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Research Article

Alleviated Shifted Gegenbauer Spectral Method for Ordinary and Fractional Differential Equations

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Abstract: This article introduces two numerical methods to address boundary value problems associated with secondorder and fractional differential equations. These methods employ two parameters related to shifted Gegenbauer polynomials as their basis functions. The process involves establishing a differentiation operational matrix for the shifted Gegenbauer polynomials. Subsequently, the initial/boundary value problems for ordinary and fractional differential equations are transformed into a system of equations through the Galerkin, collocation, and tau methods. The convergence analysis is ensured by leveraging theorems pertaining to the shifted Gegenbauer polynomials. To validate the accuracy of the approach, numerous numerical examples are presented.

***Keywords*:** shifted gegenbauer polynomials, boundary value problems, fractional differential equations, bagley-trovik equation, spectral methods, convergence analysis

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1. Introduction

The many beneficial characteristics of orthogonal polynomials, like Gegenbauer, highlight their usefulness in the numerical solution of various differential equations. Gegenbauer polynomials are very appropriate for a wide range of applications because they have orthogonality, exponential accuracy, and a single parameter that may greatly affect the range of approximations. Their regular usage in several research attests to their efficacy. For example, the authors of [1] used shifted Gegenbauer polynomials to solve the space fractional diffusion equation problems. Furthermore, as the authors of [2] show, the Laguerre and Gegenbauer polynomials together provided the basis for a meshless Petrov-Galerkin method that was especially used for spinning Rayleigh beams. Gegenbauer polynomials are also used in harmonic analysis and potential theory, where they easily emerge as Legendre polynomial extensions. Also employed in the theory of the positive-definite functions. When a function is enlarged by the Gegenbauer basis in spectral approaches to solving fractional differential equations, the derivative operator transforms into a diagonal matrix, allowing for quick banded matrix solutions to large problems.

As noted in [3] and [4], second-order boundary value problems (BVPs) have important uses in the field of mathematical modeling in situations such as the displacement analysis of cantilever beams under concentrated tension,

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plate perversion theories, and the investigation of ray distortion. Several writers have explored the numerical solution of second-order BVPs, demonstrating the variety of methods in this field [5, 6]. In [5], for example, Walsh wavelets were used to improve the numerical approach to solving second-order BVPs under both Dirichlet and Neumann boundary conditions. This comprehensive investigation of numerical solutions and mathematical models attests to the adaptability and usefulness of different polynomial approaches in solving a range of differential problems.

With the introduction of the groundbreaking field known as fractional calculus, differential equations and fractional calculus have solidified their positions as essential elements of mathematical analysis. Because of its wide range of applications in the biological, engineering, and industrial sectors, this subject has seen a rise in prominence [7, 8]. Fractional calculus has developed into a significant area of mathematical study over time, with applications in many different domains, including physics, fluid dynamics, and biology. The significance lies in the fact that fractional differential equations can emulate a multitude of phenomena in these fields. Given the analytical challenges associated with acquiring solutions for many of these equations, the pursuit of numerical estimates becomes imperative, for recent advances in fractional calculus, the interested reader can see [9–11].

The field of fractional differential equations has seen significant advancements thanks to the work of several academics, which has resulted in the creation of numerous numerical techniques for solving them. Taylor collocation method [12], tau method [13], and improved collocation method [14], as well as the Petrov-Galerkin method [15] are notable techniques. In the broader context of numerical solutions for differential equations, spectral methods, like those employing smooth Gegenbauer polynomials, have gained widespread adoption [16–18]. This study focuses on employing three distinct spectral methods-tau, collocation, and Galerkin-to numerically solve linear and non-linear boundary value problems as well as fractional differential equations. These methods draw on a variety of polynomial expansions, including shifted Gegenbauer polynomials.

The paper follows the following structure: Shifted Gegenbauer polynomials are introduced in detail in Section 2 which also covers their basic relations. The operational matrix for shifted Gegenbauer polynomials, Section 3 is devoted to solving second-order boundary value problems, both linear and non-linear. Section 4 is for the derivation of the operational matrix designed for fractional differential equations. This matrix is important in the proposed methodology and facilitates a systematic and efficient approach to solving fractional differential equations. Section 5 offers a detailed examination of the convergence and error analysis of the proposed method. This section assesses the reliability and accuracy of the method. Section 6 presents numerical results and comparisons. Section 7 provides concluding remarks to report the key findings of the proposed method.

2. Some helpful formulae for shifted gegenbauer polynomials

In this section, some basic relations of the shifted Gegenbauer polynomials' properties and some helpful formulae are also shown. The explicit formula given below can be used to compute the Gegenbauer polynomials $C_q^{(\alpha)}(\chi)$, which are based on the interval $[-1,\ 1]$:

$$\begin{aligned} C_q^{(\alpha)}(\chi) &= \sum_{m=0}^{\lfloor \frac{q}{2} \rfloor} \frac{(-1)^m\,\Gamma(q-m+\alpha)}{m!\,\Gamma(\alpha)\,(q-2m)!}\,(2\chi)^{q-2m}, \\ C_q^{(\alpha)}(1) &= \frac{\Gamma(q+2\alpha)}{q!\,\Gamma(2\alpha)}. \end{aligned} \tag{1}$$

The recurrence equation of the Gegenauer polynomials:

$$(2\alpha+2q)\chi C_q^{(\alpha)}(\chi) = qC_{q-1}^{(\alpha)}(\chi)+(2\alpha+q)C_{q+1}^{(\alpha)}(\chi), \quad q=1,2,\ldots, \tag{2}$$

where

$$C_0^{(\alpha)}(\chi)=1, \quad C_1^{(\alpha)}(\chi)=\chi.$$

In Eqs. (1) and (2), by adding the new variable $\chi=\frac{2\chi}{L}-1$, so $\tilde{C}_q^{(\alpha)}(\chi)=C_q^{(\alpha)}(\frac{2\chi}{L}-1)$. Then we may construct the shifted Gegenbauer polynomials [19] and the recurrence relation of the shifted Gegenbauer polynomials and use them on the interval $[0, L]$. An explicit formula presented can be used to find the qth-order shifted Gegenbauer polynomials

$$\begin{aligned} \tilde{C}_q^{(\alpha)}(\chi) &= \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(2\alpha)}\sum_{m=0}^{q}\frac{(-1)^{q-m}L^{-m}\Gamma(q+m+2\alpha)}{m!\,\Gamma(\alpha+m+\frac{1}{2})\,(q-m)!}\chi^m, \\ \tilde{C}_q^{(\alpha)}(0) &= \frac{(-1)^q\Gamma(2\alpha+q)}{q!\,\Gamma(2\alpha)}, \end{aligned} \tag{3}$$

the recurrence equation of the shifted Gegenbauer polynomials:

$$(2q+2\alpha)(\frac{2\chi}{L}-1)\tilde{C}_q^{(\alpha)}(\chi) = q\tilde{C}_{q-1}^{(\alpha)}(\chi)+(2\alpha+q)\tilde{C}_{q+1}^{(\alpha)}(\chi), \quad q=1,2,\ldots,$$

with

$$\tilde{C}_0^{(\alpha)}(\chi)=1, \quad \tilde{C}_1^{(\alpha)}(\chi)=\frac{2\chi}{L}-1.$$

According to L^2-space on the range $[0, L]$, the shifted Gegenbauer polynomials $\tilde{C}_q^{(\alpha)}(\chi)$ provided in Eq. (3) are orthogonal [19].

$$\int_0^L \tilde{C}_q^{(\alpha)}(\chi)\tilde{C}_p^{(\alpha)}(\chi)\,\theta_L^{(\alpha)}(\chi)\,d\chi = \begin{cases} h_q^{(\alpha)}, & q=p, \\ 0, & q\neq p, \end{cases}$$

where $h_q^{(\alpha)}$ and $\theta_L^{(\alpha)}(\chi)$ signify the normalizing factor and the weight function, respectively, and are provided as

$$h_q^{(\alpha)} = \frac{\pi L^{2\alpha}2^{1-4\alpha}\Gamma(q+2\alpha)}{\Gamma^2(2\alpha)\,q!\,(q+\alpha)},$$

$$\theta_L^{(\alpha)}(\chi) = (L\chi-\chi^2)^{\alpha-\frac{1}{2}}.$$

This polynomial represents the shifted Legendre polynomial $Z_q(\chi) = \tilde{C}_q^{(\frac{1}{2})}(\chi)$, the first kind of shifted Chebyshev polynomial $T_q(\chi) = \tilde{C}_q^{(0)}(\chi)$, and the second kind of the shifted Chebyshev polynomial $U_q(\chi) = (q+1)\tilde{C}_q^{(1)}(\chi)$.

