

## Research Article

# New Lifetime Regression Model with Application to Prostate Cancer Data

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**Abstract:** A new five-parameter extended fatigue lifetime model named the Weibull generalized gamma distribution is introduced, which generalizes different distributions widely used in survival and reliability analysis. Different mathematical properties are presented, such as stochastic representation, quantiles, minimum, stochastic orders, closed-form expressions for the expectation, and Kullback-Leibler divergence. We estimate the model parameters by maximum likelihood. A Monte Carlo simulation is performed to study the asymptotic normality of the estimates. Further, we propose an extended regression model based on the logarithm of this distribution with two systematic components suitable for censored data, especially in the oncology area, as shown in the analysis of a prostate cancer dataset.

**Keywords:** censored data, generalized gamma distribution, prostate cancer, regression model

**MSC:** 47N30, 62J05, 62J20

## 1. Introduction

The three-parameter generalized gamma (GG) distribution, due to Stacy [1], includes important special cases such as gamma, lognormal, and Weibull, and its density and hazard rate functions present a large variety of shapes. It can be used to determine which parametric model is more appropriate for lifetime data.

The GG distribution has the probability density function (pdf)

$$g(x) = g(x; \alpha, \beta, \delta) = \frac{\beta}{\alpha\Gamma(\delta)} \left(\frac{x}{\alpha}\right)^{\beta\delta-1} \exp\left[-\left(\frac{x}{\alpha}\right)^\beta\right], \quad (1)$$

where  $\alpha > 0$  is a scale,  $\beta > 0$  and  $\delta > 0$  are shape parameters, and  $\Gamma(\delta)$  is the gamma function. Its cumulative distribution function (cdf) is

$$G(x) = G(x; \alpha, \beta, \delta) = \frac{\gamma(\delta, (x/\alpha)^\beta)}{\Gamma(\delta)} = \gamma_1(\delta, (x/\alpha)^\beta), \quad (2)$$

where  $\gamma(\delta, x) = \int_0^x w^{\delta-1} e^{-w} dw$  is the lower incomplete gamma function.

Previous extended classes of the GG distribution include the exponentiated-GG by Cordeiro et al. [2], Kumaraswamy-GG by Pascoa et al. [3], beta-GG by Cordeiro et al. [4], and Marshall-Olkin-GG by Barriga et al. [5].

In this article, the *Weibull generalized gamma* (WGG) distribution is studied. The parameters of the main functions are omitted to simplify the notation. So, the parent cdf is  $G(x) = G(x; \boldsymbol{\xi})$ , where  $\boldsymbol{\xi}$  is the parameter vector of  $G$ .

The paper is organized as follows. The WGG is defined in Section 2, and some structural properties are reported in Section 3. A linear representation of the proposed density is derived in Section 4, and other structural properties are given in Section 5. The estimation of the parameters is addressed in Section 6, and a new regression model is constructed in Section 7. Some properties of the log transform of the WGG distribution and a new regression model are discussed in Section 8. Some simulations are given in Section 9, and three real applications of the proposed models are given in Section 10. We present a detailed analysis of the recurrence time of prostate cancer after radical prostatectomy using a new regression model that can be adopted for other types of cancers. Section 11 concludes the paper with some remarks.

## 2. The WGG distribution

The cdf of the Weibull-G (W-G) family ([6]) is

$$F(x) = F(x; \theta, \lambda, \boldsymbol{\xi}) = 1 - \exp \left\{ -\theta \left[ \frac{G(x)}{\overline{G}(x)} \right]^\lambda \right\}, \quad (3)$$

where  $\overline{G}(x) = 1 - G(x)$ .

Let  $G(x)$  be the GG cdf. The cdf of the WGG distribution can be determined by inserting  $G(x)$  into Equation (3). Hence, the cdf and pdf of the WGG distribution with five positive parameters ( $\alpha, \beta, \delta, \theta$ , and  $\lambda$ ) are given by

$$F(x) = F(x; \alpha, \beta, \delta, \theta, \lambda) = 1 - \exp \left\{ -\theta \left[ \frac{\gamma_1[\delta, (x/\alpha)^\beta]}{1 - \gamma_1[\delta, (x/\alpha)^\beta]} \right]^\lambda \right\} \quad (4)$$

and

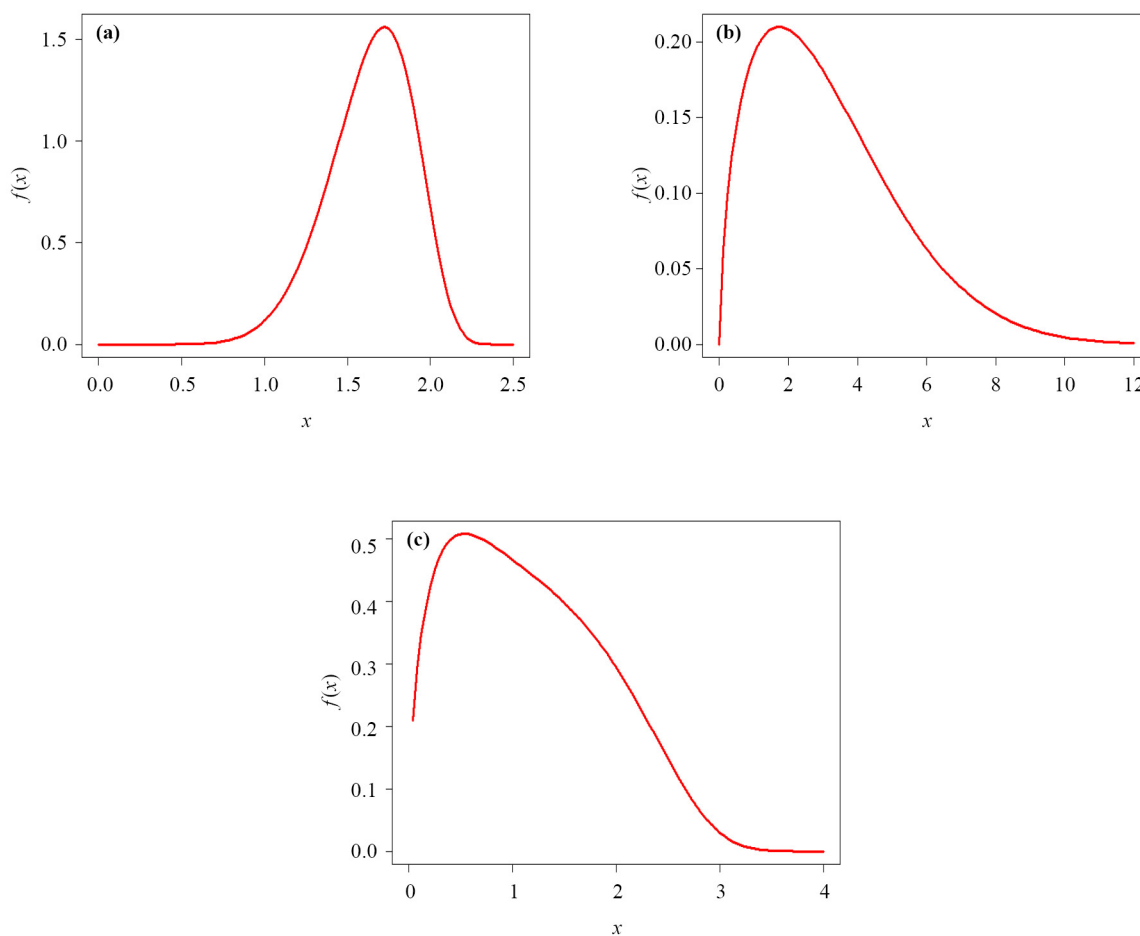
$$\begin{aligned} f(x) &= f(x; \alpha, \beta, \delta, \theta, \lambda) \\ &= \frac{\beta \lambda \theta}{\alpha \Gamma(\delta)} \left( \frac{x}{\alpha} \right)^{\beta \delta - 1} \exp \left\{ -\theta \left[ \frac{\gamma_1[\delta, (x/\alpha)^\beta]}{1 - \gamma_1[\delta, (x/\alpha)^\beta]} \right]^\lambda - \left( \frac{x}{\alpha} \right)^\beta \right\} \times \\ &\quad \left[ \frac{1}{1 - \gamma_1[\delta, (x/\alpha)^\beta]} \right]^2 \left[ \frac{\gamma_1[\delta, (x/\alpha)^\beta]}{1 - \gamma_1[\delta, (x/\alpha)^\beta]} \right]^{\lambda - 1}, \end{aligned} \quad (5)$$

respectively, where  $\alpha$  is a scale and the other are shape parameters. Henceforth, let  $X \sim \text{WGG}(\alpha, \beta, \delta, \theta, \lambda)$  have density (5).

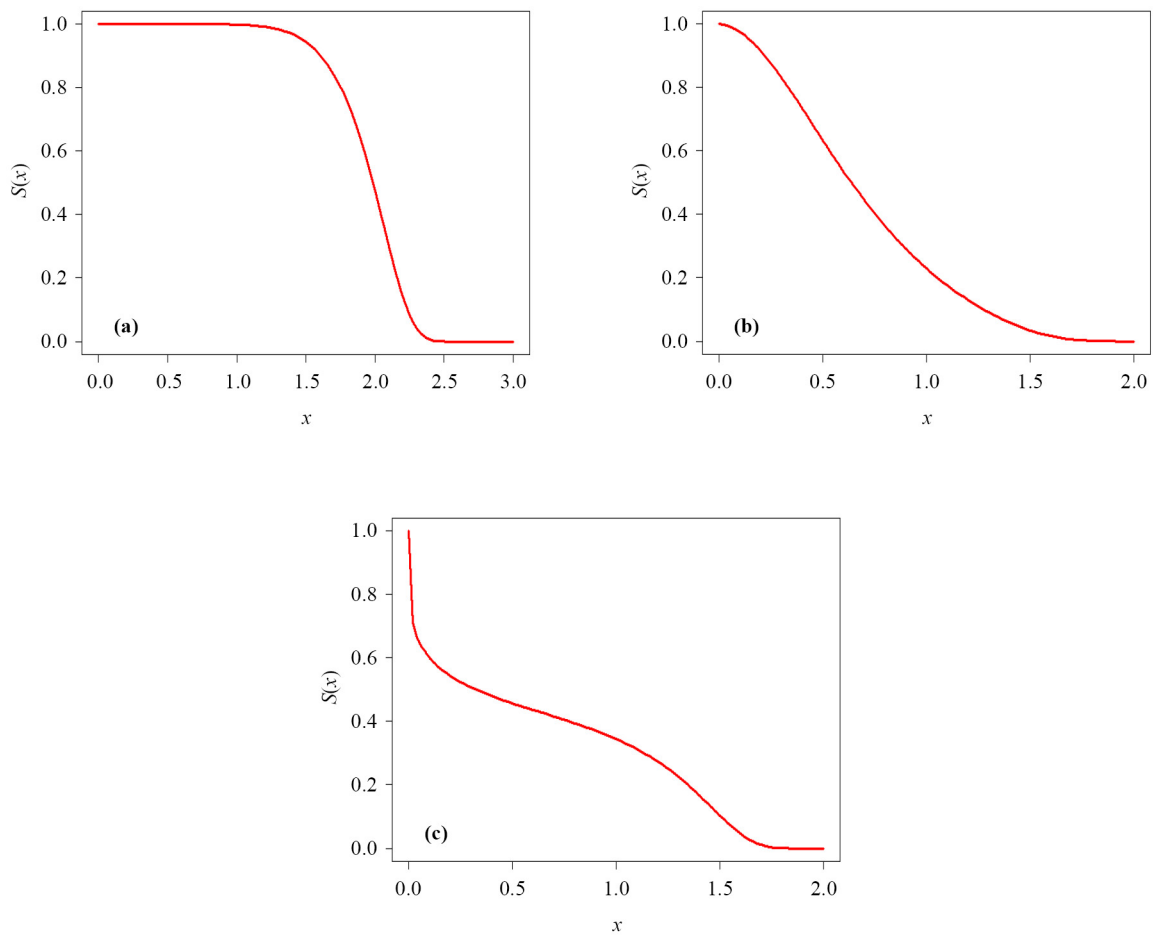
Equation (5) encompasses some special distributions: for  $\delta = 1$ , the Weibull exponential (WE) and Weibull Rayleigh (WR) follow when  $\beta = 1$  and  $\beta = 2$ , respectively; the Weibull gamma (WG) refers to  $\beta = 1$ ; for  $\delta = 1/2$ , the Weibull chi-square (WCS), Weibull half-normal (WHN), Weibull generalized half-normal (WGHN), and Weibull folded normal (WFN) are found when  $\alpha = 2, \beta = 1, \alpha = 2^{1/2}\rho, \beta = 2, \alpha = 2^{1/(2\gamma)}\rho, \beta = 2\gamma$ , and  $\delta \rightarrow \infty$ , respectively; and the Weibull Maxwell (WM) is obtained if  $\delta = 3/2$  and  $\beta = 2$ .

