $n_1 = n_2 = 1$ and $d_1 = d_2 = 2$. Since $\dim \sigma_3(X) = 7$[14-16], a general $q \in \sigma_3(X)$ has $\dim \mathcal{S}(Y, q) = 1$. By [11, Ex. II.3.22, part (b)] every $q \in \mathbb{P}^8$ with $r_X(q) = 3$ has $\dim \mathcal{S}(X, q) \ge 1$.

Remark 5.3 Take $k = 3$, $n_1 = n_2 = n_3 = 1$ and

$(d_1, d_2, d_3) \in \{(2, 1, 1), (1, 2, 1), (1, 1, 2)\}$.

Since $\dim \sigma_3(X) = 10$[5,7,15], a general $q \in \sigma_3(X)$ has $\dim \mathcal{S}(X, q) = 1$. [11, Ex. II.3.22, part (b)] every $q \in \mathbb{P}^r$ with $r_X(q) = 3$ has $\dim \mathcal{S}(X, q) \geq 1$.

Remark 5.4 Take $k = 4$, $n_1 = n_2 = n_3 = n_4 = 1$ and $d_1 = d_2 = d_3 = d_4 = 1$. Since $\dim \sigma_3(X) = 13$[6,15], a general $q \in \mathbb{P}^r$ has $\dim \mathcal{S}(X, q) = 1$. By [11, Ex. II.3.22, part (b)] every $q \in \mathbb{P}^r$ with $r_X(q) = 3$ has $\dim \mathcal{S}(X, q) \geq 1$.

Remark 5.5 The case $k = 1$, i.e. the case of Veronese embedding, is easy for points q with $r_X(q) = 3$. Note the existence of points of rank > 1 implies $d_1 \geq 2$. Since $r_X(q) = 3$, concision gives $n_1 \in \{1, 2\}$. for each $q \in \mathbb{P}^2 \setminus X$ ($\mathcal{S}(Y, q)$ is the set of all lines of $\mathbb{P}^2$ through q and not tangent to X. Sylvester's theorem[9,12] says that there are no cases with $n_1 = 1$ and $d_1 \neq 4$.

Claim 1: There is no q with $r_X(q) = 3$ and $\#\mathcal{S}(Y, q) > 1$ with $n_1 = 2$, $q \notin \langle \nu_d(L) \rangle$ for any line $L \subset \mathbb{P}^2$ and $\mathrm{d} - 1 \geq 4$.

Proof of Claim 1: Assume the existence of q with $A, B \in \mathcal{S}(Y, q)$ and $A \neq B$. Set $S := A \cup B$. Take a line $L \subset \mathbb{P}^2$ containing at least 2 points of A. We get $h^1(\mathcal{I}_{S \setminus S \cap L}(d - 1)) > 0$[2-3]. Since $\#(S \setminus S \cap L) \leq 4$ and $d_1 - 1 \geq 3$, this is false.

Thus we get the following cases:

(1) $n_1 = 1$, $d_1 = 4$, q sufficiently general in $\mathbb{P}^4$ with $\dim \mathcal{S}(Y, q) = 1$; by [11, Ex. II.3.22] every $q \in \mathbb{P}^4$ with $r_X(q) = 3$ has $\dim \mathcal{S}(X, q) \geq 1$

(2) $n_2 = 2$, $d_1 = 3$, q sufficiently general in $\mathbb{P}^5$ with $\dim \mathcal{S}(Y, q) = 3$; by [11, Ex. II.3.22] every $q \in \mathbb{P}^5$ with $r_X(q) = 3$ has $\dim \mathcal{S}(X, q) \geq 3$.

Example 5.6 Take $n_1 = 2$, $n_2 = 1$, $d_1 = 1$ and $d_2 = 2$ (the same proof works for the case $(n_1, n_2, d_1, d_2) = (1, 2, 2, 1)$). We have $r = 8$. Since $\sigma_3(X) = \mathbb{P}^8$, we have $\dim \mathcal{S}(Y, q) = 3$ for a general $q \in \mathbb{P}^8$. By [11, Ex. II.3.22] every $q \in \mathbb{P}^8$ with $r_X(q) = 3$ has $\dim \mathcal{S}(X, q) \geq 1$.