3. Shifted gegenbauer operational matrix of BVPs of second-order

This section's main goal is to establish the operational matrix derivatives of the shifted Gegenbauer polynomials and then use those values to solve linear and non-linear BVPs.

3.1 *Linear boundary value problems of second-order*

The following second-order BVP in one dimension is as follows:

$$\omega''(\chi) + g_1(\chi)\,\omega'(\chi) + g_2(\chi)\,\omega(\chi) = g_3(\chi), \tag{4}$$

controlled by non-homogeneous boundary conditions:

$$\omega(0) = \gamma, \quad \omega(L) = \zeta. \tag{5}$$

According to the subsequent transformation [20]:

$$\omega(\chi) = u(\chi) + \frac{\gamma(L-\chi) + \zeta\,\chi}{L}, \tag{6}$$

It is simple to demonstrate that (4)-(5) may be adjusted to become the following:

$$u''(\chi) + g_1(\chi)\,u'(\chi) + g_2(\chi)\,u(\chi) = \varepsilon(\chi), \tag{7}$$

where

$$\varepsilon(\chi) = \varepsilon_1(\chi) - \left(\frac{\zeta-\gamma}{L}\right) g_1(\chi) - \left(\frac{\gamma(L-\chi)+\zeta\,\chi}{L}\right) g_2(\chi), \tag{8}$$

according to the next homogeneous boundary conditions

$$u(0) = u(L) = 0. \tag{9}$$

The numerical expression for (7) is as follows:

$$u(\chi) \approx u_M(\chi) = \sum_{q=0}^{M} c_q\,\Phi_q(\chi) = \mathbf{C}^T\,\boldsymbol{\Phi}(\chi),$$

in which

$$\mathbf{C} = [c_0, c_1, \ldots, c_M]^T,$$

and

$$\boldsymbol{\Phi}(\chi) = [\Phi_0(\chi), \Phi_1(\chi), \ldots, \Phi_M(\chi)]^T.$$

In this research, we define the basis function as a compact combination of shifted Gegenbauer polynomials, for instance:

$$\Phi_q(\chi) = P_q(\chi) - P_{q+2}(\chi), \tag{10}$$

where

$$P_q(\chi) = \frac{q!\,\tilde{C}_q^{(\alpha)}(\chi)}{(2\alpha)_q}. \tag{11}$$

Theorem 1 For all $q \geq 0$, there exist an alternative form for the shifted Gegenbauer polynomials $\Phi_q(\chi)$:

$$\Phi_q(\chi) = \frac{\alpha\, q!\,(1+q+\alpha)\,(2\alpha)}{(2+q+2\alpha)}\,\chi\,(L-\chi)\,(\frac{4}{L})^2\,\tilde{C}_q^{(\alpha+1)}(\chi). \tag{12}$$

Proof. From relation (10) and Eq. (11), we get

$$\Phi_q(\chi) = \frac{q!\,\tilde{C}_q^{(\alpha)}(\chi)}{(2\alpha)_q} - \frac{(q+2)!\,\tilde{C}_{q+2}^{(\alpha)}(\chi)}{(2\alpha)_{q+2}}, \tag{13}$$

The Pochhammer symbol is defined as

$$\begin{aligned}(z)_w &= \frac{(z+w)}{(z)}\\ &= z\,(z+1)\ldots(z+w-1).\end{aligned} \tag{14}$$

If we employ the last relation (14) in (13), we get

$$\begin{aligned}\Phi_q(\chi) =& \frac{q!\,(2\,\alpha)\,\tilde{C}_q^{(\alpha)}(\chi)}{(2\,\alpha+q)} - \frac{(q+2)!\,(2\,\alpha)\,\tilde{C}_{q+2}^{(\alpha)}(\chi)}{(2\,\alpha+q+2)} \\ =& q!\,(2\,\alpha)\left(\frac{\tilde{C}_q^{(\alpha)}(\chi)}{(2\,\alpha+q)} - \frac{(q+2)\,(q+1)\tilde{C}_{q+2}^{(\alpha)}(\chi)}{(2\,\alpha+q+2)}\right) \\ =& \frac{q!\,(2\,\alpha)}{(q+2\,\alpha+2)}\left((q+2\,\alpha+1)\,(q+2\,\alpha)\,\tilde{C}_q^{(\alpha)}(\chi) - (q+2)\,(q+1)\,\tilde{C}_{q+2}^{(\alpha)}(\chi)\right) \\ =& \frac{\alpha\,q!\,(1+q+\alpha)\,(2\,\alpha)}{(2+q+2\,\alpha)}\,\chi\,(L-\chi)\,(\frac{4}{L})^2\,\tilde{C}_q^{(\alpha+1)}(\chi).\end{aligned}$$

□

Theorem 2 The orthogonality on the range $[0,\ L]$ of $\Phi_q(\chi)$ in (10) provides by

$$\int_0^L \Phi_p(\chi)\,\Phi_q(\chi)\,\tilde{\theta}_L^{(\alpha)}(\chi)\,d\chi = \begin{cases}\tilde{h}_q^{(\alpha)}, & q=p, \\ 0, & q \neq p,\end{cases}$$

where

$$\tilde{h}_q^{(\alpha)} = \frac{8\,L^{-2+2\,\alpha}\,(1+p+\alpha)\,(1+q)^2(\frac{1}{2}+\alpha)}{(2+q+2\,\alpha)},$$

and the weight function

$$\tilde{\theta}_L^{(\alpha)}(\chi) = (L\chi-\chi^2)^{\alpha-\frac{3}{2}}.$$

Proof. The formula $\Phi_q(\chi)$ in (12) yields

$$\Phi_p(\chi)\,\Phi_q(\chi)\,\tilde{\theta}_L^{(\alpha)}(\chi) = \frac{256\,(L\chi-\chi^2)^{\alpha+\frac{1}{2}}\,\alpha^2\,(1+q+\alpha)\,(1+p+\alpha)\,q!\,p!^2(2\,\alpha)\,\tilde{C}_q^{(\alpha+1)}(\chi)\,\tilde{C}_p^{(\alpha+1)}(\chi)}{L^4\,(2+q+2\,\alpha)\,(2+p+2\,\alpha)}. \tag{15}$$

When we integrate both sides in (15) from 0 to L, we get

$$\int_0^L \Phi_p(\chi)\,\Phi_q(\chi)\,\tilde{\theta}_L^{(\alpha)}(\chi)\,d\chi = \frac{256\,\alpha^2\,(1+q+\alpha)\,(1+p+\alpha)\,q!\,p!^2(2\,\alpha)}{L^4\,(2+q+2\,\alpha)\,(2+p+2\,\alpha)}$$

$$\int_0^L \left((L\chi-\chi^2)^{\alpha+\frac{1}{2}}\,\tilde{C}_q^{(\alpha+1)}(\chi)\,\tilde{C}_p^{(\alpha+1)}(\chi)\right)\,d\chi$$

$$= \frac{256\,\alpha^2\,(1+q+\alpha)\,(1+p+\alpha)\,q!\,p!^2(2\,\alpha)}{L^4\,(2+q+2\,\alpha)\,(2+p+2\,\alpha)}$$

$$\begin{cases} \dfrac{L^{2+2\,\alpha}\,(2+q+2\,\alpha)^2(\alpha+\frac{1}{2})\,(\alpha+1)}{32\,(1+q+\alpha)\,q!\,p!\,\alpha^{2\,2}(2\,\alpha)}, & q=p, \\ 0, & q\neq p, \end{cases}$$

$$= \begin{cases} \dfrac{8\,L^{-2+2\,\alpha}\,(1+p+\alpha)\,(1+q)^2(\frac{1}{2}+\alpha)}{(2+q+2\,\alpha)}, & q=p, \\ 0, & q\neq p, \end{cases}$$

$$= \begin{cases} \tilde{h}_q^{(\alpha)}, & q=p, \\ 0, & q\neq p. \end{cases}$$

□

When $g_1(\chi)$ and $g_2(\chi)$ are constants in Eq. (4), they can be represented as k and v, respectively. Thus, we possess the matrices of the shifted Gegenbauer polynomials and their derivatives.

