Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

Coupled 2D-Fractional Wavelet Transform

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Received: 19 March 2024; **Revised:** 8 July 2024; **Accepted:** 30 July 2024

Abstract: We introduce a new fractional wavelet transform using the convolution for the coupled fractional Fourier transform. We derive that the fractional wavelet transform satisfies all the expected properties such as Parseval identity, inversion formula, convolution theorem. We also characterize the range of the fractional wavelet transform on $\mathscr{L}^2(\mathbb{R}^2)$. Finally, we establish the uncertainty principle for the fractional wavelet transform.

***Keywords*:** Coupled Fractional Fourier Transform, fractional wavelet transform, convolution, uncertainty principle

MSC: 44A35, 44A15, 42C40

Abbreviation

CFRFT	Coupled Fractional Fourier Transform
CFRWT	Coupled Fractional Wavelet Transform

1. Introduction

Condon [1] and Namias [2] introduced the fractional Fourier transform independently. For $\alpha \in \mathbb{R}$, the one-dimensional fractional Fourier transform of $f \in \mathscr{L}^1(\mathbb{R})$ is defined by $\mathscr{F}_\alpha(f)(u) = \int_{\mathbb{R}} f(x) K_\alpha(x, u)dx,$ where

$$K_\alpha(x, u) = \begin{cases} \sqrt{\frac{1-i\cot\alpha}{2\pi}} \cdot \mathrm{e}^{-i\left[\frac{\cot\alpha}{2}(x^2+u^2)-xu\csc\alpha\right]}, & \alpha \notin \pi\mathbb{Z}, \\ \boldsymbol{\delta}(x-u), & \alpha \in 2\pi\mathbb{Z}, \\ \boldsymbol{\delta}(x+u), & \alpha+\pi \in 2\pi\mathbb{Z}, \end{cases}$$

and $\boldsymbol{\delta}$ is the Dirac delta function. The inverse of $\mathscr{F}_\alpha$ is same as $\mathscr{F}_{-\alpha}$.

In 1982, Morlet discovered the idea of the wavelet transform which gives a new mathematical tool for seismic wave analysis. A wavelet ϕ is a function in $\mathscr{L}^2(\mathbb{R})$ which satisfies the condition $0 \neq C_\phi = \int_{\mathbb{R}} \frac{|\hat{\phi}(w)|^2}{|w|} dw < \infty,$ where $\hat{\phi}$ is the

DOI: https://doi.org/10.37256/cm.5320244615

Fourier transform of ϕ. The wavelet transform of $f \in \mathscr{L}^2(\mathbb{R})$ with respect to the wavelet ϕ is defined by $W_\phi f(x, s) = \frac{1}{\sqrt{s}} \int_{\mathbb{R}} f(t) \overline{\phi\left(\frac{t-x}{s}\right)} dt$, $\forall x \in \mathbb{R}$, $s \in \mathbb{R} \setminus \{0\}$, where $\overline{\phi\left(\frac{t-x}{a}\right)}$ is the complex conjugate of $\phi\left(\frac{t-x}{a}\right)$. The inversion formula for the wavelet transform is given by $f(t) = \frac{1}{2\pi C_\phi} \int_{\mathbb{R}^*} \int_{\mathbb{R}} W_\phi f(x, s) \phi\left(\frac{t-x}{s}\right) \frac{dxds}{s^2}$. The one-dimensional fractional wavelet transform was first introduced by Mendlovic et al. [3] and is used to study the optical signals. Shi et al. [4] proposed a new fractional wavelet transform of $x(t)$ by $W_x^\alpha(a, b) = x(t)\Theta_\alpha\left(a^{-\frac{1}{2}} \psi^*\left(\frac{-t}{a}\right)\right)$ where ψ^* is the complex conjugate of ψ, $x(t)\Theta_\alpha h(t) = [\omega(t)]^{-1} [(x\omega)(t) * h(t)]$, and $\omega(t) = \mathrm{e}^{\frac{i}{2} t^2 \cot\theta}$. The properties of the fractional wavelet transform are derived in [4–6]. Dai et al. [7] introduced yet another a fractional wavelet transform using a convolution and discussed the multiresolution analysis associated with this transform. The fractional wavelet transform is applied on optics [3], denoising [4], signal processing [5, 8] and image processing [9].

Recently, Zayed [10] introduced a two dimensional coupled fractional Fourier transform whose kernel is not a tensor product of kernels of one-dimensional fractional Fourier transform. The coupled fractional Fourier transform satisfies Parseval's identity, additive property, convolution theorem, and uncertainty principle [10–13]. Using the theory of coupled fractional Fourier transform, short-time coupled fractional Fourier transform [12] and two-dimensional fractional Stockwell transform [14] were introduced and their properties were obtained. In this paper, we introduce a $2D$-coupled fractional wavelet transform using the convolution associated with the coupled fractional Fourier transform.

This paper is organized as follows: In section 2, we recall some definitions and theorems which are required to follow this paper. In Section 3, we introduce a new fractional wavelet transform and derive the properties including the Parseval's identity, inversion formula, range characterization, convolution theorem. In section 4, we discuss the uncertainty principle for the proposed transform.

2. Preliminaries

Let $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^*$, and $\mathbb{C}$ denote the sets of integers, reals, non-zero reals, and complex numbers, respectively. Throughout this paper, $\mathbf{x} = (x_1, x_2)^t$ is viewed as a column vector in $\mathbb{R}^2$ as well as a complex number. For any $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^2$, we denote the complex multiplication of $\mathbf{x}$ and $\mathbf{y}$ by $\mathbf{xy}$ and the scalar product of $\mathbf{x}$ and $\mathbf{y}$ by $\mathbf{x} \cdot \mathbf{y}$.

For $p = 1, 2$, the norm of $f \in \mathscr{L}^p(\mathbb{R}^2)$ is defined by $\|f\|_p := \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} |f(\mathbf{x})|^p d\mathbf{x}\right)^{1/p}$, where $d\mathbf{x} = dx_1 dx_2$. The inner product on $\mathscr{L}^2(\mathbb{R}^2)$ is defined by $\langle f, g \rangle = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(\mathbf{x}) \overline{g(\mathbf{x})} d\mathbf{x}$, $\forall f, g \in \mathscr{L}^2(\mathbb{R}^2)$.

We denote by $\mathscr{L}^2(\mathbb{R}^2 \times \mathbb{R}^*)$, the Hilbert space of all square integrable complex valued functions on $\mathbb{R}^2 \times \mathbb{R}^*$ with the inner product $\langle F, G \rangle_{\mathbb{R}^2 \times \mathbb{R}^*} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^*} F(\mathbf{x}, s) \overline{G(\mathbf{x}, s)} \frac{d\mathbf{x}ds}{|s|^3}$, $\forall F, G \in \mathscr{L}^2(\mathbb{R}^2 \times \mathbb{R}^*)$.

Definition 1 For $\mathbf{t} \in \mathbb{R}^2$, $\lambda \in \mathbb{R}^*$ and $f \in \mathscr{L}^p(\mathbb{R}^2)$, we define

$$\check{f}(\mathbf{x}) = f(-\mathbf{x}), \quad (\tau_{\mathbf{t}} f)(\mathbf{x}) = f(\mathbf{x} - \mathbf{t}), D_\lambda f(\mathbf{x}) = \frac{1}{|\lambda|} f\left(\frac{\mathbf{x}}{\lambda}\right), \ M_{\mathbf{t}} f(\mathbf{x}) = \mathrm{e}^{-i\mathbf{t}\cdot\mathbf{x}} f(\mathbf{x}), \forall \mathbf{x} \in \mathbb{R}^2.$$

The Fourier transform of $f \in \mathscr{L}^1(\mathbb{R}^2)$ is defined by $\hat{f}(\mathbf{u}) = \dfrac{1}{2\pi} \int_{\mathbb{R}^2} f(\mathbf{x}) \mathrm{e}^{-i\mathbf{x}\cdot\mathbf{u}} d\mathbf{x}$, $\forall \mathbf{u} \in \mathbb{R}^2$.

Before defining the coupled fractional Fourier transform $\mathscr{F}_{\alpha, \beta}$, we introduce the following notations: For α, $\beta \in \mathbb{R}$ with $\alpha + \beta \notin 2\pi\mathbb{Z}$, let

$$\gamma = \frac{\alpha + \beta}{2}, \ \delta = \frac{\alpha - \beta}{2}, \ \tilde{a}_\gamma = \frac{\cot\gamma}{2}, \ \tilde{b} = \tilde{b}(\gamma, \delta) = \frac{\cos\delta}{\sin\gamma}, \ \tilde{c} = \tilde{c}(\gamma, \delta) = \frac{\sin\delta}{\sin\gamma}, \ d_\gamma = \frac{i\mathrm{e}^{-i\gamma}}{2\pi \sin\gamma}, \ \boldsymbol{\zeta} = (\tilde{b}, -\tilde{c})^t. \quad (1)$$

Definition 2 ([10]) For α, $\beta \in \mathbb{R}$ with $\alpha + \beta \notin 2\pi\mathbb{Z}$, the coupled fractional Fourier transform (CFRFT) of $f \in \mathscr{L}^1(\mathbb{R}^2)$ is defined by

$$\mathscr{F}_{\alpha,\beta}(f)(\mathbf{u}) = \int_{\mathbb{R}^2} f(\mathbf{x}) K_{\alpha,\beta}(\mathbf{x},\mathbf{u})\, d\mathbf{x},\ \forall \mathbf{u} \in \mathbb{R}^2 \tag{2}$$

where $K_{\alpha,\beta}(\mathbf{x},\mathbf{u}) = d_\gamma \exp\{-i[\tilde{a}_\gamma(\|\mathbf{x}\|^2 + \|\mathbf{u}\|^2) - \mathbf{x}\cdot\boldsymbol{\zeta}\mathbf{u}]\}$, and $\boldsymbol{\zeta}\mathbf{u}$ is the complex multiplication of $\boldsymbol{\zeta}$ and $\mathbf{u}$. The CFRFT of $f \in \mathscr{L}^2(\mathbb{R})$ is defined by $\mathscr{F}_{\alpha,\beta}(f) = \lim_{n\to\infty} \mathscr{F}_{\alpha,\beta}(f_n)$, where (f_n) is a sequence from $\mathscr{L}^1(\mathbb{R}^2) \cap \mathscr{L}^2(\mathbb{R}^2)$ such that $f_n \to f$ in $\mathscr{L}^2(\mathbb{R}^2)$ as $n \to \infty$.

