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Research Article

Functional Train Algebras of Rank ≤ 3

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Abstract: In this paper we show that every baric algebra satisfying a functional train identity of rank ≤ 3 and admitting an idempotent is a special train algebra. The functional train equation of train algebras of rank 3 is given. Some examples are also given.

***Keywords*:** baric algebra, train algebra, idempotent element, nilpotent element, Peirce decomposition

MSC: 17D92, 17A30

1. Introduction

A special train algebra is a train algebra in which the nilideal, consisting of elements of weight zero, is nilpotent and all its powers are ideals. Abraham showed that every principal train algebra of rank 3 is a special train algebra and the principal train algebras of rank 4 are not necessarily special train algebras [1]. Earlier, Etherington showed that every commutative principal train algebra of rank 3 is a special train algebra [2, 3]. A baric algebra is one that admits a non-trivial homomorphism ω into its coefficient field.

In this paper, we study the baric algebras satisfying the *functional train* identity of rank ≤ 3:

$$L_x^3-(1+\alpha+\beta)\omega(x)L_x^2+\alpha\omega(x)^2L_x+\beta\omega(x)^3id_A=0,\ \ \forall x\in A \tag{1}$$

with $L_x : A \longrightarrow A$, $y \longmapsto xy$ and α, β are constants in the coefficients field. It is a subclass of principal train algebras of rank 4 characterized by the identity $x^4-(1+\alpha+\beta)\ \omega\ (x)\ x^3+\alpha\omega(x)^2x^2+\beta\omega(x)^3x=0$, $\forall x\in A$. We show that such algebras are special train algebras under some additional conditions.

This paper is structured as follows: Section 1 provides an introduction to the paper. Section 2 is devoted to reminders of some basic notions. In Sections 3 and 4, we obtain the Peirce decomposition of the algebra with respect to an element of weight 1. We then show that functional train algebras of rank ≤ 2 and functional train algebras of rank 3 admitting an idempotent are special train algebras. In Section 5, we give the functional train equation for train algebras of rank 3. Finally, in Section 6 we give two conjectures, the first of which holds that in finite dimension, the assumption on idempotent existence can be omitted; the second gives the relation between the minimal polynomial M_A and the minimal train polynomial m_A of a train algebra A.

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2. Preliminary

Let K be an infinite field of characteristic $\neq 2$ and A be a commutative algebra over K. The principal powers of an element $x \in A$ are defined by

$$x^1 = x, \;\; x^{k+1} = xx^k, \;\; k \geq 1.$$

An element $x \neq 0$ of A is said to be nilpotent of nil-index $m \geq 2$ if $x^m = 0$ and $x^{m-1} \neq 0$. If $x^2 = x$ then x is said to be an idempotent element. The principal powers of A are defined by

$$A^1 = A, \;\; A^{k+1} = AA^k, \;\; k \geq 1.$$

The algebra A is said to be nilpotent of nil-index $m \geq 2$ if $A^m = 0$ and $A^{m-1} \neq 0$.

The methods of linearization developed by Osborn in [4] are often used in our investigation. For more information, see also [5, 6].

Definition 1 ([7, 8]) A finite dimensional baric algebra $(A, \; \omega)$ is said to be a train algebra if there are scalars α_1, α_2, $\cdots$, α_n in K such that the characteristic polynomial of L_x, for all x in A, can be written as

$$det(\lambda I - L_x) = \lambda^n - \alpha_1 \omega(x) \lambda^{n-1} + \; \cdots \; + (-1)^n \alpha_n \omega(x)^n.$$

Remark 1 If A is a train algebra then by the Cayley-Hamilton theorem, $P(\lambda) = det(\lambda I - L_x)$ is an annihilator polynomial of L_x for all $x \in A$.

For a baric algebra $(A, \; \omega)$ we denote by H the hyperplane unit $H = \{x \in A \mid \omega(x) = 1\}$.

Lemma 1 ([9], Lemma 3) A baric algebra $(A, \; \omega)$ is a train algebra if and only if there exists a polynomial $p(\lambda)$ such that for all $x \in H$,

$$p(L_x) \equiv 0.$$

Definition 2 The *minimal polynomial* of a train algebra A is the monic generator M_A of the ideal of the annihilator polynomials of L_x for all $x \in H$.

Let $P = X^n + \alpha_{n-1} X^{n-1} + \alpha_{n-2} X^{n-2} + \; \cdots \; + \alpha_1 X + \alpha_0$ be a polynomial of degree n. Let's put $\widehat{P} = XP$.

Definition 3 Let $(A, \; \omega)$ be a train algebra. A polynomial P is a *train polynomial* of A if for any $x \in H = \{x \in A \mid \omega(x) = 1\}$, we have

$$\widehat{P}(x) = P(L_x)(x) = 0. \tag{2}$$

Definition 4 Let $(A, \; \omega)$ be a train algebra. The monic generator m_A of the ideal of the train polynomials of A is called the *minimal train polynomial* of A.

Proposition 1 Let $(A, \; \omega)$ be a train algebra. Every annihilator polynomial is a train polynomial of A.

