Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

Gauss-Seidel Type Iterative Algorithm for a Generalized System of Extended Non-linear Variational Inequalities

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Received: 15 March 2024; **Revised:** 24 April 2024; **Accepted:** 29 April 2024

Abstract: This study is focused on a new generalized system of extended non-linear variational inequalities (GSENVI, for short) involving $3k$-distinct non-linear relaxed cocoercive operators. We give the equivalent formulation of GSENVI in a more convenient form by using auxiliary principal technique. Through the projection technique, we demonstrate that the non-linear projection equations are analogous to equivalent form of GSENVI. By the use of alternative fixed point formulations, we proposed the k-steps Gauss-Seidel type iterative algorithms to obtain an approximate solution of the GSENVI. Further, we discuss the convergence of proposed k-step Gauss Seidel type iterative algorithms. Several special cases of GSENVI are discussed for the reliability of our findings.

Keywords: generalized system of extended non-linear variational inequalities (GSENVI), relaxed (α, β)-cocoercitivity, lipschitz continuity, fixed point, projection technique, k-steps gauss-seidel type iterative algorithm

MSC: 65P40, 47J20, 49J40

1. Introduction

The mathematical theory of variational inequalities was initially developed to deal with equilibrium problems, precisely the Signorini problem: in that model problem, the functional involved was obtained as the first variation of the involved potential energy. Therefore, it has a variational origin, recalled by the name of the general abstract problem. In recent years, variational inequalities have received a lot of attention due to its applications. Since non-linear variational inequalities may be used to create a unified, innovative and general framework to explore a broad range of issues that come up in applied sciences, transportation, network analysis, elasticity, finance, economics, optimization, etc., see for example [1–21]. It blends new developments in theory and algorithms with a fresh set of applications. We currently have a range of methods to recommend and evaluate different iterative algorithms for resolving variational inequalities and related optimization issues as a result of interaction between several mathematics and engineering scientific disciplines. The projection technique is one feasible and effective way among many iterative algorithms for obtaining numerical solutions of variational inequalities. The Wiener-Hopf equations and the projection method, both of which have their roots in Lions and Stampacchia [2], are valuable resources for estimating the approximate solution of variational inequalities. The primary aim of this method is to use the concept of projection to show that the variational inequality and the fixed-point

DOI: https://doi.org/10.37256/cm.5320244595

problem are equivalent. It is widely recognized that for projection methods to converge, the operator needs to be both strongly monotone and Lipschitz continuous. These convergence conditions ensure the effectiveness of the projection technique in obtaining accurate numerical solutions in various practical scenarios.

In recent studies, several researchers, including Hao et al. [3], Noor et al. [4, 5], Verma [6, 7], and Zhang [8], have explored the application of two and three-step iterative update schemes, double projection and projection type methods. These schemes have been utilized to establish the existence and approximate solutions for systems of non-linear variational inequalities.

However, these sequential iterative methods are only suitable for implementing on the traditional single-processor computer. They may not fully meet the practical demands of modern multiprocessor systems. To satisfy this practical requirements of modern multiprocessor systems, efficient iterative methods having parallel characteristics need to be further developed for the system of variational inequalities. In 2010, Yang et al. [9] introduced and studied the system of nonlinear variational inequalities involving two different nonlinear operators. Using the parallel projection technique, they suggested and analyzed an iterative method for this system of variational inequalities.

In 2009, Noor [10] introduced and studied the extended general variational inequality problem involving three non-linear univariate strongly monotone operators and discuss the projection iterative method for solving extended general variational inequality problem in [11]. In 2016, Noor et al. [12] presented an application wherein they demonstrated that the optimality conditions of a nonconvex minimax problem can be analyzed through a system of extended general variational inequalities involving six non-linear univariate strongly monotone operators. Furthermore, they utilized an equivalent fixed point formulation to address this problem. In 2018, Kim [13] considered a system of extended general variational inequalities featuring six nonlinear univariate operators. Specifically, he focused on relaxed cocoercive operators, which are more generalized than strongly monotone operators, thereby expanding the analytical framework and enhancing the versatility of the approach.

Motivated by above recent advancements in the field, we introduce and investigate a novel generalized system of extended non-linear variational inequalities (GSENVI) by using k-steps Gauss-Seidel type iterative algorithm. Specifically, our focus lies in establishing connections with $3k$-distinct non-linear multivariate and univariate operators. Notably, we explore the application of $3k$-relaxed (r, s)-cocoercive operators, a more generalized class compared to strongly monotone operators. We go over how the GSENVI is made up of several systems of variational inequalities as well as a class of variational inequalities as special cases. We give the equivalent form of GSENVI in a more convenient form by using auxiliary principal technique. Through the projection technique, we show that the non-linear projection equations are analogue to GSENVI. By the use of alternative fixed point formulations, we proposed the k-steps Gauss-Seidel type iterative algorithm to obtain an approximate solution of the GSENVI. Further, we discuss the convergence of proposed k-step Gauss Seidel type iterative algorithms. We have given an example in support of our main result. Importantly, our work serves as an extension, refinement, and improvement of well-established results discussed in previous works such as [10–13]. It is encouraged that those with an interest in the pure and applied sciences look for innovative, original, and novel approaches to use variational inequalities and optimization problems. Researchers should also go in the direction of implementing the novel approaches that are suggested in this work. The remaining part of this paper is organized as follows.

In the next section, we will explore fundamental concepts and established findings necessary for our discussion moving forward. We'll also introduce the GSENVIP and discuss the special cases of the GSENVIP. In section 3, we will concentrate on establishing the equivalence between GSENVIP and a fixed-point problem while also outlining iterative algorithms designed to approximate solutions to GSENVIP. In the final section, we'll furnish a proof of the existence of solutions for GSENVIP and analyze the convergence criteria for sequences generated by the iterative algorithms outlined earlier. Lastly, we'll conclude the paper with a summary of our findings and potential implications.

2. Pertinent terminology and properties

Hereafter, we consider $\mathscr{Z}$ as a real Hilbert space and its norm and inner product are presented by $\|.\|$ and $\langle .,. \rangle$. Let $\mathscr{D}$ be a closed and convex subset of real Hilbert space $\mathscr{Z}$.

Let $\mathscr{A}, f, g : \mathscr{Z} \to \mathscr{Z}$ be the non-linear operators, then a generalized extended variational inequality problem (GEVIP, in short) is to find $q^* \in \mathscr{Z}$: $g(q^*) \in \mathscr{D}$ such that

$$\langle \mathscr{A}(q^*), f(q) - g(q^*) \rangle \geq 0, \; \forall\, q \in \mathscr{Z}, \; f(q) \in \mathscr{D}. \tag{1}$$

The GEVIP (1) is equivalent to a problem (2), which becomes more important in application point of view of finding $q^* \in \mathscr{Z}$: $g(q^*) \in \mathscr{D}$ such that

$$\langle \mathscr{A}(q^*) + g(q^*) - f(q^*), f(q) - g(q^*) \rangle \geq 0, \; \forall\, q \in \mathscr{Z}, \; f(q) \in \mathscr{D}. \tag{2}$$

Lemma 2.1 [14] Let $\mathscr{D}$ be a closed and convex subset of $\mathscr{Z}$. It is given $p^* \in \mathscr{Z}$, then $q^* \in \mathscr{D}$ satisfies

$$\langle q^* - p^*, q - q^* \rangle \; \geq \; 0, \;\; q \in \mathscr{D},$$

iff $q^* \; = \; \mathscr{P}_{\mathscr{D}}(p^*)$. Here $\mathscr{P}_{\mathscr{D}}$ is the projection operator from $\mathscr{H}$ onto $\mathscr{D}$.

Nonexpansive is the well known property of projection operator $\mathscr{P}_{\mathscr{D}}$, that is,

$$\|\mathscr{P}_{\mathscr{D}}(q) - \mathscr{P}_{\mathscr{D}}(q^*)\| \leq \|q - q^*\|, \; \forall\, q, q^* \in \mathscr{Z}.$$

Lemma 2.2 [10] The function $q^* \in \mathscr{Z} : f(q^*) \in \mathscr{D}$ is a solution of the GEVIP (2) if and only if $q^* \in \mathscr{Z} : f(q^*) \in \mathscr{D}$ satisfies $g(q^*) = \mathscr{P}_{\mathscr{D}}[f(q^*) - \eta \mathscr{A} q^*]$, where $\mathscr{P}_{\mathscr{D}}$ is the projection operator and $\eta > 0$ is a constant.

It is easy to show that q^* is a solution of GEVIP (2) iff q^* is a fixed point of $I - g - \mathscr{P}_{\mathscr{D}}[f - \eta \mathscr{A}]$, where I is the identity mapping.

If $f = g$, then the GEVIP (2) is equivalent to find $q^* \in \mathscr{H}$: $g(q^*) \in \mathscr{D}$ such that

$$\langle \mathscr{A}(q^*), \; g(q) - g(q^*) \rangle \geq 0, \; \forall\, q \in \mathscr{Z}, \; g(q) \in \mathscr{D}. \tag{3}$$

If $g = f = I$, then GEVIP (3) coincides with (4) to find $q^* \in \mathscr{D}$ as

$$\langle \mathscr{A}(q^*), \; q - q^* \rangle \geq 0, \; \forall q \in \mathscr{D}. \tag{4}$$

Problem (4) represents the classical variational inequality which was considered and studied by Stampacchia [15].