For a given $\gamma \in \mathbb{R}\setminus\pi\mathbb{Z}$, we define $E_\gamma(\mathbf{x}) = e^{i\tilde{a}_\gamma\|\mathbf{x}\|^2}$, $\forall \mathbf{x} \in \mathbb{R}^2$. The coupled fractional Fourier transform (2) can be written in terms of the Fourier transform as follows.

$$\mathscr{F}_{\alpha,\beta}(f)(\mathbf{u}) = 2\pi d_\gamma E_\gamma^{-1}(\mathbf{u})(fE_\gamma^{-1})\hat{}\,(-\boldsymbol{\zeta}\mathbf{u}),\ \text{where } \gamma \text{is as introduced in (1).}$$

In the above equation, $(fE_\gamma^{-1})\hat{}$ means the Fourier transform of fE_γ^{-1}. The Parseval's identity $\|\mathscr{F}_{\alpha,\beta}(f)\|_2 = \|f\|_2$ and inversion formula $\mathscr{F}_{-\alpha,-\beta}\mathscr{F}_{\alpha,\beta}(f) = f$ for CFRFT are established in [11, 10].

Definition 3 ([10]) Let $\gamma \in \mathbb{R}\setminus\pi\mathbb{Z}$. The fractional convolution of $f \in \mathscr{L}^p(\mathbb{R}^2)$, $p = 1, 2$ and $g \in \mathscr{L}^1(\mathbb{R}^2)$ is defined by

$$f \circledast_\gamma g = 2\pi d_\gamma E_\gamma[fE_\gamma^{-1} * gE_\gamma^{-1}],$$

where $*$ is the usual convolution operator defined by $(f * g)(\mathbf{x}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} f(\mathbf{x}-\mathbf{y})g(\mathbf{y})\,d\mathbf{y},\ \forall\, \mathbf{x} \in \mathbb{R}^2$.

This fractional convolution is a generalization of the usual convolution as $f \circledast_{\frac{\pi}{2}} g = f * g$ and it satisfies the following properties [11]:

$$f \circledast_\gamma g = g \circledast_\gamma f, \tag{3}$$

$$f \circledast_\gamma (g_1 \circledast_\gamma g_2) = (f \circledast_\gamma g_1) \circledast_\gamma g_2, \tag{4}$$

for all $f \in \mathscr{L}^p(\mathbb{R}^2)$, $p = 1, 2$ and $g, g_1, g_2 \in \mathscr{L}^1(\mathbb{R}^2)$.

Theorem 1 (Convolution theorem for CFRFT, [10]) If $f \in \mathscr{L}^p(\mathbb{R}^2)$, $p = 1, 2$ and $g \in \mathscr{L}^1(\mathbb{R}^2)$ then $\mathscr{F}_{\alpha,\beta}(f \circledast_\gamma g) = E_\gamma \mathscr{F}_{\alpha,\beta} f \mathscr{F}_{\alpha,\beta}\, g$, holds for every $\alpha, \beta \in \mathbb{R}$ satisfying $\frac{\alpha+\beta}{2} = \gamma$.

3. 2*D*-coupled fractional wavelet transform

We first prove some identities involving the fractional convolution $\circledast_\gamma$, which will be useful to discuss properties of the coupled fractional wavelet transform.

Theorem 2 Let $f, f_1, f_2 \in \mathscr{L}^p(\mathbb{R}^2)$, $p = 1, 2$, $g, g_1, g_2 \in \mathscr{L}^1(\mathbb{R}^2)$, $\mathbf{t} \in \mathbb{R}^2$, $z \in \mathbb{C}$ and $\lambda \in \mathbb{R}^*$. Then, we have

1. $(f_1 + f_2) \circledast_\gamma g = (f_1 \circledast_\gamma g) + (f_2 \circledast_\gamma g)$.
2. $z(f \circledast_\gamma g) = (zf) \circledast_\gamma g = f \circledast_\gamma (zg)$.
3. $(f \circledast_\gamma g)\check{} = (\check{f} \circledast_\gamma \check{g})$.
4. $M_\mathbf{t}(f \circledast_\gamma g) = (M_\mathbf{t} f \circledast_\gamma M_\mathbf{t} g)$.
5. $\tau_\mathbf{t}(f \circledast_\gamma g) = (\tau_\mathbf{t} f \circledast_\gamma M_{2\tilde{a}_\gamma \mathbf{t}}\, g) = (M_{2\tilde{a}_\gamma \mathbf{t}} f \circledast_\gamma \tau_\mathbf{t} g)$.

6. $D_\lambda(f \circledast_\gamma g) = \frac{d_\gamma}{|\lambda| d_{\gamma'}}(D_\lambda f \circledast_{\gamma'} D_\lambda g),$ where $\gamma' = \cot^{-1}\left(\frac{\cot\gamma}{\lambda^2}\right)$ and $d_{\gamma'} = \frac{ie^{-i\gamma'}}{2\pi\sin\gamma'}$.
7. Let $P_k \colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $P_k(\mathbf{x}) = x_k,$ for $k = 1, 2$.
   (a) If $P_k f, \partial_{x_k} f \in \mathscr{L}^1(\mathbb{R}^2) \cup \mathscr{L}^2(\mathbb{R}^2)$, then $\partial_{x_k}(f \circledast_\gamma g) = 2i\tilde{a}_\gamma[P_k(f \circledast_\gamma g) - P_k f \circledast_\gamma g] + \partial_{x_k} f \circledast_\gamma g$.
   (b) If $g, \partial_{x_k} g \in \mathscr{L}^1(\mathbb{R}^2)$ then $\partial_{x_k}(f \circledast_\gamma g) = 2i\tilde{a}_\gamma[P_k(f \circledast_\gamma g) - f \circledast_\gamma P_k g] + f \circledast_\gamma \partial_{x_k} g$.

Proof. We can easily prove the first two identities using the properties of the usual convolution.
1. Applying the well known property $(f * g)\check{} = (\check{f} * \check{g})$ of the convolution $*$, we get

$$\begin{aligned}
(f \circledast_\gamma g)\check{}(\mathbf{x}) &= 2\pi d_\gamma E_\gamma(-\mathbf{x})(fE_\gamma^{-1} * gE_\gamma^{-1})\check{}(\mathbf{x}) = 2\pi d_\gamma E_\gamma(\mathbf{x})((fE_\gamma^{-1})\check{} * (gE_\gamma^{-1})\check{})(\mathbf{x}) \\
&= 2\pi d_\gamma E_\gamma(\mathbf{x})(\check{f}E_\gamma^{-1} * \check{g}E_\gamma^{-1})(\mathbf{x}) = (\check{f} \circledast_\gamma \check{g})(\mathbf{x}).
\end{aligned}$$

2. By direct calculation, we obtain that

$$\begin{aligned}
M_\mathbf{t}(f \circledast_\gamma g)(\mathbf{x}) &= e^{-i\mathbf{x}\cdot\mathbf{t}}(f \circledast_\gamma g)(\mathbf{x}) \\
&= d_\gamma e^{-i\mathbf{x}\cdot\mathbf{t}} E_\gamma(\mathbf{x}) \int_{\mathbf{R}^2} f(\mathbf{x}-\mathbf{z})E_\gamma^{-1}(\mathbf{x}-\mathbf{z})g(\mathbf{z})E_\gamma^{-1}(\mathbf{z})d\mathbf{z} \\
&= d_\gamma E_\gamma(\mathbf{x}) \int_{\mathbf{R}^2} e^{-i(\mathbf{x}-\mathbf{z})\cdot\mathbf{t}} f(\mathbf{x}-\mathbf{z})E_\gamma^{-1}(\mathbf{x}-\mathbf{z})e^{-i\mathbf{z}\cdot\mathbf{t}} g(\mathbf{z})E_\gamma^{-1}(\mathbf{z})d\mathbf{z} \\
&= d_\gamma E_\gamma(\mathbf{x}) \int_{\mathbf{R}^2} (M_\mathbf{t} f)(\mathbf{x}-\mathbf{z})E_\gamma^{-1}(\mathbf{x}-\mathbf{z})(M_\mathbf{t} g)(\mathbf{z})E_\gamma^{-1}(\mathbf{z})d\mathbf{z} \\
&= (M_\mathbf{t} f \circledast_\gamma M_\mathbf{t} g).
\end{aligned}$$

3. We first observe that $E_\gamma(\mathbf{x}-\mathbf{t})E_\gamma^{-1}(\mathbf{z}-\mathbf{t}) = E_\gamma(\mathbf{x})E_\gamma^{-1}(\mathbf{z})e^{-i2\tilde{a}_\gamma(\mathbf{x}-\mathbf{z})\cdot\mathbf{t}}$. Indeed,

$$\begin{aligned}
E_\gamma(\mathbf{x}-\mathbf{t})E_\gamma^{-1}(\mathbf{z}-\mathbf{t}) &= e^{i\tilde{a}_\gamma(\|\mathbf{x}-\mathbf{t}\|^2 - \|\mathbf{z}-\mathbf{t}\|^2)} \\
&= e^{i\tilde{a}_\gamma((x_1-t_1)^2+(x_2-t_2)^2-(z_1-t_1)^2-(z_2-t_2)^2)} \\
&= e^{i\tilde{a}_\gamma(x_1^2+x_2^2-z_1^2-z_2^2-2x_1t_1-2x_2t_2+2z_1t_1+2z_2t_2)} \\
&= e^{i\tilde{a}_\gamma(|x_1|^2+|x_2|^2)}\, e^{-i\tilde{a}_\gamma(|z_1|^2+|z_2|^2)}\, e^{-2i\tilde{a}_\gamma((x_1-z_1)t_1+(x_2-z_2)t_2)} \\
&= E_\gamma(\mathbf{x})E_\gamma^{-1}(\mathbf{z})e^{-i2\tilde{a}_\gamma(\mathbf{x}-\mathbf{z})\cdot\mathbf{t}}.
\end{aligned}$$