Proof. If P is an annihilator polynomial then, $P(L_x) \equiv 0$ for all $x \in H$. So, $\widehat{P}(x) = P(L_x)(x) = 0$. □

Corollary 1 The minimal train polynomial m_A is a divisor of the minimal polynomial M_A.

Proof. The annihilator polynomial M_A is a train polynomial, then m_A divides M_A. □

The following example shows that the converse of the above proposition is not true.

Example 1 Let A be an algebra of dimension 3 over the real number field and $\{e,\ t,\ u\}$ be a basis of A in which the nonzero products are as follows: $e^2 = e+t$, $et = \frac{1}{2}t$, $eu = \frac{1}{2}u$, $u^2 = t$. We show that $(A,\ \omega)$ is a baric algebra where ω is linear application such that $\omega(e) = 1$ and $\omega(t) = \omega(u) = 0$. For $x = \omega(x)e + \alpha t + \beta u$, $x^2 = \omega(x)^2(e+t) + \omega(x)(\alpha t + \beta u) + \beta^2 t = \omega(x)x + (\omega(x)^2 + \beta^2)t$. So $x(x^2 - \omega(x)x) = \frac{1}{2}\omega(x)\left(x^2 - \omega(x)x\right)$. We get $x^3 - \frac{3}{2}\omega(x)x^2 + \frac{1}{2}\omega(x)^2 x = 0$, and $m_A = X^2 - \frac{3}{2}X + \frac{1}{2}$. For $x = e+2u$, $m_A(L_x)(e) = 2t \neq 0$. So, m_A is not an annihilator of L_x. But, we have $x(x(xy)) - 2\,\omega(x)x(xy) + \frac{5}{4}\,\omega(x)^2 xy - \frac{1}{4}\omega(x)^3 y = 0$ and $M_A = X^3 - 2X^2 + \frac{5}{4}X - \frac{1}{4}$. Let's note that $M_A = \left(X - \frac{1}{2}\right)m_A$.

In the following example we have $M_A = m_A$.

Example 2 In dimension 2, the Osborn algebra defined in a basis $\{e,\ t\}$ by $e^2 = e+t$, $et = \frac{1}{2}t$ is an algebra without idempotent element and it satisfies the following identity: $x(xy) - \frac{3}{2}\omega(x)xy + \frac{1}{2}\omega(x)^2 y = 0$. Furthermore, it is a train algebra of rank 3 with train equation: $x^3 - \frac{3}{2}\omega(x)x^2 + \frac{1}{2}\omega(x)^2 x = 0$ and $M_A = m_A = X^2 - \frac{3}{2}X + \frac{1}{2}$.

We know that the polynomial P, defined above, is an annihilator of L_x for all $x \in H$ if an only if

$$P(L_x)(y) = 0\ \forall x \in H \ \text{ and } \ \forall y \in A. \tag{3}$$

The identity (3), called *functional train equation* by Etherington [2], is equivalent to the following:

$$\left(L_x^n + \alpha_{n-1}\omega(x)L_x^{n-1} + \alpha_{n-2}\omega(x)^2 L_x^{n-2} + \cdots + \alpha_1\omega(x)^{n-1}L_x + \alpha_0\omega(x)^n id_A\right)(y) = 0.$$

Any baric algebra that satisfies (3) is a train algebra of rank $\leq n+1$ and satisfying

$$x^{n+1} + \alpha_{n-1}\omega(x)x^n + \alpha_{n-2}\omega(x)^2 x^{n-1} + \cdots + \alpha_1\omega(x)^{n-1}x^2 + \alpha_0\omega(x)^n x = 0, \tag{4}$$

for all $x \in A$. The converse is not generally true.

Definition 5 A baric algebra $(A,\ \omega)$ is said to be a *special train algebra* if $N := \ker\omega$ is nilpotent and N^k is an ideal of A for every integer $k \geq 1$.

3. Minimal polynomial of degree ≤ 2

The case $M_A = X - 1$ leads to $xy = \omega(x)y\ \forall x,\ y \in A$. By commutativity $\omega(x)y = \omega(y)x$ for all $x,\ y \in A$. So, for an idempotent element e, $A = Ke$.

Proposition 2 Let $(A,\ \omega)$ be a train algebra of rank 2. Then $m_A = X - 1$ and $(A,\ \omega)$ satisfies the identity:

$$x(xy) - \frac{3}{2}\omega(x)xy + \frac{1}{2}\omega(x)^2 y = 0.$$

Proof. In fact, we know that $(A,\ \omega)$ satisfies $x^2 - \omega(x)x = 0$. By linearization, the identity $2xy = \omega(x)y + \omega(y)x$ holds in A. So $(2L_x - \omega(x)id_A)(y) = \omega(y)x$. So, $((L_x - \omega(x)id_A) \circ (2L_x - \omega(x)id_A))(y) = \omega(y)(L_x - \omega(x)id_A)(x) = \omega(y)(x^2 - \omega(x)x) = 0$ for all $x,\ y \in A$. □

The case $M_A = X^2-(1+\lambda)X+\lambda$ gives the identity

$$x(xy)-(1+\lambda)\omega(x)xy+\lambda\,\omega(x)^2y=0. \tag{5}$$

Theorem 1 Let $(A,\ \omega)$ be an algebra satisfying (5) and $N=\ker\omega$. Then N^2 is an ideal of A and $N^3=0$. In particular, N is associative.