The hazard rate function (hrf) of  $X$  can be easily computed numerically from  $h(x) = f(x)/[1 - F(x)]$ . Figures 1-3 report plots of the density, survival, and hrf of  $X$  for some parameters, thus indicating the distribution flexibility to adapt and represent a wide range of data patterns and behaviors.

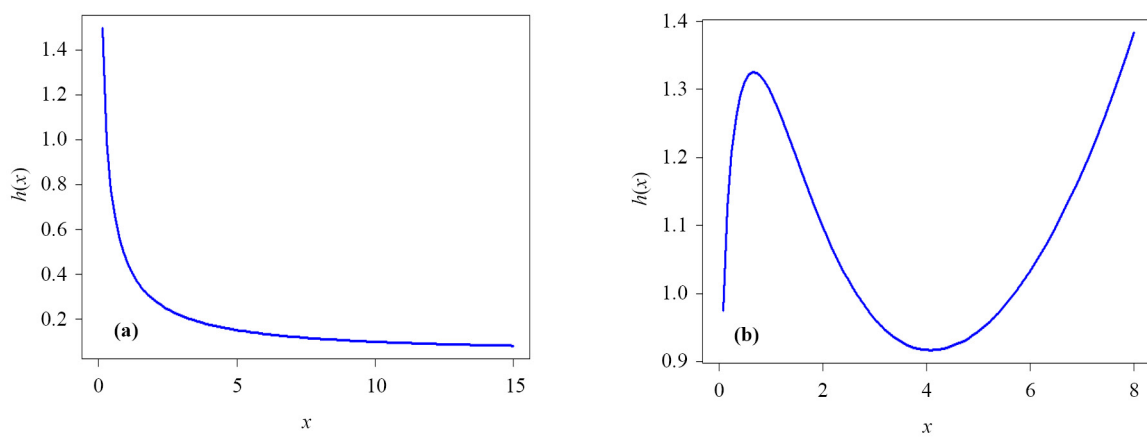
The quantile function (qf) of the GG distribution, say  $Q_{GG}(u)$ , can be obtained from R. By inverting  $F(x) = u$  from Equation (4), the qf of  $X$  can be expressed as  $x = Q_X(u) = F^{-1}(u) = Q_{GG}(u^*)$ , where  $u^* = \{1 + \theta^{-1}[-\log(1 - u)]^{1/\lambda}\}^{-1}$ .

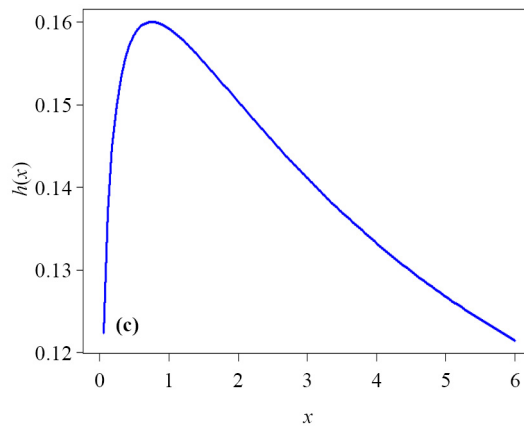


**Figure 1.** Plots of the density of  $X$  for some parameters  $\alpha = 1$ : (a)  $\beta = 2, \delta = 4, \theta = 2, \lambda = 1.25$ ; (b)  $\beta = 0.5, \delta = 3, \theta = 3, \lambda = 1.25$ ; (c)  $\beta = 1.5, \delta = 2, \theta = 1, \lambda = 0.5$  variable



**Figure 2.** Plots of the survival function of  $X$  for some parameters  $\alpha = 1$ : (a)  $\beta = 2, \delta = 4, \theta = 0.5, \lambda = 1.5$ ; (b)  $\beta = 3, \delta = 2, \theta = 2, \lambda = 0.3, \lambda = 1.25$ ; (c)  $\beta = 5, \delta = 0.2, \theta = 0.9, \lambda = 0.1, \lambda = 0.5$  variable





**Figure 3.** Plots of the hrf of  $X$  for some parameters  $\alpha = 1$ : (a)  $\beta = 0.5, \delta = 4, \theta = 3, \lambda = 0.25$ ; (b)  $\beta = 1, \delta = 5, \theta = 5, \lambda = 0.25, \lambda = 1.25$ ; (c)  $\beta = 0.5, \delta = 5, \theta = 2.5, \lambda = 0.5$  variable

### 3. Properties of the WGG model

Some structural properties of  $X \sim \text{WGG}(\alpha, \beta, \delta, \theta, \lambda)$  are discussed here, namely stochastic representation, quantiles, minima, stochastic orders, closed-form expressions for the expectation, and Kullback-Leibler divergence.

#### 3.1 Stochastic representation

From now on  $W \sim \text{Weibull}(\theta, \lambda)$ . We write

$$T(x) = T(x; \alpha, \beta, \delta) = \frac{G(x)}{\overline{G}(x)}, \quad (6)$$

where  $G(x)$  is given in (2). We have  $(\forall x)$  from Equation (4)

$$F(x) = F(x; \alpha, \beta, \delta, \theta, \lambda) = F_W(T(x)) = F_{T^{-1}(W)}(x) = F_{G^{-1}\left(\frac{W}{1+W}\right)}(x), \quad (7)$$

where  $G^{-1}(x)$  is the inverse function of  $G(x)$  and  $T^{-1}(w) = G^{-1}(w/(1+w))$ ,  $w > 0$ , is the inverse function of  $T$ . Then,  $X$  admits the stochastic representation

$$X \stackrel{d}{=} G^{-1}\left(\frac{W}{1+W}\right), \quad (8)$$

where  $\stackrel{d}{=}$  denotes equality in distribution.

**Proposition 1** We can write

1.  $cX \sim \text{WGG}(c\alpha, \beta, \delta, \theta, \lambda)$ ,  $c > 0$ .
2.  $X^k \sim \text{WGG}(\alpha^k, \beta/k, \delta, \theta, \lambda)$ ,  $k > 0$ .

**Proof.** Note that the cdf of  $cX$ , say  $F_{cX}$ , is

$$F_{cX}(t) = \mathbb{P}(cX \leq t) = \mathbb{P}(X \leq t/c) = F_W(T(t/c)),$$

where the result in (7) is used for  $W \sim \text{Weibull}(\theta, \lambda)$ . Hence,

$$T(t/c) = T(t/c; \alpha, \beta, \delta) = T(t; c\alpha, \beta, \delta),$$

and

$$F_{cX}(t) = F_W(T(t; c\alpha, \beta, \delta)) = F(t; c\alpha, \beta, \delta, \theta, \lambda),$$

where in the last line the result (7) is used again. In other words,  $cX \sim \text{WGG}(c\alpha, \beta, \delta, \theta, \lambda)$ ,  $c > 0$ . This shows the statement of Item (1).

In order to prove the second item, the cdf of  $X^k$ , say  $F_{X^k}$ , becomes by (7)

$$F_{X^k}(t) = \mathbb{P}(X^k \leq t) = \mathbb{P}(X \leq t^{1/k}) = F(t^{1/k}) = F_W(T(t^{1/k})), \quad W \sim \text{Weibull}(\theta, \lambda).$$

Since

$$T(x^{1/k}; \alpha, \beta, \delta) = T(x; \alpha^k, \beta/k, \delta),$$

the function  $F_{X^k}$  is

$$F_{X^k}(t) = F_W(T(t; \alpha^k, \beta/k, \delta)) = F(t; \alpha^k, \beta/k, \delta, \theta, \lambda),$$

where again we have used (7). So,  $X^k \sim \text{WGG}(\alpha^k, \beta/k, \delta, \theta, \lambda)$ ,  $k > 0$ , which completes the proof of Item (2).  $\square$

### 3.2 Quantiles

If  $Q_X(p)$ ,  $p \in [0, 1]$ , is the  $p$ -quantile of  $X$ , by the identity  $F(x) = F_W(T(x))$  in (7), the result holds

$$p = F(Q_X(p)) = F_W(T(Q_X(p))), \quad W \sim \text{Weibull}(\theta, \lambda),$$

where  $T(x) = T(x; \alpha, \beta, \delta) = G(x)/\bar{G}(x)$ . The above identities lead to

$$Q_W(p) = T(Q_X(p)).$$

Thus, the quantile function is equivariant under the increasing transformation  $T$ . By using  $T^{-1}(w) = G^{-1}(w/(1+w))$ ,  $w > 0$ , it follows

$$Q_X(p) = T^{-1}(Q_W(p)) = G^{-1}\left(\frac{Q_W(p)}{1+Q_W(p)}\right) = G^{-1}\left(\frac{\left[-\frac{\log(1-p)}{\theta}\right]^{1/\lambda}}{1+\left[-\frac{\log(1-p)}{\theta}\right]^{1/\lambda}}\right), \quad (9)$$

since  $Q_W(p) = [-\log(1-p)/\theta]^{1/\lambda}$ .

### 3.3 Minima

Let  $X_1, \dots, X_n$  be independent and identically distributed WGG random variables with parameter vector  $(\alpha, \beta, \delta, \theta, \lambda)^\top$ . If the minimum of these random variables is  $Z = \min(X_1, \dots, X_n)$ , then, by (7), the cdf of  $Z$  is

$$F_Z(z) = 1 - \exp[-n\theta T^\lambda(z)],$$

where  $T(x) = T(x; \alpha, \beta, \delta)$  is given in (6). In other words,  $Z$  will be also WGG distributed with parameter vector  $(\alpha, \beta, \delta, n\theta, \lambda)^\top$ .

### 3.4 The usual stochastic order

Let  $X$  and  $Y$  be two random variables such that

$$\mathbb{P}(X > x) \leq \mathbb{P}(Y > x), \quad \forall x. \quad (10)$$

Then,  $X$  is said to be smaller than  $Y$  in the usual stochastic order (say  $X \leq_{\text{st}} Y$ ).

**Proposition 2** If  $X \sim \text{WGG}(\alpha, \beta, \delta, \theta, \lambda)$ , then  $X \leq_{\text{st}} Y$  with  $Y = G^{-1}(W)$ .