Using the above observation and the well known property $\tau_\mathbf{t}(f * g) = (\tau_\mathbf{t} f * g)$ of the usual convolution, we get that

$$
\begin{aligned}
\tau_{\mathbf{t}}(f \circledast_\gamma g)(\mathbf{x}) &= 2\pi d_\gamma E_\gamma(\mathbf{x}-\mathbf{t})\,\tau_{\mathbf{t}}(fE_\gamma^{-1} * gE_\gamma^{-1})(\mathbf{x}) \\
&= 2\pi d_\gamma E_\gamma(\mathbf{x}-\mathbf{t})\,(\tau_{\mathbf{t}}(fE_\gamma^{-1}) * gE_\gamma^{-1})(\mathbf{x}) \\
&= d_\gamma E_\gamma(\mathbf{x}-\mathbf{t}) \int_{\mathbb{R}^2} [\tau_{\mathbf{t}} f](\mathbf{z}) E_\gamma^{-1}(\mathbf{z}-\mathbf{t})\, g(\mathbf{x}-\mathbf{z}) E_\gamma^{-1}(\mathbf{x}-\mathbf{z}) d\mathbf{z} \\
&= d_\gamma E_\gamma(\mathbf{x}) \int_{\mathbb{R}^2} [\tau_{\mathbf{t}} f](\mathbf{z}) E_\gamma^{-1}(\mathbf{z}) \mathrm{e}^{-i2\tilde{a}_\gamma(\mathbf{x}-\mathbf{z})\cdot\mathbf{t}} g(\mathbf{x}-\mathbf{z}) E_\gamma^{-1}(\mathbf{x}-\mathbf{z}) d\mathbf{z} \\
&= d_\gamma E_\gamma(\mathbf{x}) \int_{\mathbb{R}^2} [\tau_{\mathbf{t}} f](\mathbf{z}) E_\gamma^{-1}(\mathbf{z}) (M_{2\tilde{a}_\gamma \mathbf{t}}\, g)(\mathbf{x}-\mathbf{z}) E_\gamma^{-1}(\mathbf{x}-\mathbf{z})\, d\mathbf{z} \\
&= 2\pi d_\gamma E_\gamma(\mathbf{x})([\tau_t f] E_\gamma^{-1} * M_{2\tilde{a}_\gamma \mathbf{t}}\, g E_\gamma^{-1})(\mathbf{x}) \\
&= ([\tau_t f] \circledast_\gamma M_{2\tilde{a}_\gamma \mathbf{t}}\, g)(\mathbf{x}). \qquad (5)
\end{aligned}
$$

Using (3) and (5), one can get that $\tau_{\mathbf{t}}(f \circledast_\gamma g) = \tau_{\mathbf{t}}(g \circledast_\gamma f) = (\tau_{\mathbf{t}} g \circledast_\gamma M_{2\tilde{a}_\gamma \mathbf{t}} f) = (M_{2\tilde{a}_\gamma \mathbf{t}} f \circledast_\gamma \tau_{\mathbf{t}} g)$.
4. By a direct computation, we get that

$$
\begin{aligned}
D_\lambda(f \circledast_\gamma g)(\mathbf{x}) &= \frac{2\pi d_\gamma E_\gamma\left(\frac{\mathbf{x}}{\lambda}\right)}{|\lambda|} \left(fE_\gamma^{-1} * gE_\gamma^{-1}\right)\left(\frac{\mathbf{x}}{\lambda}\right) \\
&= \frac{d_\gamma E_\gamma\left(\frac{\mathbf{x}}{\lambda}\right)}{|\lambda|} \int_{\mathbb{R}^2} (fE_\gamma^{-1})\left(\frac{\mathbf{x}}{\lambda} - \mathbf{z}\right)(gE_\gamma^{-1})(\mathbf{z}) d\mathbf{z} \\
&= \frac{d_\gamma E_\gamma\left(\frac{\mathbf{x}}{\lambda}\right)}{|\lambda|^3} \int_{\mathbb{R}^2} (fE_\gamma^{-1})\left(\frac{\mathbf{x}}{\lambda} - \frac{\mathbf{t}}{\lambda}\right)(gE_\gamma^{-1})\left(\frac{\mathbf{t}}{\lambda}\right) d\mathbf{t} \text{ (Substituting } \mathbf{z} = \tfrac{\mathbf{t}}{\lambda}) \\
&= \frac{d_\gamma E_\gamma\left(\frac{\mathbf{x}}{\lambda}\right)}{|\lambda|} \int_{\mathbb{R}^2} (D_\lambda f)(\mathbf{x}-\mathbf{t}) E_\gamma^{-1}\left(\frac{\mathbf{x}-\mathbf{t}}{\lambda}\right)(D_\lambda g)(\mathbf{t}) E_\gamma^{-1}\left(\frac{\mathbf{t}}{\lambda}\right) d\mathbf{t} \\
&= \frac{d_\gamma E_{\gamma'}(\mathbf{x})}{|\lambda|} \int_{\mathbb{R}^2} (D_\lambda f)(\mathbf{x}-\mathbf{t}) E_{\gamma'}^{-1}(\mathbf{x}-\mathbf{t})(D_\lambda g)(\mathbf{t}) E_{\gamma'}^{-1}(\mathbf{t}) d\mathbf{t} \text{ (using } \gamma' = \cot^{-1}\left(\tfrac{\cot\gamma}{\lambda^2}\right)) \\
&= \frac{d_\gamma}{|\lambda|} \frac{(D_\lambda f \circledast_{\gamma'} D_\lambda g)(\mathbf{x})}{d_{\gamma'}}.
\end{aligned}
$$

5. We prove the first identity and the second one follows by a similar argument.

$$
\begin{aligned}
\partial_{x_k}(f \circledast_\gamma g)(\mathbf{x}) &= \partial_{x_k}[2\pi d_\gamma E_\gamma(\mathbf{x})(fE_\gamma^{-1} * gE_\gamma^{-1})(\mathbf{x})] \\
&= 2\pi d_\gamma \partial_{x_k}(E_\gamma(\mathbf{x}))\,(fE_\gamma^{-1} * gE_\gamma^{-1})(\mathbf{x}) + 2\pi d_\gamma E_\gamma(\mathbf{x})\partial_{x_k}(fE_\gamma^{-1} * gE_\gamma^{-1})(\mathbf{x}) \\
&= 2\pi d_\gamma (2i\tilde{a}_\gamma x_k \times E_\gamma(\mathbf{x}))\,(fE_\gamma^{-1} * gE_\gamma^{-1})(\mathbf{x}) + 2\pi d_\gamma E_\gamma(\mathbf{x})(\partial_{x_k}(fE_\gamma^{-1}) * gE_\gamma^{-1})(\mathbf{x}) \\
&= 2i\tilde{a}_\gamma P_k(\mathbf{x})(f \circledast_\gamma g)(\mathbf{x}) + 2\pi d_\gamma E_\gamma(\mathbf{x})[((\partial_{x_k} f)E_\gamma^{-1} - 2i\tilde{a}_\gamma P_k f E_\gamma^{-1}) * gE_\gamma^{-1}](\mathbf{x}) \\
&= 2i\tilde{a}_\gamma[P_k(f \circledast_\gamma g) - (P_k f \circledast_\gamma g)](\mathbf{x}) + (\partial_{x_k} f \circledast_\gamma g)(\mathbf{x}).
\end{aligned}
$$

Hence, the theorem follows. □

Corollary 1 For $f \in \mathscr{L}^p(\mathbb{R}^2),\ p = 1,\ 2,\ g \in \mathscr{L}^1(\mathbb{R}^2) \cap \mathscr{L}^2(\mathbb{R}^2)$ and $\lambda \in \mathbb{R}^*,\ (D_\lambda f \circledast_\gamma g)(\mathbf{x}) = \frac{|\lambda| d_\gamma}{d_\eta} D_\lambda(f \circledast_\eta D_{\frac{1}{\lambda}} g)(\mathbf{x}),$ where $\eta = \cot^{-1}(\lambda^2 \cot\gamma)$ and $d_\eta = \frac{\mathrm{i}e^{-i\eta}}{2\pi \sin\eta}$.

Proof. Replacing γ with η in the sixth identity of Theorem 2 and applying that $\eta' = \cot^{-1}\left(\frac{\cot\eta}{\lambda^2}\right) = \gamma$, we get

$$
D_\lambda(f \circledast_\eta g) = \frac{d_\eta}{|\lambda| d_\gamma}(D_\lambda f \circledast_\gamma D_\lambda g). \tag{6}
$$

Replacing g with $D_{\frac{1}{\lambda}} g$ in (6), we get $(D_\lambda f \circledast_\gamma g) = \frac{|\lambda| d_\gamma}{d_\eta} D_\lambda(f \circledast_\eta D_{\frac{1}{\lambda}} g)$. □

Definition 4 For $0 \neq \psi \in \mathscr{L}^1(\mathbb{R}^2) \cap \mathscr{L}^2(\mathbb{R}^2)$ and $\gamma \in \mathbb{R} \setminus \pi\mathbb{Z}$, the $2D$-coupled fractional wavelet transform (CFRWT) of $f \in \mathscr{L}^2(\mathbb{R}^2)$ is defined by $W_\psi^\gamma f(\mathbf{x}, s) = [f \circledast_\gamma E_\gamma D_s(\check{\bar{\psi}} E_\gamma)](\mathbf{x}),\ \forall \mathbf{x} \in \mathbb{R}^2,\ s \in \mathbb{R}^*$.

It is easy to observe that the CFRWT W_ψ^γ is equal to the two dimensional wavelet transform when $\gamma = \frac{\pi}{2}$.

Lemma 1 If $\gamma \in \mathbb{R} \setminus \pi\mathbb{Z}$ and $\psi \in \mathscr{L}^1(\mathbb{R}^2)$ then $\mathscr{F}_{\alpha,\beta}(E_\gamma D_s(\check{\bar{\psi}} E_\gamma))(\mathbf{u}) = -|s| \mathrm{e}^{-2i\gamma} E_\gamma^{-1}(\mathbf{u}) E_\gamma^{-1}(s\mathbf{u}) \overline{\mathscr{F}_{\alpha,\beta}\psi(s\mathbf{u})}$, for all $\mathbf{u} \in \mathbb{R}^2$ and $\alpha, \beta \in \mathbb{R}$ such that $\frac{\alpha+\beta}{2} = \gamma$.