Proof. By linearizing the identity (5) we get $z(xy)+x(yz)-(1+\lambda)[\omega(z)xy+\omega(x)yz]+2\lambda\,\omega(xz)y=0$, for all x, y, $z\in A$. We know that there is an element c_0 of weight 1 in A such that $A=Kc_0\oplus N$. For $z=c_0$ and x, $y\in N$ we get from the previous identity $c_0(xy)=-x(c_0y)+(1+\lambda)xy\in N^2$. Thus N^2 is an ideal of A because $N^3\subset N^2$. Moreover, $x(xy)=0$ for all x, $y\in N$. So $x^3=0$ and this leads to $2x(xy)+x^2y=0$. We conclude that $x^2y=0$ for all x, $y\in N$ and $N^3=0$. In particular, since A is commutative, we get $x(yz)=0$ and $(xy)z=0$ for all x, y, $z\in N$. Therefore, N is associative. □

Theorem 2 Let $(A,\ \omega)$ be a baric algebra. Then the following assertions are equivalent:

(i) For all x and y in A, $x(xy)-(1+\lambda)\omega(x)xy+\lambda\,\omega(x)^2y=0$;

(ii) There are $e\in A$ and $t\in\ker\omega$ such that $e^2=e+t$, $\omega(e)=1$ and $A=Ke\oplus\ker\omega$ with $(\ker(\omega))^2=0$, $ex=\lambda x$ $\forall x\in\ker\omega$;

(iii) For all x and y in A, $\omega(y)x^2-\omega(x)xy+\lambda\,\omega(x)^2y-\lambda\,\omega(xy)x=0$.

Proof. $(i)\Longrightarrow(ii)$. If A satisfies (i) then for all $x=c_0\in H$ and $y\in\ker\omega$, $((R_{c_0}-\lambda I)\circ(R_{c_0}-I))(y)=0$ where R_{c_0} is the restriction of the multiplication L_{c_0} to $\ker\omega$.

The following cases can be distinguished:

Case 1: $\lambda\neq 1$.

Then $A=Kc_0\oplus A_1(c_0)\oplus A_\lambda(c_0)$ with $A_\lambda(c_0)=\{x\in\ker\omega\mid c_0x=\lambda x\}$.

The following identity is obtained by linearization of (5):

$$z(xy)+x(yz)-(1+\lambda)[\omega(z)xy+\omega(x)yz]+2\lambda\,\omega(xz)y=0. \tag{6}$$

Let's permute y and z in (6). Then we get

$$y(xz)+x(yz)-(1+\lambda)[\omega(y)xz+\omega(x)yz]+2\lambda\,\omega(xy)z=0. \tag{7}$$

By difference the identities (6) and (7) give:

$$z(xy)-y(xz)-(1+\lambda)[\omega(z)xy-\omega(y)xz]+2\lambda[\omega(xz)y-\omega(xy)z]=0. \tag{8}$$

For $y=c_0$ and $x\in A_1(c_0)$ in (5), $x^2=0$ is valid. Thus $A_1(c_0)^2=0$. For $x=c_0$ and y, $z\in A_\lambda(c_0)$ in (6), we have $c_0(yz)=yz$. So, $A_\lambda(c_0)^2\subset A_1(c_0)$. For $y=c_0$ and $x\in A_\lambda(c_0)$ in (5), we have $\lambda x^2=0$. Thus $A_\lambda(c_0)^2=0$ if $\lambda\neq 0$. If $\lambda=0$ then A satisfies $x^3-\omega(x)x^2=0$. So, $2x(xy)+x^2y-\omega(y)x^2-2\omega(x)xy=0$ is valid. For $x=c_0$ and $y\in A_1(c_0)$, $c_0^2y=0$. Since, in this case, the algebra A contains an idempotent element [10], we can choose c_0 as an idempotent element. Then $y=0$ and $A_1(c_0)=0$. So, $A_0(c_0)^2=0$. It follows that $A_\lambda(c_0)^2=0$ for all $\lambda\neq 1$. For $x=c_0$, $y\in A_1(c_0)$ and $z\in A_\lambda(c_0)$ in (8), we have $(1-\lambda)yz=0$. Since $\lambda\neq 1$, then $yz=0$ and thus $A_1(c_0)A_\lambda(c_0)=0$. We can conclude that for $\lambda\neq 1$, we get $N^2=0$ with $N=\ker\omega$.