Definition 2.1 [6] Let us consider a non-linear operator $\mathscr{A} : \mathscr{Z} \to \mathscr{Z}$. Then:

(i) An operator $\mathscr{A}$ is called monotone if

$$\langle \mathscr{A} q^1 - \mathscr{A} q^{1*}, q^1 - q^{1*} \rangle \geq 0, \ \forall q^1, q^{1*} \in \mathscr{Z}.$$

(ii) An operator $\mathscr{A}$ is called μ-strongly monotone if there exists non-negative μ as

$$\langle \mathscr{A} q^1 - \mathscr{A} q^{1*}, q^1 - q^{1*} \rangle \geq \mu \|q^1 - q^{1*}\|^2, \ \forall q^1, q^{1*} \in \mathscr{Z}.$$

This implies that $\mathscr{A}$ is μ-expansive that is

$$\|\mathscr{A} q^1 - \mathscr{A} q^{1*}\| \geq \mu \|q^1 - q^{1*}\|, \ \forall q^1, q^{1*} \in \mathscr{Z}.$$

If $\mu = 1$, then $\mathscr{A}$ is expansive.
(iii) An operator $\mathscr{A}$ is called $\bar{t}$-Lipschitz continuous if there exists non-negative $\bar{t}$ as

$$\|\mathscr{A} q^1 - \mathscr{A} q^{1*}\| \leq \bar{t} \|q^1 - q^{1*}\|, \ \forall q^1, q^{1*} \in \mathscr{Z}.$$

(iv) An operator $\mathscr{A}$ is called relaxed $\bar{r}$-cocoercive if there exists non-negative $\bar{r}$ as

$$\langle \mathscr{A} q^1 - \mathscr{A} q^{1*}, q^1 - q^{1*} \rangle \geq -\bar{r} \|\mathscr{A} q^1 - \mathscr{A} q^{1*}\|^2, \ \forall q^1, q^{1*} \in \mathscr{Z}.$$

It is evident that every $\bar{r}$-cocoercive operator $\mathscr{A}$ is $\frac{1}{\bar{r}}$-Lipschitz continuous.
(v) An operator $\mathscr{A}$ is called relaxed $(\bar{r}, \ \bar{s})$-cocoercive if there exist non-negative $\bar{r}, \ \bar{s}$ as

$$\langle \mathscr{A} q^1 - \mathscr{A} q^{1*}, q^1 - q^{1*} \rangle \geq -\bar{r} \|\mathscr{A} q^1 - \mathscr{A} q^{1*}\|^2 + \bar{s} \|q^1 - q^{1*}\|^2, \ \forall q^1, q^{1*} \in \mathscr{Z}.$$

If $\bar{r} = 0$, then $\mathscr{A}$ is s-strongly monotone. The class of relaxed $(\bar{r}, \ \bar{s})$-cocoercive operators is more generalized than the class of $\bar{r}$-strongly monotone operators. It is easy to observe that the $\bar{r}$-strong monotonicity implies the relaxed $(\bar{r}, \ \bar{s})$-cocoercitivity.

For each $i \in \{1, 2, ..., k\}$, let $\mathscr{A}_i : \underbrace{\mathscr{Z} \times \mathscr{Z} \times ... \times \mathscr{Z}}_{(k \ times)} \to \mathscr{Z}$ and $f_i, g_i : \mathscr{Z} \to \mathscr{Z}$ be $3k$-distinct non-linear operators. Then generalized system of extended non-linear variational inequalities problem (GSENVIP) is to find $(q^1, q^2, ..., q^k) \in \underbrace{\mathscr{Z} \times \mathscr{Z} \times ... \times \mathscr{Z}}_{(k \ times)} : g_1(q^2) \in \mathscr{D}_1, g_2(q^3) \in \mathscr{D}_2, ..., g_{k-1}(q^k) \in \mathscr{D}_{k-1}, g_k(q^1) \in \mathscr{D}_k$ such that

$$\begin{cases} \langle \mathscr{A}_1(q^2, q^3, ..., q^k, q^1), f_1(q) - g_1(q^1) \rangle \geq 0, \ \forall q \in \mathscr{Z}, \ f_1(q) \in \mathscr{D}_1, \\ \langle \mathscr{A}_2(q^3, q^4, ..., q^1, q^2), f_2(q) - g_2(q^2) \rangle \geq 0, \ \forall q \in \mathscr{Z}, \ f_2(q) \in \mathscr{D}_2, \\ : \\ : \\ \langle \mathscr{A}_k(q^1, q^2, ..., q^{k-1}, q^k), f_k(q) - g_k(q^k) \rangle \geq 0, \ \forall q \in \mathscr{Z}, \ f_k(q) \in \mathscr{D}_k. \end{cases} \quad (5)$$

Now, we discuss some special cases of GSENVIP (5):

(I) If $\mathscr{A}_i$ for each $i \in \{1,\ 2\}$, is univariate non-linear mapping, then GSENVIP (5) reduces to extended general variational inequality (6) with 6 non-linear operators to find $q^1q^2 \in \mathscr{X} : f_2(q^2) \in \mathscr{D}_1,\ g_2(q^1) \in \mathscr{D}_2$ such that

$$\begin{cases} \langle \mathscr{A}_1(q^1),\ f_1(q) - f_2(q^2) \rangle \geq 0,\ \forall q \in \mathscr{X},\ f_1(q) \in \mathscr{D}_1, \\ \langle \mathscr{A}_2(q^2),\ g_1(q) - g_2(q^1) \rangle \geq 0,\ \forall q \in \mathscr{X},\ g_1(q) \in \mathscr{D}_2. \end{cases} \tag{6}$$

Problem (6) was studied by Noor et al. [12] and Kim [13].

(II) If $f_1 = g_1 = f,\ f_2 = g_2 = g,\ \mathscr{D}_1 = \mathscr{D}_2 = \mathscr{D}$, is univariate non-linear mapping, then GSENVIP (5) reduces to extended general variational inequality (7) with 4 non-linear operators to find $q^1q^2 \in \mathscr{X} : g_1(q^2),\ g_2(q^1) \in \mathscr{D}$ such that

$$\begin{cases} \langle \mathscr{A}_1(q^1),\ f(q) - g(q^2) \rangle \geq 0,\ \forall q \in \mathscr{X},\ f(q) \in \mathscr{D}, \\ \langle \mathscr{A}_2(q^2),\ f(q) - g(q^1) \rangle \geq 0,\ \forall q \in \mathscr{X},\ f(q) \in \mathscr{D}. \end{cases} \tag{7}$$

Problem (7) was studied by Noor et al. [12].

(III) If $\mathscr{A}_i$ is univariate non-linear mapping with $\mathscr{A}_i = \mathscr{A},\ \mathscr{D}_i = \mathscr{D}$ for each $i \in \{1\}$, then GSENVIP (5) reduces to extended general variational inequality (8) to find $q^1 \in \mathscr{X} : g(q^1) \in \mathscr{D}$ such that

$$\langle \mathscr{A}_1(q^1),\ f(q) - g(q^1) \rangle \geq 0,\ \forall q \in \mathscr{X},\ f(q) \in \mathscr{D}. \tag{8}$$

Problem (8) was studied by Noor et al. [10, 11].

Several new and well-known classes of variational inequalities can be obtained with proper operators and spaces chosen. Regarding contemporary implementations, existence theory, iterative techniques, sensitivity analysis, and various facets of problem (8), consult [10, 11, 16–31] and the references provided therein.

By the auxiliary principle technique of Glowinski and lions [32], as developed by Noor [10, 21], we can rewrite the GSENVIP (5) equivalently to (9) to find $(q^1,\ q^2,\ \ldots,\ q^k) \in \underbrace{\mathscr{X} \times \mathscr{X} \times \ldots \times \mathscr{X}}_{(k\ times)} : g_1(q^2) \in \mathscr{D}_1,\ g_2(q^3) \in \mathscr{D}_2,\ \ldots,\ g_{k-1}(q^k) \in \mathscr{D}_{k-1},\ g_k(q^1) \in \mathscr{D}_k$ such that

$$\begin{cases} \langle \eta_1 \mathscr{A}_1(q^2, q^3,\ \ldots,\ q^k,\ q^1) + g_1(q^1) - f_1(q^2),\ f_1(q) - g_1(q^1) \rangle \geq 0,\ \forall q \in \mathscr{X},\ f_1(q) \in \mathscr{D}_1, \\ \langle \eta_2 \mathscr{A}_2(q^3, q^4, \ldots, q^1, q^2) + g_2(q^2) - f_2(q^3),\ f_2(q) - g_2(q^2) \rangle \geq 0,\ \forall q \in \mathscr{X},\ f_2(q) \in \mathscr{D}_2, \\ : \\ : \\ \langle \eta_k \mathscr{A}_k(q^1, q^2, \ldots, q^{k-1}, q^k) + g_k(q^k) - f_k(q^1),\ f_k(q) - g_k(q^k) \rangle \geq 0,\ \forall q \in \mathscr{X},\ f_k(q) \in \mathscr{D}_k, \\ \text{for each } \eta_i > 0, i \in \{1, 2, \ldots, k\}. \end{cases} \tag{9}$$

3. k-Steps gauss seidel type iterative algorithms

Here, we introduce a k-steps Gauss-Seidel type iterative algorithm to approximate the solution of the GSENVIP (9) (12), utilizing its alternative fixed-point problem denoted by equation (13).