Proof. For arbitrary $\psi \in \mathscr{L}^1(\mathbb{R}^2)$, we have

$$
\begin{aligned}
\mathscr{F}_{\alpha,\beta}(E_\gamma D_s(\check{\bar{\psi}} E_\gamma))(\mathbf{u}) &= \int_{\mathbb{R}^2} E_\gamma(\mathbf{t}) D_s(\check{\bar{\psi}} E_\gamma)(\mathbf{t}) K_{\alpha,\beta}(\mathbf{t}, \mathbf{u}) d\mathbf{t} \\
&= \frac{d_\gamma E_\gamma^{-1}(\mathbf{u})}{|s|} \int_{\mathbb{R}^2} \overline{\psi\left(\frac{-\mathbf{t}}{s}\right)} E_\gamma^{-1}\left(\frac{\mathbf{t}}{s}\right) \mathrm{e}^{i\mathbf{t}\cdot\boldsymbol{\zeta}\mathbf{u}} d\mathbf{t} \\
&= \frac{d_\gamma E_\gamma^{-1}(\mathbf{u})}{|s|} \int_{\mathbb{R}^2} \overline{\psi(-\mathbf{w})} E_\gamma^{-1}(\mathbf{w}) \mathrm{e}^{is\mathbf{w}\cdot\boldsymbol{\zeta}\mathbf{u}} |s|^2 d\mathbf{w} \quad \left(\text{Putting } \tfrac{\mathbf{t}}{s} = \mathbf{w}\right) \\
&= |s| d_\gamma E_\gamma^{-1}(\mathbf{u}) \int_{\mathbb{R}^2} \overline{\psi(-\mathbf{w})} E_\gamma^{-1}(\mathbf{w}) \mathrm{e}^{i\mathbf{w}\cdot\boldsymbol{\zeta}(s\mathbf{u})} d\mathbf{w} \\
&= 2\pi |s| d_\gamma E_\gamma^{-1}(\mathbf{u}) (\check{\bar{\psi}} E_\gamma)\hat{}\,(-\boldsymbol{\zeta}(s\mathbf{u}))
\end{aligned}
$$

$$= \quad 2\pi|s|d_\gamma E_\gamma^{-1}(\mathbf{u})\overline{(\psi E_\gamma^{-1}\hat{)}(-\boldsymbol{\zeta}(s\mathbf{u}))}$$

$$= \quad \frac{|s|d_\gamma}{\overline{d}} E_\gamma^{-1}(\mathbf{u})E_\gamma^{-1}(s\mathbf{u}) \times [2\pi\overline{d_\gamma} E_\gamma(s\mathbf{u})\overline{(\psi E_\gamma^{-1}\hat{)}(-\boldsymbol{\zeta}(s\mathbf{u}))}]$$

$$= \quad -|s|\mathrm{e}^{-2i\gamma} E_\gamma^{-1}(\mathbf{u})E_\gamma^{-1}(s\mathbf{u})\overline{\mathscr{F}_{\alpha,\beta}\psi(s\mathbf{u})}.$$

□

Theorem 3 (Parseval's identity for CFRWT) Let $\alpha, \beta \in \mathbb{R}$ such that $\frac{\alpha+\beta}{2} = \gamma \in \mathbb{R}\setminus\pi\mathbb{Z}$ and $\psi_1, \psi_2 \in \mathscr{L}^1(\mathbb{R}^2)\cap\mathscr{L}^2(\mathbb{R}^2)$. If $0 \neq C_{\psi_1,\psi_2} = \int_{\mathbb{R}^*}\overline{\mathscr{F}_{\alpha,\beta}(\psi_1)(r\mathbf{w})}\mathscr{F}_{\alpha,\beta}(\psi_2)(r\mathbf{w})\frac{dr}{|r|} < \infty$, for almost all $\mathbf{w}\in\mathbb{R}^2$ with $\|\mathbf{w}\| = 1$, then $\langle W_{\psi_1}^\gamma f_1, W_{\psi_2}^\gamma f_2\rangle_{\mathbb{R}^2\times\mathbb{R}^*} = C_{\psi_1,\psi_2}\langle f_1, f_2\rangle, \forall f_1, f_2 \in \mathscr{L}^2(\mathbb{R}^2)$.

Proof. For arbitrary $f_1, f_2 \in \mathscr{L}^2(\mathbb{R}^2)$, we have

$$2\pi\langle W_{\psi_1}^\gamma f_1, W_{\psi_2}^\gamma f_2\rangle_{\mathbb{R}^2\times\mathbb{R}^*} = \int_{\mathbb{R}^*}\int_{\mathbb{R}^2} W_{\psi_1}^\gamma f_1(\mathbf{x},s)\overline{W_{\psi_2}^\gamma f_2(\mathbf{x},s)}\frac{d\mathbf{x}ds}{|s|^3}$$

$$= \int_{\mathbb{R}^*}\int_{\mathbb{R}^2}[f_1 \circledast_\gamma E_\gamma D_s(\check{\psi}_1 E_\gamma)](\mathbf{x})\,\overline{[f_2 \circledast_\gamma E_\gamma D_s(\check{\psi}_2 E_\gamma)](\mathbf{x})}\,\frac{d\mathbf{x}ds}{|s|^3}$$

$$= \int_{\mathbb{R}^*}\int_{\mathbb{R}^2}\mathscr{F}_{\alpha,\beta}(f_1 \circledast_\gamma E_\gamma D_s(\check{\psi}_1 E_\gamma))(\mathbf{x})\,\overline{\mathscr{F}_{\alpha,\beta}(f_2 \circledast_\gamma E_\gamma D_s(\check{\psi}_2 E_\gamma))(\mathbf{x})}\,\frac{d\mathbf{x}ds}{|s|^3}$$

(Invoking the Parseval's identity for CFRFT.)

$$= \int_{\mathbb{R}^*}\int_{\mathbb{R}^2} E_\gamma(\mathbf{u})\mathscr{F}_{\alpha,\beta}f_1(\mathbf{u})\,\mathscr{F}_{\alpha,\beta}(E_\gamma D_s(\check{\psi}_1 E_\gamma))(\mathbf{u})$$

$$\times E_\gamma^{-1}(\mathbf{u})\overline{\mathscr{F}_{\alpha,\beta}f_2(\mathbf{u})}\,\overline{\mathscr{F}_{\alpha,\beta}(E_\gamma D_s(\check{\psi}_2 E_\gamma))(\mathbf{u})}\frac{d\mathbf{u}ds}{|s|^3}$$

(Using the convolution theorem for CFRFT)

$$= \int_{\mathbb{R}^2}\mathscr{F}_{\alpha,\beta}f_1(\mathbf{u})\,\overline{\mathscr{F}_{\alpha,\beta}f_2(\mathbf{u})}$$

$$\times\left(\int_{\mathbb{R}^*}\mathscr{F}_{\alpha,\beta}(E_\gamma D_s(\check{\psi}_1 E_\gamma))(\mathbf{u})\overline{\mathscr{F}_{\alpha,\beta}(E_\gamma D_s(\check{\psi}_2 E_\gamma))(\mathbf{u})}\frac{ds}{|s|^3}\right)d\mathbf{u}$$

(Applying the Fubini's theorem)

$$= \int_{\mathbb{R}^2} \mathscr{F}_{\alpha,\beta} f_1(\mathbf{u}) \overline{\mathscr{F}_{\alpha,\beta} f_2(\mathbf{u})} \left(\int_{\mathbb{R}^*} \overline{\mathscr{F}_{\alpha,\beta}(\psi_1)(s\mathbf{u})} \mathscr{F}_{\alpha,\beta}(\psi_2)(s\mathbf{u}) \frac{ds}{|s|} \right) d\mathbf{u}$$

(By Lemma 1)

$$= \int_{\mathbb{R}^2} \mathscr{F}_{\alpha,\beta} f_1(\mathbf{u}) \overline{\mathscr{F}_{\alpha,\beta} f_2(\mathbf{u})} \left(\int_{\mathbb{R}^*} \overline{\mathscr{F}_{\alpha,\beta}(\psi_1)(s\|\mathbf{u}\|\mathbf{v})} \mathscr{F}_{\alpha,\beta}(\psi_2)(s\|\mathbf{u}\|\mathbf{v}) \frac{ds}{|s|} \right) d\mathbf{u}$$

(Substituting $\mathbf{u} = \|\mathbf{u}\|\mathbf{v}$ with $\mathbf{v} \in \mathbb{R}^2$ and $\|\mathbf{v}\| = 1$)

$$= \int_0^\infty \mathscr{F}_{\alpha,\beta} f_1(\mathbf{u}) \overline{\mathscr{F}_{\alpha,\beta} f_2(\mathbf{u})} \left(\int_0^\infty \overline{\mathscr{F}_{\alpha,\beta}(\psi_1)(s\|\mathbf{u}\|\mathbf{v})} \mathscr{F}_{\alpha,\beta}(\psi_2)(s\|\mathbf{u}\|\mathbf{v}) \frac{ds}{s} \right.$$

$$\left. - \int_{-\infty}^0 \overline{\mathscr{F}_{\alpha,\beta}(\psi_1)(s\|\mathbf{u}\|\mathbf{v})} \mathscr{F}_{\alpha,\beta}(\psi_2)(s\|\mathbf{u}\|\mathbf{v}) \frac{ds}{s} \right) d\mathbf{u}$$

$$= \int_{\mathbb{R}^2} \mathscr{F}_{\alpha,\beta} f_1(\mathbf{u}) \overline{\mathscr{F}_{\alpha,\beta} f_2(\mathbf{u})} \left(\int_{\mathbb{R}^*} \overline{\mathscr{F}_{\alpha,\beta}(\psi_1)(r\mathbf{v})} \mathscr{F}_{\alpha,\beta}(\psi_2)(r\mathbf{v}) \frac{dr}{|r|} \right) d\mathbf{u}$$

(Putting $\|\mathbf{u}\|s = r$)

$$= C_{\psi_1,\psi_2} \int_{\mathbb{R}^*} \mathscr{F}_{\alpha,\beta} f_1(\mathbf{u}) \overline{\mathscr{F}_{\alpha,\beta} f_2(\mathbf{u})} d\mathbf{u} = 2\pi C_{\psi_1,\psi_2} \langle f_1, f_2 \rangle.$$

Hence, the theorem follows. □

Theorem 4 For $\psi \in \mathscr{L}^1(\mathbb{R}^2) \cap \mathscr{L}^2(\mathbb{R}^2)$ with $0 \neq C_\psi := C_{\psi,\psi} < \infty$, we have

$$\langle W_\psi^\gamma f_1, W_\psi^\gamma f_2 \rangle_{\mathbb{R}^2 \times \mathbb{R}^*} = C_\psi \langle f_1, f_2 \rangle, \ \forall f_1, f_2 \in \mathscr{L}^2(\mathbb{R}^2).$$

Remark 1 If $f_1 = f_2$, then from Theorem 4, we have $\frac{1}{2\pi} \int_{\mathbb{R}^*} \int_{\mathbb{R}^2} |W_\psi^\gamma f(\mathbf{x}, s)|^2 \frac{d\mathbf{x} ds}{|s|^3} = C_\psi \|f\|_2^2$.

Lemma 2 ([15, p.43]) Let Ω be a function defined on $\mathbb{R}^2 \times \mathbb{R}^*$ and $\Omega(\mathbf{t}, s) \in \mathscr{L}^2(\mathbb{R}^2)$, $\forall \mathbf{t} \in \mathbb{R}^2$, $s \in \mathbb{R}^*$. If the mapping $T(h) = \int_{\mathbb{R}^*} \int_{\mathbb{R}^2} \langle \Omega(\mathbf{t}, s), h \rangle \frac{d\mathbf{t} ds}{|s|^3}$ is a bounded conjugate linear functional on $\mathscr{L}^2(\mathbb{R}^2)$ then there exists a unique element $\phi = \int_{\mathbb{R}^*} \int_{\mathbb{R}^2} \Omega(\mathbf{t}, s) \frac{d\mathbf{t} ds}{|s|^3} \in \mathscr{L}^2(\mathbb{R}^2)$ which satisfies $T(h) = \int_{\mathbb{R}^*} \int_{\mathbb{R}^2} \langle \Omega(\mathbf{t}, s), h \rangle \frac{d\mathbf{t} ds}{|s|^3} = \langle \phi, h \rangle$, $\forall h \in \mathscr{L}^2(\mathbb{R}^2)$.