The shifted Gegenbauer polynomials can be written by matrix as:

$$\Phi_n(\chi) = \sum_{k=0}^{n} a_{k,\,n}\,\Phi_k(\chi), \quad n\geq 0,$$

with

$$a_{k,\,n} = \begin{cases} \dfrac{L^{2\,\alpha}\,\Gamma(1+n)\,\Gamma^2(\frac{1}{2}+\alpha)}{2\,(k+\alpha)\,\Gamma(k+2\,\alpha)}, & \text{if } n-k=0, \\ \dfrac{L^{2\,\alpha}\,\Gamma(1+n)\,\Gamma^2(\frac{1}{2}+\alpha)}{2\,(2+k+\alpha)\,\Gamma(2+k+2\,\alpha)}, & \text{if } n-k=2, \\ 0, & \text{otherwise.} \end{cases}$$

For the shifted Gegenbauer polynomials, the first derivative can be described as:

$$D\,\Phi_n(\chi) = \sum_{k=0}^{n-1} b_{k,\,n}\,\Phi_k(\chi), \quad n\geq 1,$$

whereas

$$b_{k,\,n}=\begin{cases}-\dfrac{2L^{-1+2\alpha}\,\Gamma(2+n)\,\Gamma^2(\frac{1}{2}+\alpha)}{\Gamma(2+k+2\alpha)}, & \text{if } n-k=1,\\ \dfrac{4L^{2\alpha-1}\,(1+k+\alpha)\,(2\alpha-1)\,\Gamma(1+k)\,\Gamma^2(\frac{1}{2}+\alpha)}{\Gamma(2+k+2\alpha)}, & \text{if } k>n,\ (n-k) \text{ odd},\\ 0, & \text{otherwise.}\end{cases}$$

The second derivative of the shifted Gegenbauer polynomials can be stated as:

$$D^2\,\Phi_n(\chi)=\sum_{k=0}^{n-2} d_{k,\,n}\,\Phi_k(\chi),\quad n\geq 2,$$

where

$$d_{k,\,n}=-\frac{4L^{2\alpha-2}\,(1+k+\alpha)\,(-4-3k\,(2+k)+n^2+2\,(k-n)\,(n+k-1)\,\alpha+4\,(k-n)\,\alpha^2)\,\Gamma(k+1)\,\Gamma^2(\alpha+\frac{1}{2})}{\Gamma(2+k+2\alpha)}.$$

Therefore, we can calculate the residual of Equation (7) by

$$R(\chi)=\sum_{n=0}^{M} c_n\,\left(d_{k,\,n}+k\,b_{k,\,n}+\nu\,a_{k,\,n}\right)+\int_0^L \varepsilon(\chi)\,P_j(\chi)\,\theta_L^{\alpha}(\chi)\,d\chi. \tag{16}$$

By the tau method, we get

$$\int_0^L R_M(\chi)\,\Phi_n(\chi)\,\theta_L^{\alpha}(\chi)d\chi=0,\quad n=0,1,\ldots,M-q, \tag{17}$$

if we apply the boundary conditions, then we obtain that

$$\mathbf{C}^T\,\boldsymbol{\Phi}(0)=0 \text{ and } \mathbf{C}^T\,\boldsymbol{\Phi}(L)=0. \tag{18}$$

Now, we construct a linear system of (M + 1) equations in the unknowns $\{c_n\}_{0\leq n\leq M}$ by combining (17) and (18). Then solve them by the Gauss elimination method.

3.2 *Non-linear boundary value problems of second-order*

The following second-order non-linear boundary value problems:

$$\omega''(\chi)=g\left(\chi,\,\omega(\chi),\,\omega'(\chi)\right), \tag{19}$$

where the homogeneous boundary conditions

$$\omega(0) = \omega(L) = 0.$$

The residual of Eq. (19)

$$R_M(\chi) = \omega''(\chi) - g\left(\chi, \omega(\chi), \omega'(\chi)\right), \tag{20}$$

if we apply the collocation method, we have

$$R_M(\chi_n) = 0, \quad n = 0, 1, \ldots, M-2, \tag{21}$$

and the boundary conditions

$$\mathbf{C}^T \boldsymbol{\Phi}(0) = \mathbf{C}^T \boldsymbol{\Phi}(L) = 0. \tag{22}$$

By combining (21) and (22), we get the non-linear system of $(M + 1)$ equations in the unknowns $\mathbf{C}^T$.

Algorithm 1 Coding algorithm for our method to the linear and non-linear BVPs.

Input M.

Step 1. Use transformation (6) to convert Eqs. (4)-(5) into Eqs. (7), (8), and (9).

Step 2. Suppose an approximate solution is $u_M(\chi) = \mathbf{C}^T \boldsymbol{\Phi}(\chi)$.

Step 3. Get the elements of the matrices $a_{k,\, n}$, $b_{k,\, n}$, and $d_{k,\, n}$.

Step 4. Get the residual of the Eqs. (16) or (20).

Step 5. Use the Galerkin method for obtaining the system in (17) and (18) or in (21) and (22).

Step 6. To solve the system (17) and (18) or in (21) and (22), apply the *NSolve* command to obtain **C**.

Step 7. Establish $u_M(\chi) = \mathbf{C}^T \boldsymbol{\Phi}(\chi)$.

Output $\omega_M(\chi)$.

4. Shifted gegenbauer operational matrix of FDE

In this section, some basic definitions of the theory of fractional calculus will be provided. Then we study the effect of fractional calculus on the shifted Gegenbauer polynomials.

4.1 *Some aspects of fractional calculus*

Definition 1 In terms of order $\iota > 0$, $t > 0$, and $n \in \mathbb{N}$, the Caputo fractional derivative is defined as [21]:

$$(D^{\iota} f)(\chi) = \begin{cases} \dfrac{d^n}{d\chi^n} f(\chi), & \iota = n, \\ \dfrac{1}{(n-\iota)} \displaystyle\int_0^{\chi} f^{(n)}(\zeta)\,(\chi - \zeta)^{n-1-\iota}\, d\zeta, & n-1 < \iota < n. \end{cases}$$

Definition 2 For all $n \in \mathbb{N}$ and order $\iota > 0$, the Riemann-Liouville fractional derivative is defined as [21]:

$$\begin{cases} (D_*^\iota f)(\chi) = \dfrac{1}{(n-\iota)} \dfrac{d^n}{d\chi^n} \displaystyle\int_0^\chi f(\zeta)\,(\chi-\zeta)^{n-1-\iota}\, d\zeta,\ n-1 \le \iota < n, \\ (D_*^0 f)(\chi) = f(\chi). \end{cases}$$

The Riemann-Liouville and Caputo operators' relationship:

$$D_*^\iota f(\chi) = D^\iota f(x) + \sum_{s=0}^{n-1} \frac{\chi^{s-\iota}\, f^{(s)}(0)}{(s+1-\iota)}.$$

Some features of the fractional derivatives:

- $D^\iota \chi^\beta = \dfrac{(1+\beta)}{(1+\iota+\beta)}\, \chi^{\beta+\iota}$,
- $D_*^\iota D^\iota f(\chi) = f(\chi)$,
- $D^\iota D_*^\iota f(\chi) = f(\chi) - \displaystyle\sum_{s=0}^{n-1} \frac{\chi^s f^{(s)}(0^+)}{s!}$.

See [22] for further information on fractional calculus.

4.2 *The tau method for handling linear fractional differential equations*

This subsection discusses the numerical solution of linear fractional differential equations using the tau procedure. Consider the next linear fractional differential equations:

$$D^{m_q}\,\omega(\chi) + \sum_{p=1}^{q-1} \xi_p(\chi)\, D^{m_p}\,\omega(\chi) + \varepsilon(\chi)\,\omega(\chi) = \rho(\chi), \tag{23}$$

managed by the initial conditions:

$$\omega^{(\mu)}(0) = z_\mu, \quad \mu = 0, 1, \ldots, q-1, \quad q \ge 1, \tag{24}$$

or the boundary conditions:

$$\omega^{(\mu)}(0) = z_\mu, \quad \omega^{(\mu)}(1) = v_\mu, \quad \mu = 0, 1, \ldots, q-1, \tag{25}$$

where $m_1 < m_2 < \ldots < m_q$ and $m_j \in (j-1,\ j];\ j \in \mathbb{N}$ and $\xi_p(\chi),\ \varepsilon(\chi),$ and $\rho(\chi)$ are continuous functions.