**Proof.** By using the inequality  $x/(1-x) \geq x$ ,  $x \in [0, 1]$ , then  $T(x) \geq G(x)$ , and consequently by (7),

$$\mathbb{P}(X > x) = 1 - F(x) = 1 - F_W(T(x)) \leq 1 - F_W(G(x)) = \mathbb{P}(Y > x), \quad \forall x, \quad (11)$$

which completes the proof. □

### 3.5 Closed-form expressions for the mean value

Let  $X_1$  and  $X_2$  be two random variables with distribution functions  $F_1$  and  $F_2$ , respectively. The Fortret-Mourier-Wasserstein distance [7] between the finite mean random variables  $X_1$  and  $X_2$  has the form

$$d(X_1, X_2) = \inf_{(X_1^*, X_2^*)} \mathbb{E}|X_2^* - X_1^*|, \quad X_1^* \stackrel{d}{=} F_1, \quad X_2^* \stackrel{d}{=} F_2,$$

where the infimum is taken over all random vectors  $(X_1^*, X_2^*)$  with marginal distributions  $F_1$  and  $F_2$ , respectively.

The next result provides a formula for the mean value of  $X \sim \text{WGG}(\alpha, \beta, \delta, \theta, \lambda)$  as a function of this distance between  $X$  and  $Y = G^{-1}(W)$ , where  $W \sim \text{Weibull}(\theta, \lambda)$ .

**Proposition 3** If  $X \sim \text{WGG}(\alpha, \beta, \delta, \theta, \lambda)$ , then

$$\mathbb{E}(X) = \mathbb{E}(Y) - d(X, Y),$$

where  $Y = G^{-1}(W)$  and  $W \sim \text{Weibull}(\theta, \lambda)$ .

**Proof.** By using the well-known formula  $\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X > x) dx$ ,  $X > 0$ , we have from (11)  $\mathbb{E}(X) \leq \mathbb{E}(Y)$ . Hence, applying Theorem 1.A.11 of [7], the proof of the proposition holds.  $\square$

By using the formula of Dorea and Ferreira [8]

$$d(X, Y) = \int_0^1 |F_X^{-1}(u) - F_Y^{-1}(u)| du,$$

the expression for  $\mathbb{E}(X)$  in Proposition 3 reduces to

**Proposition 4** If  $X \sim \text{WGG}(\alpha, \beta, \delta, \theta, \lambda)$ , then

$$\mathbb{E}(X) = \int_0^1 F_Y^{-1}(u) du - \int_0^1 |F_X^{-1}(u) - F_Y^{-1}(u)| du,$$

where  $Y = G^{-1}(W)$  and  $W \sim \text{Weibull}(\theta, \lambda)$ .

Adding  $\mathbb{E}(X) = \int_0^1 F_X^{-1}(u) du$  to both sides of the identity in Proposition 4 and using the well-known formula  $\min\{x, y\} = (x + y - |x - y|)/2$ , the next result follows.

**Proposition 5** If  $X \sim \text{WGG}(\alpha, \beta, \delta, \theta, \lambda)$ , then

$$\mathbb{E}(X) = \int_0^1 \min\{F_X^{-1}(u), F_Y^{-1}(u)\} du,$$

where  $Y = G^{-1}(W)$  and  $W \sim \text{Weibull}(\theta, \lambda)$ .

### 3.6 The likelihood ratio order

Let  $X$  and  $Y$  be continuous random variables with densities  $f$  and  $g$ , respectively, such that

$$x \mapsto \frac{g(x)}{f(x)}$$

increases in  $x$  over the union of the supports of  $X$  and  $Y$ . Then,  $X$  is said to be smaller than  $Y$  in the likelihood ratio order (say  $X \leq_{lr} Y$ ) (see reference [7]).

**Proposition 6** If  $X \sim \text{WGG}(\alpha, \beta, \delta, \theta_1, \lambda)$  and  $Y \sim \text{WGG}(\alpha, \beta, \delta, \theta_2, \lambda)$ , then  $Y \leq_{lr} X$  provided  $\theta_1 < \theta_2$ , and  $X \leq_{lr} Y$  provided  $\theta_2 < \theta_1$ .

**Proof.** Let  $f_X$  and  $f_Y$  be the densities of  $X$  and  $Y$ , respectively. As  $T(x)$  in (6) is positive and increasing, we can note from (5) that

$$\frac{f_X(x; \alpha, \beta, \delta, \theta_1, \lambda)}{f_Y(x; \alpha, \beta, \delta, \theta_2, \lambda)} = \frac{\theta_1}{\theta_2} \exp\{-(\theta_1 - \theta_2)T(x)\} \quad (12)$$



increases in  $x$  for  $\theta_1 < \theta_2$ . Hence,  $Y \leq_{lr} X$ . Similarly, it is proved that  $X \leq_{lr} Y$  provided  $\theta_2 < \theta_1$ .  $\square$

By applying Theorem 1.C.1. of [7], we obtain from Proposition 6

**Proposition 7** If  $X \sim \text{WGG}(\alpha, \beta, \delta, \theta_1, \lambda)$  and  $Y \sim \text{WGG}(\alpha, \beta, \delta, \theta_2, \lambda)$ , then  $Y \leq_{st} X$  provided  $\theta_1 < \theta_2$ , and  $X \leq_{st} Y$  provided  $\theta_2 < \theta_1$ , where  $\leq_{st}$  is the stochastic order defined in Section 3.4

### 3.7 Kullback-Leibler divergence

The Kullback-Leibler divergence (see [9]) is very useful to measure the difference between two probability distributions. If  $f_X$  and  $f_Y$  are the pdfs of  $X \sim \text{WGG}(\alpha, \beta, \delta, \theta_1, \lambda)$  and  $Y \sim \text{WGG}(\alpha, \beta, \delta, \theta_2, \lambda)$ , respectively, and  $\theta_1 \neq \theta_2$ , then their Kullback-Leibler divergence has the form

$$D_{\text{KL}}(f_X \| f_Y) = \int_0^\infty f_X(x; \alpha, \beta, \delta, \theta_1, \lambda) \log \left( \frac{f_X(x; \alpha, \beta, \delta, \theta_1, \lambda)}{f_Y(x; \alpha, \beta, \delta, \theta_2, \lambda)} \right) dx.$$

From (12) the above integral is

$$\begin{aligned} & \log(\theta_1) - \log(\theta_2) - (\theta_1 - \theta_2) \int_0^\infty T(x) f_X(x; \alpha, \beta, \delta, \theta_1, \lambda) dx \\ &= \log(\theta_1) - \log(\theta_2) - (\theta_1 - \theta_2) \int_0^\infty T(x) dF_X(x; \alpha, \beta, \delta, \theta_1, \lambda) \\ &= \log(\theta_1) - \log(\theta_2) - (\theta_1 - \theta_2) \int_0^\infty T(x) dF_W(T(x)), \end{aligned}$$

where  $T(x) = T(x; \alpha, \beta, \delta)$  is given in (6),  $W \sim \text{Weibull}(\theta_1, \lambda)$ , and the identity in (7) is adopted in the last line. Changing the variable  $w = T(x)$  in the previous integral it is clear that  $D_{\text{KL}}(f_X \| f_Y)$  can be expressed as

$$\begin{aligned} D_{\text{KL}}(f_X \| f_Y) &= \log(\theta_1) - \log(\theta_2) - (\theta_1 - \theta_2) \int_0^\infty w dF_W(w) \\ &= \log(\theta_1) - \log(\theta_2) - (\theta_1 - \theta_2) \mathbb{E}(W) \\ &= \log(\theta_1) - \log(\theta_2) - (\theta_1 - \theta_2) \theta_1^{-1/\lambda} \Gamma \left( 1 + \frac{1}{\lambda} \right), \end{aligned}$$

since  $W \sim \text{Weibull}(\theta_1, \lambda)$ .

## 4. Linear representation

We derive a new linear representation for the W-G family, since the previous one in Bourguignon et al. [6] involves much more algebraic calculations. For simplicity, let  $z = G(x)$  be the parent cdf. We can write using Wolfram Mathematica software

$$\left(\frac{z}{1-z}\right)^\lambda = \sum_{i=0}^{\infty} p_i z^{i+\lambda}, \quad (13)$$

where  $p_0 = 1$ ,  $p_1 = \lambda$ ,  $p_2 = \lambda(\lambda + 1)/2$ ,  $p_3 = \lambda(\lambda^2 + 3\lambda + 2)/6$ ,  $p_4 = \lambda(\lambda^3 + 6\lambda^2 + 11\lambda + 6)/24$ , etc.

Based on the generalized binomial expansion twice and changing the sums, we obtain

$$z^{i+\lambda} = \sum_{k=0}^{\infty} s_{i,k} z^k, \quad (14)$$

where

$$s_{i,k} = s_{i,k}(\theta, \lambda) = \theta \sum_{j=k}^{\infty} (-1)^{k+j} p_i \binom{\lambda+i}{j} \binom{j}{k}.$$

By inserting (14) in Equation (13) and the final expression in (3) gives

$$F(x) = 1 - \exp\left(\sum_{k=0}^{\infty} w_k z^k\right),$$

where  $w_k = w_k(\theta, \lambda) = \theta \sum_{i=0}^{\infty} p_i s_{i,k}$ .

The exponential of a power series is a power series ([10], p.36). Setting  $z = G(x)$

$$F(x) = 1 - \sum_{k=0}^{\infty} v_k G(x)^k, \quad (15)$$

where, for  $k \geq 1$ ,

$$v_k = v_k(\theta, \lambda) = k^{-1} \sum_{m=1}^k m w_m v_{k-m},$$

and  $v_0 = e^{w_0}$ .

By differentiating (15), and changing indices, we can write

$$f(x) = \sum_{k=0}^{\infty} t_{k+1} \pi_{k+1}(x), \quad (16)$$

where (for  $k \geq 0$ )  $t_{k+1} = -v_{k+1}$ , and  $\pi_{k+1}(x) = (k+1)G(x)^k g(x)$  is the exponentiated-G (exp-G) density with power parameter  $k$ .

Equation (16) gives a new linear representation for the W-G family density, which is a simpler expression than that one derived by Bourguignon et al. [6].

We now obtain a linear representation of  $\pi_{k+1}(x)$  when  $G(x)$  is the GG cdf. Based on Equation (21) in Cordeiro et al. [11] and a power series raised to an integer power (Gradshteyn and Ryzhik [12], Section 0.314), we can write

$$\gamma_1(\delta, z)^k = \frac{1}{\Gamma(\delta)^k} \sum_{r=0}^{\infty} c_{k,r} z^{r+k\delta}. \quad (17)$$

Here, the coefficients  $c_{k,r}$  (for  $k = 1, 2, \dots$ ) obey the recurrence relation

$$c_{k,r} = c_{k,r}(\delta) = (ra_0)^{-1} \sum_{j=1}^r [j(k+1) - r] a_j c_{k,r-j}, \quad (18)$$

where  $a_j = a_j(\delta) = (-1)^j / [(\delta + j)j!]$  (for  $j \geq 0$ ), and  $c_{k,0} = a_0^k$ .