Lemma 3.1 The GSENVIP (5) has a solution $(q^1, q^2, \ldots, q^k) \in \underbrace{\mathscr{X} \times \mathscr{X} \times \ldots \times \mathscr{X}}_{(k\ times)} : g_1(q_0^2) \in \mathscr{D}_1 \subset f_1(\mathscr{X}), g_1(\mathscr{X}),$

$g_2(q_0^3) \in \mathscr{D}_2 \subset f_2(\mathscr{Z}), g_2(\mathscr{Z}), ..., g_{k-1}(q_0^k) \in \mathscr{D}_{k-1} \subset f_{k-1}(\mathscr{Z}), g_{k-1}(\mathscr{Z}), g_k(q_0^1) \in \mathscr{D}_k \subset f_k(\mathscr{Z}), g_k(\mathscr{Z})$ iff, $(q^1, q^2, ..., q^k) \in \underbrace{\mathscr{Z} \times \mathscr{Z} \times ... \times \mathscr{Z}}_{(k\ times)} : g_1(q_0^2) \in \mathscr{D}_1, g_2(q_0^3) \in \mathscr{D}_2, ..., g_{k-1}(q_0^k) \in \mathscr{D}_{k-1}, g_k(q_0^1) \in \mathscr{D}_k$ satisfies the relations

$$\begin{cases} g_1(q^1) = \mathscr{P}_{\mathscr{D}_1}[f_1(q^2) - \eta_1 \mathscr{A}_1(q^2, q^3, ..., q^k, q^1)], \\ g_k(q^k) = \mathscr{P}_{\mathscr{D}_k}[f_k(q^1) - \eta_k \mathscr{A}_k(q^1, q^2, ..., q^{k-1}, q^k)], \\ : \\ : \\ g_2(q^2) = \mathscr{P}_{\mathscr{D}_2}[f_2(q^3) - \eta_2 \mathscr{A}_2(q^3, q^4, ..., q^1, q^2)], \\ \text{for each } \eta_i > 0, i \in \{1, 2, ..., k\}. \end{cases} \tag{10}$$

Proof. The first variaional inequity of (9) is can be written as

$$\langle \eta_1 \mathscr{A}_1(q^2, q^3, ..., q^k, q^1) + g_1(q^1) - f_1(q^2), f_1(q) - g_1(q^1) \rangle \geq 0, \ \forall q \in \mathscr{Z}, \ f_1(q) \in \mathscr{D}_1, \ \eta_1 > 0.$$

By Lemma 2.1 and nonexpansive property of projection operator, the above inequality is equivalent to

$$g_1(q^1) = \mathscr{P}_{\mathscr{D}_1}[f_1(q^2) - \eta_1 \mathscr{A}_1(q^2, q^3, ..., q^k, q^1)], \ \eta_1 > 0.$$

Similarly, the rest variational inequalities are equivalent to the following projection formula:

$$g_k(q^k) = \mathscr{P}_{\mathscr{D}_k}[f_k(q^1) - \eta_k \mathscr{A}_k(q^1, q^2, ..., q^{k-1}, q^k)], \ \eta_k > 0,$$

$$:$$

$$g_3(q^3) = \mathscr{P}_{\mathscr{D}_3}[f_3(q^4) - \eta_3 \mathscr{A}_3(q^4, q^5, ..., q^2, q^3)], \ \eta_3 > 0,$$

$$g_2(q^2) = \mathscr{P}_{\mathscr{D}_2}[f_2(q^3) - \eta_2 \mathscr{A}_2(q^3, q^4, ..., q^1, q^2)], \ \eta_2 > 0.$$

This completes the proof of Lemma 3.1.

Lemma 3.1 allows us to give equivalent fixed point problems (10) corresponding to GSENVIP (9). This alternative fixed point formulation (10) has significant theoretical and numerical implications. Using the formulations (10), we propose few iterative algorithms. Now system (10) can be rewritten as:

$$\begin{cases} q^1 = (1 - \varepsilon_n^1)q^1 + \varepsilon_n^1 \left\{ q^1 - g_1(q^1) + \mathscr{P}_{\mathscr{D}_1}[f_1(q^2) - \eta_1 \mathscr{A}_1(q^2, q^3, ..., q^k, q^1)] \right\}, \\ q^k = (1 - \varepsilon_n^1)q^k + \varepsilon_n^k \left\{ q^k - g_k(q^k) + \mathscr{P}_{\mathscr{D}_k}[f_k(q^1) - \eta_k \mathscr{A}_k(q^1, q^2, ..., q^{k-1}, q^k)] \right\}, \\ : \\ : \\ q^2 = (1 - \varepsilon_n^2)q^2 + \varepsilon_n^2 \left\{ q^2 - g_2(q^2) + \mathscr{P}_{\mathscr{D}_2}[f_2(q^3) - \eta_2 \mathscr{A}_2(q^3, q^4, ..., q^1, q^2)] \right\}, \end{cases} \tag{11}$$

for each $\eta_i > 0, i \in \{1, 2, ..., k\}$.

Algorithm 3.1 For any $(q_0^1, q_0^2, ..., q_0^k) \in \underbrace{\mathscr{Z} \times \mathscr{Z} \times ... \times \mathscr{Z}}_{(k\ times)} : g_1(q_0^2) \in \mathscr{D}_1,\ g_2(q_0^3) \in \mathscr{D}_2,\ ...,\ g_{k-1}(q_0^k) \in \mathscr{D}_{k-1},\ g_k(q_0^1)$ $\in \mathscr{D}_k$, compute the sequences $\{q_{n+1}^1\},\ \{q_{n+1}^2\},\ ...,\ \{q_{n+1}^k\}$ by

$$\begin{cases} q_{n+1}^1 = (1-\varepsilon_n^1)q_n^1 + \varepsilon_n^1\{q_n^1 - g_1(q_n^1) + \mathscr{P}_{\mathscr{D}_1}[f_1(q_n^2) - \eta_1\mathscr{A}_1(q_n^2, q_n^3, ..., q_n^k, q_n^1)]\}, \\ q_{n+1}^k = (1-\varepsilon_n^1)q_n^k + \varepsilon_n^k\left\{q_n^k - g_k(q_n^k) + \mathscr{P}_{\mathscr{D}_k}[f_k(q_{n+1}^1) - \eta_k\mathscr{A}_k(q_{n+1}^1, q_n^2, ..., q_n^{k-1}, q_n^k)]\right\}, \\ \vdots \\ \vdots \\ q_{n+1}^2 = (1-\varepsilon_n^2)q_n^2 + \varepsilon_n^2\left\{q_n^2 - g_2(q_n^2) + \mathscr{P}_{\mathscr{D}_2}[f_2(q_{n+1}^3) - \eta_2\mathscr{A}_2(q_{n+1}^3, q_n^4, ..., q_n^1, q_n^2)]\right\}, \end{cases} \tag{12}$$

where $\eta_i > 0$ and sequence $\varepsilon_n^i \in [0,1],\ i \in \{1, 2, ..., k\}$ for all $n \geq 0$.

For each $i \in \{1, 2, ..., k\}$, let $g_i(q^i) = \mathscr{P}_{\mathscr{D}_i}(z^i)$, where $z^i = f_i(q^i) - \eta_i\mathscr{A}_i(q^{i+1}, q^{i+2}, ..., q^i)$. Then problem (13) coincides with GSENVIP (9).

From Lemma 3.1, we can easily observed that, the GSENVIP (9) has a solution $(q^1, q^2, ..., q^k) \in \underbrace{\mathscr{Z} \times \mathscr{Z} \times ... \times \mathscr{Z}}_{(k\ times)}$ with $g_1(q_0^2) \in \mathscr{D}_1, g_2(q_0^3) \in \mathscr{D}_2, ..., g_{k-1}(q_0^k) \in \mathscr{D}_{k-1}, g_k(q_0^1) \in \mathscr{D}_k$ iff, $(q^1, q^2, ..., q^k) \in \underbrace{\mathscr{Z} \times \mathscr{Z} \times ... \times \mathscr{Z}}_{(k\ times)}$ with $g_1(q_0^2) \in \mathscr{D}_1, g_2(q_0^3) \in \mathscr{D}_2, ..., g_{k-1}(q_0^k) \in \mathscr{D}_{k-1}, g_k(q_0^1) \in \mathscr{D}_k$ satisfies the relations

$$\begin{cases} g_1(q^1) = \mathscr{P}_{\mathscr{D}_1}(z^1), \\ g_k(q^k) = \mathscr{P}_{\mathscr{D}_k}(z^k), \\ \vdots \\ \vdots \\ g_2(q^2) = \mathscr{P}_{\mathscr{D}_2}(z^2), \\ z^1 = f_1(q^2) - \eta_1\mathscr{A}_1(q^2, q^3, ..., q^k, q^1), \\ z^k = f_k(q^1) - \eta_k\mathscr{A}_k(q^1, q^2, ..., q^{k-1}, q^k), \\ \vdots \\ \vdots \\ z^2 = f_2(q^3) - \eta_2\mathscr{A}_2(q^3, q^4, ..., q^1, q^2), \end{cases} \tag{13}$$

for each $\eta_i > 0, i \in \{1, 2, ..., k\}$. Equivalent problem (13) can be utilize to suggest and analyze the following iteration process to solve GSENVIP (9).

For each $i \in \{1, 2, ..., k\}$, let $g_i(q^i) = \mathscr{P}_{\mathscr{D}_i}(z_n^i)$, where $z_n^i = f_i(q_n^i) - \eta_i\mathscr{A}_i(q_n^{i+1}, q_n^{i+2}, ..., q_n^i)$. Then Algorithm 3.1 coincides with the Algorithm 3.2.

Algorithm 3.2 For any $(q_0^1, q_0^2, ..., q_0^k) \in \underbrace{\mathscr{Z} \times \mathscr{Z} \times ... \times \mathscr{Z}}_{(k\ times)} : g_1(q_0^2) \in \mathscr{D}_1, g_2(q_0^3) \in \mathscr{D}_2, ..., g_{k-1}(q_0^k) \in \mathscr{D}_{k-1}, g_k(q_0^1) \in$ $\mathscr{D}_k$, compute the sequences $\{q_{n+1}^1\}, \{q_{n+1}^2\}, ..., \{q_{n+1}^k\}$ by

$$
\begin{cases}
q^1_{n+1} = (1-\varepsilon^1_n)q^1_n + \varepsilon^1_n \left\{q^1_n - g_1(q^1_n) + \mathscr{P}_{\mathscr{D}_1}(z^1_n)\right\}, \\
q^k_{n+1} = (1-\varepsilon^1_n)q^k_n + \varepsilon^k_n \left\{q^k_n - g_k(q^k_n) + \mathscr{P}_{\mathscr{D}_k}({z^k}_n)\right\}, \\
\vdots \\
\vdots \\
q^2_{n+1} = (1-\varepsilon^2_n)q^2_n + \varepsilon^2_n \left\{q^2_n - g_2(q^2_n) + \mathscr{P}_{\mathscr{D}_2}(z^2_n)\right\}, \\
z^1_n = f_1(q^2_n) - \eta_1 \mathscr{A}_1(q^2_n, q^3_n, \ldots, q^k_n, q^1_n), \\
z^k_n = f_k(q^1_{n+1}) - \eta_k \mathscr{A}_k(q^1_{n+1}, q^2_n, \ldots, q^{k-1}_n, q^k_n), \\
\vdots \\
\vdots \\
z^2_n = f_2(q^3_{n+1}) - \eta_2 \mathscr{A}_2(q^3_{n+1}, q^4_n, \ldots, q^1_n, q^2_n),
\end{cases} \tag{14}
$$

where $\eta_i > 0$ and sequence $\varepsilon^i_n \in [0, 1]$, $i \in \{1, 2, \ldots, k\}$ for all $n \geq 0$.