To derive a numerical solution for Eq. (23), we assume that

$$\omega(\chi) \approx \omega_M(\chi) = \sum_{n=0}^{M} c_n\, \Phi_n(\chi) = \mathbf{C}^T\, \boldsymbol{\Phi}(\chi), \tag{26}$$

then the residual of Eq. (23) can be obtained by

$$R_M(\chi) = \sum_{n=\lceil m_q \rceil}^{M} c_n D^{m_q} \Phi_n(\chi) + \sum_{p=1}^{q-1} \sum_{n=\lceil m_p \rceil}^{M} c_n \xi_p(\chi) D^{m_p} \Phi_n(\chi) + \sum_{n=0}^{M} c_n \varepsilon(\chi) \Phi_n(\chi) - \rho(\chi).$$

Now, we apply the tau method

$$\int_0^L R_M(\chi) \Phi_n(\chi) \theta_L^{\alpha}(\chi) d\chi = 0, \quad n = 0, 1, \ldots, M-q. \tag{27}$$

If we employ relation (26), then using the initial conditions Equation (24) produces

$$\mathbf{C}^T \boldsymbol{\Phi}(0) = z_\mu, \quad \mu = 0, 1, \ldots, q-1, \tag{28}$$

furthermore, if we apply relation (26), the boundary conditions Equation (25) suggests that

$$\mathbf{C}^T \boldsymbol{\Phi}(0) = z_\mu, \quad \mathbf{C}^T \boldsymbol{\Phi}(1) = v_\mu, \quad \mu = 0, 1, \ldots, q-1. \tag{29}$$

By combining (27) and (28) or (27) and (29), we have a system of linear equations of dimension (M + 1) in the unknowns $\{c_n\}_{0 \le n \le M}$. Furthermore, the resulting problem can be solved using a numerical algebraic procedure.

4.3 *The collocation method for handling non-linear fractional differential equations*

In this subsection, we apply the collocation method to shifted Gegenbauer polynomials to solve fractional differential equations of non-linear:

$$D^{m_q} \omega(\chi) = S\left(\chi, \omega(\chi), D^{m_1} \omega(\chi), D^{m_2} \omega(\chi), \ldots, D^{m_{q-1}} \omega(\chi)\right), \quad \chi \in (0, 1), \tag{30}$$

subject to the initial conditions (24) or the boundary conditions (25), such that $m_1 < m_2 < \ldots < m_q$ and $m_j \in (j-1, j]$; $j \in \mathbb{N}$, and S is a continuous non-linear function, then the residual from Equation (30) can be obtained by

$$R_M(\chi) = D^{m_q} \omega(\chi) - S\left(\chi, \omega(\chi), D^{m_1} \omega(\chi), D^{m_2} \omega(\chi), \ldots, D^{m_{q-1}} \omega(\chi)\right).$$

The collocation points are selected to be the first $(M-q+1)$ roots of $\Phi_{M+1}(\chi)$, and the collocation method is used to obtain

$$R_M(\chi_n) = 0, \quad n = 0, 1, \ldots, M-q. \tag{31}$$

Equations (31) and (28) or (31) and (29), comprise a system of non-linear (M + 1) algebraic equations with (M + 1) unknowns $\{c_n\}_{0 \le n \le M}$. The approximate solution may be determined using Newton's iterative procedure.

Algorithm 2 Coding algorithm for our method to the linear and non-linear BVPs.
Input M.
Step 1. Suppose an approximate solution is $u_M(\chi) = \mathbf{C}^T\, \boldsymbol{\Phi}(\chi)$.
Step 2. Get the residual of the Eqs. (23) or (30).
Step 3. Use the Galerkin method for obtaining the system in (23)-(25) or in (30), (24) and (25).
Step 4. To solve the system in (23)-(25) or in (30), (24) and (25), apply the *NSolve* command to obtain **C**.
Step 5. Establish $u_M(\chi) = \mathbf{C}^T\, \boldsymbol{\Phi}(\chi)$.
Output $\omega_M(\chi)$.

5. Examining the convergence and error analysis

This section thoroughly examines the shifted Gegenbauer polynomial expansion's convergence and error analysis. As a result, several necessary lemmas are employed in this research.

Lemma 1 For the combination of shifted Gegenbauer polynomials $\Phi_q(\chi)$, the next inequality holds:

$$|\Phi_q(\chi)| \leq 32\,(qL)^{-2}, \quad q>0, \quad \chi \in [0,\, L].$$

Proof. From the alternative form of $\Phi_q(\chi)$ in (12), we have

$$\begin{aligned}|\Phi_q(\chi)| &= \left|\frac{\alpha\, q!\,(1+q+\alpha)\,(2\,\alpha)}{(2+q+2\,\alpha)}\,\chi\,(L-\chi)\,(\frac{4}{L})^2\,\tilde{C}_q^{(\alpha+1)}(\chi)\right| \\ &= \frac{16}{L^2}\left|\frac{\alpha\, q!\,(1+q+\alpha)\,(2\,\alpha)}{(2+q+2\,\alpha)}\,\chi\,(L-\chi)\tilde{C}_q^{(\alpha+1)}(\chi)\right| \\ &\leq 32\,(qL)^{-2}.\end{aligned}$$

□

Lemma 2 The following inequality is a valid one for the shifted Gegenbauer polynomials:

$$|\tilde{C}_q^{(\alpha+1)}(\chi)| \leq \frac{32\,(\alpha+\frac{3}{2})\,q!\,L^{-2}}{(2\,\alpha+2)}, \quad q>0, \quad \chi \in [0,\, L].$$

Proof. From relation (3), we have

$$\tilde{C}_q^{(\alpha+1)}(\chi) = \frac{\Gamma(\alpha+\frac{3}{2})}{\Gamma(2\,\alpha+2)}\sum_{m=0}^{q}\frac{(-1)^{q-m}L^{-m}\Gamma(q+m+2\,\alpha+2)}{m!\,\Gamma(\alpha+1+m+\frac{1}{2})\,(q-m)!}\,\chi^m,$$

$$|\tilde{C}_q^{(\alpha+1)}(\chi)| = \frac{\Gamma(\alpha+\frac{3}{2})}{\Gamma(2\alpha+2)} \left| \sum_{m=0}^{q} \frac{(-1)^{q-m} L^{-m} \Gamma(q+m+2\alpha+2)}{m!\,\Gamma(\alpha+1+m+\frac{1}{2})\,(q-m)!} \chi^m \right|$$

$$\leq \frac{32\,(\alpha+\frac{3}{2})\,q!\,L^{-2}}{(2\alpha+2)}.$$

□

Theorem 3 If $\omega(\chi)$ represents the analytic solution for (4), (19), (23), or (30), and $\omega_M(\chi) = \sum_{q=0}^{M} c_q\,\Phi_q(\chi)$ denotes the approximate solution, then the expansion coefficients fulfill the following approximation

$$|c_q| \leq 4\,L^{-4-2\alpha}\,q!\,N.$$

Proof. We start with the analytical solution

$$\omega(\chi) = \sum_{q=0}^{\infty} c_q\,\Phi_q(\chi). \tag{32}$$

Multiplying both sides of (32) by $\Phi_p(\chi)\,\tilde{\theta}_L^{(\alpha)}(\chi)$, then we have after integrating each side from 0 to L

$$\begin{aligned} c_q &= \frac{1}{\tilde{h}_q^{(\alpha)}} \int_0^L \omega(\chi)\,\Phi_q(\chi)\,\tilde{\theta}_L^{(\alpha)}(\chi)\,d\chi \\ &= \frac{(2+q+2\alpha)}{8\,L^{-2+2\alpha}\,(1+q+\alpha)\,(1+q)^2(\frac{1}{2}+\alpha)} \int_0^L \omega(\chi)\,\Phi_q(\chi)\,\tilde{\theta}_L^{(\alpha)}(\chi)\,d\chi, \end{aligned} \tag{33}$$

If we replace $\omega(\chi)$ in (33) with the value obtained from relation (6), then we obtain

$$c_q = \frac{(2+q+2\alpha)}{8\,L^{-2+2\alpha}\,(1+q+\alpha)\,(1+q)^2(\frac{1}{2}+\alpha)}$$

$$\int_0^L \frac{16\,(L\chi-\chi^2)^{\alpha-\frac{1}{2}}\,\alpha\,(1+q+\alpha)\,q!\,(2\alpha)\,(\zeta\,\chi+(L-\chi)\,\gamma+L u(\chi))}{L^3\,(2+q+2\alpha)} \tilde{C}_q^{(\alpha+1)}(\chi)\,d\chi$$

$$= \frac{4^{\alpha}\,L^{-1-2\alpha}\,(1+\alpha)}{\sqrt{\pi}\,(\alpha+\frac{1}{2})} \int_0^L (L\chi-\chi^2)^{\alpha-\frac{1}{2}}\,(\zeta\,\chi+(L-\chi)\,\gamma+L u(\chi))\,\tilde{C}_q^{(\alpha+1)}(\chi)\,d\chi.$$