By inserting (1) and (17) in  $\pi_{k+1}(x) = (k+1) \gamma_1(\delta, (x/\alpha)^\beta)^k g(x)$ ,

$$\pi_{k+1}(x) = \frac{(k+1)\beta}{\alpha \Gamma(\delta)^{k+1}} \sum_{r=0}^{\infty} c_{k,r} \left(\frac{x}{\alpha}\right)^{[r+(k+1)\delta]\beta-1} \exp\left(\left[\frac{x}{\alpha}\right]^\beta\right).$$

Setting  $\delta_{k,r}^* = r + (k+1)\delta$ , we can rewrite  $\pi_{k+1}(x)$  after some algebra

$$\pi_{k+1}(x) = \sum_{r=0}^{\infty} d_{k,r} g(x; \alpha, \beta, \delta_{k,r}^*), \quad (19)$$

where  $d_{k,r} = d_{k,r}(\delta) = (k+1) \Gamma(\delta_{k,r}^*) c_{k,r} / \Gamma(\delta)^{k+1}$ , and  $g(x; \alpha, \beta, \delta_{k,r}^*)$  is given in (1) changing  $\delta$  by  $\delta_{k,r}^*$ .

By inserting (19) in Equation (16), the pdf of  $X$  can be expressed as

$$f(x) = \sum_{k,r=0}^{\infty} e_{k,r} g(x; \alpha, \beta, \delta_{k,r}^*), \quad (20)$$

where  $e_{k,r} = e_{k,r}(\delta, \theta, \lambda) = d_{k,r} t_{k+1}$  (for  $k, r \geq 0$ ).

Equation (20) reveals that the WGG density is a double-linear combination of GG densities that provides its properties.

## 5. Other properties

The formulae in this section can be handled in most computation software. First, we obtain the ordinary moments of  $X$ . The  $n$ th ordinary moment of the GG model in (1) is well-known

$$\mu'_{n,GG} = \frac{\alpha^n \Gamma(\delta + n/\beta)}{\Gamma(\delta)}.$$

Then, the corresponding moment of  $X$  follows from (20) as

$$\mu'_n = \mathbb{E}(X^n) = \alpha^n \sum_{k,r=0}^{\infty} e_{k,r} \frac{\Gamma(\delta_{k,r}^* + n/\beta)}{\Gamma(\delta_{k,r}^*)}. \quad (21)$$

Second, the central moments, cumulants, skewness and kurtosis of  $X$  can be found from Equation (21) from well-known relations.

Third, simpler expressions for the skewness and kurtosis of  $X$  can be based on quantile measures

$$B = \frac{Q_X(3/4) - 2Q_X(1/2) + Q_X(1/4)}{Q_X(3/4) - Q_X(1/4)}$$

and

$$M = \frac{Q_X(7/8) - Q_X(5/8) + Q_X(3/8) - Q_X(1/8)}{Q_X(6/8) - Q_X(2/8)},$$

respectively. Plots of  $B$  and  $M$  in Figures 4 and 5 show how these measures vary for some shape parameters.

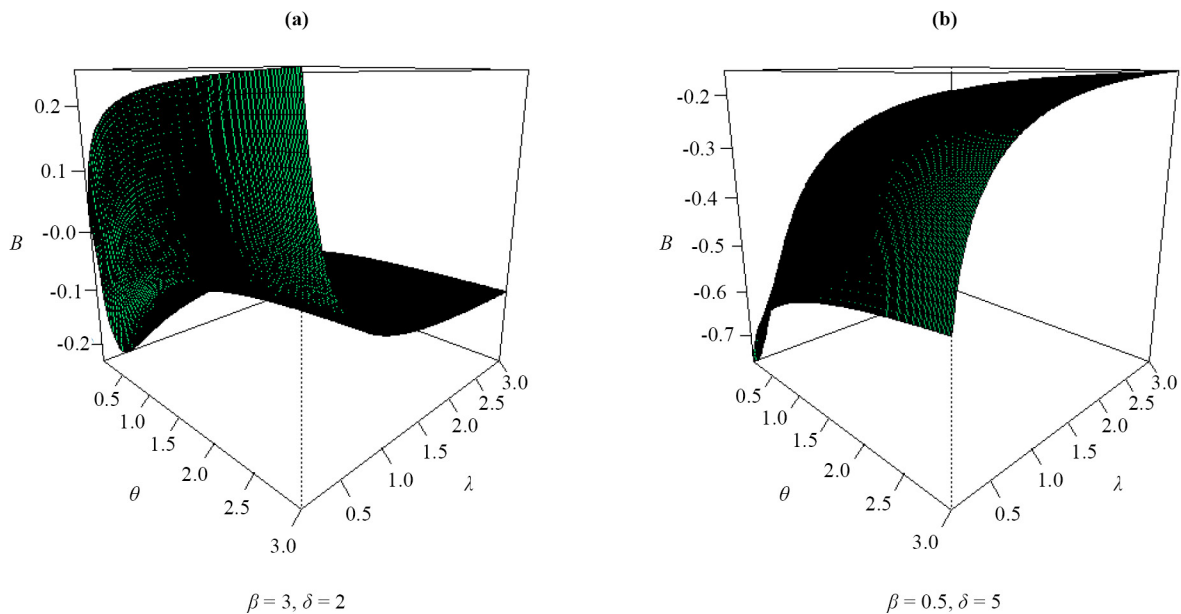
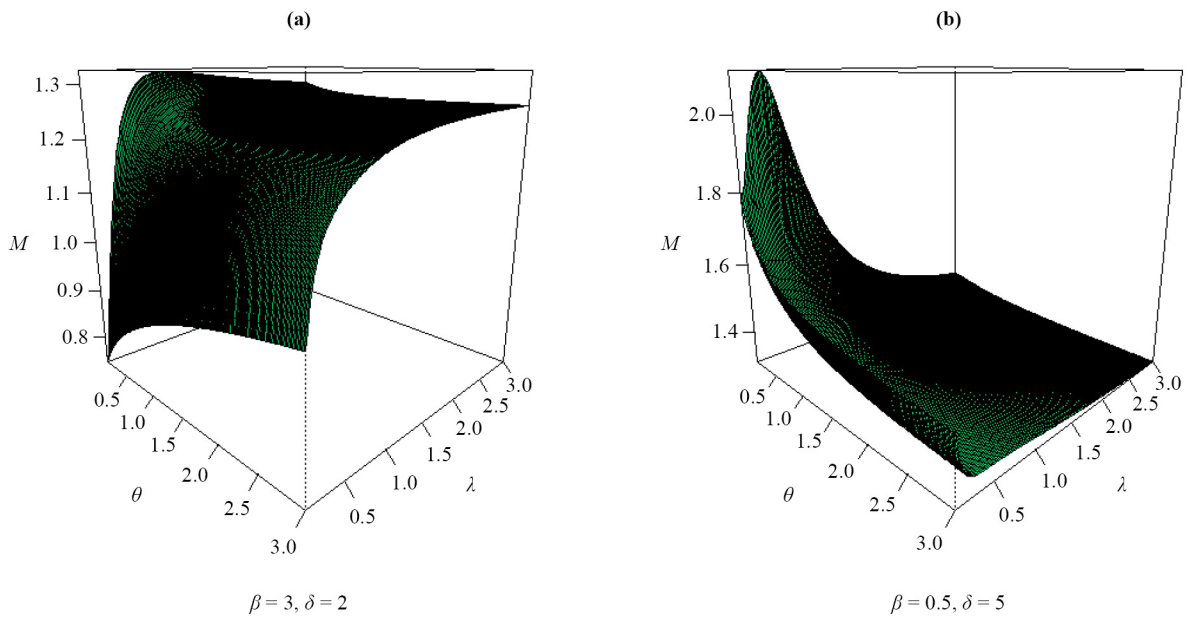


Figure 4. Bowley skewness ( $B$ ) for the WGG model ( $\alpha = 1$ )



**Figure 5.** Moors' kurtosis ( $M$ ) for the WGG model ( $\alpha = 1$ )

Fourth, we derive the  $n$ th incomplete moment of  $X$ , say  $m_n(w) = \int_0^w x^n f(x) dx$ . We can write from Equation (1)

$$\int_0^w \left(\frac{x}{\alpha}\right)^{\beta\delta-1} \exp\left[-\left(\frac{x}{\alpha}\right)^\beta\right] = \frac{\alpha\Gamma(\delta)}{\beta} \gamma_1\left(\delta, \left[\frac{x}{\alpha}\right]^\beta\right).$$

Thus, the  $n$ th incomplete moment of the GG distribution can be determined from (1) and the previous expression as

$$\kappa_n(w; \alpha, \beta, \delta) = \frac{\alpha\Gamma(n/\beta + \delta)}{\Gamma(\delta)} \gamma_1\left(\frac{n}{\beta} + \delta, \left[\frac{x}{\alpha}\right]^\beta\right). \quad (22)$$

Finally, we write from Equation (20)

$$m_n(w) = \sum_{k,r=0}^{\infty} e_{k,r} \kappa_n(w; \alpha, \beta, \delta_{k,r}^*), \quad (23)$$

where  $\kappa_n(w; \alpha, \beta, \delta_{k,r}^*)$  comes from (22). Setting  $n = 1$  in (23), we can construct the Bonferroni and Lorentz curves of  $X$ .

## 6. Estimation

Suppose that  $x_1, \dots, x_n$  is a random sample from the WGG distribution. The log-likelihood function for  $\Theta = (\alpha, \beta, \delta, \theta, \lambda)^\top$  follows from Equation (5) as

$$\begin{aligned} \ell(\Theta) &= \sum_{i=1}^n \log f(x_i) \\ &= n \log \left[ \frac{\beta \lambda \theta}{\alpha \Gamma(\delta)} \right] + (\delta - 1/\beta) \sum_{i=1}^n \log u_i - (\lambda + 1) \sum_{i=1}^n \log \left[ 1 - \gamma_1(\delta, u_i) \right] \\ &\quad + (\lambda - 1) \sum_{i=1}^n \log \gamma_1(\delta, u_i) - \theta \sum_{i=1}^n \left[ \frac{\gamma_1(\delta, u_i)}{1 - \gamma_1(\delta, u_i)} \right]^\lambda - \sum_{i=1}^r u_i, \end{aligned}$$

where  $u_i = (x_i/\alpha)^\beta$ .

The maximum likelihood estimates (MLEs) can be found via the AdequacyModel library (Marinho et al., [13]) of the R software by selecting any method available.