Now let's put $c_0^2=c_0+x_1+x_\lambda$ where $x_1\in A_1(c_0)$ and $x_\lambda\in A_\lambda(c_0)$. Then $c_0^3=c_0^2+x_1+\lambda x_\lambda=c_0+x_1+x_\lambda+x_1+\lambda x_\lambda=c_0+2x_1+(1+\lambda)x_\lambda$. Furthermore, we get from (5): $c_0^3=(1+\lambda)c_0^2-\lambda c_0=(1+\lambda)(c_0+x_1+x_\lambda)-\lambda c_0=c_0+(1+\lambda)x_1+(1+\lambda)x_\lambda$. So $x_1=0$ and it follows that $c_0^2=c_0+x_\lambda$.

Let $t \in A_\lambda(c_0)$ be a fixed element such that $c_0^2 = c_0 + t$. If $x = c_0 + x_1 + x_\lambda$ is an idempotent element of A of weight 1 then $(1-2\lambda)x_\lambda = t$ and $x_1 = 0$.

(a) $\lambda = 1/2$:

If $t = 0$ then $c_0^2 = c_0$ and c_0 is an idempotent element. Otherwise, A contains no idempotent elements.

(b) $\lambda \neq 1/2$:

Then $x_\lambda = \dfrac{1}{1-2\lambda}t$ and in this case $e = c_0 + \dfrac{1}{1-2\lambda}t$ is an idempotent element of A.

Now, let $x = e + x_1 + x_\lambda$ and $y = e + y_1 + y_\lambda$ be two elements in A. We get the equalities one by one: $xy = e + t + y_1 + \lambda y_\lambda + x_1 + \lambda x_\lambda$, $x(xy) = e + t + \lambda t + y_1 + \lambda^2 y_\lambda + x_1 + \lambda^2 x_\lambda + x_1 + \lambda x_\lambda$ and $x(xy) - (1+\lambda)xy + \lambda y = (1-\lambda)x_1$. Since $x(xy) - (1+\lambda)xy + \lambda y = 0$ we get $x_1 = 0$. So, $A_1(c_0) = 0$. It follows that, if $\lambda \neq 1$ then there exists an element e of weight 1 such that $e^2 = e + t$ with $et = \lambda t$ and $A = Ke \oplus \ker\omega$, $(\ker(\omega))^2 = 0$, $ex = \lambda x \; \forall x \in \ker\omega$.

Case 2: $\lambda = 1$.

Then (5) is written $x(xy) - 2\omega(x)xy + \omega(x)^2 y = 0$. This algebra is a power-associative algebra containing and idempotent element e and satisfying the identity $x^3 - 2\omega(x)x^2 + \omega(x)^2 x = 0$ [11]. It follows that for all $y \in A$, $2e(e(ey)) - 3e(ey) + ey = 0$. We know that $e(ey) - 2ey + y = 0$. So $ey = y$ for all $y \in A$. From (5), for all $x \in \ker\omega$ and $y = e$, $x^2 = 0$. So, $A = Ke \oplus \ker\omega$ with $ex = x$ and $x^2 = 0 \; \forall x \in \ker\omega$.

$(ii) \implies (iii)$. Now, let's assume that (ii) is satisfied. Let $x = \omega(x)e + x_\lambda$ and $y = \omega(y)e + y_\lambda$. Then we have: $xy = \omega(xy)(e+t) + \lambda(\omega(x)y_\lambda + \omega(y)x_\lambda)$, $x^2 = \omega(x)^2(e+t) + 2\lambda\omega(x)x_\lambda$ and $y^2 = \omega(y)^2(e+t) + 2\lambda\omega(y)y_\lambda$. So we get $\omega(y)x^2 - \omega(x)xy + \lambda\omega(x)^2 y - \lambda\omega(xy)x = 0$. So, we just established (iii).

$(iii) \implies (i)$. If (iii) is valid then substituting xy for y yields the identity $\omega(y)x^2 - x(xy) + \lambda\omega(x)xy - \lambda\omega(xy)x = 0$. From (iii) $\omega(y)x^2 - \lambda\omega(xy)x = \omega(x)xy - \lambda\omega(x)^2 y$. So (i) is valid. □

Remark 2 If $(A,\ \omega)$ is a train algebra with equation $x^3 - 2\omega(x)x^2 + \omega(x)^2 x = 0$, it is known that the idempotents are of the form $e_a = 2a - a^2$ with $\omega(a) = 1$ [10].

We get the following proposition.

Proposition 3 If the degree of the minimal polynomial is ≤ 2 then $(A,\ \omega)$ is special train algebra.

Proof. We know that $N^2 = (\ker\omega)^2 = 0$. So, the powers of N are ideals of A. □

Corollary 2 ([12]) If $M_A = X^2 - \dfrac{3}{2}X + \dfrac{1}{2}$ then $(A,\ \omega)$ is a Lie triple algebra whose equation is $2x(x(xy)) - 3x(x^2 y) + x^3 y = 0$.