We will use Lemma 2, then we get

$$|c_q| \leq \frac{4^{\alpha} L^{-1-2\alpha} (1+\alpha)}{\sqrt{\pi} (\alpha+\frac{1}{2})} \frac{32 (\alpha+\frac{3}{2}) q! L^{-2}}{(2\alpha+2)} \left| \int_0^L (L\chi-\chi^2)^{\alpha-\frac{1}{2}} \left(\zeta\chi+(L-\chi)\gamma+Lu(\chi)\right) d\chi \right|$$

$$\leq \frac{16 L^{-3-2\alpha} (1+q)}{(\alpha+\frac{1}{2})} \left(\int_0^L \left| (L\chi-\chi^2)^{\alpha-\frac{1}{2}} (\zeta\chi+(L-\chi)\gamma) \right| d\chi + \int_0^L \left| (L\chi-\chi^2)^{\alpha-\frac{1}{2}} Lu(\chi) \right| d\chi \right).$$

Upon performing integration by parts on the right-hand side of the last relation, then we obtain

$$|c_q| \leq \frac{16 L^{-2-2\alpha} (1+q) (\alpha-\frac{1}{2})}{(\alpha+\frac{1}{2})} \int_0^L \left| (L\chi-\chi^2)^{\alpha-\frac{3}{2}} (L-2\alpha) I(u(\chi)) \right| d\chi;$$

$$|I(u(\chi))| \leq N.$$

Then

$$|c_q| \leq 4 L^{-4-2\alpha} q! N.$$

□

Theorem 4 The following is the error bound, if $\omega_M(\chi) = \mathbf{C}^T \boldsymbol{\Phi}(\chi)$ is the best approximation of $\omega(\chi)$

$$||\omega_M(\chi)-\omega(\chi)||_2 \leq \frac{\rho}{(m+1)!} \sqrt{\frac{2\lambda^{2m+3}}{2m+3}},$$

where $\rho = \max_{0\leq\chi<L} |\omega^{(m+1)}(\chi)|$ and $\lambda = \max(L-\chi_0,\ \chi_0)$.

Proof. Initially, we take up the Taylor's formula

$$\bar{\omega}(\chi) = \omega(\chi_0) + \omega^{(1)}(\chi_0)(\chi-\chi_0) + \cdots + \omega^{(m)}(\chi_0) \frac{(\chi-\chi_0)^m}{m!}.$$

Furthermore, we are aware that

$$|\bar{\omega}(\chi)-\omega(\chi)| \leq \left| \omega^{(m+1)}(\eta) \right| \frac{(\chi-\chi_0)^{m+1}}{(m+1)!}; \quad \eta \in (0,\ L).$$

Through the application of the best approximation definition, we have

$$||\omega_M(\chi)-\omega(\chi)||_2^2 \le ||\bar{\omega}(\chi)-\omega(\chi)||_2^2$$

$$= \int_0^L (\bar{\omega}(\chi)-\omega(\chi))^2 \, d\chi$$

$$= \int_0^L \left(\left| \omega^{(m+1)}(\eta) \right| \frac{(\chi-\chi_0)^{m+1}}{(m+1)!} \right)^2 d\chi$$

$$\le \left(\frac{\rho}{(m+1)!} \right)^2 \int_0^L (\chi-\chi_0)^{2m+2} \, d\chi$$

$$\le \left(\frac{\rho}{(m+1)!} \right)^2 \frac{2\lambda^{2m+3}}{2m+3},$$

finally, we have

$$||\omega_M(\chi)-\omega(\chi)||_2 \le \frac{\rho}{(m+1)!} \sqrt{\frac{2\lambda^{2m+3}}{2m+3}}.$$

□

6. Test examples and comparisons

In this section, the test examples considered in this paper and the CPU (central processing unit) running times were all solved in all examples using the Mathematica program.

Example 1 Consider the linear boundary value problem as [23–25]:

$$\omega''(\chi)+\chi\,\omega(\chi)-(3-\chi-\chi^2+\chi^3)\sin\chi-4\chi\cos\chi=0, 0\le\chi\le 1,$$

where the boundary conditions:

$$\omega(0)=\omega(1)=0.$$

Actual solution of previous example is $\omega(\chi)=(\chi^2-1)\sin\chi$. Considering the maximum absolute error shown in Table 1 with different values of M, we observed that compared to previous approaches, our method performs better in small modes of M. Table 2 depicts CPU running times for Example 1 by our method in seconds with $M=8$, $M=16$, and $M=32$. In Figure 1, we compare approximate solutions to Example 1 in various values of α and $M=3$, we note that the approximate solutions when $\alpha=1$, $\alpha=2$, *and* $\alpha=3$ are extremely near to each other. The graphic in Figure 2 shows how absolute error behaves when $M=3$ and $\alpha=1$.

Table 1. Comparison of the MAE for Example 1

Methods	$M = 8$	$M = 16$	$M = 32$
QSM [23]	4.50×10^{-2}	3.08×10^{-3}	7.71×10^{-4}
CSM [23]	1.15×10^{-2}	2.88×10^{-3}	7.21×10^{-4}
NPSM [23]	2.67×10^{-3}	3.24×10^{-4}	3.99×10^{-5}
Method in [24]	2.22×10^{-4}	5.05×10^{-6}	1.63×10^{-7}
MLTM [25]	1.35×10^{-7}	1.11×10^{-16}	1.08×10^{-12}
Our method	2.22×10^{-16}	1.07×10^{-15}	1.57×10^{-9}

Table 2. CPU running times for Example 1 in seconds

M	8	16	32
Our method	0.656	0.827	1.936

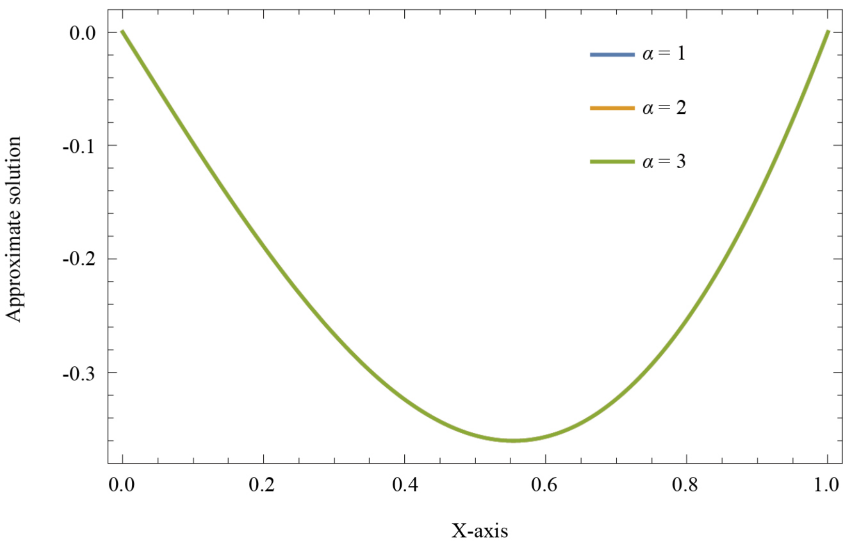

Figure 1. Approximate solution for various values of α and $M = 3$ of Example 1

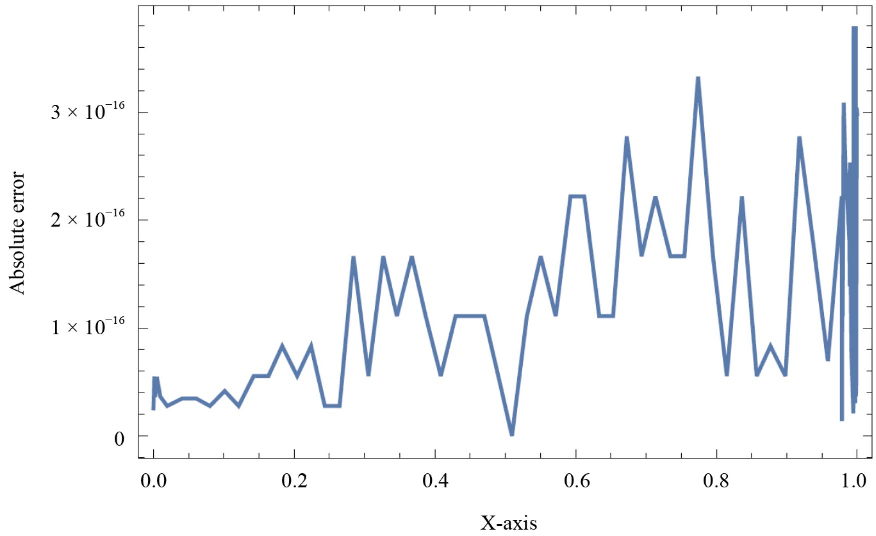

Figure 2. Absolute error with $M = 3$ and $\alpha = 1$ of Example 1

Example 2 Consider the singular initial value problem as [25–27]:

$$\chi\,\omega''(\chi) + 2\,\omega'(\chi) - (4\chi^3 + 6\chi)\,\omega(\chi) = 0, \quad 0 < \chi \leq 1, \tag{34}$$

where the initial conditions:

$$\omega(0) = 1, \quad \omega'(0) = 0. \tag{35}$$

The analytic solution to Example 2 is $\omega(\chi) = e^{\chi^2}$. In Table 3, we choose $\alpha = 1$ and a variety of values of M in Example 2 by the proposed method, the maximum absolute error reaches 10^{-16} with $M = 8$ then this value rises as M is raised. Table 4 depicts the CPU running times in seconds with different values of M for Example 2. For $\alpha = 1$ and $M = 3$, Figure 3 shows the absolute error behavior.