Proposition 5.7 Set $d_1 := 2$. Fix an integer $k \geq 2$ and take positive integers $d_2, \ldots, d_k$. Fix $n_1 \in \{1, 2\}$ and set $n_i := 1$ for all $i = 2, \ldots, k$. Fix a line $L \subseteq \mathbb{P}^{n_1}$ and take $o_1 \in \mathbb{P}^{n_1}$; if $n_1 = 2$ assume $o_1 \notin L$. Fix $e_i, o_i \in \mathbb{P}^1$, $i = 2, \ldots, k$ such that $e_i \neq o_i$ for all i. Set $Y := \mathbb{P}^{n_1} \times (\mathbb{P}^1)^{k-1}$, $Y' := L \times \{e_2\} \times \cdots \times \{e_k\} \subset Y$, $o := (o_1, \ldots, o_k)$, $X := \nu_{(d_1, \ldots, d_k)}(Y)$ and $X' := \nu_{(d_1, \ldots, d_k)}(Y')$. Fix $q' \in \langle X' \rangle \setminus X'$ and take $q \in \langle \{q', \nu_{(d_1,\ldots,d_k)}(o)\} \rangle \setminus \langle \{q', \nu_{(d_1,\ldots,d_k)}(o)\} \rangle$. Then

(1) Y (resp. Y') is the minimal multiprojective space containing q (resp. q').

(2) $r_X(q') = r_X(q') = 2$, $\mathcal{S}(X, q') = \mathcal{S}(X', q')$ is isomorphic to $\mathbb{P}^1$ minus 2 points.

(3) $2 \leq r_X(q) \leq 3$.

(4) Assume either $d_2 \geq 2$ or $k \geq 3$. Then $r_X(q) = 3$ and $\dim \mathcal{S}(X, q) > 0$.

Proof. Since $\langle X' \rangle \cong \mathbb{P}^2$ and $q' \notin X'$, q' is as in case (1) of Theorem 1.1 and hence $\mathcal{S}(X', q')$ is isomorphic to $\mathbb{P}^1$ minus 2 points. By Autarky $r_X(q') = r_{X'}(q)$. By our choice of o, Y is the minimal multiprojective space containing Y' and o. Since q is in the linear span of q' and a point of X, $1 \leq r_X(q) \leq 3$. Thus to prove part (3) it is sufficient to prove that $r_X(q) > 1$. Assume $r_X(q) = 1$, i.e. assume $q = \nu_{(d_1, \ldots, d_k)}(a)$ for some $a \in Y$. Since $q' \in \langle \{q, \{o\}\} \rangle$ and $r_X(q') = 2$, we get $\{a, o\} \in \mathcal{S}(X, q')$, contradicting Autarky and the assumption $o_2 \neq e_2$.

(a) Assume $d_2 \geq 2$.

Assume $r_X(q) = 2$ and take $B \in \mathcal{S}(X, q)$. Fix $A \in \mathcal{S}(X', q')$ and set $S := A \cup B\{o\}$. Since $q \in \langle \nu_{(d_1,\ldots,d_k)}(A) \rangle \cap_{(d_1,\ldots,d_k)} (B) \rangle$, $q \notin \langle \nu_{(d_1,\ldots,d_k)}(B') \rangle$ for any $B' \subsetneq B$ and $B \nsubseteq A$, $h^1(\mathcal{I}_S(d_1,\ldots,d_k)) > 0$[1] (Lemma 1). Let M be the only element of $|\mathcal{O}_Y(\varepsilon_2)|$ such that $\pi_2(M) = \{e_2\}$. Consider the residual exact sequence of M:

$$0 \to \mathcal{I}_{S \setminus S \cap M}(d_1, d_2 - 1, \ldots, d_k) \to \mathcal{I}_S(d_1, \ldots, d_k) \to \mathcal{I}_{S \cap M, M}(d_1, \ldots, d_k) \to 0 \qquad (2)$$