The score components for the parameters are

$$\begin{aligned} U_\alpha &= -\frac{n}{\alpha} + \frac{1 - \beta \delta}{\alpha} - \frac{\beta(1 + \lambda)}{\alpha \Gamma(\delta)} \sum_{i=1}^n \frac{u_i^\delta e^{-u_i}}{1 - \gamma_1(\delta, u_i)} \\ &\quad + \frac{\beta(1 - \lambda)}{\alpha \Gamma(\delta)} \sum_{i=1}^n \frac{u_i^\delta e^{-u_i}}{\gamma_1(\delta, u_i)} + \frac{\theta \beta}{\alpha \Gamma(\delta)} \sum_{i=1}^n \frac{[\gamma_1(\delta, u_i)]^{\lambda-1} u_i^\delta e^{-u_i}}{[1 - \gamma_1(\delta, u_i)]^{\lambda+1}} - \frac{\beta}{\alpha^2} \sum_{i=1}^n u_i^{1-1/\beta}, \\ U_\beta &= \frac{n}{\beta} + \frac{\delta}{\beta} \sum_{i=1}^n \log u_i + \frac{\lambda + 1}{\beta \Gamma(\delta)} \sum_{i=1}^n \frac{u_i^{\beta \delta} e^{-u_i} \log u_i}{1 - \gamma_1(\delta, u_i)} \\ &\quad + \frac{\lambda - 1}{\beta \Gamma(\delta)} \sum_{i=1}^n \frac{u_i^{\beta \delta} e^{-u_i} \log u_i}{\gamma_1(\delta, u_i)} - \frac{\theta \lambda}{\beta \Gamma(\delta)} \sum_{i=1}^n \frac{[\gamma_1(\delta, u_i)]^{\lambda-1} u_i^{\beta \delta} e^{-u_i} \log u_i}{[1 - \gamma_1(\delta, u_i)]^{\lambda+1}} - \frac{1}{\beta} \sum_{i=1}^r u_i \log u_i, \\ U_\delta &= -n \psi(\delta) + \sum_{i=1}^n \log u_i + \frac{\lambda + 1}{\Gamma(\delta)} \sum_{i=1}^n \frac{\dot{\gamma}(\delta, u_i) - \psi(\delta) \Gamma(\delta) \gamma_1(\delta, u_i)}{1 - \gamma_1(\delta, u_i)} \\ &\quad + \frac{\lambda - 1}{\Gamma(\delta)} \sum_{i=1}^n \frac{\dot{\gamma}(\delta, u_i) - \psi(\delta) \Gamma(\delta) \gamma_1(\delta, u_i)}{\gamma_1(\delta, u_i)} - \frac{\theta \lambda}{\Gamma(\delta)} \sum_{i=1}^n \frac{[\dot{\gamma}(\delta, u_i) - \psi(\delta) \Gamma(\delta) \gamma_1(\delta, u_i)] [\gamma_1(\delta, u_i)]^{\lambda-1}}{[1 - \gamma_1(\delta, u_i)]^{\lambda+1}}, \\ U_\theta &= \frac{n}{\theta} - \sum_{i=1}^n \left[ \frac{\gamma_1(\delta, u_i)}{1 - \gamma_1(\delta, u_i)} \right]^\lambda, \\ U_\lambda &= \frac{n}{\lambda} - \sum_{i=1}^n \log [1 - \gamma_1(\delta, u_i)] + \sum_{i=1}^r \log \gamma_1(\delta, u_i) - \theta \sum_{i=1}^n \left[ \frac{\gamma_1(\delta, u_i)}{1 - \gamma_1(\delta, u_i)} \right]^\lambda \log \left[ \frac{\gamma_1(\delta, u_i)}{1 - \gamma_1(\delta, u_i)} \right], \end{aligned}$$

where  $\psi(\cdot)$  is the digamma function and  $\dot{\gamma}(\delta, u_i) = \int_0^{u_i} x^{\delta-1} e^{-x} \log(x) dx$ .

## 7. The log Weibull generalized gamma regression model

Following the idea of Stacy and Mihram [14], we define an extended form of (5) (for  $x > 0$ ), where  $\beta \neq 0$  and the other parameters are positive, namely

$$f(x) = \frac{|\beta| \lambda \theta}{\alpha \Gamma(\delta)} \left(\frac{x}{\alpha}\right)^{\beta \delta - 1} \exp \left\{ -\theta \left[ \frac{\gamma_1[\delta, (x/\alpha)^\beta]}{1 - \gamma_1[\delta, (x/\alpha)^\beta]} \right]^\lambda - \left(\frac{x}{\alpha}\right)^\beta \right\} \frac{\{\gamma_1[\delta, (x/\alpha)^\beta]\}^{\lambda-1}}{\{1 - \gamma_1[\delta, (x/\alpha)^\beta]\}^{\lambda+1}}. \quad (24)$$

For  $\beta > 0$ , we clearly obtain equation (5).

Next, let  $X \sim \text{WGG}(\alpha, \beta, \delta, \theta, \lambda)$  have density (24). The cdf of  $X$  has the form

$$F(x) = \begin{cases} 1 - \exp \left\{ -\theta \left[ \frac{\gamma_1[\delta, (x/\alpha)^\beta]}{1 - \gamma_1[\delta, (x/\alpha)^\beta]} \right]^\lambda \right\} & \text{if } \beta > 0, \\ \exp \left\{ -\theta \left[ \frac{1 - \gamma_1[\delta, (x/\alpha)^\beta]}{\gamma_1[\delta, (x/\alpha)^\beta]} \right]^\lambda \right\} & \text{if } \beta < 0. \end{cases} \quad (25)$$

Henceforth, let  $X$  be a random variable having the WGG density function (24) and  $Y = \log(X)$ . By setting  $\delta = q^{-2}$ ,  $\beta = (\sigma \sqrt{\delta})^{-1}$ , and  $\alpha = \exp[\mu - \beta^{-1} \log(\delta)]$ , the density function of  $Y$  follows as

$$f(y) = \frac{\theta \lambda |q| (q^{-2})^{q^{-2}}}{\sigma \Gamma(q^{-2})} \exp \left\{ q^{-1} \left( \frac{y - \mu}{\sigma} \right) - q^{-2} \exp \left[ q \left( \frac{y - \mu}{\sigma} \right) \right] \right\} \times \frac{\gamma_1^{\lambda-1}(q^{-2}; q^{-2} \exp[q(\frac{y-\mu}{\sigma})])}{\{1 - \gamma_1(q^{-2}; q^{-2} \exp[q(\frac{y-\mu}{\sigma})])\}^{\lambda+1}} \exp \left\{ -\theta \left[ \frac{\gamma_1(q^{-2}; q^{-2} \exp[q(\frac{y-\mu}{\sigma})])}{1 - \gamma_1(q^{-2}; q^{-2} \exp[q(\frac{y-\mu}{\sigma})])} \right]^\lambda \right\},$$

where  $y, \mu \in \mathbb{R}$ ,  $\sigma > 0$ ,  $\lambda > 0$ ,  $\theta > 0$  and  $q \neq 0$ .

By including the case  $q = 0$  (Lawless [15]), the density of  $Y$  reduces to

$$f(y) = \begin{cases} \frac{\theta \lambda |q| (q^{-2})^{q^{-2}}}{\sigma \Gamma(q^{-2})} \exp \left\{ q^{-1} \left( \frac{y - \mu}{\sigma} \right) - q^{-2} \exp \left[ q \left( \frac{y - \mu}{\sigma} \right) \right] \right\} \frac{\gamma_1^{\lambda-1}(q^{-2}; q^{-2} \exp[q(\frac{y-\mu}{\sigma})])}{\{1 - \gamma_1(q^{-2}; q^{-2} \exp[q(\frac{y-\mu}{\sigma})])\}^{\lambda+1}} \times \\ \exp \left\{ -\theta \left[ \frac{\gamma_1(q^{-2}; q^{-2} \exp[q(\frac{y-\mu}{\sigma})])}{1 - \gamma_1(q^{-2}; q^{-2} \exp[q(\frac{y-\mu}{\sigma})])} \right]^\lambda \right\} & \text{if } q \neq 0, \\ \frac{\theta \lambda}{\sigma \sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left( \frac{y - \mu}{\sigma} \right)^2 \right\} \frac{\Phi^{\lambda-1}(\frac{y-\mu}{\sigma})}{[1 - \Phi(\frac{y-\mu}{\sigma})]^{\lambda+1}} \exp \left\{ -\theta \left[ \frac{\Phi(\frac{y-\mu}{\sigma})}{1 - \Phi(\frac{y-\mu}{\sigma})} \right]^\lambda \right\} & \text{if } q = 0, \end{cases} \quad (26)$$

where  $\Phi(\cdot)$  is the standard normal cdf.

Equation (26) is a five-parameter family centered on the Weibull-normal distribution (for  $q = 0$ ). It refers to the *log-Weibull generalized gamma* (LWGG) distribution, say  $Y \sim \text{LWGG}(\mu, \sigma, q, \theta, \lambda)$ , where  $\mu \in \mathbb{R}$  is the location parameter,  $\sigma > 0$  is the scale and  $q, \theta$  and  $\lambda$  are shape parameters. Thus,

if  $X \sim \text{WGG}(\alpha, \beta, \delta, \theta, \lambda)$  then  $Y = \log(X) \sim \text{LWGG}(\mu, \sigma, q, \theta, \lambda)$ .

The survival function of  $Y$ , say  $S(y) = \mathbb{P}(Y \geq y)$ , has the form

$$S(y) = \begin{cases} \exp \left\{ -\theta \left[ \frac{\gamma_1(q^{-2}; q^{-2} \exp[q(\frac{y-\mu}{\sigma})])}{1-\gamma_1(q^{-2}; q^{-2} \exp[q(\frac{y-\mu}{\sigma})])} \right]^\lambda \right\} & \text{if } q > 0, \\ 1 - \exp \left\{ -\theta \left[ \frac{1-\gamma_1(q^{-2}; q^{-2} \exp[q(\frac{y-\mu}{\sigma})])}{\gamma_1(q^{-2}; q^{-2} \exp[q(\frac{y-\mu}{\sigma})])} \right]^\lambda \right\} & \text{if } q < 0, \\ \exp \left\{ -\theta \left[ \frac{\Phi(\frac{y-\mu}{\sigma})}{1-\Phi(\frac{y-\mu}{\sigma})} \right]^\lambda \right\} & \text{if } q = 0. \end{cases} \quad (27)$$

The pdf of  $Z = (Y - \mu)/\sigma \sim \text{LWGG}(0, 1, q, \theta, \lambda)$  reduces to

$$f(z) = \begin{cases} \frac{\theta \lambda |q|(q^{-2})^{q^{-2}}}{\Gamma(q^{-2})} \exp [q^{-1}z - q^{-2} \exp(qz)] \frac{\gamma_1^{\lambda-1}(q^{-2}; q^{-2} \exp(qz))}{\{1-\gamma_1(q^{-2}; q^{-2} \exp(qz))\}^{\lambda+1}} \times \\ \exp \left\{ -\theta \left[ \frac{\gamma_1(q^{-2}; q^{-2} \exp(qz))}{1-\gamma_1(q^{-2}; q^{-2} \exp(qz))} \right]^\lambda \right\} & \text{if } q \neq 0, \\ \frac{\theta \lambda}{\sqrt{2\pi}} \exp \left\{ -\frac{1}{2}z^2 \right\} \frac{\Phi^{\lambda-1}(z)}{[1-\Phi(z)]^{\lambda+1}} \exp \left\{ -\theta \left[ \frac{\Phi(z)}{1-\Phi(z)} \right]^\lambda \right\} & \text{if } q = 0. \end{cases} \quad (28)$$

## 8. Some LWGG properties and a new regression model

Here, some structural properties of the LWGG distribution are reported. We consider  $\mu = 0$  and  $\sigma = 1$ , because in the general case it is enough to define  $Y = \mu + \sigma Y^*$ , where  $Y^* \sim \text{LWGG}(0, 1, q, \theta, \lambda)$ .

### 8.1 Stochastic representation

It follows from (8) that  $Y \sim \text{LWGG}(0, 1, q, \theta, \lambda)$  admits the stochastic representation

$$Y \stackrel{d}{=} \log G^{-1} \left( \frac{W}{1+W} \right), \quad W \sim \text{Weibull}(\theta, \lambda),$$

where  $G(x)$  is given in (2).

### 8.2 Quantiles

Let  $Y \sim \text{LWGG}(0, 1, q, \theta, \lambda)$ ,  $X \sim \text{WGG}(\alpha, \beta, \delta, \theta, \lambda)$ , with  $\alpha$ ,  $\beta$  and  $\delta$  defined some lines below (25), and  $p \in [0, 1]$ . Let  $Q_Y(p)$  denote the  $p$ th quantile of  $Y$ . The invariance of quantiles under positive monotone transformations and (9) gives



$$Q_Y(p) = Q_{\log(X)}(p) = \log(Q_X(p)) = \log \left\{ \frac{G\left(\left[-\frac{\log(1-p)}{\theta}\right]^{1/\lambda}\right)}{1 + G\left(\left[-\frac{\log(1-p)}{\theta}\right]^{1/\lambda}\right)} \right\}.$$

### 8.3 Minima

Let  $Y_1, \dots, Y_n$  be independent and identically distributed LWGG random variables with parameter vector  $(0, 1, q, \theta, \lambda)^\top$ . If the minimum of these random variables is  $Z = \min(Y_1, \dots, Y_n)$ , the cdf of  $Z$  follows from (7)

$$F_Z(z) = 1 - \exp[-n\theta T^\lambda(e^z)],$$

where  $T(x) = T(x; \alpha, \beta, \delta)$  is given in (6). So,  $Z$  is also LWGG distributed with parameter vector  $(0, 1, q, n\theta, \lambda)^\top$ .