The lemma given below is an analog of Lemma 2 in terms of convolution.

Lemma 3 Let Ω be a function defined on $\mathbb{R}^2 \times \mathbb{R}^*$ and $\Omega(\mathbf{t}, s) \in \mathscr{L}^2(\mathbb{R}^2)$, $\forall \mathbf{t} \in \mathbb{R}^2$, $s \in \mathbb{R}^*$. If the mapping $I(g) = \int_{\mathbb{R}^*} \int_{\mathbb{R}^2} (\Omega(\mathbf{t}, s) * \check{\bar{g}})(0) \frac{d\mathbf{t} ds}{|s|^3}$ is a bounded conjugate linear functional on $\mathscr{L}^2(\mathbb{R}^2)$ then there exists a unique element $\phi = \int_{\mathbb{R}^*} \int_{\mathbb{R}^2} \Omega(\mathbf{t}, s) \frac{d\mathbf{t} ds}{|s|^3} \in \mathscr{L}^2(\mathbb{R}^2)$ which satisfies $I(g) = \int_{\mathbb{R}^*} \int_{\mathbb{R}^2} (\Omega(\mathbf{t}, s) * \check{\bar{g}})(0) \frac{d\mathbf{t} ds}{|s|^3} = (\phi * \check{\bar{g}})(0)$, $\forall g \in \mathscr{L}^2(\mathbb{R}^2)$.

Lemma 4 Let $\psi \in \mathscr{L}^1(\mathbb{R}^2) \cap \mathscr{L}^2(\mathbb{R}^2)$ such that $0 \neq C_\psi < \infty$. For fixed $\mathbf{x} \in \mathbb{R}^2$ and $F \in \mathscr{L}^2(\mathbb{R}^2 \times \mathbb{R}^*)$, define $\varphi(\mathbf{z}) = \int_{\mathbb{R}^*} (F(\cdot, s) \circledast_\gamma E_\gamma D_s(\psi E_\gamma^{-1}))(\mathbf{z}) \frac{ds}{|s|^3}$. Then, we have $\varphi \in \mathscr{L}^2(\mathbb{R}^2)$ and $\left[\left(\int_{\mathbb{R}^*} F(\cdot, s) \circledast_\gamma E_\gamma D_s(\psi E_\gamma^{-1}) \frac{ds}{|s|^3} \right) \circledast_\gamma E_\gamma \check{\bar{h}} \right](\mathbf{x}) = \int_{\mathbb{R}^*} [F(\cdot, s) \circledast_\gamma (E_\gamma D_s(\psi E_\gamma^{-1}) \circledast_\gamma E_\gamma \check{\bar{h}})](\mathbf{x}) \frac{ds}{|s|^3}$, $\forall h \in \mathscr{L}^1(\mathbb{R}^2) \cap \mathscr{L}^2(\mathbb{R}^2)$.

Proof. If $\Omega_{\mathbf{t},s}(\mathbf{z}) = d_\gamma F(\mathbf{t}, s)E_\gamma^{-1}(\mathbf{t})E_\gamma(\mathbf{z})\tau_{\mathbf{t}} D_s(\psi E_\gamma^{-1})(\mathbf{z})$, then we get $\Omega_{\mathbf{t},s}E_\gamma^{-1} \in \mathscr{L}^2(\mathbb{R}^2)$, $\forall \mathbf{t} \in \mathbb{R}^2$, $s \in \mathbb{R}^*$. For fixed $\mathbf{x} \in \mathbb{R}^2$, we define $I_{\mathbf{x}}(g) = \int_{\mathbb{R}^*}\int_{\mathbb{R}^2} (\Omega(\mathbf{t}, s)E_\gamma^{-1} * \check{\bar{g}})(\mathbf{x})\frac{d\mathbf{t}ds}{|s|^3}$, $\forall g \in \mathscr{L}^2(\mathbb{R}^2)$. Then,

$$
\begin{aligned}
|I_{\mathbf{x}}(g)| &= \left| \int_{\mathbb{R}^*}\int_{\mathbb{R}^2} (\Omega_{\mathbf{t},s}E_\gamma^{-1} * \check{\bar{g}})(\mathbf{x})\frac{d\mathbf{t}ds}{|s|^3} \right| \\
&= \left| \int_{\mathbb{R}^*}\int_{\mathbb{R}^2} d_\gamma F(\mathbf{t}, s)E_\gamma^{-1}(\mathbf{t})(\tau_{\mathbf{t}} D_s(\psi E_\gamma^{-1}) * \check{\bar{g}})(\mathbf{x})\frac{d\mathbf{t}ds}{|s|^3} \right| \\
&= \left| \int_{\mathbb{R}^*}\int_{\mathbb{R}^2} d_\gamma F(\mathbf{t}, s)E_\gamma^{-1}(\mathbf{t})(D_s(\psi E_\gamma^{-1}) * \check{\bar{g}})(\mathbf{x}-\mathbf{t})\frac{d\mathbf{t}ds}{|s|^3} \right| \\
&\quad \text{(Using the property } (f * \tau_{\mathbf{t}} g) = \tau_{\mathbf{t}}(f * g)) \\
&= \left| \int_{\mathbb{R}^*}\int_{\mathbb{R}^2} d_\gamma F(\mathbf{t}, s)E_\gamma^{-1}(\mathbf{t})\overline{(D_s(\check{\bar{\psi}} E_\gamma) * g)(\mathbf{t}-\mathbf{x})}\frac{d\mathbf{t}ds}{|s|^3} \right| \\
&\quad \text{(Using the properties } (f * \check{g})(\mathbf{x}) = (\check{f} * g)(-\mathbf{x}) \text{ and} \\
&\quad (\bar{f} * \bar{g})(\mathbf{x}) = \overline{(f * g)(\mathbf{x})}, \forall f, g \in \mathscr{L}^2(\mathbb{R}^2)) \\
&= \left| \frac{d_\gamma}{2\pi \overline{d_\gamma}} \int_{\mathbb{R}^*}\int_{\mathbb{R}^2} F(\mathbf{t}, s)E_\gamma^{-1}(\mathbf{t})E_\gamma(\mathbf{t}-\mathbf{x})\overline{W_\psi^\gamma(gE_\gamma)(\mathbf{t}-\mathbf{x}, s)}\frac{d\mathbf{t}ds}{|s|^3} \right| \\
&\leq \frac{1}{2\pi}\int_{\mathbb{R}^*}\int_{\mathbb{R}^2} |F(\mathbf{t}, s)|\,|\overline{W_\psi^\gamma(gE_\gamma)(\mathbf{t}-\mathbf{x}, s)}|\frac{d\mathbf{x}ds}{|s|^3} \\
&\leq \frac{1}{2\pi}\|F\|_{\mathscr{L}^2(\mathbb{R}^2\times\mathbb{R}^*)}\|W_\psi^\gamma(gE_\gamma)\|_{\mathscr{L}^2(\mathbb{R}^2\times\mathbb{R}^*)} \\
&= \frac{1}{2\pi}\|F\|_{\mathscr{L}^2(\mathbb{R}^2\times\mathbb{R}^*)} \times \sqrt{2\pi C_\psi}\|gE_\gamma\|_2 \text{ (Using Remark 1)} \\
&= \sqrt{\frac{C_\psi}{2\pi}}\|F\|_{\mathscr{L}^2(\mathbb{R}^2\times\mathbb{R}^*)}\|g\|_2.
\end{aligned}
$$

Therefore, $I_{\mathbf{x}}$ is a bounded conjugate linear functional on $\mathscr{L}^2(\mathbb{R}^2)$. For arbitrary $h \in \mathscr{L}^1(\mathbb{R}^2) \cap \mathscr{L}^2(\mathbb{R}^2)$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^*}[F(\cdot,s)\circledast_\gamma(E_\gamma D_s(\psi E_\gamma^{-1})\circledast_\gamma E_\gamma\check{\bar{h}})](\mathbf{x})\frac{ds}{|s|^3}\\
=&\int_{\mathbb{R}^*} d_\gamma E_\gamma(\mathbf{x})\left(\int_{\mathbb{R}^2}F(\mathbf{t},s)E_\gamma^{-1}(\mathbf{t})(E_\gamma D_s(\psi E_\gamma^{-1})\circledast_\gamma E_\gamma\check{\bar{h}})(\mathbf{x}-\mathbf{t})E_\gamma^{-1}(\mathbf{x}-\mathbf{t})d\mathbf{t}\right)\frac{ds}{|s|^3}\\
=&\ 2\pi d_\gamma^2 E_\gamma(\mathbf{x})\int_{\mathbb{R}^*}\int_{\mathbb{R}^2}F(\mathbf{t},s)E_\gamma^{-1}(\mathbf{t})(D_s(\psi E_\gamma^{-1})*\check{\bar{h}})(\mathbf{x}-\mathbf{t})\frac{d\mathbf{t}ds}{|s|^3}\\
=&\ 2\pi d_\gamma^2 E_\gamma(\mathbf{x})\int_{\mathbb{R}^*}\int_{\mathbb{R}^2}F(\mathbf{t},s)E_\gamma^{-1}(\mathbf{t})\tau_{-\mathbf{x}}(\tau_{\mathbf{t}}(D_s(\psi E_\gamma^{-1})*\check{\bar{h}}))(0)\frac{d\mathbf{t}ds}{|s|^3}\\
=&\ 2\pi d_\gamma^2 E_\gamma(\mathbf{x})\int_{\mathbb{R}^*}\int_{\mathbb{R}^2}F(\mathbf{t},s)E_\gamma^{-1}(\mathbf{t})(\tau_{\mathbf{t}}D_s(\psi E_\gamma^{-1})*\tau_{-\mathbf{x}}\check{\bar{h}})(0)\frac{d\mathbf{t}ds}{|s|^3}\\
=&\ 2\pi d_\gamma E_\gamma(\mathbf{x})\int_{\mathbb{R}^*}\int_{\mathbb{R}^2}(\Omega_{\mathbf{t},s}E_\gamma^{-1}*\check{\overline{\tau_{\mathbf{x}}h}})(0)\frac{d\mathbf{t}ds}{|s|^3}. \qquad (7)
\end{aligned}$$