Remark 1

1. For each $i \in \{1, 2\}$, let A_i be univariate mapping, then Algorithm 3.1 coincides with Algorithm 1 obtained by Kim [13].

2. For each $i \in \{1, 2\}$, let A_i be univariate and A_i, f_i, g_i are strongly monotone mappings, then Algorithm 3.1 coincides with Algorithm 1 obtained by Noor et al. [12].

3. For each $i \in \{1\}$, let A_i is univariate and A_i, f_i, g_i are strongly monotone mappings, then Algorithm 3.1 coincides with Algorithm 1 obtained by Noor et al. [10, 11].

Definition 3.1 A mapping $\mathscr{A} : \underbrace{\mathscr{X} \times \mathscr{X} \times \ldots \times \mathscr{X}}_{k\ times} \to \mathscr{X}$

(i) is said to be β-strongly monotone in j^{th} argument, if there exist non-negative β such that

$$
\left\langle \mathscr{A}\left(q^1, \ldots, q^{j-1}, \tilde{q}^j, q^{j+1}, \ldots, q^k\right) - \mathscr{A}\left(q^{1*}, \ldots, q^{j-1*}, \tilde{q}^{j*}, q^{j+1*}, \ldots, q^{k*}\right), \tilde{q}^j - \tilde{q}^{j*} \right\rangle
$$

$$
\geq \beta \left\|\tilde{q}^j - \tilde{q}^{j*}\right\|^2, \forall\, \tilde{q}^j, \tilde{q}^{j*}, q^2, \ldots, q^k, q^{2*}, \ldots, q^{k*} \in \mathscr{X}.
$$

(ii) is said to be relaxed (α, β)-cocoercive in j^{th} argument, if there exist non-negative α, β such that

$$
\left\langle \mathscr{A}\left(q^1, \ldots, q^{j-1}, \tilde{q}^j, q^{j+1}, \ldots, q^k\right) - \mathscr{A}\left(q^{1*}, \ldots, q^{j-1*}, \tilde{q}^{j*}, q^{j+1*}, \ldots, q^{k*}\right), \tilde{q}^j - \tilde{q}^{j*} \right\rangle
$$

$$
\geq -\alpha \left\| \mathscr{A}\left(q^1, \ldots, q^{j-1}, \tilde{q}^j, q^{j+1}, \ldots, q^k\right) - \mathscr{A}\left(q^{1*}, \ldots, q^{j-1*}, \tilde{q}^{j*}, q^{j+1*}, \ldots, q^{k*}\right) \right\|^2 + \beta \left\|\tilde{q}^j - \tilde{q}^{j*}\right\|^2,
$$

$$
\forall\, \tilde{q}^j, \tilde{q}^{j*}, q^2, \ldots, q^k, q^{2*}, \ldots, q^{k*} \in \mathscr{X}.
$$

(iii) is said to be κ-Lipschitz continuous in j^{th} argument, if there exists non-negative κ such that

$$\left\|\mathscr{A}\left(q^1, ..., q^{j-1}, \tilde{q}^j, q^{j+1}, ..., q^k\right) - \mathscr{A}\left(q^{1*}, ..., q^{j-1*}, \tilde{q}^{j*}, q^{j+1*}, ..., q^{k*}\right)\right\|$$

$$\leq \kappa\left\|\tilde{q}^j - \tilde{q}^{j*}\right\|, \quad \forall\, \tilde{q}^j, \tilde{q}^{j*}, q^2, ..., q^k, q^{2*}, ..., q^{k*} \in \mathscr{X}.$$

Lemma 3.2 [33] If $\{\alpha_n\}_0^\infty$ is a non-negative sequence satisfying the following inequality:

$$\alpha_{n+1} \leq (1-\lambda_n)\,\alpha_n + \beta_n, \quad \forall\, n \geq 0, \tag{15}$$

with $\lambda_n \in [0, 1]$, $\sum_{n=0}^{\infty} \lambda_n = \infty$, and $\beta_n = o(\lambda_n)$. Then $\lim_{n\to\infty} \alpha_n = 0$.

4. Convergence analysis

In this section, we examine GSENVIP (9) and its equivalence to a system of fixed-point problems. This alternative equivalent problem is employed to formulate iteration schemes for solving GSENVIP (9).

To simplify the computation steps easily, let us given the following equations which are used in Theorem 4.1, Corollary 4.1 and Theorem 4.2.

$$\mathbf{A1}: \begin{cases} 1-\nu = \varepsilon_n^1(k_1^* + k_1^{**}) \geq 0 & \text{such that } \sum_{n=0}^{\infty} \varepsilon_n^1(k_1^* + k_1^{**}) = \infty; \\ 1-\varepsilon^1 = (1-k_1)\varepsilon_n^1 - (k_k^* + k_k^{**})\varepsilon_n^k \geq 0 & \text{such that } \sum_{n=0}^{\infty}(1-k_1)\varepsilon_n^1 - (k_k^* + k_k^{**})\varepsilon_n^k = \infty; \\ 1-\varepsilon^2 = (1-k_2)\varepsilon_n^2 \geq 0 & \text{such that } \sum_{n=0}^{\infty}(1-k_2)\varepsilon_n^2 = \infty; \\ \vdots \\ \vdots \\ 1-\varepsilon^k = (1-k_k)\varepsilon_n^k - (k_{k-1}^* + k_{k-1}^{**})\varepsilon_n^{k-1} \geq 0 & \text{such that } \sum_{n=0}^{\infty}(1-k_k)\varepsilon_n^k - (k_k^* + k_k^{**})\varepsilon_n^k = \infty. \end{cases}$$

$$\mathbf{A2}: \begin{cases} u^{1*} = (q^{1*}, q^{2*}, ..., q^{k*}), \; u_{n+1}^1 = (q_{n+1}^1, q_n^2, ..., q_n^k) \\ u^{2*} = (q^{2*}, q^{3*}, ..., q^{1*}), \; u_{n+1}^2 = (q_{n+1}^2, q_n^3, ..., q_n^1) \\ \vdots \\ \vdots \\ u^{k*} = (q^{k*}, q^{1*}, ..., q^{(k-1)*}), \; u_{n+1}^k = (q_{n+1}^k, q_n^1, ..., q_n^{k-1}). \end{cases}$$

We now examine the convergence analysis of the Algorithm 3.1 which is the core of our following result.

Theorem 4.1 For $i \in \{1, 2, ..., k\}$, let $\mathscr{A}_i : \underbrace{\mathscr{X} \times \mathscr{X} \times ... \times \mathscr{X}}_{k\ times} \to \mathscr{X}$ be relaxed (α_i, β_i)-cocoercive and κ_i-Lipschitzian in the first argument. Let $f_i, g_i : \mathscr{X} \to \mathscr{X}$ be relaxed $(\bar{r}_i, \bar{s}_i)$-cocoercive and $\bar{t}_i$-Lipschitzian, (r_i, s_i)-cocoercive and t_i-Lipschitzian, respectively. If the following conditions (16)-(18) are satisfied with condition (A1):

$$k_i = [1 + 2r_i t_i^2 - 2s_i + t_i^2]^{1/2} \text{ with } 2s_i - (2r_i t_i^2 + t_i^2) < 1; \tag{16}$$

$$k_i^* = [1 + 2\eta_i \alpha_i \kappa_i^2 - 2\eta_i \beta_i + \eta_i^2 \kappa_i^2]^{1/2} \text{ with } 0 < k' < 1; \tag{17}$$

$$k_i^{**} = [1 + 2\bar{r}_i\bar{t}_i^2 - 2\bar{s}_i + \bar{t}_i^2]^{1/2} \text{ with } 2\bar{s}_i - (2\bar{r}_i\bar{t}_i^2 + \bar{t}_i^2) < 1; \tag{18}$$

and $\varepsilon_n^i \in [0, 1]$ for each $i \in \{1, 2, ..., k\}$, and $n \geq 0$. Then sequences $\{q_n^1\}, \{q_n^2\}, ..., \{q_n^k\}$ generated from Algorithm 3.1 strongly converge to the solution $(q^{1*}, q^{2*}, ..., q^{k*})$ of GSENVIP (9).