4. Minimal polynomial of degree 3

If $M_A = X^3 - (1+\alpha+\beta)X^2 + \alpha X + \beta$, then the following identity is valid in A:

$$x(x(xy)) - (1+\alpha+\beta)\omega(x)x(xy) + \alpha\omega(x)^2 xy + \beta\omega(x)^3 y = 0. \tag{9}$$

We have $P = (X-1)(X^2 - (\alpha+\beta)X - \beta)$. In a suitable extension of the field K we have $P = (X-1)(X-r)(X-s) = X^3 - (1+r+s)X^2 + (r+s+rs)X - rs$ and (9) becomes

$$x(x(xy)) - (1+r+s)\omega(x)x(xy) + (r+s+rs)\omega(x)^2 xy - rs\omega(x)^3 y = 0. \tag{10}$$

In the following, we assume that the roots 1, r and s are all different. Then for an element c_0 of weight 1 $(R_{c_0} - I)(R_{c_0} - rI)(R_{c_0} - sI) = 0$ where R_{c_0} is the multiplication operator by c_0. Thus $A = Kc_0 \oplus A_1(c_0) \oplus A_r(c_0) \oplus A_s(c_0)$ with $A_\lambda(c_0) = \{x \in \ker\omega \mid c_0 x = \lambda x\}$, for $\lambda = 1,\ r,\ s$.

Theorem 3 For $\mu,\ \nu \in \{1,\ r,\ s\}$, we have:

$$A_\mu(c_0)A_\nu(c_0) \subset \bigoplus_{\lambda\in\{1,\, r,\, s\}\setminus\{\mu,\, \nu\}} A_\lambda(c_0). \tag{11}$$

Proof. Partial linearization of (10) yields

$$\begin{aligned} z(x(xy)) + x(z(xy)) + x(x(zy)) &= (1+r+s)[\omega(z)x(xy) + \omega(x)z(xy) + \omega(x)x(zy)] \\ &\quad - (r+s+rs)[2\omega(xz)xy + \omega(x)^2 zy] + 3rs\omega(x^2z)y. \end{aligned} \tag{12}$$

For $x = c_0$, $y \in A_\mu(c_0)$ and $z \in A_\nu(c_0)$ in (12), we get

$$c_0(c_0(yz)) + (\mu - 1 - r - s)c_0(yz) + [\mu^2 - (1+r+s)\mu + (r+s+rs)]yz = 0. \tag{13}$$

Let's put $yz = (yz)_1 + (yz)_r + (yz)_s$ in (13) where $(yz)_\lambda \in A_\lambda(c_0)$. Then

$$c_1(\mu)(yz)_1 + c_r(\mu)(yz)_r + c_s(\mu)(yz)_s = 0, \tag{14}$$

where

$$c_1(\mu) = (\mu - s)(\mu - r),\ c_r(\mu) = (\mu - 1)(\mu - s),\ c_s(\mu) = (\mu - 1)(\mu - r). \tag{15}$$

The inclusions in (11) are due to the fact that the scalars 1, r, s are different. □

Corollary 3 If $A_1(c_0) = 0$ then

(*i*) $A_r(c_0)^2 \subset A_s(c_0)$, $A_s(c_0)^2 \subset A_r(c_0)$, $A_r(c_0)A_s(c_0) = 0$,

(*ii*) $\left(A_r(c_0) \oplus A_s(c_0)\right)^2$ is an ideal of A,

(*iii*) $\left(A_r(c_0) \oplus A_s(c_0)\right)^3 = 0$.

In this case A is a special train algebra.

Proof. Assertion (*i*) follows from (11) under the assumption $A_1(c_0) = 0$. Assertion (*ii*) is a consequence of (*i*). We will now prove assertion (*iii*). Linearizing (12) yields the following identity:

$$\begin{aligned} & z(t(xy)) + z(x(ty)) + t(z(xy)) + x(z(ty)) + t(x(zy)) + x(t(zy)) \\ &= (1+r+s)[\omega(z)t(xy) + \omega(z)x(ty) + \omega(t)z(xy) + \omega(x)z(ty) + \omega(t)x(zy) \\ &\quad + \omega(x)t(zy)] - (r+s+rs)[2\omega(tz)xy + 2\omega(xz)ty + 2\omega(xt)zy] + 6rs\omega(xzt)y. \end{aligned} \tag{16}$$

For $x = c_0, y, z \in A_r(c_0)$ and $t \in A_s(c_0)$, we have: $(2r+s)t(zy) = (1+r+s)t(zy)$. Thus $(r-1)t(zy) = 0$. Since $r \neq 1$ it follows that $t(zy) = 0$, so $A_s(c_0)A_r(c_0)^2 = 0$. By symmetry we get $A_r(c_0)A_s(c_0)^2 = 0$. Since $A_r(c_0)^3 = A_s(c_0)^3 = 0$, we deduce (*iii*). □

Proposition 4 If c_0 is an idempotent of A then $A_1(c_0) = 0$.