Applying Lemma 3 in (7), there exists a unique function $\phi\in\mathscr{L}^2(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^*}\int_{\mathbb{R}^2}(\Omega_{\mathbf{t},s}E_\gamma^{-1}*\check{\bar{g}})(0)\frac{d\mathbf{t}ds}{|s|^3}=(\phi*\check{\bar{g}})(0),\ \forall g\in\mathscr{L}^2(\mathbb{R}^2)$. If $\varphi=E_\gamma\phi$ then $\varphi\in\mathscr{L}^2(\mathbb{R}^2)$ and

$$\begin{aligned}
\int_{\mathbb{R}^*}[F(\cdot,s)\circledast_\gamma(E_\gamma D_s(\psi E_\gamma^{-1})\circledast_\gamma E_\gamma\check{\bar{h}})](\mathbf{x})\frac{ds}{|s|^3} &= 2\pi d_\gamma E_\gamma(\mathbf{x})(\phi*\check{\overline{\tau_{\mathbf{x}}h}})(0)\\
&= 2\pi d_\gamma E_\gamma(\mathbf{x})(\varphi E_\gamma^{-1}*\tau_{-\mathbf{x}}\check{\bar{h}})(0)\\
&= 2\pi d_\gamma E_\gamma(\mathbf{x})(\varphi E_\gamma^{-1}*\check{\bar{h}})(\mathbf{x})=(\varphi\circledast_\gamma E_\gamma\check{\bar{h}})(\mathbf{x})\\
&= \left[\left(\int_{\mathbb{R}^*}F(\cdot,s)\circledast_\gamma E_\gamma D_s(\psi E_\gamma^{-1})\frac{ds}{|s|^3}\right)\circledast_\gamma E_\gamma\check{\bar{h}}\right](\mathbf{x}).
\end{aligned}$$

This completes the proof. □

Lemma 5 If $\gamma\in\mathbb{R}\setminus\pi\mathbb{Z}$ and $\psi\in\mathscr{L}^1(\mathbb{R}^2)$ then $\mathscr{F}_{\alpha,\beta}(E_\gamma D_s(\psi E_\gamma^{-1}))(\mathbf{u})=|s|E_\gamma^{-1}(\mathbf{u})E_\gamma(s\mathbf{u})\mathscr{F}_{\alpha,\beta}\psi(s\mathbf{u})$, for all $\alpha,\beta\in\mathbb{R}$ such that $\frac{\alpha+\beta}{2}=\gamma$.

Proof. For arbitrary $\mathbf{u}\in\mathbb{R}^2$, we get

$$
\begin{aligned}
\mathscr{F}_{\alpha,\beta}(E_\gamma D_s(\psi E_\gamma^{-1}))(\mathbf{u}) &= \int_{\mathbb{R}^2} E_\gamma(\mathbf{t}) D_s(\psi E_\gamma^{-1})(\mathbf{t}) K_{\alpha,\beta}(\mathbf{t},\mathbf{u}) d\mathbf{t} \\
&= \frac{d_\gamma E_\gamma^{-1}(\mathbf{u})}{|s|} \int_{\mathbb{R}^2} \psi\left(\frac{\mathbf{t}}{s}\right) E_\gamma^{-1}\left(\frac{\mathbf{t}}{s}\right) \mathrm{e}^{i\mathbf{t}\cdot\boldsymbol{\zeta}\mathbf{u}} d\mathbf{t} \\
&= \frac{d_\gamma E_\gamma^{-1}(\mathbf{u})}{|s|} \int_{\mathbb{R}^2} \psi(\mathbf{w}) E_\gamma^{-1}(\mathbf{w}) \mathrm{e}^{is\mathbf{w}\cdot\boldsymbol{\zeta}\mathbf{u}} s^2 d\mathbf{w} \text{ (Substituting } \tfrac{\mathbf{t}}{s} = \mathbf{w}) \\
&= |s| d_\gamma E_\gamma^{-1}(\mathbf{u}) \int_{\mathbb{R}^2} \psi(\mathbf{w}) E_\gamma^{-1}(\mathbf{w}) \mathrm{e}^{i\mathbf{w}\cdot\boldsymbol{\zeta}(s\mathbf{u})} d\mathbf{w} \\
&= |s| E_\gamma^{-1}(\mathbf{u}) E_\gamma(s\mathbf{u}) \int_{\mathbb{R}^2} \psi(\mathbf{w}) K_{\alpha,\beta}(\mathbf{w}, s\mathbf{u}) d\mathbf{w} \\
&= |s| E_\gamma^{-1}(\mathbf{u}) E_\gamma(s\mathbf{u}) \mathscr{F}_{\alpha,\beta}\psi(s\mathbf{u}).
\end{aligned}
$$

□

Theorem 5 Let $\psi \in \mathscr{L}^1(\mathbb{R}^2) \cap \mathscr{L}^2(\mathbb{R}^2)$ such that $0 \neq C_\psi < \infty$. If $f \in \mathscr{L}^2(\mathbb{R}^2)$ then

$$
f(\mathbf{x}) = \frac{-\mathrm{e}^{2i\gamma}}{C_\psi} \int_{\mathbb{R}^*} [W_\psi^\gamma f(\cdot, s) \circledast_\gamma E_\gamma D_s(\psi E_\gamma^{-1})](\mathbf{x}) \frac{ds}{|s|^3},
$$

holds weakly in $\mathscr{L}^2(\mathbb{R}^2)$.

Proof. Using the Parseval's identity for W_ψ^γ (Theorem 4), we have

$$
\begin{aligned}
\langle f, g \rangle &= \frac{1}{C_\psi} \langle W_\psi^\gamma f, W_\psi^\gamma g \rangle_{\mathbb{R}^2 \times \mathbb{R}^*} = \frac{1}{2\pi C_\psi} \int_{\mathbb{R}^*} \int_{\mathbb{R}^2} W_\psi^\gamma f(\mathbf{x}, s) \overline{W_\psi^\gamma g(\mathbf{x}, s)} \frac{d\mathbf{x} ds}{|s|^3} \\
&= \frac{1}{2\pi C_\psi} \int_{\mathbb{R}^*} \int_{\mathbb{R}^2} \mathscr{F}_{\alpha,\beta}(W_\psi^\gamma f(\cdot, s))(\mathbf{u}) \overline{\mathscr{F}_{\alpha,\beta}(W_\psi^\gamma g(\cdot, s))(\mathbf{u})} \frac{d\mathbf{u} ds}{|s|^3} \\
&= \frac{1}{2\pi C_\psi} \int_{\mathbb{R}^*} \int_{\mathbb{R}^2} \mathscr{F}_{\alpha,\beta}(W_\psi^\gamma f(\cdot, s))(\mathbf{u}) \overline{\mathscr{F}_{\alpha,\beta}(g \circledast_\gamma E_\gamma D_s(\check{\bar{\psi}} E_\gamma))(\mathbf{u})} \frac{d\mathbf{u} ds}{|s|^3} \\
&= \frac{1}{2\pi C_\psi} \int_{\mathbb{R}^*} \int_{\mathbb{R}^2} \mathscr{F}_{\alpha,\beta}(W_\psi^\gamma f(\cdot, s))(\mathbf{u}) E_\gamma^{-1}(\mathbf{u}) \overline{\mathscr{F}_{\alpha,\beta} g(\mathbf{u})} \\
&\quad \times \overline{\mathscr{F}_{\alpha,\beta}(E_\gamma D_s(\check{\bar{\psi}} E_\gamma))(\mathbf{u})} \frac{d\mathbf{u} ds}{|s|^3} \text{ (Using Theorem 1)}
\end{aligned}
$$

$$= \frac{-e^{2i\gamma}}{2\pi C_\psi}\int_{\mathbb{R}^*}\int_{\mathbb{R}^2}\mathscr{F}_{\alpha,\beta}(W_\psi^\gamma f(\cdot,s))(\mathbf{u})\overline{\mathscr{F}_{\alpha,\beta}g(\mathbf{u})}E_\gamma(s\mathbf{u})\mathscr{F}_{\alpha,\beta}\psi(s\mathbf{u})\frac{d\mathbf{u}ds}{|s|^2}$$

(Using Lemma 1)

$$= \frac{-e^{2i\gamma}}{2\pi C_\psi}\int_{\mathbb{R}^*}\int_{\mathbb{R}^2}E_\gamma(\mathbf{u})\mathscr{F}_{\alpha,\beta}(W_\psi^\gamma f(\cdot,s))(\mathbf{u})$$

$$\times\mathscr{F}_{\alpha,\beta}(E_\gamma D_s(\psi E_\gamma^{-1}))(\mathbf{u})\overline{\mathscr{F}_{\alpha,\beta}g(\mathbf{u})}\frac{d\mathbf{u}ds}{|s|^3}\quad \text{(Using Lemma 5)}$$

$$= \frac{-e^{2i\gamma}}{2\pi C_\psi}\int_{\mathbb{R}^*}\int_{\mathbb{R}^2}\mathscr{F}_{\alpha,\beta}(W_\psi^\gamma f(\cdot,s)\circledast_\gamma E_\gamma D_s(\psi E_\gamma^{-1}))(\mathbf{u})\overline{\mathscr{F}_{\alpha,\beta}g(\mathbf{u})}\frac{d\mathbf{u}ds}{|s|^3}$$

$$= \frac{-e^{2i\gamma}}{2\pi C_\psi}\int_{\mathbb{R}^*}\int_{\mathbb{R}^2}(W_\psi^\gamma f(\cdot,s)\circledast_\gamma E_\gamma D_s(\psi E_\gamma^{-1}))(\mathbf{x})\overline{g(\mathbf{x})}\frac{d\mathbf{x}ds}{|s|^3}$$

$$= \frac{-e^{2i\gamma}}{C_\psi}\int_{\mathbb{R}^*}\left\langle W_\psi^\gamma f(\cdot,s)\circledast_\gamma E_\gamma D_s(\psi E_\gamma^{-1}),\, g\right\rangle\frac{ds}{|s|^3}$$

$$= \left\langle\frac{-e^{2i\gamma}}{C_\psi}\int_{\mathbb{R}^*}W_\psi^\gamma f(\cdot,s)\circledast_\gamma E_\gamma D_s(\psi E_\gamma^{-1})\frac{ds}{|s|^3},\, g\right\rangle.$$

Thus, $f = \frac{-e^{2i\gamma}}{C_\psi}\int_{\mathbb{R}^*}W_\psi^\gamma f(\cdot,s)\circledast_\gamma E_\gamma D_s(\psi E_\gamma^{-1})\frac{ds}{|s|^3}$, holds weakly in $\mathscr{L}^2(\mathbb{R}^2)$. □

Theorem 6 (Inversion formula for CFRWT) Let $\psi \in \mathscr{L}^1(\mathbb{R}^2)\cap\mathscr{L}^2(\mathbb{R}^2)$ such that $0 \neq C_\psi < \infty$. If $f \in \mathscr{L}^2(\mathbb{R}^2)$, $\alpha, \beta \in \mathbb{R}$ such that $\alpha+\beta \neq 2n\pi$ for any $n \in \mathbb{Z}$, and $\mathscr{F}_{\alpha,\beta}f \in \mathscr{L}^1(\mathbb{R}^2)$ then

$$f(\mathbf{x}) = \frac{-e^{2i\gamma}}{C_\psi}\int_{\mathbb{R}^*}\left(W_\psi^\gamma f(\cdot,s)\circledast_\gamma E_\gamma D_s(\psi E_\gamma^{-1})\right)(\mathbf{x})\frac{ds}{|s|^3},$$

almost everywhere in $\mathbb{R}^2$.