Proof. Let $(q^{1*}, q^{2*}, ..., q^{k*})$ be the solution of GSENVIP (9). From (11), it follows that

$$\begin{cases} q^{1*} = (1-\varepsilon_n^1)q^{1*} + \varepsilon_n^1\left\{q^{1*} - g_1(q^{1*}) + \mathscr{P}_{\mathscr{D}_1}[f_1(q^{2*}) - \eta_1\mathscr{A}_1(u^{2*})]\right\}, \\ q^{k*} = (1-\varepsilon_n^1)q^{k*} + \varepsilon_n^k\left\{q^{k*} - g_k(q^{k*}) + \mathscr{P}_{\mathscr{D}_k}[f_k(q^{1*}) - \eta_k\mathscr{A}_k(u^{1*})]\right\}, \\ : \\ : \\ q^{2*} = (1-\varepsilon_n^2)q^{2*} + \varepsilon_n^2\left\{q^{2*} - g_2(q^{2*}) + \mathscr{P}_{\mathscr{D}_2}[f_2(q^{3*}) - \eta_2\mathscr{A}_2(u^{3*})]\right\}, \\ \text{for each } \eta_i > 0, i \in \{1, 2, ..., k\}. \end{cases} \tag{19}$$

Using Algorithm 3.1 and Lemma 3.1, we obtain

$$\begin{aligned} \|q_{n+1}^1 - q^{1*}\| &= \|(1-\varepsilon_n^1)q_n^1 + \varepsilon_n^1\left\{q_n^1 - g_1(q_n^1) + \mathscr{P}_{\mathscr{D}_1}[f_1(q_{n+1}^2) - \eta_1\mathscr{A}_1(u_{n+1}^2)]\right\} \\ &\quad - (1-\varepsilon_n^1)q^{1*} - \varepsilon_n^1\left\{q^{1*} - g_1(q^{1*}) + \mathscr{P}_{\mathscr{D}_1}[f_1(q^{2*}) - \eta_1\mathscr{A}_1(u^{2*})]\right\}\| \\ &\leq (1-\varepsilon_n^1)\|q_n^1 - q_n^{1*}\| + \varepsilon_n^1\|q_n^1 - q^{1*} - (g_1(q_n^1) - g_1(q^{1*}))\| \\ &\quad + \varepsilon_n^1\|f_1(q_{n+1}^2) - f_1(q^{2*}) - \eta_1(\mathscr{A}_1(u_{n+1}^2) - \eta_1\mathscr{A}_1(u^{2*}))\| \\ &\leq (1-\varepsilon_n^1)\|q_n^1 - q^{1*}\| + \varepsilon_n^1\|q_n^1 - q^{1*} - (g_1(q_n^1) - g_1(q^{1*}))\| \\ &\quad + \varepsilon_n^1\|f_1(q_{n+1}^2) - f_1(q^{2*}) - \eta_1(\mathscr{A}_1(u_{n+1}^2) - \mathscr{A}_1(u^{2*}))\| \\ &\leq (1-\varepsilon_n^1)\|q_n^1 - q^{1*}\| + \varepsilon_n^1\|q_{n+1}^2 - q^{2*} - \eta_1(\mathscr{A}_1(u_{n+1}^2) - \mathscr{A}_1(u^{2*}))\| \\ &\quad + \varepsilon_n^1\|q_n^1 - q^{1*} - (g_1(q_n^1) - g_1(q^{1*}))\| + \varepsilon_n^1\|q_{n+1}^2 - q^{2*} - (f_1(q_{n+1}^2) - f_2(q^{2*})\|. \end{aligned} \tag{20}$$

Since $\mathscr{A}_i$ is relaxed (α_i, β_i)-cocoercive as well as κ_i-Lipschitz continuous in the first argument for $i = 1$, then it follows that

$$\begin{aligned} &\|q_{n+1}^2 - q^{2*} - \eta_1(\mathscr{A}_1(u_{n+1}^2) - \mathscr{A}_1(u^{2*}))\|^2 \\ &= \|q_{n+1}^2 - q^{2*}\|^2 - 2\eta_1\langle\mathscr{A}_1(u_{n+1}^2) - \mathscr{A}_1(u^{2*}),\ q_{n+1}^2 - q^{2*}\rangle + \eta_1^2\ \|\mathscr{A}_1(u_{n+1}^2) - \mathscr{A}_1(u^{2*})\|^2 \end{aligned}$$

$$
\begin{aligned}
&\leq \|q_{n+1}^2 - q^{2*}\|^2 - 2\eta_1\ [-\alpha_1\|\mathscr{A}_1(u_{n+1}^2) - \mathscr{A}_1(u^{2*})\|^2 + \beta_1\|q_{n+1}^2 - q^{2*}\|^2]\ +\ \eta_1^2\ \|\mathscr{A}_1(u_{n+1}^2) - \mathscr{A}_1(u^{2*})\|^2 \\
&\leq \|q_{n+1}^2 - q^{2*}\|^2 + 2\eta_1\alpha_1\kappa_1^2\|q_{n+1}^2 - q^{2*}\|^2 - 2\eta_1\beta_1\|q_{n+1}^2 - q^{2*}\|^2 + \eta_1^2\kappa_1^2\|q_{n+1}^2 - q^{2*}\|^2 \\
&= [1\ +\ 2\eta_1\alpha_1\kappa_1^2 - 2\eta_1\beta_1\ +\ \eta_1^2\kappa_1^2]\|q_{n+1}^2 - q^{2*}\|^2 \\
&= k_1^*\|q_{n+1}^2 - q^{2*}\|^2,
\end{aligned}
\tag{21}
$$

where k_1^* is given by (17) for $i = 1$.

Since g_i is relaxed (r_i, s_i)-cocoercive as well as t_i-Lipschitz continuous in the first argument for $i = 1$, then it follows that

$$
\|q_n^1 - q^{1*} - (g_1(q_n^1) - g_1(q^{1*}))\|^2\ \leq\ k_1\|q_n^1 - q^{1*}\|, \tag{22}
$$

where k_1 is given by (22) for $i = 1$.

Since f_i is relaxed $(\bar{r}_i, \bar{s}_i)$-cocoercive as well as $\bar{t}_i$-Lipschitz continuous in the first argument for $i = 1$, then it follows that

$$
\|q_{n+1}^2 - q^{2*} - (f_1(q_{n+1}^2) - f_1(q^{2*}))\|^2\ \leq\ k_1^{**}\|q_{n+1}^2 - q^{2*}\|, \tag{23}
$$

where k_1^{**} is given by (18) for $i = 1$.

Using (21)-(23) in (20), we have

$$
\|q_{n+1}^1 - q^{1*}\|\ \leq\ [1 - (1 - k_1)\varepsilon_n^1]\|q_n^1 - q^{1*}\|\ +\ \varepsilon_n^1(k_1^* + k_1^{**})\|q_{n+1}^2 - q^{2*}\|. \tag{24}
$$

In the light of (20), we have

$$
\begin{aligned}
\|q_{n+1}^2 - q^{2*}\| = &\ \|(1 - \varepsilon_n^2)q_n^2 + \varepsilon_n^2\left\{q_n^2 - g_2(q_n^2) + \mathscr{P}_{\mathscr{D}_2}[f_2(q_n^3) - \eta_1\mathscr{A}_2(u_n^3)]\right\} \\
&- (1 - \varepsilon_n^2)q^{2*} - \varepsilon_n^2\left\{q^{2*} - g_2(q^{2*}) + \mathscr{P}_{\mathscr{D}_2}[f_2(q^{3*}) - \eta_2\mathscr{A}_2(u^{3*})]\right\}\| \\
\leq &\ (1 - \varepsilon_n^2)\|q_n^2 - q^{2*}\| + \varepsilon_n^2\|q_n^3 - q^{3*} - \eta_2(\mathscr{A}_2(u_n^3) - \mathscr{A}_2(u^{3*}))\| \\
&+ \varepsilon_n^2\|q_n^2 - q^{2*} - (g_2(q_n^2) - g_2(q^{2*}))\| + \varepsilon_n^2\|q_n^3 - q^{3*} - (f_2(q_n^3) - f_2(q^{3*})\|.
\end{aligned}
\tag{25}
$$

Similarly using the given condition on g_i, f_i and A_i for $i = 2$, we have

$$\|q_n^3 - q^{3*} - \eta_1(\mathscr{A}_2(u_n^3) - \mathscr{A}_2(u^{3*}))\|^2 \leq k_2^* \|q_n^3 - q^{3*}\|^2, \tag{26}$$

where k_2^* is given by (17) for $i = 2$.

$$\|q_n^2 - q^{2*} - (g_2(q_n^2) - g_2(q^{2*}))\|^2 \ \leq \ k_2 \|q_n^2 - q^{2*}\|, \ k_2 \text{ given by (16) for } i = 2. \tag{27}$$

$$\|q_n^3 - q^{3*} - (f_2(q_n^3) - f_2(q^{3*}))\|^2 \ \leq \ k_2^{**} \|q_n^3 - q^{3*}\|, \tag{28}$$

where k_2^{**} is given by (18) for $i = 2$.

Using (26)-(27) in (25), we get

$$\|q_{n+1}^2 - q^{2*}\| \ \leq \ [1 - (1 - k_2)\varepsilon_n^2] \|q_n^2 - q^{2*}\| \ + \ \varepsilon_n^2 (k_2^* + k_2^{**}) \|q_n^3 - q^{3*}\|. \tag{29}$$

Now, we can compute the following (30) by using the above similar steps

$$\begin{aligned} \|q_{n+1}^k - q^{k*}\| \leq (1 - \varepsilon_n^k) \|q_n^k - q^{k*}\| + \varepsilon_n^k \|q_n^k - q^{k*} - \eta_k(\mathscr{A}_k(u_n^1) - \mathscr{A}_k(u^{1*}))\| \\ + \varepsilon_n^k \|q_n^k - q^{k*} - (g_k(q_n^k) - g_k(q^{k*}))\| + \varepsilon_n^k \|q_n^1 - q^{1*} - (f_k(q_n^1) - f_k(q^{1*})\|. \end{aligned} \tag{30}$$

Similarly using the given conditions on g_i, f_i and A_i for $i = k$, we have

$$\|q_n^1 - q^{1*} - \eta_k(\mathscr{A}_k(u_n^1) - \mathscr{A}_k(u^{1*}))\|^2 \leq k_k^* \|q_n^k - q^{k*}\|^2, \tag{31}$$

where k_k^* is given by (18) for $i = k$.