Table 3. Comparison of *MAE* with various values of *M* for Example 2

M	*MLTM* [25]	*ADM* [26]	*VIM* [26]	*HWCM* [26]	*HWAGM* [27]	Our method
8	6.11×10^{-16}	6.31×10^{-4}	7.30×10^{-4}	3.72×10^{-4}	1.18×10^{-4}	2.22×10^{-16}
16	1.83×10^{-15}	9.45×10^{-4}	1.09×10^{-3}	1.19×10^{-4}	3.29×10^{-5}	3.89×10^{-16}
32	1.19×10^{-11}	1.15×10^{-3}	1.33×10^{-3}	3.14×10^{-5}	8.00×10^{-6}	4.17×10^{-9}
64	7.08×10^{-7}	1.27×10^{-3}	1.47×10^{-3}	7.91×10^{-6}	2.00×10^{-6}	1.18×10^{-4}

Table 4. CPU running times for Example 2 in seconds

M	3	8	16	32	64
Our method	0.562	0.656	0.891	4.955	11.922

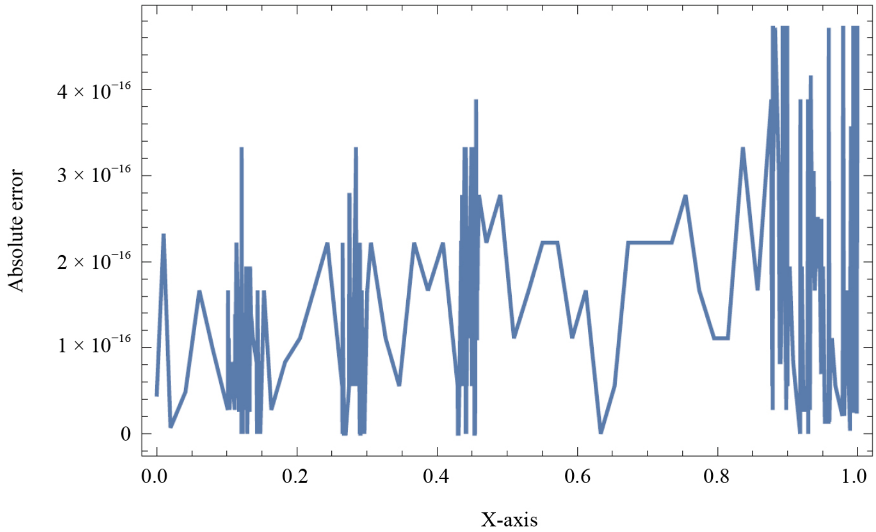

Figure 3. Absolute error with $M = 3$ and $\alpha = 1$ of Example 2

Example 3 Consider the non-linear boundary value problem as [28–30]:

$$\omega''(\chi) = \frac{1}{2}(\omega(\chi)+\chi+1)^3, \quad 0 < \chi < 1,$$

with the boundary conditions:

$$\omega(0) = \omega(1) = 0.$$

The exact solution of Example 3 is $\omega(\chi) = -1-\chi+\frac{2}{2-\chi}$. With $M = 4$ and ($\alpha = 1$ (shifted Chebyshev polynomials of the second kind), $\alpha = 2$, and $\alpha = 3$), we used the collocation approach to our method for this problem, and the approximate solutions with the CPU running times are displayed in Table 5. In Table 6, when comparing our method's maximum absolute error (MAE) with other methods for various values of M, we found that our method's MAE reached 10^{-17} when $M = 4$. The absolute error with $\alpha = \frac{1}{2}$ (shifted Legendre polynomials) and $M = 2$ is shown in Figure 4.

Table 5. The approximate solutions and CPU running times in seconds by our method ($M = 4$) for Example 3 with $\alpha = 1$, $\alpha = 2$, and $\alpha = 3$

χ	Exact solution	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$
0.1	-0.04736842	-0.04736842	-0.04736842	-0.04736842
0.2	-0.08888888	-0.08888888	-0.08888888	-0.08888888
0.3	-0.12352941	-0.12352941	-0.12352941	-0.12352941
0.4	-0.15	-0.15	-0.15	-0.15
0.5	-0.16666666	-0.16666666	-0.16666666	-0.16666666
0.6	-0.17142857	-0.17142857	-0.17142857	-0.17142857
0.7	-0.16153846	-0.16153846	-0.16153846	-0.16153846
0.8	-0.13333333	-0.13333333	-0.13333333	-0.13333333
0.9	-0.08181818	-0.08181818	-0.08181818	-0.08181818
CPU		0.545	0.548	0.547

Table 6. Comparison of the *MAEs* of Example 3

Method	Our method	*S3CPCM* [28]	*S4CPCM* [28]	Method in [29]	Sinc-collocation [30]	Sinc-Galerkin [30]
M	4	20	20	22	130	130
MAE	3.47×10^{-17}	4.44×10^{-17}	4.44×10^{-17}	5.31×10^{-17}	9.16×10^{-16}	9.99×10^{-16}

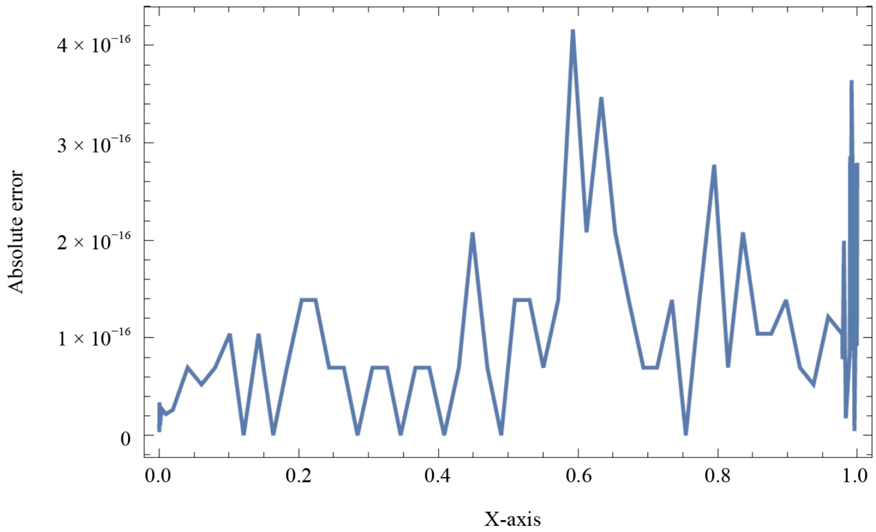

Figure 4. Absolute error with $M = 2$ and $\alpha = 0.5$ of Example 3

Example 4 Consider the non-homogeneous Bagley-Trovik equation shown below [31, 32]:

$$D^2\omega(\chi) + D^{1.5}\omega(\chi) + \omega(\chi) - g(\chi) = 0, \quad \chi \in (0,\ 1),$$

where the initial conditions:

$$\omega(0) = 0, \quad \omega'(0) = \gamma,$$

where $g(\chi)$ is selected so that Example 4 has an exact solution $\omega(\chi) = \sin(\gamma\chi)$. In Table 7, our method uses a variety of M values to determine the maximum absolute error in two cases $\gamma = 1$ and $\gamma = 4\pi$. the results show that our method is perfect in small values of M about the Chebyshev spectral method (CSM) and tau Lucas matrix method (TLMM). Table 8 shows CPU running times in seconds with different values of M by our method. Figure 5 shows that the comparison between the exact and approximate solutions with $M = 4$, $M = 8$, $M = 16$, and $M = 32$.