Proof. Using the inversion formula for CFRFT, we get

$$\frac{-e^{2i\gamma}}{C_\psi}\int_{\mathbb{R}^*}\left(W_\psi^\gamma f(\cdot,s)\circledast_\gamma E_\gamma D_s(\psi E_\gamma^{-1})\right)(\mathbf{x})\frac{ds}{|s|^3}$$

$$= \frac{-e^{2i\gamma}}{C_\psi}\int_{\mathbb{R}^*}\mathscr{F}_{-\alpha,-\beta}\left(\mathscr{F}_{\alpha,\beta}\left(\left(f\circledast_\gamma E_\gamma D_s(\check{\bar{\psi}}E_\gamma)\right)\circledast_\gamma E_\gamma D_s(\psi E_\gamma^{-1})\right)\right)(\mathbf{x})\frac{ds}{|s|^3}$$

$$
\begin{aligned}
&= \frac{-e^{2i\gamma}}{C_\psi}\int_{\mathbb{R}^*}\mathscr{F}_{-\alpha,\,-\beta}\left(E_\gamma\mathscr{F}_{\alpha,\,\beta}(f\circledast_\gamma E_\gamma D_s(\check{\bar{\psi}}E_\gamma))\mathscr{F}_{\alpha,\,\beta}(E_\gamma D_s(\psi E_\gamma^{-1}))\right)(\mathbf{x})\frac{ds}{|s|^3}\\
&= \frac{-e^{2i\gamma}}{C_\psi}\int_{\mathbb{R}^*}\mathscr{F}_{-\alpha,\,-\beta}\left(E_\gamma^2\mathscr{F}_{\alpha,\,\beta}(f)\mathscr{F}_{\alpha,\,\beta}(E_\gamma D_s(\check{\bar{\psi}}E_\gamma))\mathscr{F}_{\alpha,\,\beta}(E_\gamma D_s(\psi E_\gamma^{-1}))\right)(\mathbf{x})\frac{ds}{|s|^3}\\
&= \frac{-e^{2i\gamma}}{C_\psi}\int_{\mathbb{R}^*}\int_{\mathbb{R}^2}E_\gamma^2(\mathbf{u})\mathscr{F}_{\alpha,\,\beta}(f)(\mathbf{u})\mathscr{F}_{\alpha,\,\beta}(E_\gamma D_s(\check{\bar{\psi}}E_\gamma))(\mathbf{u})\\
&\quad\times\mathscr{F}_{\alpha,\,\beta}(E_\gamma D_s(\psi E_\gamma^{-1}))(\mathbf{u})\,K_{-\alpha,\,-\beta}(\mathbf{u},\,\mathbf{x})\frac{d\mathbf{u}ds}{|s|^3}\\
&= \frac{1}{C_\psi}\int_{\mathbb{R}^*}\left(\int_{\mathbb{R}^2}\mathscr{F}_{\alpha,\,\beta}f(\mathbf{u})|\mathscr{F}_{\alpha,\,\beta}\psi(s\mathbf{u})|^2K_{-\alpha,\,-\beta}(\mathbf{u},\,\mathbf{x})d\mathbf{u}\right)\frac{ds}{|s|}
\end{aligned}
$$

(Using Lemma 1 and Lemma 5)

$$
= \frac{1}{C_\psi}\int_{\mathbb{R}^2}\mathscr{F}_{\alpha,\,\beta}f(\mathbf{u})K_{-\alpha,\,-\beta}(\mathbf{u},\,\mathbf{x})\left(\int_{\mathbb{R}^*}|\mathscr{F}_{\alpha,\,\beta}\psi(s\mathbf{u})|^2\frac{ds}{|s|}\right)d\mathbf{u}
$$

(Using Fubini's theorem)

$$
= \int_{\mathbb{R}^2}\mathscr{F}_{\alpha,\,\beta}f(\mathbf{u})K_{-\alpha,\,-\beta}(\mathbf{u},\,\mathbf{x})d\mathbf{u} = f(\mathbf{x}).
$$

Hence, the theorem follows. □

Theorem 7 (Characterization of range of W_ψ^γ) Let $\Phi\in\mathscr{L}^2(\mathbb{R}^2\times\mathbb{R}^*)$. Then $\Phi\in W_\psi^\gamma(\mathscr{L}^2(\mathbb{R}^2))$ if and only if

$$
\Phi(\mathbf{x}',\,s') = \frac{-e^{2i\gamma}}{C_\psi}\int_{\mathbb{R}^*}\left(\Phi(\cdot,\,s)\circledast_\gamma K_\psi^{\alpha,\,\beta}(s,\,s')\right)(\mathbf{x}')\frac{ds}{|s|^3}, \tag{8}
$$

where $K_\psi^{\alpha,\,\beta}(s,\,s') = E_\gamma D_s(\psi E_\gamma^{-1})\circledast_\gamma E_\gamma D_{s'}(\check{\bar{\psi}}E_\gamma)$.

Proof. We first assume that $\Phi\in W_\psi^\gamma(\mathscr{L}^2(\mathbb{R}^2))$. Then there exists $f\in\mathscr{L}^2(\mathbb{R}^2)$ such that $W_\psi^\gamma f=\Phi$. Using Theorem 5, we get

$$
\begin{aligned}
W_\psi^\gamma f(\mathbf{x}',\,s') &= [f\circledast_\gamma E_\gamma D_{s'}(\check{\bar{\psi}}E_\gamma)](\mathbf{x}')\\
&= \left[\left(\frac{-e^{2i\gamma}}{C_\psi}\int_{\mathbb{R}^*}[W_\psi^\gamma f(\cdot,\,s)\circledast_\gamma E_\gamma D_s(\psi E_\gamma^{-1})]\frac{ds}{|s|^3}\right)\circledast_\gamma E_\gamma D_{s'}(\check{\bar{\psi}}E_\gamma)\right](\mathbf{x}')
\end{aligned}
$$

$$= \frac{-e^{2i\gamma}}{C_\psi}\int_{\mathbb{R}^*}\left(W_\psi^\gamma f(\cdot, s)\circledast_\gamma [E_\gamma D_s(\psi E_\gamma^{-1})\circledast_\gamma E_\gamma D_{s'}(\check{\bar{\psi}}E_\gamma)]\right)(\mathbf{x}')\frac{ds}{|s|^3} \text{ (Using Lemma 4)}$$

$$= \frac{-e^{2i\gamma}}{C_\psi}\int_{\mathbb{R}^*}\left(W_\psi^\gamma f(\cdot, s)\circledast_\gamma K_\psi^{\alpha,\beta}(s, s')\right)(\mathbf{x}')\frac{ds}{|s|^3}.$$

Conversely, we assume that Φ satisfies the equation (8). If $f(\mathbf{x}) = \frac{-e^{2i\gamma}}{C_\psi}\int_{\mathbb{R}^*}\left(\Phi(\cdot, s)\circledast_\gamma E_\gamma D_s(\psi E_\gamma^{-1})\right)(\mathbf{x})\frac{ds}{|s|^3}$, then by Lemma 4, we get that $f\in\mathscr{L}^2(\mathbb{R}^2)$. Therefore,

$$\begin{aligned} W_\psi^\gamma(f)(\mathbf{x}', s') &= [f\circledast_\gamma E_\gamma D_{s'}(\check{\bar{\psi}}E_\gamma)](\mathbf{x}') \\ &= \left[\left(\frac{-e^{2i\gamma}}{C_\psi}\int_{\mathbb{R}^*}\Phi(\cdot, s)\circledast_\gamma E_\gamma D_s(\psi E_\gamma^{-1})\frac{ds}{|s|^3}\right)\circledast_\gamma E_\gamma D_{s'}(\check{\bar{\psi}}E_\gamma)\right](\mathbf{x}') \\ &= \frac{-e^{2i\gamma}}{C_\psi}\int_{\mathbb{R}^*}\left(\Phi(\cdot, s)\circledast_\gamma\left(E_\gamma D_s(\psi E_\gamma^{-1})\circledast_\gamma E_\gamma D_{s'}(\check{\bar{\psi}}E_\gamma)\right)\right)(\mathbf{x}')\frac{ds}{|s|^3} \\ &= \frac{-e^{2i\gamma}}{C_\psi}\int_{\mathbb{R}^*}\left(\Phi(\cdot, s)\circledast_\gamma K_\psi^{\alpha,\beta}(s, s')\right)(\mathbf{x}')\frac{ds}{|s|^3} = \Phi(\mathbf{x}', s'). \end{aligned}$$

This completes the proof. □

Theorem 8 (Convolution theorem for CFRWT) If $f\in\mathscr{L}^2(\mathbb{R}^2)$ and $g\in\mathscr{L}^1(\mathbb{R}^2)\cap\mathscr{L}^2(\mathbb{R}^2)$ then

$$W_\psi^\gamma(f\circledast_\gamma g)(\mathbf{x}, s) = (f\circledast_\gamma W_\psi^\gamma g(\cdot, s))(\mathbf{x}) = (W_\psi^\gamma f(\cdot, s)\circledast_\gamma g)(\mathbf{x}),$$

for all $\mathbf{x}\in\mathbb{R}^2$, $s\in\mathbb{R}^*$.