$$\|q_n^k - q^{k*} - (g_k(q_n^k) - g_k(q^{2*}))\|^2 \ \leq \ k_k \|q_n^k - q^{k*}\|, \ k_k \text{ given by (16) for } i = k. \tag{32}$$

$$\|q_n^1 - q^{1*} - (f_k(q_n^1) - f_k(q^{k*}))\|^2 \ \leq \ k_k^{**} \|q_n^1 - q^{1*}\|, \ k_k^{**} \text{ given by (16) for } i = k. \tag{33}$$

Using (31)-(33) in (30), we get

$$\|q_{n+1}^k - q^{k*}\| \ \leq \ [1 - (1 - k_k)\varepsilon_n^k] \|q_n^k - q^{k*}\| + \varepsilon_n^k (k_k^* + k_k^{**}) \|q_n^1 - q^{1*}\|. \tag{34}$$

We compute

$$\|q_{n+1}^1 - q^{1*}\| + \|q_{n+1}^2 - q^{2*}\| + ... + \|q_{n+1}^k - q^{k*}\|$$

$$\leq [1-(1-k_1)\varepsilon_n^1 + (k_k^* + k_k^{**})\varepsilon_n^k]\|q_n^1 - q^{1*}\|$$

$$+ [1-(1-k_2)\varepsilon_n^2]\|q_n^2 - q^{2*}\| + (k_1^* + k_1^{**})\varepsilon_n^1\|q_{n+1}^2 - q^{2*}\|$$

$$\vdots$$

$$\vdots$$

$$+ [1-(1-k_k)\varepsilon_n^k + (k_{k-1}^* + k_{k-1}^{**})\varepsilon_n^{k-1}]\|q_n^2 - q^{2*}\|$$

$$\|q_{n+1}^1 - q^{1*}\| + [1-\varepsilon_n^1(k_1^* + k_1^{**})]\|q_{n+1}^2 - q^{2*}\| + ... + \|q_{n+1}^k - q^{k*}\|$$

$$\leq [1-(1-k_1)\varepsilon_n^1 + (k_k^* + k_k^{**})\varepsilon_n^k]\|q_n^1 - q^{1*}\|$$

$$+ [1-(1-k_2)\varepsilon_n^2]\|q_n^2 - q^{2*}\|$$

$$\vdots$$

$$\vdots$$

$$+ [1-(1-k_k)\varepsilon_n^k + (k_{k-1}^* + k_{k-1}^{**})\varepsilon_n^{k-1}]\|q_n^2 - q^{2*}\|.$$

Thus, we have

$$\|q_{n+1}^1 - q^{1*}\| + \nu\|q_{n+1}^2 - q^{2*}\| + ... + \|q_{n+1}^k - q^{k*}\|$$

$$\leq \max(\varepsilon^1 + \varepsilon^2 + ... + \varepsilon^k)\left\{|q_n^1 - q^{1*}\| + \|q_n^2 - q^{2*}\| + ... + \|q_n^k - q^{k*}\|\right\}$$

$$\leq \varepsilon\left\{\|q_n^1 - q^{1*}\| + \|q_n^2 - q^{2*}\| + ... + \|q_n^k - q^{k*}\|\right\},$$

where

$$\nu = 1 - \varepsilon_n^1(k_1^* + k_1^{**})$$

$$\varepsilon^1 = 1-(1-k_1)\varepsilon_n^1 + (k_k^* + k_k^{**})\varepsilon_n^k$$

$$\varepsilon^2 = 1-(1-k_2)\varepsilon_n^2$$

$$\vdots$$

$$\vdots$$

$$\varepsilon^k = 1-(1-k_k)\varepsilon_n^k + (k_{k-1}^* + k_{k-1}^{**})\varepsilon_n^{k-1}$$

$$\varepsilon = \max(\varepsilon^1 + \varepsilon^2 + \dots + \varepsilon^k).$$

From (16)-(18) and (A1), we can obtain $\varepsilon < 1$.

Using Lemma 3.2, we have

$$\lim_{n\to\infty} \|q_{n+1}^1 - q^{1*}\| + \nu\|q_{n+1}^2 - q^{2*}\| + \dots + \|q_{n+1}^k - q^{k*}\| = 0. \tag{35}$$

This implies that

$$\lim_{n\to\infty} \|q_{n+1}^1 - q^{1*}\| = \lim_{n\to\infty} \|q_{n+1}^2 - q^{2*}\| = \dots = \lim_{n\to\infty} \|q_{n+1}^k - q^{k*}\| = 0. \tag{36}$$

This completes the proof.

Remark 4.1

(i) For each $i \in \{1, 2\}$ if A_i is univariate mapping then we can obtain the convergence result for GSENVIP (9) which is equivalent to (6) by using Algorithms 3.1, studied in [12, 13].

(ii) For each $i \in \{1, 2\}$ if $f_i = f$ and $g_i = g$ then we can obtain the convergence result for GSENVIP (9) which is equivalent to (7) by using Algorithms 3.1, studied in [12].

(iii) For each $i \in \{1\}$ if $f_i = f$ and $g_i = g$, then we can obtain the convergence result for GSENVIP (9) which is equivalent to (8) by using Algorithms 3.1, studied in [10, 11].

Corollary 4.1 For $i \in \{1, 2, \dots, k\}$, let $\mathscr{A}_i : \underbrace{\mathscr{X} \times \mathscr{X} \times \dots \mathscr{X}}_{k\ times} \to \mathscr{X}$ be β_i-strongly monotone and κ_i-Lipschitzian in the first argument. Let $f_i, g_i : \mathscr{X} \to \mathscr{X}$ be $\bar{s}_i$-strongly monotone and $\bar{t}_i$-Lipschitzian, s_i-strongly monotone and t_i-Lipschitzian, respectively. If the following conditions (37)-(39) are satisfied with condition (A1):

$$k_i = [1 - 2s_i + t_i^2]^{1/2} \text{ with } 2s_i - t_i^2 < 1; \tag{37}$$

$$k_i^* = [1 - 2\eta_i\beta_i + \eta_i^2\kappa_i^2]^{1/2} \text{ with } 0 < k_i' < 1; \tag{38}$$

$$k_i^{**} = [1 - 2\bar{s}_i + \bar{t}_i^2]^{1/2} \text{ with } 2\bar{s}_i - \bar{t}_i^2 < 1; \tag{39}$$

and $\varepsilon_n^i \in [0, 1]$ for each $i \in \{1, 2, ..., k\}$ and $n \geq 0$.

Then sequences $\{q_n^1\}$, $\{q_n^2\}$, ..., $\{q_n^k\}$ generated from Algorithm 3.1 strongly converges to the solution of $(q^{1*}, q^{2*}, ..., q^{k*})$ of GSENVIP (9).

In Theorem 4.1, using Definition 2.1 (i) and (iii), we can choose $\alpha_i = s_i = \bar{s}_i$ for each $i \in \{1, 2, ..., k\}$ to obtain the result of Corollary 4.1.

Theorem 4.2 For $i \in \{1, 2, ..., k\}$, let $\mathscr{A}_i : \underbrace{\mathscr{X} \times \mathscr{X} \times ... \times \mathscr{X}}_{k\ times} \to \mathscr{X}$ be relaxed (α_i, β_i)-cocoercive and κ_i-Lipschitzian in the first argument. Let $f_i, g_i : \mathscr{X} \to \mathscr{X}$ be relaxed $(\bar{r}_i, \bar{s}_i)$-cocoercive and $\bar{t}_i$-Lipschitzian, (r_i, s_i)-cocoercive and t_i-Lipschitzian, respectively. If the following conditions (40)-(42) are satisfied with condition (A1):

$$k_i = [1 + 2r_it_i^2 - 2s_i + t_i^2]^{1/2} \text{ with } 2s_i - (2r_it_i^2 + t_i^2) < 1; \tag{40}$$

$$k_i^* = [1 + 2\eta_i\alpha_i\kappa_i^2 - 2\eta_i\beta_i + \eta_i^2\kappa_i^2]^{1/2} \text{ with } 0 < k^* < 1; \tag{41}$$

$$k_i^{**} = [1 + 2\bar{r}_i\bar{t}_i^2 - 2\bar{s}_i + \bar{t}_i^2]^{1/2} \text{ with } 2\bar{s}_i - (2\bar{r}_i\bar{t}_i^2 + \bar{t}_i^2) < 1; \tag{42}$$

and $\varepsilon_n^i \in [0, 1]$ for each $i \in \{1, 2, ..., k\}$ and $n \geq 0$.

Then iterative $\{q_n^1\}$, $\{q_n^2\}$, ..., $\{q_n^k\}$ generated from Algorithm 3.2 strongly converges to the solution $(q^{1*}, q^{2*}, ..., q^{k*})$ of GSENVIP (9).