Table 7. Comparison of the MAE for Example 4

	$\gamma = 1$			$\gamma = 4\pi$		
M	Our method	$TLMM$ [32]	CSM [31]	Our method	$TLMM$ [32]	CSM [31]
4	3.1×10^{-17}	1.0×10^{-4}	3.4×10^{-4}	3.5×10^{-14}	2.1×10^{-2}	3.9×10^{0}
8	1.6×10^{-17}	2.3×10^{-7}	4.3×10^{-7}	1.2×10^{-13}	5.2×10^{-6}	4.7×10^{-1}
16	1.4×10^{-17}	7.5×10^{-11}	1.8×10^{-8}	1.5×10^{-13}	3.9×10^{-10}	3.5×10^{-5}
32	3.7×10^{-13}	3.7×10^{-13}	7.1×10^{-10}	4.9×10^{-11}	6.1×10^{-13}	1.4×10^{-6}

Table 8. CPU running times for Example 4 in seconds

M	4	8	16	32
Our method	1.36	4.97	14.71	31.42

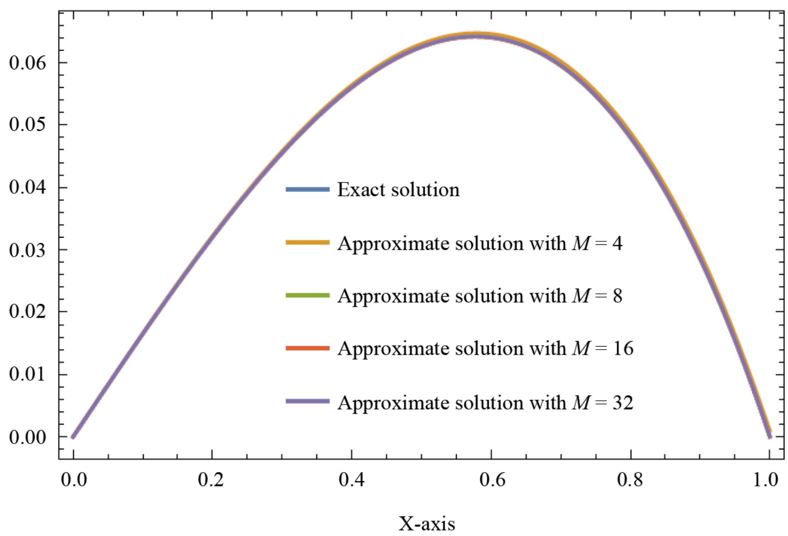

Figure 5. Comparison between exact and approximate solutions of Example 4

Example 5 The non-linear Riccati fractional differential equation is shown below as [32, 33]:

$$D^r\omega(\chi)+\omega^2(\chi)=1, \quad \chi\in(0,\ 1), \quad r\in(0,\ 1],$$

subject to the initial condition:

$$\omega(0)=0.$$

When $r=1$, the exact solution to this example is $\omega(\chi)=\tanh\chi$. We applied our method to solve Example 5 for two cases $r=0.7$ and $r=0.9$, and then we compared the approximate results obtained with other methods (Bernoulli wavelet and collocation Lucas matrix) and the exact solution for case $r=1$, which are displayed in Table 9. These results demonstrate that the approximation solution approach to the exact solution. Table 10 explains the CPU running times in seconds by our method with $M=1$, $M=2$, and $M=3$. Also, the comparison between the exact solution when $r=1$ and the approximate results when $r=0.9$ for various values of M is plotted in Figure 6. From this graph, we can see that the approximate solution when $M=1$ is very close to the exact solution about the approximate solution with $M=2$ and $M=3$.

Table 9. Comparison of the approximate solution for Example 5

	$r=0.7$			$r=0.9$			
χ	Our method	*CLMM* [32]	Bernoulli wavelet [33]	Our method	*CLMM* [32]	Bernoulli wavelet [33]	Exact solution
0.1	0.0996678	0.171333	0.209216	0.0996679	0.101874	0.129138	0.099668
0.2	0.197375	0.296421	0.335973	0.197375	0.200768	0.238981	0.197375
0.3	0.291312	0.393889	0.429549	0.291312	0.295195	0.336448	0.291313
0.4	0.379948	0.472838	0.500339	0.379949	0.383814	0.422741	0.379949
0.5	0.462116	0.537862	0.556331	0.462117	0.465588	0.498915	0.462117
0.6	0.537048	0.591777	0.603099	0.537049	0.539863	0.565851	0.53705
0.7	0.604366	0.636781	0.643854	0.604367	0.606359	0.624307	0.604368
0.8	0.664035	0.674632	0.679183	0.664036	0.665128	0.674869	0.664037
0.9	0.716296	0.706583	0.707567	0.716297	0.716482	0.717972	0.716298

Table 10. CPU running times for Example 5 in seconds

M	1	2	3
Our method	2.12	4.95	7.36

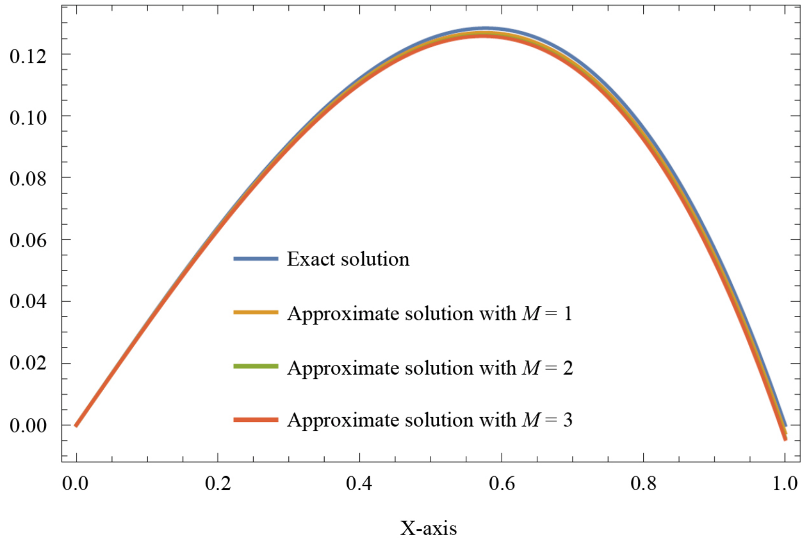

Figure 6. Comparison between exact solution ($r = 1$) and approximate solution ($r = 0.9$) of Example 5

Example 6 The non-linear fractional initial value problem is shown below as [34]:

$$D^r\omega(\chi) = 9 + \chi + \frac{2\sinh^{-1}(\frac{\sqrt{\chi}}{3})}{\sqrt{(\chi+9)\pi}} - e^{\omega(\chi)}, \quad \chi \in (0,\ 1),$$

with

$$\omega(0) = \ln 9.$$

Table 11. Comparison of the approximate solution and the CPU running times for $r = 0.5$ for Example 6 with various values of M

χ	$M = 2$	$M = 4$	$M = 6$
0.1	6.25693×10^{-16}	6.47594×10^{-16}	4.96348×10^{-16}
0.2	1.10112×10^{-15}	9.9237×10^{-16}	1.12757×10^{-15}
0.3	5.99564×10^{-16}	6.5269×10^{-16}	7.71952×10^{-16}
0.4	4.38885×10^{-16}	7.02563×10^{-16}	5.07623×10^{-16}
0.5	5.31476×10^{-16}	7.47015×10^{-16}	5.96311×10^{-16}
0.6	7.13188×10^{-16}	5.96745×10^{-16}	8.87962×10^{-16}
0.7	1.18959×10^{-15}	7.66965×10^{-16}	1.15554×10^{-15}
0.8	1.20433×10^{-15}	9.10188×10^{-16}	7.72169×10^{-16}
0.9	8.14398×10^{-16}	1.0749×10^{-15}	9.39461×10^{-16}
CPU	4.57	12.64	19.37

When $r = 0.5$, the exact solution is $\ln(\chi + 9)$. In Table 11, for the case with a value r = 0.5, we employ our method to compare the approximate solution and the CPU running times in seconds with (M = 2, M = 4, and M = 6), then we found a minor variation in the approximate results. Figure 7 indicates the absolute error to Example 6 with M = 2, where the maximum absolute error reached 10^{-15}. Figure 8 compares the approximate results with multiple r values and $M = 2$ for Example 6. Table 12 compares the maximum absolute error between our method and the method shifted fifth-kind Chebyshev modified tau method (S5CMTQM) in [34].