### 8.4 Stochastic orders

The proofs of the following two stochastic orders (Propositions 8 and 9) can be determined analogously from those Propositions 2 and 6. Hence, due to space limitations, we omit the corresponding proofs.

**Proposition 8** If  $Y \sim \text{LWGG}(0, 1, q, \theta, \lambda)$ , then  $Y \leq_{st} R$ , where  $R = \log(G^{-1}(W))$  and  $W \sim \text{Weibull}(\theta, \lambda)$ .

**Proposition 9** If  $Y_1 \sim \text{LWGG}(0, 1, q, \theta_1, \lambda)$  and  $Y_2 \sim \text{LWGG}(0, 1, q, \theta_2, \lambda)$ , then  $Y_2 \leq_{lr} Y_1$  provided  $\theta_1 < \theta_2$ , and  $Y_1 \leq_{lr} Y_2$  provided  $\theta_2 < \theta_1$ .

### 8.5 Closed-form expressions for the mean value

Based on the stochastic orders between  $Y$  and  $R$  (Proposition 8) we obtain the closed-form expressions for the mean value of  $Y \sim \text{LWGG}(0, 1, q, \theta, \lambda)$ .

**Proposition 10** If  $Y \sim \text{LWGG}(0, 1, q, \theta, \lambda)$ , then

$$\begin{aligned} \mathbb{E}(Y) &= \int_0^1 F_R^{-1}(u) du - \int_0^1 |F_Y^{-1}(u) - F_R^{-1}(u)| du, \\ &= \int_0^1 \min\{F_Y^{-1}(u), F_R^{-1}(u)\} du, \end{aligned}$$

where  $R = \log(G^{-1}(W))$ .

### 8.6 Kullback-Leibler divergence

If  $f_{Y_1}$  and  $f_{Y_2}$  are the pdfs of  $Y_1 \sim \text{LWGG}(0, 1, q, \theta_1, \lambda)$  and  $Y_2 \sim \text{LWGG}(0, 1, q, \theta_2, \lambda)$ , respectively, and  $\theta_1 \neq \theta_2$ , their Kullback-Leibler divergence reduces to

$$D_{\text{KL}}(f_{Y_1} \| f_{Y_2}) = \int_{-\infty}^{\infty} f_{Y_1}(x; 0, 1, q, \theta_1, \lambda) \log \left( \frac{f_{Y_1}(x; 0, 1, q, \theta_1, \lambda)}{f_{Y_2}(x; 0, 1, q, \theta_2, \lambda)} \right) dx.$$

Since this divergence measure is invariant under invertible transforms, we have

$$D_{\text{KL}}(f_{Y_1} \| f_{Y_2}) = D_{\text{KL}}(f_{\log(X_1)} \| f_{\log(X_2)}) = D_{\text{KL}}(f_{X_1} \| f_{X_2}),$$

where  $X_1 \sim \text{WGG}(\alpha, \beta, \delta, \theta_1, \lambda)$  and  $X_2 \sim \text{WGG}(\alpha, \beta, \delta, \theta_2, \lambda)$ . By using the formula of the Kullback-Leibler divergence  $D_{\text{KL}}(f_{X_1} \| f_{X_2})$  given in Section 3.7, we obtain

$$D_{\text{KL}}(f_{Y_1} \| f_{Y_2}) = \log(\theta_1) - \log(\theta_2) - (\theta_1 - \theta_2) \theta_1^{-1/\lambda} \Gamma\left(1 + \frac{1}{\lambda}\right).$$

Further, we propose a linear location-scale regression model linking the response variable  $y_i$  and the explanatory variable vector  $\mathbf{v}_i^\top = (\mathbf{v}_{i1}, \dots, \mathbf{v}_{ip})$  under two systematic components (for  $i = 1, \dots, n$ )

$$y_i = \mu_i + \sigma_i z_i, \tag{29}$$

where the random error  $z_i$  has density function (28),

$$\mu_i = \mathbf{v}_i^\top \boldsymbol{\beta}_1 \quad \text{and} \quad \log(\sigma_i) = \mathbf{v}_i^\top \boldsymbol{\beta}_2,$$

$\boldsymbol{\beta}_1 = (\beta_{11}, \dots, \beta_{1p})^\top$  and  $\boldsymbol{\beta}_2 = (\beta_{21}, \dots, \beta_{2p})^\top$  are functionally independent. The systematic component for  $\sigma$  can model heteroscedastic data with non-proportional risks.

Consider independent observations  $(y_1, \mathbf{v}_1), \dots, (y_n, \mathbf{v}_n)$ , where  $y_i = \min\{\log(x_i), \log(c_i)\}$ . Here, the  $x_i$ 's are the failure times and the  $c_i$ 's are the censored times assumed independent. Let  $F$  and  $C$  be the sets for the log lifetimes and log-censoring times, respectively. The log-likelihood function for  $\tau = (\boldsymbol{\beta}_1^\top, \boldsymbol{\beta}_2^\top, q, \theta, \lambda)^\top$  from model (29) has the form

$$l(\tau) = \begin{cases} r \log \left[ \frac{\theta \lambda q (q^{-2})^{q^{-2}}}{\Gamma(q^{-2})} \right] - \sum_{i \in F} \log(\sigma_i) + q^{-1} \sum_{i \in F} z_i - \\ q^{-2} \sum_{i \in F} \exp(q z_i) + \sum_{i \in F} \log \left\{ \frac{\gamma_1^{\lambda-1} (q^{-2}, q^{-2} \exp(q z_i))}{[1 - \gamma_1 (q^{-2}, q^{-2} \exp(q z_i))]^{\lambda+1}} \right\} - \\ \theta \sum_{i \in F} \left[ \frac{\gamma_1 (q^{-2}, q^{-2} \exp(q z_i))}{1 - \gamma_1 (q^{-2}, q^{-2} \exp(q z_i))} \right]^\lambda - \theta \sum_{i \in C} \left[ \frac{\gamma_1 (q^{-2}, q^{-2} \exp(q z_i))}{1 - \gamma_1 (q^{-2}, q^{-2} \exp(q z_i))} \right]^\lambda & \text{if } q > 0, \\ r \log \left[ \frac{\theta \lambda (-q) (q^{-2})^{q^{-2}}}{\Gamma(q^{-2})} \right] - \sum_{i \in F} \log(\sigma_i) + q^{-1} \sum_{i \in F} z_i - \\ q^{-2} \sum_{i \in F} \exp(q z_i) + \sum_{i \in F} \log \left\{ \frac{\gamma_1^{\lambda-1} (q^{-2}, q^{-2} \exp(q z_i))}{[1 - \gamma_1 (q^{-2}, q^{-2} \exp(q z_i))]^{\lambda+1}} \right\} - \theta \sum_{i \in F} \left[ \frac{\gamma_1 (q^{-2}, q^{-2} \exp(q z_i))}{1 - \gamma_1 (q^{-2}, q^{-2} \exp(q z_i))} \right]^\lambda + \\ \sum_{i \in C} \log \left\{ 1 - \exp \left[ -\theta \left\{ \frac{\gamma_1 (q^{-2}, q^{-2} \exp(q z_i))}{1 - \gamma_1 (q^{-2}, q^{-2} \exp(q z_i))} \right\}^\lambda \right] \right\} & \text{if } q < 0, \\ r \log \left( \frac{\theta \lambda}{\sqrt{2\pi}} \right) - \sum_{i \in F} \log(\sigma_i) - \frac{1}{2} \sum_{i \in F} z_i^2 + \\ \sum_{i \in F} \log \left\{ \frac{\Phi^{\lambda-1}(z_i)}{[1 - \Phi(z_i)]^{\lambda+1}} \right\} - \theta \sum_{i \in F} \left[ \frac{\Phi(z_i)}{1 - \Phi(z_i)} \right]^\lambda - \theta \sum_{i \in C} \left[ \frac{\Phi(z_i)}{1 - \Phi(z_i)} \right]^\lambda & \text{if } q = 0. \end{cases} \tag{30}$$

where  $r$  is the number of failures, and  $z_i = (y_i - \mu_i)/\sigma_i$ .

The MLE  $\hat{\tau}$  can be calculated by maximizing Equation (30). Initial parameter values are found by fitting the log-GG regression model.

To find the MLEs, several authors using equation (30) for example: Ortega et al. [16] presented the generalized log-gamma regression models with cure fraction, Cancho et al. [17] considered the Conway-Maxwell-Poisson-generalized gamma regression model with long-term survivors, Ortega et al. [18] introduced the log-exponentiated generalized gamma regression model for censored data, Pascoa et al. [19] introduced the log-Kumaraswamy generalized gamma regression model with application to chemical dependency data, Hashimoto et al. [20] discussed on estimation and diagnostics analysis in log-generalized gamma regression model for interval-censored data, and Pratavia et al. [21] presented the heteroscedastic odd log-logistic generalized gamma regression model for censored data.

## 9. Simulations

A simulation study is conducted to evaluate the effectiveness of the MLEs of the WGG distribution when  $\alpha = 1$ . The mean square errors (MSEs) and absolute biases (ABs) of these estimates for different scenarios are reported in Table 1 using 1,000 samples. The results reveal a reduction in both ABs and MSEs when  $n$  increases. So, the MLEs converge toward the true parameters.

**Table 1.** Simulation results for the WGG distribution

Parameters	$n$	ABs				MSEs			
		$\hat{\beta}$	$\hat{\delta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\theta}$	$\hat{\lambda}$
$\beta = 0.5$	20	0.084	0.091	0.087	0.116	0.183	0.244	0.331	0.381
$\delta = 0.5$	50	0.043	0.083	0.062	0.076	0.103	0.132	0.167	0.181
$\theta = 0.5$	100	0.025	0.078	0.064	0.041	0.013	0.063	0.100	0.096
$\lambda = 0.5$	150	0.005	0.012	0.049	0.003	0.002	0.008	0.093	0.054
$\beta = 0.5$	20	0.148	0.113	0.292	0.268	0.148	0.213	0.320	0.161
$\delta = 0.5$	50	0.120	0.092	0.290	0.224	0.120	0.062	0.258	0.135
$\theta = 2$	100	0.029	0.037	0.077	0.137	0.026	0.041	0.035	0.019
$\lambda = 2$	150	0.009	0.001	0.018	0.002	0.012	0.006	0.031	0.012
$\beta = 2$	20	0.145	0.119	0.150	0.262	1.423	1.036	0.951	0.824
$\delta = 2$	50	0.082	0.111	0.060	0.109	1.241	0.961	0.932	0.508
$\theta = 2$	100	0.026	0.040	0.025	0.051	0.021	0.062	0.106	0.092
$\lambda = 2$	150	0.023	0.004	0.005	0.012	0.012	0.010	0.032	0.069
$\beta = 2$	20	0.522	0.126	0.152	0.164	0.623	0.708	0.120	1.846
$\delta = 2$	50	0.433	0.102	0.025	0.110	0.325	0.239	0.112	0.612
$\theta = 3$	100	0.012	0.002	0.022	0.080	0.196	0.164	0.073	0.035
$\lambda = 3$	150	0.005	0.002	0.007	0.023	0.026	0.074	0.004	0.029
$\beta = 2$	20	0.327	0.185	0.177	0.341	0.626	0.862	0.654	0.618
$\delta = 3$	50	0.173	0.156	0.124	0.166	0.318	0.236	0.456	0.212
$\theta = 5$	100	0.108	0.014	0.079	0.144	0.147	0.063	0.156	0.035
$\lambda = 2$	150	0.007	0.011	0.030	0.103	0.054	0.012	0.139	0.016
$\beta = 0.5$	20	0.148	0.494	0.098	0.399	0.565	0.924	0.271	1.099
$\delta = 3$	50	0.105	0.304	0.065	0.157	0.184	0.816	0.182	0.697
$\theta = 0.5$	100	0.010	0.096	0.006	0.067	0.160	0.556	0.134	0.392
$\lambda = 3$	150	0.009	0.050	0.004	0.005	0.064	0.024	0.101	0.100