Proof. If A satisfies (9) then for all $x \in A$,

$$x^4 - (1+\alpha+\beta)\omega(x)x^3 + \alpha\omega(x)^2x^2 + \beta\omega(x)^3x = 0. \tag{17}$$

Let e be an idempotent element of A and $y \in \ker\omega$. The linearization of (17) gives us $2e(e(ey)) + e(ey) + ey - (1+\alpha+\beta)(2e(ey)+ey) + 2\alpha ey + \beta y = 0$. So $2e(e(ey)) - (1+2\alpha+2\beta)e(ey) + (\alpha-\beta)ey + \beta y = 0$. Also $e(e(ey)) - (1+\alpha+\beta)e(ey) + \alpha ey + \beta y = 0$. The previous two identities can be combined to form $e(ey) - (\alpha+\beta)ey - \beta y = 0$. Since $\alpha = r+s+rs$ and $\beta = -rs$, we have $e(ey) - (r+s)ey + rsy = 0$. Let $y \in A_1(c_0)$. Then $(1-r)(1-s)y = 0$ and $y = 0$ because $r \neq 1$ and $s \neq 1$. So $A_1(c_0) = 0$. □

Proposition 5 Let c_0 be an idempotent element. Then the set of idempotents of A is given by:

(*i*) $I_p(A) = \{c_0\}$ if $r \neq 1/2$ and $s \neq 1/2$,

(*ii*) $I_p(A) = \left\{c_0 + x_s + \dfrac{1}{1-2s}x_s^2 \mid x_s \in A_s(c_0)\right\}$ if $r = 1/2$,

(*iii*) $I_p(A) = \left\{c_0 + x_r + \dfrac{1}{1-2r}x_r^2 \mid x_r \in A_r(c_0)\right\}$ if $s = 1/2$.

Proof. Let $f = c_0 + x_r + x_s$ be an idempotent of A. We have $f^2 = f$. Then $c_0 + 2rx_r + 2sx_s + x_r^2 + x_s^2 = c_0 + x_r + x_s$ because $x_rx_s = 0$ by using the Corollary 3. Thus $x_r^2 = (1-2r)x_s$ and $x_s^2 = (1-2s)x_r$. The three announced cases are again obtained thanks to the Corollary 3. Indeed, $0 = x_sx_r^2 = (1-2r)x_s^2 = (1-2r)(1-2s)x_r$ and $0 = x_rx_s^2 = (1-2s)x_r^2 = (1-2r)(1-2s)x_s$. So, if $r \neq 1/2$ and $s \neq 1/2$, then $x_r = x_s = 0$ and we get (*i*). If $r = 1/2$ then $s \neq 1/2$. So, $x_r^2 = 0$ and $x_r = \dfrac{1}{1-2s}x_s^2$. Therefore (*ii*) is obtained. Using the same method, we get (*iii*). □

Corollary 4 (*i*) If $r = 1/2$ and $c_0' = c_0 + x_s + \dfrac{1}{1-2s}x_s^2$ is another idempotent element then $A_{1/2}(c_0) = A_{1/2}(c_0')$ and $A_s(c_0') = \{u_s + u_sx_s \mid u_s \in A_s(c_0)\}$.

(*ii*) If $s = 1/2$ and $c_0' = c_0 + x_r + \dfrac{1}{1-2r}x_r^2$ is another idempotent element then $A_{1/2}(c_0) = A_{1/2}(c_0')$ and $A_r(c_0') = \{u_r + u_rx_r \mid u_r \in A_r(c_0)\}$.

Theorem 4 If A contains an idempotent element then A is special train algebra.

Proof. This result follows from Corollary 3 and Proposition 4. □

A train algebra with minimal polynomial of degree ≥ 4 is not necessarily a special train algebra. In fact, below we give an example of a train algebra with minimal polynomial of degree 4 and which is not a special train algebra.

Example 3 Let N be a commutative algebra of dimension 5 over a commutative field K, whose nonzero products according to the basis $\{c_1 \ \dots \ c_5\}$ are $c_1c_3 = c_4$, $c_1c_5 = -c_3$, $c_2c_3 = c_5$, $c_2c_4 = c_3$. Let $A = Ke \oplus N$ be the K-algebra obtained by adjoining a unit element e. Then, the linear application $\omega : A \longrightarrow K$, $\lambda e + n \longmapsto \lambda$ is a weight function of A. Let $x = e + \sum_{i=1}^5 \alpha_ic_i \in A$ be an element with weight 1. The matrix of the multiplication $L_x : A \longrightarrow A$, $y \mapsto xy$ according to the basis $\{e,\ c_1,\ \dots,\ c_5\}$ of A is written:

$$L_x = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \alpha_1 & 1 & 0 & 0 & 0 & 0 \\ \alpha_2 & 0 & 1 & 0 & 0 & 0 \\ \alpha_3 & -\alpha_5 & \alpha_4 & 1 & \alpha_2 & -\alpha_1 \\ \alpha_4 & \alpha_3 & 0 & \alpha_1 & 1 & 0 \\ \alpha_5 & 0 & \alpha_3 & \alpha_2 & 0 & 1 \end{bmatrix}.$$

We have $(L_x - I)^4 = 0$, where I is the unit matrix of order 6. So $(A,\ \omega)$ is a functional train algebra of rank 4 which satisfies the identity:

$$x(x(x(xy))) - 4\omega(x)x(x(xy)) + 6\omega(x)^2 x(xy) - 4\omega(x)^3 xy + \omega(x)^4 y = 0.$$

On the other hand, $N^3 = N^2 = \langle c_3,\ c_4,\ c_5 \rangle$. Thus $N = \ker\omega$ is not nilpotent. It follows that $(A,\ \omega)$ is not a special train algebra.