Proof. Let $(q^{1*}, q^{2*}, ..., q^{k*})$ be the solution of GSENVIP (9). From (13), It follows that

$$\begin{cases} q^{1*} = (1-\varepsilon_n^1)q^{1*} + \varepsilon_n^1\left\{q^{1*} - g_1(q^{1*}) + \mathscr{P}_{\mathscr{D}_1}(z^{1*})\right\}, \\ q^{k*} = (1-\varepsilon_n^1)q^{k*} + \varepsilon_n^k\left\{q^{k*} - g_k(q^{k*}) + \mathscr{P}_{\mathscr{D}_k}(z^{k*})\right\}, \\ : \\ : \\ q^{2*} = (1-\varepsilon_n^2)q^{2*} + \varepsilon_n^2\left\{q^{2*} - g_2(q^{2*}) + \mathscr{P}_{\mathscr{D}_2}(z^{2*})\right\}, \\ z^{1*} = f_1(q^{2*}) - \eta_1\mathscr{A}_1(u^{2*}), \\ z^{k*} = f_k(q^{1*}) - \eta_k\mathscr{A}_k(u^{1*}), \\ : \\ : \\ z^{2*} = f_2(q^{3*}) - \eta_2\mathscr{A}_2(u^{3*}), \\ \text{for each } \eta_i > 0,\ i \in \{1, 2, ..., k\}. \end{cases} \tag{43}$$

Using Algorithm 3.2 and (13), we get

$$\begin{aligned} \|z_n^1 - z^{1*}\| &= \|f_1(q_{n+1}^2) - \eta_1\mathscr{A}_1(u_{n+1}^2) - f_1(q^{2*}) + \eta_1\mathscr{A}_1(u^{2*})\| \\ &\leq \|q_{n+1}^2 - q^{2*} - \eta_1(\mathscr{A}_1(u_{n+1}^2) - \mathscr{A}_1(u^{2*}))\| + \|q_{n+1}^2 - q^{2*} - (f_1(q_{n+1}^2) - f_1(q^{2*}))\|. \end{aligned} \tag{44}$$

Using (21) and (23) in (44), we get

$$\|z_n^1 - z^{1*}\| \ \leq \ [k_1^* + k_1^{**}]\|q_{n+1}^2 - q^{2*}\|, \tag{45}$$

where k_1^*, k_1^{**} are given by (41), (42) for $i = 1$.

$$\|q_{n+1}^1 - q^{1*}\| = (1-\varepsilon_n^1)\|q_n^1 - q^{1*}\| + \varepsilon_n^1\|q_n^1 - q^{1*} - (g_1(q_n^1) - g_1(q^{1*}))\| + \varepsilon_n^1\|\mathscr{P}_{\mathscr{D}_1}(z_n^1) - \mathscr{P}_{\mathscr{D}_1}(z^{1*})\|. \tag{46}$$

Using (22), (45) in (46), we get

$$\|q_{n+1}^1 - q^{1*}\| \leq [1-(1-k_1)\varepsilon_n^1]\|q_n^1 - q^{1*}\| + \varepsilon_n^1[k_1^* + k_1^{**}]\|q_{n+1}^2 - q^{2*}\|. \tag{47}$$

Using Algorithm 3.2 and (13) to evaluate the following

$$\|z_n^2 - z^{2*}\| \ = \|f_2(q_n^3) - \eta_2\mathscr{A}_2(u_n^3) - (f_2(q^{3*}) - \eta_2\mathscr{A}_2(u^{3*}))\| \tag{48}$$

$$\leq \|q_n^3 - q^{3*} - \eta_2(\mathscr{A}_2(u_n^3) - \mathscr{A}_2(u^{3*}))\| + \|q_n^3 - q^{3*} - (f_2(q_n^3) - f_2(q^{3*}))\|. \tag{49}$$

Using (26), (27) in (48), we get

$$\|z_n^2 - z^{2*}\| \ \leq \ [k_2^* + k_2^{**}]\|q_n^3 - q^{3*}\|, \tag{50}$$

where k_2^*, k_2^{**} are given by (41), (42) for $i = 2$.

$$\|q_{n+1}^2 - q^{2*}\| \leq (1-\varepsilon_n^2)\|q_n^2 - q^{2*}\| + \varepsilon_n^2\|q_n^2 - q^{2*} - (g_2(q_n^2) - g_2(q^{2*}))\| + \varepsilon_n^2\|\mathscr{P}_{\mathscr{D}_2}(z_n^2) - \mathscr{P}_{\mathscr{D}_2}(z^{2*})\|. \tag{51}$$

Using (26), (50) in (51), we get

$$\|q_{n+1}^2 - q^{2*}\| \leq [1-(1-k_2)\varepsilon_n^2]\|q_n^2 - q^{2*}\| + \varepsilon_n^2[k_2^* + k_2^{**}]\|q_n^3 - q^{3*}\|. \tag{52}$$

Through this similar process, we can evaluate

$$\begin{aligned}\|z_n^k - z^{k*}\| &= \|f_k(q_n^1) - \eta_k\mathscr{A}_k(u_n^1) - (f_k(q^{1*}) - \eta_2\mathscr{A}_k(u^{1*}))\| \\ &\leq \|q_n^1 - q^{1*} - \eta_k(\mathscr{A}_k(u_n^1) - \mathscr{A}_k(u^{1*}))\| \\ &\quad + \|q_n^1 - q^{1*} - (f_k(q_n^1) - f_k(q^{1*}))\|.\end{aligned} \tag{53}$$

Using (31), (33) in (53), we get

$$\|z_n^k - z^{k*}\| \ \le \ [k_k^* + k_k^{**}]\|q_n^k - q^{k*}\|, \text{ where } k_k^*, k_k^{**} \text{ are given by } (41),\ (42) \text{ for } i = k. \tag{54}$$

$$\|q_{n+1}^k - q^{k*}\| \le (1-\varepsilon_n^k)\|q_n^k - q^{k*}\| + \varepsilon_n^k\|q_n^k - q^{k*} - (g_k(q_n^k) - g_k(q^{k*}))\| + \varepsilon_n^2\|\mathscr{P}_{\mathscr{D}_2}(z_n^k) - \mathscr{P}_{\mathscr{D}_2}(z^{k*})\|. \tag{55}$$

Using (32), (54) in (55), we get

$$\|q_{n+1}^k - q^{k*}\| \le [1-(1-k_k)\varepsilon_n^k]\|q_n^k - q^{k*}\| + \varepsilon_n^k[k_k^* + k_k^{**}]\|q_n^1 - q^{1*}\| \tag{56}$$

Now, we compute

$$\begin{aligned}
&\|q_{n+1}^1 - q^{1*}\| + \|q_{n+1}^2 - q^{2*}\| + \ldots + \|q_{n+1}^k - q^{k*}\| \\
\le\ &[1-(1-k_1)\varepsilon_n^1 + (k_k^* + k_k^{**})\varepsilon_n^k]\|q_n^1 - q^{1*}\| \\
&+ [1-(1-k_2)\varepsilon_n^2]\|q_n^2 - q^{2*}\| + (k_1^* + k_1^{**})\varepsilon_n^1\|q_{n+1}^2 - q^{2*}\| \\
&\vdots \\
&\vdots \\
&+ [1-(1-k_k)\varepsilon_n^k + (k_{k-1}^* + k_{k-1}^{**})\varepsilon_n^{k-1}]\|q_n^2 - q^{2*}\| \\
&\|q_{n+1}^1 - q^{1*}\| + [1-\varepsilon_n^1(k_1^* + k_1^{**})]\|q_{n+1}^2 - q^{2*}\| + \ldots + \|q_{n+1}^k - q^{k*}\| \\
\le\ &[1-(1-k_1)\varepsilon_n^1 + (k_k^* + k_k^{**})\varepsilon_n^k]\|q_n^1 - q^{1*}\| + [1-(1-k_2)\varepsilon_n^2]\|q_n^2 - q^{2*}\| \\
&\vdots \\
&\vdots \\
&+ [1-(1-k_k)\varepsilon_n^k + (k_{k-1}^* + k_{k-1}^{**})\varepsilon_n^{k-1}]\|q_n^2 - q^{2*}\|.
\end{aligned}$$

Thus, we have

$$\|q^1_{n+1}-q^{1*}\|+\nu\|q^2_{n+1}-q^{2*}\|+...+\|q^k_{n+1}-q^{k*}\|$$

$$\leq \max(\varepsilon^1+\varepsilon^2+...+\varepsilon^k)\left\{|q^1_n-q^{1*}\|+\|q^2_n-q^{2*}\|+...+\|q^k_n-q^{k*}\|\right\}$$

$$\leq \varepsilon\left\{\|q^1_n-q^{1*}\|+\|q^2_n-q^{2*}\|+...+\|q^k_n-q^{k*}\|\right\},$$

where

$$\nu=1-\varepsilon^1_n(k^*_1+k^{**}_1)$$

$$\varepsilon^1=1-(1-k_1)\varepsilon^1_n+(k^*_k+k^{**}_k)\varepsilon^k_n$$

$$\varepsilon^2=1-(1-k_2)\varepsilon^2_n$$

$$\vdots$$

$$\vdots$$

$$\varepsilon^k=1-(1-k_k)\varepsilon^k_n+(k^*_{k-1}+k^{**}_{k-1})\varepsilon^{k-1}_n$$

$$\varepsilon=\max(\varepsilon^1+\varepsilon^2+...+\varepsilon^k).$$

From (40)-(42) and (A1), we can obtain $\varepsilon<1$. Using Lemma 2.2, we have

$$\lim_{n\to\infty}\|q^1_{n+1}-q^{1*}\|+\nu\|q^2_{n+1}-q^{2*}\|+...+\|q^k_{n+1}-q^{k*}\|=0. \tag{57}$$

This implies that

$$\lim_{n\to\infty}\|q^1_{n+1}-q^{1*}\|=\lim_{n\to\infty}\|q^2_{n+1}-q^{2*}\|=...=\lim_{n\to\infty}\|q^k_{n+1}-q^{k*}\|=0. \tag{58}$$

This completes the proof.