## 10. Applications

### 10.1 Carrol data and turbocharger lifetimes

The WGG distribution with scale  $\alpha = 1$  is adopted to model two real data sets: the Carrol data and turbocharger lifetimes. The first data set includes the total monthly rainfall (in millimeters) recorded at the Carrol rain gauge station in the Australian State of New South Wales on the east coast between January 2000 and February 2007 [22]. The second data set gives the lifetime (1,000 h) of a turbocharger of a type of engine [23]. Three classical statistics (AIC, BIC and CAIC), Anderson-Darling ( $A^*$ ) and Kolmogorov-Smirnov (KS) statistics compare the fitted WGG model with four other extended models: Marshall-Olkin generalized-gamma (MOGG) with cdf  $G(x)/[\theta + (1 - \theta)G(x)]$  [5], beta generalized-gamma (BGG) with cdf  $I_{G(x)}(\theta, \lambda)$  [24], exponentiated generalized-gamma (EGG) with cdf  $G(x)^\theta$  [2], and Kumaraswamy generalized-gamma (KwGG) with cdf  $1 - [1 - G(x)^\theta]^\lambda$  [3]. Here,  $I_z(\cdot, \cdot)$  is the incomplete beta function ratio. The findings from the fitted models are reported in Tables 2-5, where the standard errors (SEs) of the MLEs are given in parentheses.

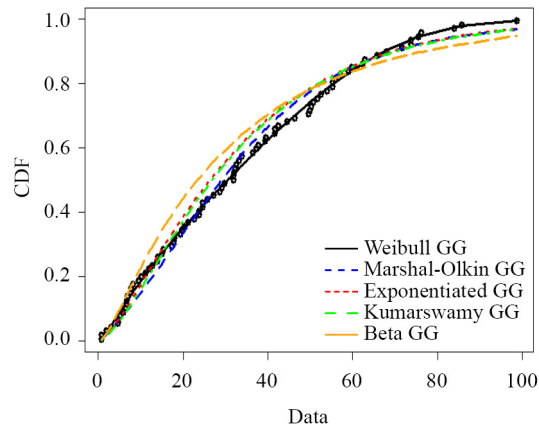
For both data sets, the WGG model provides a better fit than the other distributions. The efficiency of the fitted distributions is illustrated graphically, through a comparison of the empirical and fitted cdfs in Figures 6 and 8. The probability-probability (P-P) (Figures 7 and 9) and the quantile-quantile (Q-Q) plots (Figures 7 and 9) for the fitted WGG distribution show that the proposed distribution is a better model than the others.

**Table 2.** Estimates and their SEs for Carol data

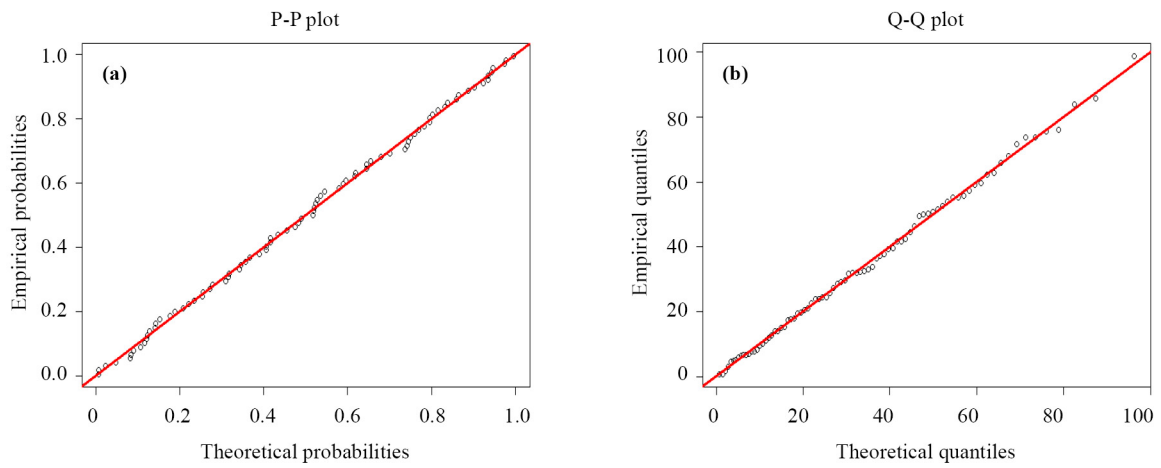
Model	$\hat{\beta}$	$\hat{\delta}$	$\hat{\theta}$	$\hat{\lambda}$
WGG	0.5833 (0.1258)	6.4540 (3.1330)	0.4864 (0.2164)	0.4733 (0.2656)
MOGG	0.4950 (0.0243)	3.1126 (0.9900)	7.8893 (6.9068)	- -
EGG	0.6382 (0.0120)	14.6952 (0.0281)	0.1836 (0.0215)	- -
KwGG	0.4615 (0.2974)	14.7271 (0.0518)	0.2576 (0.1371)	6.7161 (26.907)
BGG	0.8210 (0.0025)	3.0079 (0.0318)	0.9996 (0.2752)	0.0799 (0.0091)

**Table 3.** Statistics and the  $p$ -values of the KS statistic for Carol data

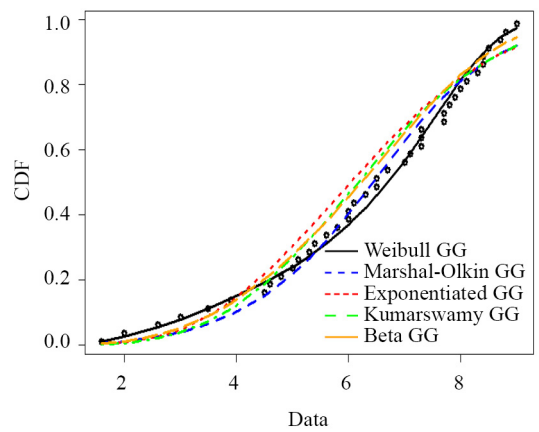
Model	AIC	BIC	CAIC	$A^*$	KS	$p$ -value
WGG	742.73	752.40	743.24	0.1065	0.038	0.9998
MOGG	748.51	755.77	748.82	0.5218	0.0717	0.7872
EGG	748.67	755.92	748.97	0.6245	0.090	0.5102
KwGG	752.31	761.98	752.82	0.7188	0.084	0.6021
BGG	758.50	768.17	759.01	1.0806	0.122	0.1660



**Figure 6.** Empirical and fitted cdfs for Carol data



**Figure 7.** PP and QQ plots for the fitted WGG model to Carol data



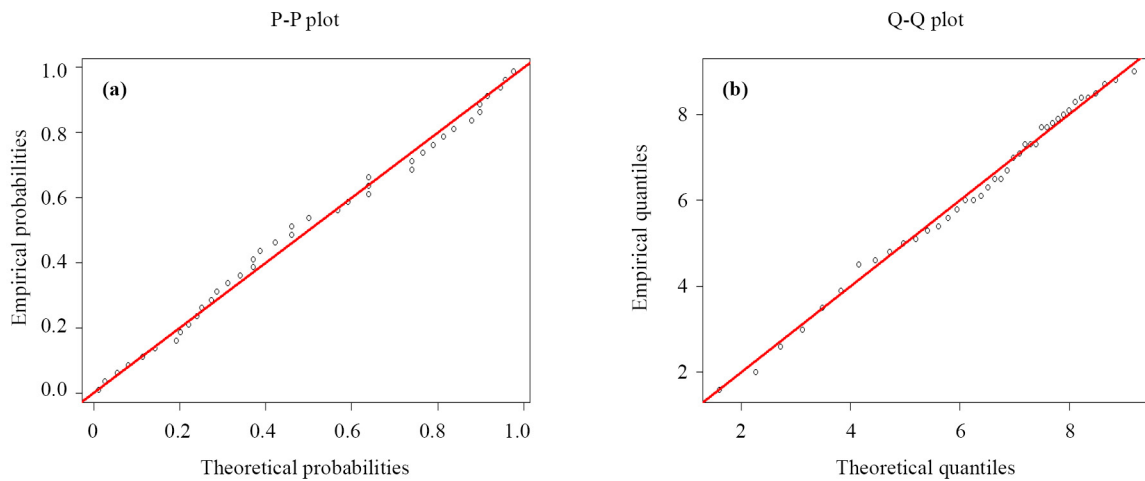
**Figure 8.** Empirical and fitted cdfs for turbocharger data

**Table 4.** Estimates and their SEs for turbocharger data

Model	$\hat{\beta}$	$\hat{\delta}$	$\hat{\theta}$	$\hat{\lambda}$
WGG	1.4150 (0.2776)	8.6175 (4.9399)	0.2090 (0.1353)	0.3549 (0.2318)
MOGG	1.0872 (0.0747)	3.8715 (1.6642)	18.9706 (23.2691)	- -
EGG	1.3841 (0.0334)	19.6289 (0.0522)	0.1922 (0.0347)	- -
KwGG	0.9402 (0.0982)	17.9986 (0.0541)	0.2943 (0.0456)	16.2560 (15.5161)
BGG	1.2019 (0.0230)	19.3895 (0.0209)	0.1859 (0.0318)	9.2041 (4.6018)

**Table 5.** Statistics and  $p$ -values of the KS statistic for turbocharger data

Model	AIC	BIC	CAIC	$A^*$	KS	$p$ -value
WGG	165.14	171.90	166.29	0.1565	0.0654	0.9955
MOGG	171.49	176.55	172.15	0.5505	0.0898	0.9033
EGG	174.81	179.88	175.48	0.9015	0.1263	0.5457
KwGG	176.25	183.00	177.39	0.8268	0.1155	0.6601
BGG	170.71	177.47	171.85	0.4617	0.1106	0.7119



**Figure 9.** PP and QQ plots for the fitted WGG model to turbocharger data

## 10.2 Prostate cancer recurrence data

This section provides an application of the LWGG regression model to prostate cancer data. The study cohort comprises 1,324 patients with clinically localized prostate cancer, between 1987 and 2003, treated by a single surgeon by open radical prostatectomy in the Cleveland Clinic. It was measured the number of months ( $y_i$ ) without detectable disease after radical prostatectomy. Uncensored observations correspond to patients having cancer recurrent time computed. The numbers of censored and uncensored observations are 1,096 and 228, respectively, from a total of 1,324 patients. The explanatory variables below are associated with each patient (for  $i = 1, \dots, 1,324$ ):

- $\delta_i$  is the event indicator, where 1 represents the event and 0 the censored observation;
- $v_{i1}$  indicates if the patient was treated with hormone therapy before the radical prostatectomy (yes = 1 and no = 0);
- $v_{i2}$  is the PSA value (in ng/mL) before the surgery;
- $v_{i3}$  is the extra-capsular extension on path report (yes = 1, no = 0);
- $v_{i4}$  is the seminal vesicle invasion on path report (yes = 1, no = 0);
- $v_{i5}$  is the lymph node involvement on path report (neg = 1, pos = 0);
- $v_{i6}$  is the Gleason score sum [4, 7) versus 7;
- $v_{i7}$  is the Gleason score sum [4, 7) versus [8, 10];
- $v_{i8}$  is the surgical margin status (yes = 1, no = 0).