Since $S \nsubseteq M$, $h^1(\mathcal{I}_{S \setminus S \cap M}(d_1, d_2 - 1, \ldots, d_k)) > 0$[2-3] (Lemma 5.1, Lemma 2.4). Since $\mathcal{O}_Y(d_1, d_2 - 1, \ldots, d_k)$ is very ample, $\#(S \setminus S \cap M) \geq 3$. Thus $\#(S \setminus S \cap M) = 3$, i.e. $(A \cup \{o\}) \cap M = \emptyset$ and $S \setminus S \cap M = A \cup \{o\}$. Since $h^1(\mathcal{I}_{S \setminus S \cap M}(d_1, d_2 - 1, \ldots, d_k)) > 0$, $\nu_{(d_1, \ldots, d_k)}(A \cup \{o\})$ is formed by 3 collinear points. Thus $\nu_{(d_1, \ldots, d_k)}(o) \in \langle X' \rangle$.

Since $\nu_{(d_1, \ldots, d_k)}(o) \notin \langle X' \rangle$, we get $r_{X'}(\nu_{(d_1, \ldots, d_k)})(o)) > r_X(\nu_{(d_1, \ldots, d_k)})(o))$, contradicting Autarky. Thus $r_X(q) = 3$. Hence $\mathcal{S}(X, q) \supseteq \{\{o\} \cup A\}_{A \in \mathcal{S}(X', q')}$. Thus $\dim \mathcal{S}(X, q) \geq 1$.

(b) Assume $d_2 = 1$ and $k \geq 3$. We take M as in step (a). We twice get $h^1(\mathcal{I}_{S \setminus S \cap M}(d_1, 0, d_2, \ldots, d_k)) > 0$. If $\#(S \setminus S \cap M) = 2$ we get that $\#\pi(S \setminus S \cap M) = 1$ for all $i \neq 1$. Since $S \setminus S \cap M$ is contained in a solution of q and $d_2 = 1$, contradicting the obvious extension (d_j arbitrary for $j \neq 2$) of [4, Remark 1.10]. Thus $r_X(q) = 3$ and hence $\mathcal{S}(X, q) \supseteq \{\{o\} \cup A\}_{A \in \mathcal{S}(X', q')}$ and in particular $\dim \mathcal{S}(X, q) \geq 1$.

In the same way we get the following result.

Proposition 5.8 Fix $(d_1, d_2) \in \{(2,1), (1, 2)\}$. Fix an integer $k \geq 3$ and take positive integers $d_3, \ldots, d_k$. Fix $n_1, n_2 \in \{1,$

2} and set $n_i := 1$ for all $i = 3, \ldots, k$. Fix lines $L_i \subseteq \mathbb{P}^{n_1}$, $i = 1, 2$, and take $o_i \in \mathbb{P}^{n_1}$; if $n_i = 2$ assume $o_2 \notin L$. Fix $e_i, o_i \in \mathbb{P}^1$, $i = 3, \ldots, k$ such that $e_i \neq o_i$ for all i. Set $Y := \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times (\mathbb{P}^1)^{k-2}$, $Y' := L_1 \times L_2 \times \{e_3\} \times \cdots \times \{e_k\} \subset Y$, $o := (o_1, \ldots, o_k)$, $X := \nu_{(d_1, \ldots, d_k)}(Y)$ and $X' := \nu_{(d_1, \ldots, d_k)}(Y')$. Fix $q' \in \langle X' \rangle \setminus X'$ and take $q \in \langle \{q', \nu_{(d_1, \ldots, d_k)}(o)\} \rangle \setminus \langle \{q', \nu_{(d_1, \ldots, d_k)}(o)\} \rangle$. Then

(1) Y (resp. Y') is the minimal multiprojective space containing q (resp. q').

(2) $r_X(q') = r_X(q') = 2$, $\mathcal{S}(X, q') = \mathcal{S}(X', q')$ and $2 \leq r_X(q) \leq 3$.

(3) $2 \leq r_X(q) \leq 3$.

(4) Assume either $d_3 \geq 2$ or $k \geq 4$. Then $r_X(q) = 3$ and $\dim \mathcal{S}(X, q) > 0$.

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