Table 12. Comparison of the *MAEs* of Example 6

Method	Our method	S5CMTQM [34]	Our method	S5CMTQM [34]
M	2	2	4	4
MAE	1.19×10^{-15}	3.55×10^{-5}	9.92×10^{-16}	6.67×10^{-8}

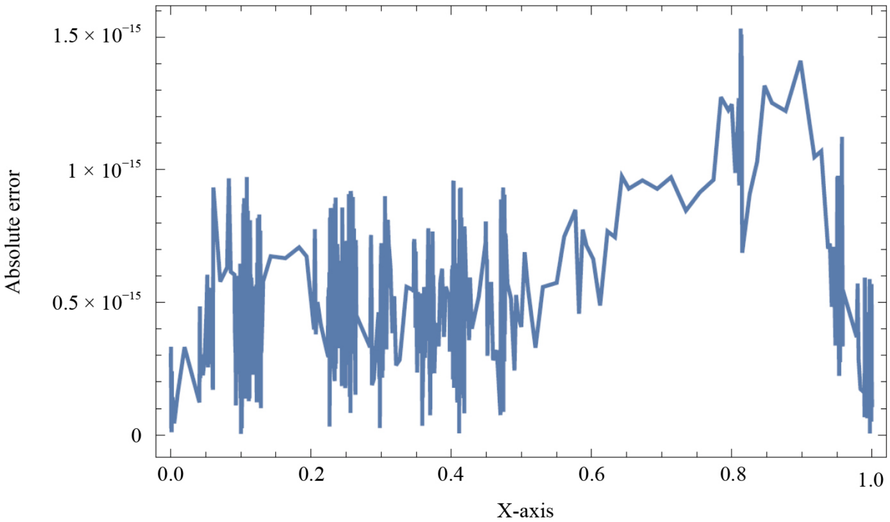

Figure 7. The absolute error of Example 6 with $M = 2$

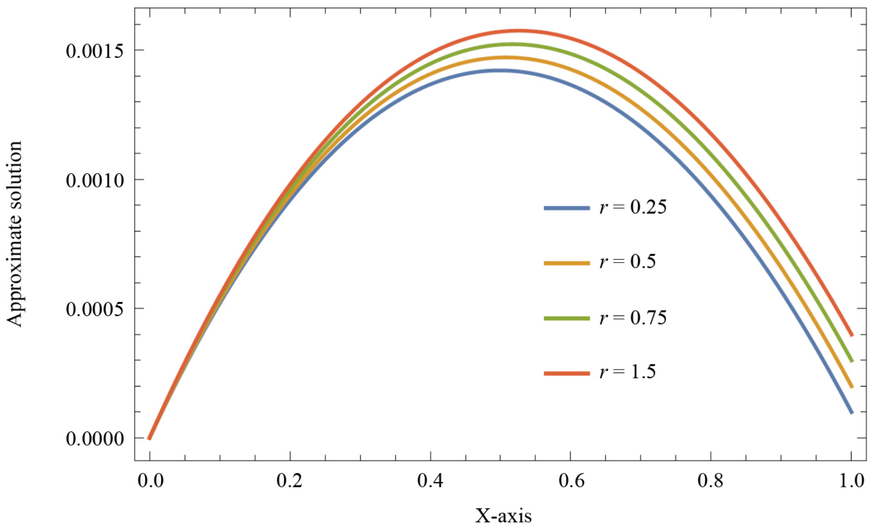

Figure 8. The comparison of the approximate solution of Example 6 with various values of r

Example 7 Consider the underlying linear initial value problem [30]:

$$\omega''(\chi) - 4\,\omega(\chi) = 4\cosh(1), \quad \chi \in (0,\ 1),$$

subject to homogeneous initial conditions:

$$\omega(0) = \omega(1) = 0.$$

The analytical solution is $\omega(\chi) = \cosh(2\chi - 1) - \cosh(1)$. With $M = 4$ and $M = 6$, we compare the absolute error and the CPU running times in seconds for Example 7 in Table 13. For various values of M, we compare the maximum absolute error by our method with the methods Sinc-collocation [30] and Sinc-Galerkin [30] in Table 14, then our method got the error on 10^{-15} when $M = 4$ while the other methods got the error on 10^{-13} with $M = 100$, which leads to taking a long time in this operation while you can get the same results by our method for 0.669 second. Figure 9 depicts the behavior of the basis of shifted Gegenbauer polynomials in calculating the absolute error for $M = 4$. Figure 10 shows the comparison of the approximate solutions for different values of M.

Table 13. Comparison of the absolute error and the CPU for Example 7 with $M = 4$ and $M = 6$

χ	$M = 4$	$M = 6$
0.1	2.2×10^{-16}	1.3×10^{-16}
0.2	1.7×10^{-16}	5.5×10^{-17}
0.3	0	1.1×10^{-16}
0.4	0	1.7×10^{-16}
0.5	0	1.7×10^{-16}
0.6	2.2×10^{-16}	4.4×10^{-16}
0.7	3.9×10^{-16}	5.5×10^{-16}
0.8	7.8×10^{-16}	7.8×10^{-16}
0.9	6.1×10^{-16}	5.8×10^{-16}
CPU	0.669	0.731

Table 14. Comparison of the *MAEs* of Example 7

Method	Our method	Sinc-collocation [30]	Sinc-Galerkin [30]
M	4	50	50
MAE	1.4×10^{-15}	1.1×10^{-9}	6.24×10^{-10}
M	6	75	75
MAE	2×10^{-15}	1.4×10^{-11}	8.69×10^{-12}
M	8	100	100
MAE	2×10^{-15}	2.5×10^{-13}	1.55×10^{-13}

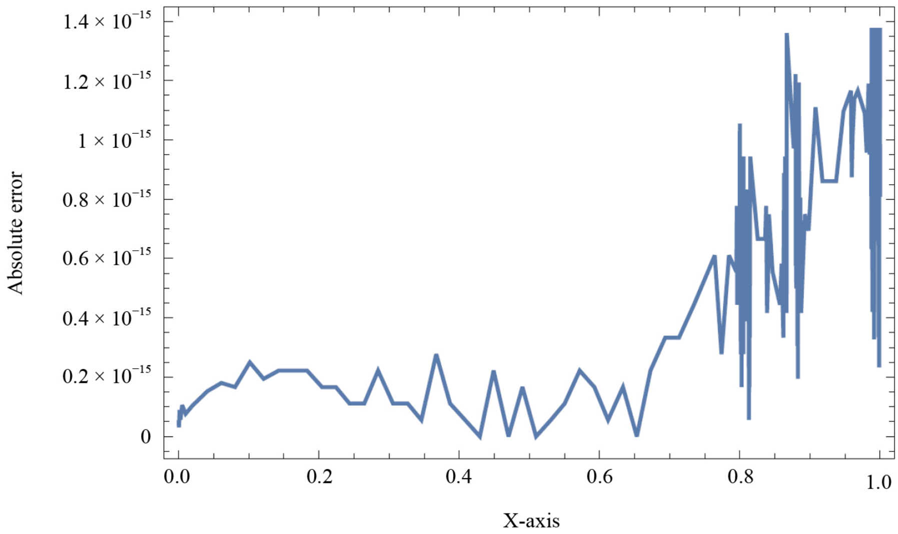

Figure 9. The absolute error of Example 7 with $M = 4$

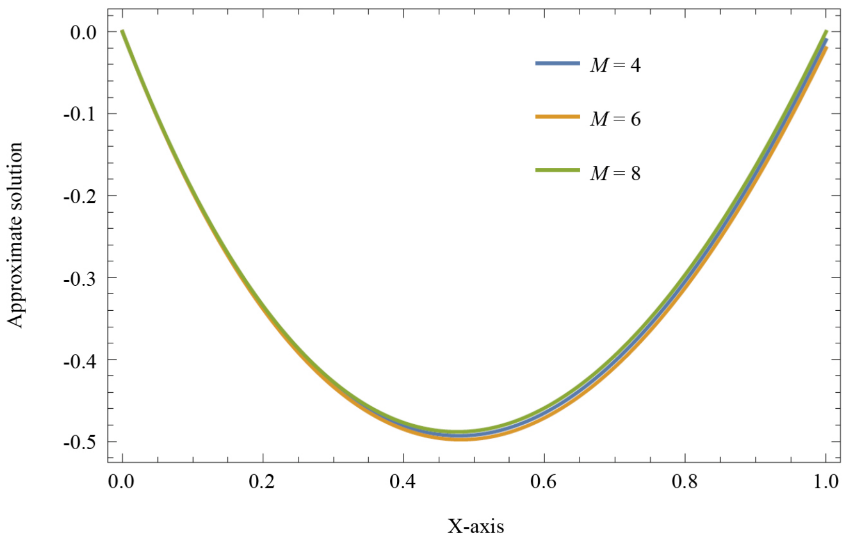

Figure 10. The comparison of the approximate solution of Example 7 with various values of M

7. Conclusions

To sum up, the shifted Gegenbauer spectral algorithm is presented to provide an efficient technique for tackling fractional differential equations with homogeneous and initial boundary conditions. The approach depends on the expansion of the unknown function in terms of polynomial basis functions obtained from the shifted Gegenbauer polynomials. The method is basically on building the matrix of the differential operators explicitly. The process of getting answers is made simpler by this explicit description, which enhances the efficacy and efficiency of the numerical solution approach. The comparison of the suggested method's outcomes with those from other approaches highlights its effectiveness.

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Confilict of interest

The authors declare no competing financial interest.

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