The two systematic components are given by

$$\mu_i = \beta_{10} + \sum_{j=1}^8 \beta_{1j} v_{ij} \quad \text{and} \quad \sigma_i = \exp \left( \beta_{20} + \sum_{j=1}^8 \beta_{2j} v_{ij} \right).$$

The MLEs of the parameters are found using the NLMixed procedure in SAS. Iterative maximization of Equation (30) starts with initial parameter values taken from the fitted LWGG regression model.

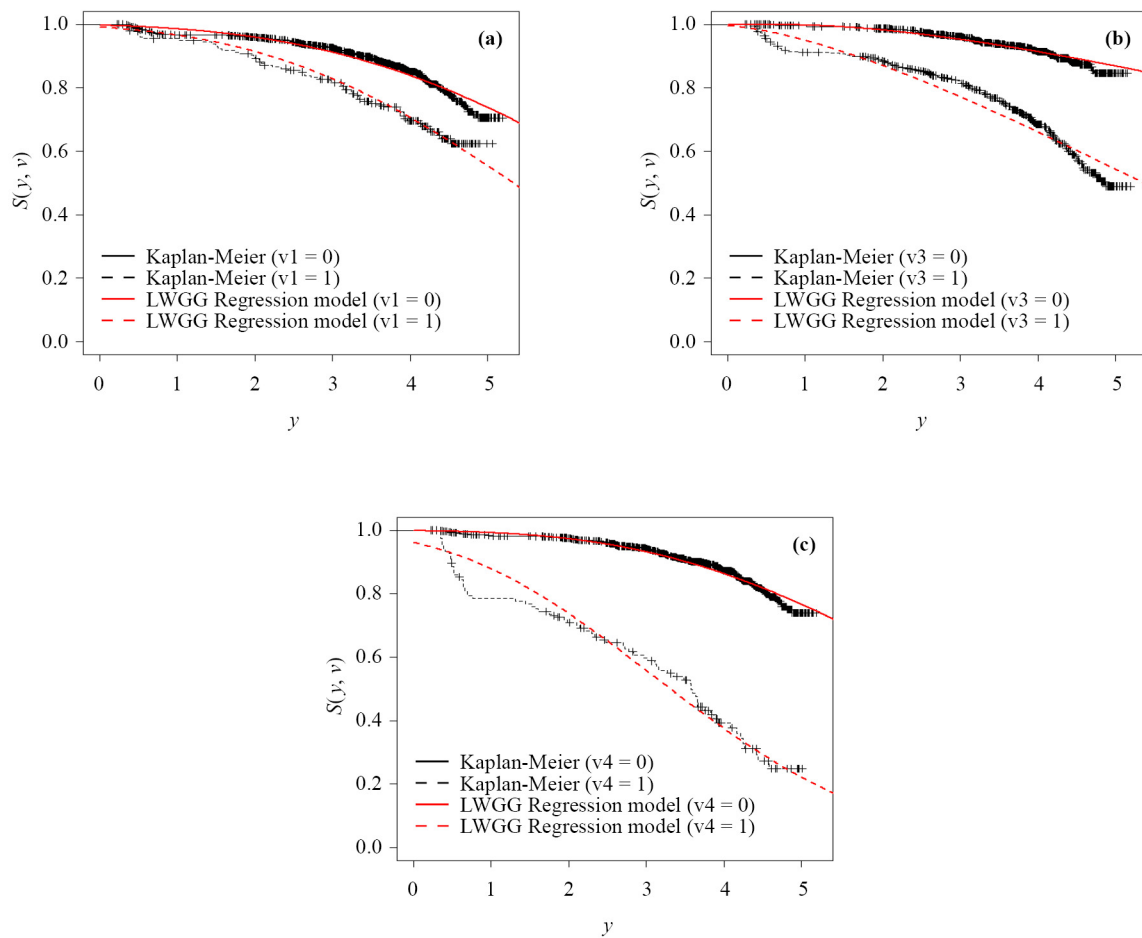
Table 6 reports the AIC, CAIC, and BIC statistics from the fitted LWGG and LGG regressions. The lowest values for the LWGG model reveal that it is an appropriate regression to explain the current data.

**Table 6.** Model selection measures

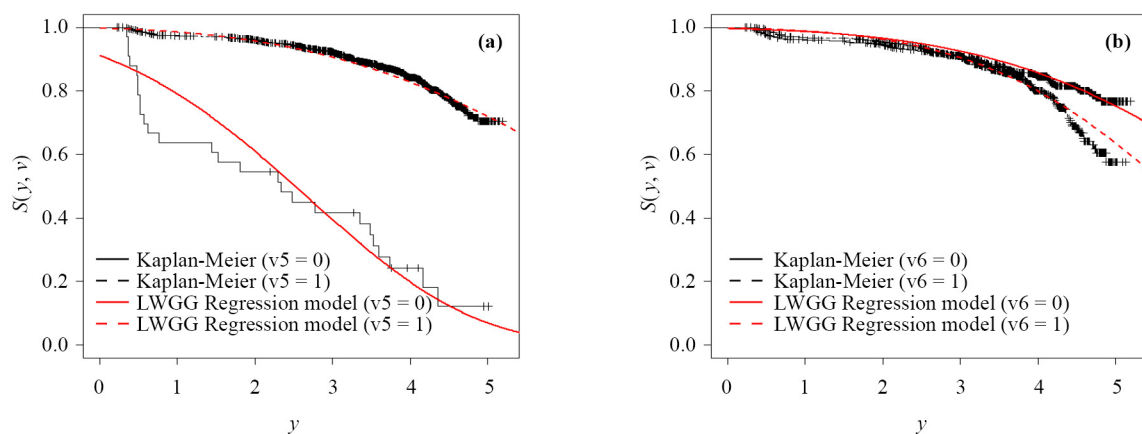
Regression model	AIC	CAIC	BIC
LWGG	1459.3	1460.0	1568.2
LGG	1855.5	1856.1	1954.1

Table 7 gives the MLEs, SEs, and  $p$ -values for the best fitted model. All covariates are significant at a significance level of 6% for the parameter  $\mu$ . Concerning the parameter  $\sigma$ , only the covariates  $v_2$  and  $v_8$  are significant at a significance level of 5%.

Further, we fit the LWGG regression model for each explanatory variable separately and plot the empirical survival function and the estimated survival function (27) for each explanatory variable in Figures 10, 11, and 12. So, this model provides a good fit for these data.

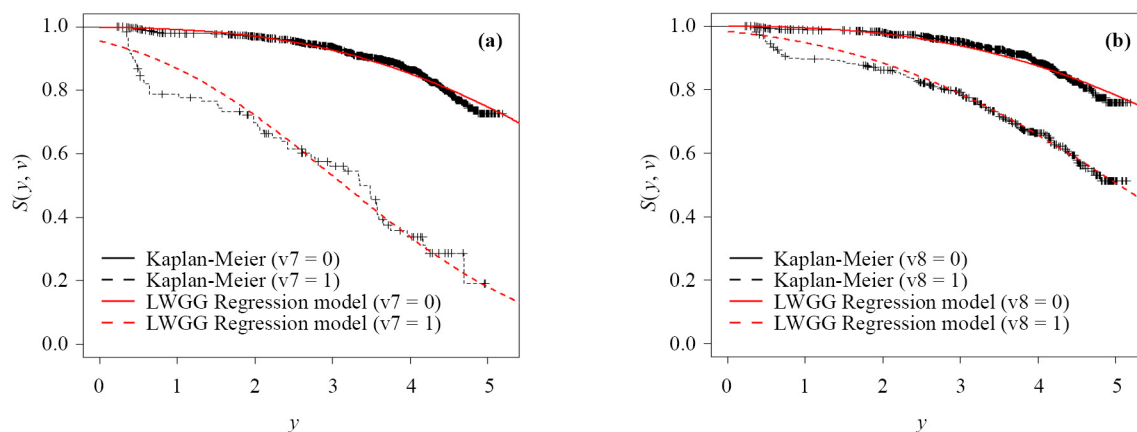


**Figure 10.** Kaplan-Meier curves stratified by explanatory variable and estimated survival functions to the recurrence prostate cancer data: (a)  $v_1$  explanatory variable; (b)  $v_3$  explanatory variable; (c)  $v_4$  explanatory variable



**Figure 11.** Kaplan-Meier curves stratified by explanatory variable and estimated survival functions to the recurrence prostate cancer data: (a)  $v_5$  explanatory variable; (b)  $v_6$  explanatory variable





**Figure 12.** Kaplan-Meier curves stratified by explanatory variable and estimated survival functions to the recurrence prostate cancer data: (a)  $v_7$  explanatory variable; (b)  $v_8$  explanatory variable

**Table 7.** Results from the fitted LWGG regression model

Parameter	Estimate	SE	$p$ -value
$\beta_{10}$	6.4742	1.8885	0.0006
$\beta_{11}$	-0.4643	0.2414	0.0547
$\beta_{12}$	-0.0394	0.0189	0.0368
$\beta_{13}$	-0.9346	0.3275	0.0044
$\beta_{14}$	-0.9083	0.2602	0.0005
$\beta_{15}$	0.7199	0.3671	0.0501
$\beta_{16}$	-1.0670	0.4250	0.0122
$\beta_{17}$	-2.3538	0.5814	<0.0001
$\beta_{18}$	-0.7971	0.3829	0.0375
$\beta_{20}$	1.3588	0.5524	0.0140
$\beta_{21}$	-0.0143	0.1065	0.8935
$\beta_{22}$	0.0071	0.0031	0.0226
$\beta_{23}$	-0.0796	0.1193	0.5048
$\beta_{24}$	0.0174	0.1501	0.9075
$\beta_{25}$	0.0624	0.1536	0.6849
$\beta_{26}$	-0.1060	0.1282	0.4085
$\beta_{27}$	-0.1747	0.1828	0.3394
$\beta_{28}$	0.2006	0.0984	0.0416
$q$	-1.6156	0.6356	
$\theta$	2.1379	5.5324	
$\lambda$	1.8609	0.4081	

## 11. Conclusions

We proposed the Weibull generalized gamma (WGG) distribution, derived a new linear representation for its density function, and determined some of its structural properties. We introduced a log-linear regression model based on the new distribution with two systematic components for censored data. We examined the maximum likelihood estimation

of the parameters. We proved that the extended regression model can be useful in the analysis of real data with more realistic fits than other special regression models. The potentiality of the new distribution and regression models was illustrated by employing three real data sets. Further, other works in a similar manner can be developed to extend several regression models in the literature, such as the COM-Poisson cure rate survival models, destructive negative binomial cure rate models, Conway-Maxwell-Poisson generalized gamma regression models, a power series beta Weibull regression model, survival models induced by discrete frailty for modeling lifetime data with long term survivors, power series cure rate models for spatially correlated interval-censored data, and long-term bivariate survival Farlie-Gumbel-Morgenstern copula models (bivariate case).

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## Conflict of interest

The authors declare no competing financial interest.

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