In support of Theorem 4.1, we construct the following example:

Example 4.1 Let $\mathscr{X}=\mathbb{R}$ with usual inner product and norm. Let $k=3$ and $f_1, f_2, f_3, g_1, g_2, q_3:\mathbb{R}\to\mathbb{R}$, are given by

$$f_1(q) = \frac{3q}{4},\ f_2(q) = \frac{3q}{5},\ f_3(q) = \frac{4q}{5},\ \forall q \in \mathbb{R},$$

$$g_1(q) = \frac{3q}{7},\ g_2(q) = \frac{4q}{7},\ g_3(q) = \frac{2q}{3},\ \forall q \in \mathbb{R}.$$

Assume that $\mathscr{A}_i : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\mathscr{A}_1(q_1, q_2, q_3) = \frac{q_1}{3} + q_2 + q_3,$$

$$\mathscr{A}_1(q_1, q_2, q_3) = q_1 + \frac{q_2}{3} + q_3,$$

$$\mathscr{A}_1(q_1, q_2, q_3) = q_1 + q_2 + \frac{q_3}{3}\ \forall q \in \mathbb{R}.$$

Now, it is easy to show that

$$\left\langle \mathscr{A}_1(q_1, q_2, q_3) - \mathscr{A}_1(\hat{q}_1, \hat{q}_2, \hat{q}_3),\ q_1 - \hat{q}_1 \right\rangle \geq -\frac{1}{4}\left\|\mathscr{A}_1(q_1, q_2, q_3) - \mathscr{A}_1(\hat{q}_1, \hat{q}_2, \hat{q}_3)\right\|^2 + \frac{1}{5}\left\|q_1 - \hat{q}_1^2\right\|^2.$$

Then it is obvious that, $\mathscr{A}_1$ is $\left(\frac{1}{4}, \frac{1}{5}\right)$-relaxed cocoercive in first argument. Similarly, we can show that, $\mathscr{A}_2$, $\mathscr{A}_3$ are also $\left(\frac{1}{4}, \frac{1}{5}\right)$-relaxed cocoercive in second and third arguments, respectively.

$$\left\|\mathscr{A}_1(q_1, q_2, q_3) - \mathscr{A}_1(\hat{q}_1, \hat{q}_2, \hat{q}_3)\right\| = \frac{1}{3}\|q_1 - \hat{q}_1\| \leq \frac{2}{5}\|q_1 - \hat{q}_1\|^2.$$

Therefore, $\mathscr{A}_1$ is $\frac{2}{5}$-Lipschitz continuous in first argument. Similarly, we can show that $\mathscr{A}_2, \mathscr{A}_3$ are also $\frac{2}{5}, \frac{2}{5}$-Lipschitz continuous in the second and third arguments, respectively.

It is easy to show that

$$\left\langle f_1(q) - f_1(\hat{q}),\ q - \hat{q} \right\rangle \geq -\frac{1}{10}\left\|f_1(q) - f_1(\hat{q})\right\|^2 + \frac{1}{2}\left\|q_1 - \hat{q}_1\right\|^2,$$

$$\left\|f_1(q) - f_1(\hat{q})\right\| = \frac{3}{4}\left\|q_1 - \hat{q}_1\right\| \leq \frac{6}{7}\left\|q_1 - \hat{q}_1\right\|.$$

Therefore, f_1 is $\left(\frac{1}{10}, \frac{1}{2}\right)$-relaxed cocoercive and $\frac{6}{7}$-Lipschitz continuous. Similarly, we can show that f_2 is $\left(\frac{1}{8}, \frac{1}{3}\right)$-relaxed cocoercive and $\frac{2}{3}$-Lipschitz continuous, and f_3 is $\left(\frac{1}{7}, \frac{1}{2}\right)$-relaxed cocoercive and $\frac{8}{9}$-Lipschitz continuous.

In this sequence, we can also show that g_1 is $\left(\frac{1}{2}, \frac{1}{3}\right)$-relaxed cocoercive and $\frac{6}{13}$-Lipschitz continuous, g_2 is $\left(\frac{1}{4}, \frac{1}{3}\right)$-relaxed cocoercive and $\frac{8}{13}$-Lipschitz continuous, and g_3 is $\left(\frac{1}{5}, \frac{1}{2}\right)$-relaxed cocoercive and $\frac{4}{5}$-Lipschitz continuous.

$$\left(\alpha_1, \beta_1\right) = \left(\frac{1}{4}, \frac{1}{5}\right),\ \left(\alpha_2, \beta_2\right) = \left(\frac{1}{4}, \frac{1}{5}\right),\ \left(\alpha_3, \beta_3\right) = \left(\frac{1}{4}, \frac{1}{5}\right),\ \kappa_1 = \frac{2}{5},\ \kappa_2 = \frac{2}{5},\ \kappa_3 = \frac{2}{5},$$

$$\left(\bar{r}_1, \bar{s}_1\right) = \left(\frac{1}{10}, \frac{1}{2}\right),\ \left(\bar{r}_2, \bar{s}_2\right) = \left(\frac{1}{8}, \frac{1}{3}\right),\ \left(\bar{r}_3, \bar{s}_3\right) = \left(\frac{1}{9}, \frac{1}{2}\right),\ \bar{t}_1 = \frac{6}{7},\ \bar{t}_2 = \frac{2}{3},\ \bar{t}_3 = \frac{8}{9},$$

$$\left(r_1, s_1\right) = \left(\frac{1}{2}, \frac{1}{3}\right),\ \left(r_2, s_2\right) = \left(\frac{1}{4}, \frac{1}{3}\right),\ \left(r_3, s_3\right) = \left(\frac{1}{5}, \frac{1}{2}\right),\ t_1 = \frac{6}{13},\ t_2 = \frac{8}{13},\ t_3 = \frac{4}{5},$$

$$\eta_1 = \eta_2 = \eta_3 = 1,\ \varepsilon_1 = 0.1,\ \varepsilon_2 = 0.2,\ \varepsilon_3 = 0.$$

In view of constants computed above, all the constants referenced in the conditions of Theorem 4.1 have been computed subsequently.

$$k_1 = [1 + 2r_1t_1^2 - 2s_1 + t_1^2]^{1/2} = 0.866 \text{ with } 2s_1 - (2r_1t_1^2 + t_1^2) < 1;$$

$$k_2 = [1 + 2r_2t_2^2 - 2s_2 + t_2^2]^{1/2} = 0.948 \text{ with } 2s_2 - (2r_2t_2^2 + t_2^2) < 1;$$

$$k_3 = [1 + 2r_3t_3^2 - 2s_3 + t_3^2]^{1/2} = 0.888 \text{ with } 2s_3 - (2r_3t_3^2 + t_3^2) < 1;$$

$$k_1^* = [1 + 2\eta_1\alpha_1\kappa_1^2 - 2\eta_1\beta_1 + \eta_1^2\kappa_1^2]^{1/2} = 0.875 \text{ with } 0 < k_1^* < 1;$$

$$k_2^* = [1 + 2\eta_2\alpha_2\kappa_2^2 - 2\eta_2\beta_2 + \eta_2^2\kappa_2^2]^{1/2} = 0.875 \text{ with } 0 < k_2^* < 1;$$

$$k_3^* = [1 + 2\eta_3\alpha_3\kappa_3^2 - 2\eta_3\beta_3 + \eta_3^2\kappa_3^2]^{1/2} = 0.875 \text{ with } 0 < k_3^* < 1;$$

$$k_1^{**} = [1 + 2\bar{r}_1\bar{t}_1^2 - 2\bar{s}_1 + \bar{t}_1^2]^{1/2} = 0.937 \text{ with } 2\bar{s}_1 - (2\bar{r}_1\bar{t}_1^2 + \bar{t}_1^2) < 1;$$

$$k_1^{**} = [1 + 2\bar{r}_2\bar{t}_2^2 - 2\bar{s}_2 + \bar{t}_2^2]^{1/2} = 0.940 \text{ with } 2\bar{s}_2 - (2\bar{r}_2\bar{t}_2^2 + \bar{t}_2^2) < 1;$$

$$k_1^{**} = [1 + 2\bar{r}_3\bar{t}_3^2 - 2\bar{s}_3 + \bar{t}_3^2]^{1/2} = 0.950 \text{ with } 2\bar{s}_i - (2\bar{r}_3\bar{t}_3^2 + \bar{t}_3^2) < 1.$$

Thus, all the conditions of Theorem 4.1 are satisfied and hence in view of Theorem 4.1, there exists a solution of generalized variational inclusion problem (4.1).

5. Conclusions

The present article described the generalized system of extended non-linear variational inequalities, GSENVI (5) and shown that it is also given as GSENVI (9) by using the auxiliary principal technique of Glowinski and Lions [32]. We designed the Gauss-seidel type terative Algorithm 3.1 by using the alternative fixed point problem (10) which is equivalent to GSNEVIP (9). In support of our main rsult Theorem 4.1, we have constructed an example to show the existence of solution of GSENVIP (9). It should be emphasized that Theorems 4.1-4.2 and Corollary 4.1 extended, enriched, improved, unified and modified numerous preceding findings in the following ways:

• By substituting $i = 1, 2$, and A_i being univariate then GSENVIP (9) convert to a system of extended general variational inequalities with six non-linear operators. Consequently, Algorithm 3.1, Algorithm 3.2 are transformed to Algorithm 1 and Algorithm 2, respectively, while and Theoorem 4.1 and Theorem 4.2 transformed to Theorem 1 and Theorem 2, respectively, as studied by Kim [13].

• If $i = 1, 2$, and A_i, g_i, f_i be univariate and strongly monotone then GSENVIP (9) involves six non-linear operators. Consequently, Algorithm 3.1, Algorithm 3.2 are transformed to Algorithm 1 and Algorith 5, respectively, while and Theoorem 4.1 and Theorem 4.2 transformed to Theorem 1 and Theorem 6, respectively, as studied by Noor et al. [12].

• If $i = 1$, and A_i, g_i, f_i be univariate and strongly monotone then GSENVIP (9) convert to extended general variational inequalities characterized by 3 non-linear operators. One can get the results of Noor [10].

• The nonconvex minimax problem can be characterized by a system of extended general variational inequalities of the type GSENVIP (5) as discussed in [12]. Utilizing the Gauss-Seidel iterative method, we approximate the solution to GSENVIP (5), which in turn approximates the solution to the nonconvex minimax problem. Thus underline algorithm would be adapted to solve optimization problems, such as minimizing or maximizing objective functions subject to constraints. This is valuable in operations research, logistics, and machine learning, among other areas.

However, it is crucial to emphasize that the implementation of these algorithms and their comparison with alternative techniques require further investigation. This research serves as a stepping stone, and we believe that the encouragement of additional studies in this domain will contribute to the ongoing advancement of knowledge.

Funding

The authors extend their appreciation to the Deanship of Research and Graduate Studies at University of Tabuk for funding this work through Research Number-S-1444-0150.

Conflict of interest

The authors declare no competing financial interest.

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