5. Functional train equation of train algebra of rank 3

In this section, we are going to consider train algebra of rank 3. We obtain its functional train equation.

Theorem 5 An arbitrary train algebra of rank 3 with the equation

$$x^3 = (1+\lambda)\omega(x)x^2 - \lambda\omega(x)^2 x, \tag{18}$$

satisfies the functional train identity of rank 4 below:

$$L_x^4 - (2+\lambda)L_x^3 + \frac{1}{4}(5+8\lambda)L_x^2 - \frac{1}{4}(5\lambda+1)L_x + \frac{1}{4}\lambda id_A = 0,\ \ \forall x \in H. \tag{19}$$

Proof. The partial linearization of (18) allows:

$$2x(xy) + x^2 y = (1+\lambda)[\omega(y)x^2 + 2\omega(x)xy] - \lambda[2\omega(xy)x + \omega(x)^2 y]. \tag{20}$$

Let's apply L_x to (20). We get:

$$x(x^2 y) = -2x(x(xy)) + (1+\lambda)[\omega(y)x^3 + 2\omega(x)x(xy)] - \lambda[2\omega(xy)x^2 + \omega(x)^2 xy]. \tag{21}$$

The linearization of (18) gives :

$$x(yz)+y(zx)+z(xy)=(1+\lambda)[\omega(x)yz+\omega(y)zx+\omega(z)xy] \\ -\lambda[\omega(xy)z+\omega(yz)x+\omega(zx)y]. \tag{22}$$

Let's put $z=x^2$ in (22):

$$x(yx^2)+yx^3+x^2(xy)=(1+\lambda)[\omega(x)yx^2+\omega(y)x^3+\omega(x)^2xy] \\ -\lambda[\omega(xy)x^2+\omega(yx^2)x+\omega(x)^3y]. \tag{23}$$

From [13], we know that (18) leads the identity below:

$$x^2x^2-(1+2\lambda)\omega(x)^2x^2+2\lambda\,\omega(x)^3x=0. \tag{24}$$

The linearization of the previous identity leads to:

$$2x^2(xy)=(1+2\lambda)[\omega(xy)x^2+\omega(x)^2xy]-\lambda[3\omega(x^2y)x+\omega(x)^3y]. \tag{25}$$

Using identities (18), (21) and (23), we have the identity

$$2x^2(xy)=4x(x(xy))-2\lambda\,\omega(x^2y)x-4(1+\lambda)\omega(x)x(xy) \\ +2\lambda\,\omega(xy)x^2+2(1+3\lambda)\omega(x)^2xy-2\lambda\,\omega(x)^3y. \tag{26}$$

So, from (25) and (26) we have

$$\omega(xy)(x^2-\lambda\,\omega(x)x)=4x(x(xy))-4(1+\lambda)\omega(x)x(xy)+(1+4\lambda)\omega(x)^2xy-\lambda\,\omega(x)^3y. \tag{27}$$

Let's apply $L_x-\omega(x)id_A$ to identity (27). We obtain

$$L_x^4-(2+\lambda)L_x^3+\frac{1}{4}(5+8\lambda)L_x^2-\frac{1}{4}(5\lambda+1)L_x+\frac{1}{4}\lambda id_A=0,\ \forall x\in H.$$

□

Example 4 ([3]) Let $\lambda\in K$. Let A_λ be a commutative algebra of dimension 4 over K, the nonzero products according a basis $\{e,\ u,\ v,\ w\}$ are $e^2=e$, $eu=\frac{1}{2}u$, $ev=\lambda v$, $ew=\frac{1}{2}w$, $uv=w$. Then, the linear application $\omega:A\longrightarrow K$, such that

$\omega(e)=1$ and $\omega(u)=\omega(v)=\omega(w)=0$ is a weight function of A. Let $x=e+\alpha_1 u+\alpha_2 v+\alpha_3 w$ be an element of weight 1 in A. The matrix of the multiplication $L_x: A\longrightarrow A$, $y\mapsto xy$ according to the basis $\{e,\ u,\ v,\ w\}$ is:

$$L_x=\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{\alpha_1}{2} & \frac{1}{2} & 0 & 0 \\ \lambda\alpha_2 & 0 & \lambda & 0 \\ \frac{\alpha_3}{2} & \alpha_2 & \alpha_1 & \frac{1}{2} \end{bmatrix}.$$

The baric algebra $(A,\ \omega)$ is a train algebra of rank 3 with train equation

$$x^3-(1+\lambda)\omega(x)x^2+\lambda\omega(x)^2x=0.$$

Let $Sp_K(L_x)$ be the set of eigenvalues of L_x.
If $\lambda=\dfrac{1}{2}$ then $Sp_K(L_x)=\{1,\ 1/2\}$ and the minimal polynomial of L_x is:

$$X^3-2X^2+\frac{5}{4}X-\frac{1}{4}.$$

If $\lambda\neq\dfrac{1}{2}$ then $Sp_K(L_x)=\{1,\ 1/2\}\cup\{\lambda\}$ and the minimal polynomial of L_x is:

$$X^4-(2+\lambda)X^3+\left(\frac{5}{4}+2\lambda\right)X^2-\left(\frac{1}{4}+\frac{5\lambda}{4}\right)X+\frac{\lambda}{4}.$$

In this example, we have

$$m_A=(X-1)(X-\lambda) \text{ and } M_A=\begin{cases} \dfrac{(X-1)(2X-1)^2}{4} & \text{if } \lambda=1/2; \\ \dfrac{(X-1)(2X-1)^2(X-\lambda)}{4} & \text{if } \lambda\neq 1/2. \end{cases}$$

Example 5 ([10]) Let $A=\langle e,\ u_1,\ u_2,\ u_3,\ u_4,\ v,\ w\rangle$ be the commutative algebra with multiplication table given by $e^2=e,\ eu_i=\dfrac{1}{2}u_i\ (i=1,\ \dots,\ 4),\ ew=w,\ u_1v=u_1w=u_3,\ u_2v=u_2w=u_4,\ u_2u_3=-u_1u_4=v+w$, other products being zero. Then A is equipped with the weight function ω such that $\omega(e)=1$, $\omega(u_i)=\omega(v)=\omega(w)=0$. By straightforward calculation, one may check that A is a power-associative train algebra of rank 4 with train equation $x^4-2\omega(x)x^3+\omega(x)^2x^2=0$.

For $x=e+\sum_{i=1}^4\alpha_i u_i+\alpha_5 v+\alpha_6 w$, the matrix of the multiplication operator L_x according to the basis $\{e,\ u_1,\ u_2,\ u_3,\ u_4,\ v,\ w\}$ is:

$$L_x = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\alpha_1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{\alpha_2}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{\alpha_3}{2} & \alpha_5+\alpha_6 & 0 & \frac{1}{2} & 0 & \alpha_1 & \alpha_1 \\ \frac{\alpha_4}{2} & 0 & \alpha_5+\alpha_6 & 0 & \frac{1}{2} & \alpha_2 & \alpha_2 \\ 0 & -\alpha_4 & \alpha_3 & \alpha_2 & -\alpha_1 & 0 & 0 \\ \alpha_6 & -\alpha_4 & \alpha_3 & \alpha_2 & -\alpha_1 & 0 & 1 \end{bmatrix}.$$

In this case, we have:

$$m_A = X(X-1)^2 \text{ and } M_A = X(X-1)^2\left(X-\frac{1}{2}\right)^2.$$

We close this paper with two conjectures.

6. Two conjectures

Conjecture 1 Any commutative baric algebra of finite dimensional satisfying the identity (9) is a special train algebra.

Conjecture 2 Let (A, ω) be a train algebra with minimal train polynomial m_A and minimal polynomial M_A. Then there exists an integer $k \geq 0$ such that

$$M_A = \left(X-\frac{1}{2}\right)^k m_A.$$

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Conflict of interest

The authors declare no competing financial interest.

References

[1] Abraham VM. A note on train algebras. *Proceedings of the Edinburgh Mathematical Society*. 1976; 20(2): 53-58.

[2] Etherington IMH. Commutative train algebras of rank 2 and 3. *Journal of the London Mathematical Society*. 1940; 15(3): 136-149.

[3] Etherington IMH. Corrigendum: Commutative train algebras of rank 2 and 3. *Journal of the London Mathematical Society*. 1945; s1-20(4): 238.

[4] Osborn JM. Varieties of algebras. *Advances in Mathematics*. 1972; 8: 163-369.

[5] Gerstenhaber M. On nilalgebras and linear varieties of nilpotent matrices. II. *Duke Mathematical Journal*. 1960; 27: 21-31.

[6] Zhevlakov KA, Slinko AM, Shestakov IP, Shirshov AI. *Rings That Are Nearly Associative*. New York: Academic Press Inc.; 1982.

[7] Wörz-Busekros A. *Algebras in Genetics*. Springer-Verlag; 1980.

[8] Lyubich YI. *Mathematical Structures in Population Genetics*. Berlin, Heidelberg: Springer; 1992.

[9] Gutiérrez Fernández JC. Principal and plenary train algebras. *Communications in Algebra*. 2000; 28(2): 653-667.

[10] Bayara J, Conseibo A, Ouattara M, Zitan F. Power-associative algebras that are train algebras. *Journal of Algebra*. 2010; 324: 1159-1176.

[11] Ouattara M. Sur les T-algèbres de Jordan. *Linear Algebra and its Applications*. 1991; 144(2): 11-21.

[12] Bayara J, Konkobo A, Ouattara M. Algèbres de lie triple sans idempotent. *Afrika Mathematika*. 2014; 25(4): 1063-1075.

[13] Walcher S. Algebras which satisfy a train equation for the first three plenary powers. *Archiv der Mathematik*. 1991; 56: 547-551.