Proof. Let $f\in\mathscr{L}^2(\mathbb{R}^2)$ and $g\in\mathscr{L}^1(\mathbb{R}^2)\cap\mathscr{L}^2(\mathbb{R}^2)$. Then, we have

$$\begin{aligned} W_\psi^\gamma(f\circledast_\gamma g)(\mathbf{x}, s) &= \left[(f\circledast_\gamma g)\circledast_\gamma E_\gamma D_s(\check{\bar{\psi}}E_\gamma)\right](\mathbf{x}) \\ &= \left[f\circledast_\gamma(g\circledast_\gamma E_\gamma D_s(\check{\bar{\psi}}E_\gamma))\right](\mathbf{x}) \text{ (Using (4))} \\ &= (f\circledast_\gamma W_\psi^\gamma g(\cdot, s))(\mathbf{x}). \end{aligned}$$

Similarly, we can prove the other equality using (3) and (4). □

Lemma 6 Let $\gamma\in\mathbb{R}\setminus\pi\mathbb{Z}$ and $\psi\in\mathscr{L}^1(\mathbb{R}^2)\cap\mathscr{L}^2(\mathbb{R}^2)$ satisfy the condition $0\neq C_\psi<\infty$, where C_ψ depends on $\alpha, \beta\in\mathbb{R}$ such that $\frac{\alpha+\beta}{2}=\gamma$. For $\lambda\in\mathbb{R}^*$, define $\eta=\cot^{-1}(\lambda^2\cot\gamma)$ and $\phi=D_{\frac{1}{\lambda}}\psi$. Then, we have $\eta\in\mathbb{R}\setminus\pi\mathbb{Z}$, $\phi\in\mathscr{L}^1(\mathbb{R}^2)\cap\mathscr{L}^2(\mathbb{R}^2)$, and $0\neq C_\phi<\infty$.

Proof. If $\eta = \cot^{-1}(\lambda^2 \cot\gamma)$ and $\phi = D_{\frac{1}{\lambda}}\psi$, then clearly we have $\eta \in \mathbb{R}\setminus\pi\mathbb{Z}$, $\|\phi\|_1 = \frac{1}{|\lambda|}\|\psi\|_1 < \infty$ and $\|\phi\|_2 = \|\psi\|_2 < \infty$. For $\delta = \frac{\alpha-\beta}{2}$, if we let $\alpha_1 = \gamma+\delta$ and $\beta_1 = \gamma-\delta$, then we get $\frac{\alpha_1+\beta_1}{2} = \eta$ and $\frac{\alpha_1-\beta_1}{2} = \delta$. As in (1), we denote by $\tilde{a}_\eta = \frac{\cot\eta}{2}$, $\tilde{b}(\eta,\delta) = \frac{\cos\delta}{\sin\eta}$, $\tilde{c}(\eta,\delta) = \frac{\sin\delta}{\sin\eta}$ and $d_\eta = \frac{ie^{-i\eta}}{2\pi\sin\eta}$. Next we find

$$\begin{aligned}
\mathscr{F}_{\alpha_1,\beta_1}\phi(\mathbf{u}) &= d_\eta e^{-i\tilde{a}_\eta\|\mathbf{u}\|^2}\int_{\mathbb{R}^2} D_{\frac{1}{\lambda}}\psi(\mathbf{x})e^{-i\tilde{a}_\eta\|\mathbf{x}\|^2}\\
&\quad\times\exp(i\tilde{b}(\eta,\delta)(x_1u_1+x_2u_2)+i\tilde{c}(\eta,\delta)(x_1u_2-x_2u_1))d\mathbf{x}\\
&= |\lambda| d_\eta E_\gamma^{-1}(\lambda\mathbf{u})\int_{\mathbb{R}^2}\psi(\lambda\mathbf{x})E_\gamma^{-1}(\lambda\mathbf{x})\\
&\quad\times\exp(i\tilde{b}(\eta,\delta)(x_1u_1+x_2u_2)+i\tilde{c}(\eta,\delta)(x_1u_2-x_2u_1))d\mathbf{x}\\
&= d_\eta E_\gamma^{-1}(\lambda\mathbf{u})\int_{\mathbb{R}^2}\psi(\mathbf{y})E_\gamma^{-1}(\mathbf{y})\\
&\quad\times\exp\left(i\tilde{b}(\eta,\delta)\left(\frac{y_1}{\lambda}u_1+\frac{y_2}{\lambda}u_2\right)+i\tilde{c}(\eta,\delta)\left(\frac{y_1}{\lambda}u_2-\frac{y_2}{\lambda}u_1\right)\right)\frac{d\mathbf{y}}{|\lambda|}\ \text{(Substituting } \lambda\mathbf{x}=\mathbf{y})\\
&= \frac{d_\eta}{|\lambda|}E_\gamma^{-1}(\lambda\mathbf{u})\int_{\mathbb{R}^2}\psi(\mathbf{y})E_\gamma^{-1}(\mathbf{y})\exp\left(i\tilde{b}(\gamma,\delta)\left(y_1\frac{u_1\sin\gamma}{\lambda\sin\eta}+y_2\frac{u_2\sin\gamma}{\lambda\sin\eta}\right)\right)\\
&\quad\times\exp\left(i\tilde{c}(\gamma,\delta)\left(y_1\frac{u_2\sin\gamma}{\lambda\sin\eta}-y_2\frac{u_1\sin\gamma}{\lambda\sin\eta}\right)\right)d\mathbf{y}\\
&= \frac{d_\eta}{|\lambda|d}E_\gamma^{-1}(\lambda\mathbf{u})E_\gamma\left(\frac{\sin\gamma}{\lambda\sin\eta}\mathbf{u}\right)\mathscr{F}_{\alpha,\beta}(\psi)\left(\frac{\sin\gamma}{\lambda\sin\eta}\mathbf{u}\right).
\end{aligned}\tag{9}$$

For $\mathbf{w}\in\mathbb{R}^2$ with $\|\mathbf{w}\| = 1$, we get

$$\begin{aligned}
C_\phi &= \int_{\mathbb{R}^*}|\mathscr{F}_{\alpha_1,\beta_1}\phi(r\mathbf{w})|^2\frac{dr}{|r|}\\
&= \left|\frac{\sin\gamma}{\lambda\sin\eta}\right|^2\int_{\mathbb{R}^*}\left|\mathscr{F}_{\alpha,\beta}\psi\left(\frac{r\sin\gamma}{\lambda\sin\eta}\mathbf{w}\right)\right|^2\frac{dr}{|r|}\ \text{(Using (9))}\\
&= \frac{\sin^2\gamma}{\lambda^2\sin^2\eta}\left(\int_0^\infty\left|\mathscr{F}_{\alpha,\beta}\psi\left(\frac{r|\sin\gamma|}{|\lambda\sin\eta|}(\pm\mathbf{w})\right)\right|^2\frac{dr}{r}-\int_{-\infty}^0\left|\mathscr{F}_{\alpha,\beta}\psi\left(\frac{r|\sin\gamma|}{|\lambda\sin\eta|}(\pm\mathbf{w})\right)\right|^2\frac{dr}{r}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sin^2\gamma}{\lambda^2\sin^2\eta}\int_{\mathbb{R}^*}|\mathscr{F}_{\alpha,\beta}\psi(p(\pm\mathbf{w}))|^2\frac{dp}{|p|}\ \text{(Putting } \tfrac{r|\sin\gamma|}{\lambda|\sin\eta|}=p)\\
&= \frac{C_\psi\sin^2\gamma}{\lambda^2\sin^2\eta}<\infty,
\end{aligned}
$$

which completes the proof. □

Theorem 9 For $f,\, f_1,\, f_2\in\mathscr{L}^2(\mathbb{R}^2)$, $\lambda\in\mathbb{R}^*$ and $z_1,\, z_2\in\mathbb{C}$, we have

1. $W_\psi^\gamma(z_1f_1+z_2f_2)=z_1W_\psi^\gamma f_1+z_2W_\psi^\gamma f_2$.
2. $W_\psi^\gamma f(\mathbf{x}-\mathbf{t},\, s)=\mathrm{e}^{2i\tilde{a}_\gamma\mathbf{x}\cdot\mathbf{t}}W_\psi^\gamma(M_{-2\tilde{a}_\gamma\mathbf{t}}(\tau_\mathbf{t}f))(\mathbf{x},\, s)$.
3. $W_\psi^\gamma(\tau_\mathbf{t}f)(\mathbf{x},\, s)=E_\gamma^{-2}(\mathbf{t})\mathrm{e}^{2i\tilde{a}_\gamma\mathbf{x}\cdot\mathbf{t}}W_\psi^\gamma(M_{-2\tilde{a}_\gamma\mathbf{t}}f)(\mathbf{x}-\mathbf{t},\, s)$.
4. $W_\psi^\gamma(D_\lambda f)(\mathbf{x},\, s)=\frac{d_\gamma}{d_\eta}W_\phi^\eta f\left(\frac{\mathbf{x}}{\lambda},\, s\right)$, where $\phi=D_{\frac{1}{\lambda}}\psi$, $\eta=\cot^{-1}(\lambda^2\cot\gamma)$ and $d_\eta=\frac{i\mathrm{e}^{-i\eta}}{2\pi\sin\eta}$.

Proof. 1. Using the first two identities of Theorem 2, we can easily prove that W_ψ^γ is linear.

2. Applying the fourth and fifth identities of Theorem 2, we get

$$
\begin{aligned}
W_\psi^\gamma f(\mathbf{x}-\mathbf{t},\, s) &= (f\circledast_\gamma E_\gamma D_s(\check{\bar{\psi}}E_\gamma))(\mathbf{x}-\mathbf{t})(\tau_\mathbf{t}f\circledast_\gamma M_{2\tilde{a}_\gamma\mathbf{t}}(E_\gamma D_s(\check{\bar{\psi}}E_\gamma))(\mathbf{x})\\
&= M_{2\tilde{a}_\gamma\mathbf{t}}M_{-2\tilde{a}_\gamma\mathbf{t}}(\tau_\mathbf{t}f\circledast_\gamma M_{2\tilde{a}_\gamma\mathbf{t}}(E_\gamma D_s(\check{\bar{\psi}}E_\gamma))(\mathbf{x})=M_{2\tilde{a}_\gamma\mathbf{t}}(M_{-2\tilde{a}_\gamma\mathbf{t}}(\tau_\mathbf{t}f)\circledast_\gamma(E_\gamma D_s(\check{\bar{\psi}}E_\gamma))(\mathbf{x})\\
&= \mathrm{e}^{-i2\tilde{a}_\gamma\mathbf{t}\cdot\mathbf{x}}(M_{-2\tilde{a}_\gamma\mathbf{t}}(\tau_\mathbf{t}f)\circledast_\gamma(E_\gamma D_s(\check{\bar{\psi}}E_\gamma))(\mathbf{x})=\mathrm{e}^{2i\tilde{a}_\gamma\mathbf{x}\cdot\mathbf{t}}W_\psi^\gamma(M_{-2\tilde{a}_\gamma\mathbf{t}}(\tau_\mathbf{t}f))(\mathbf{x},\, s).
\end{aligned}
$$

3. For $(\mathbf{x},\, s)\in\mathbb{R}^2\times\mathbb{R}^*$,

$$
\begin{aligned}
W_\psi^\gamma(\tau_\mathbf{t}f)(\mathbf{x},\, s) &= \left(\tau_\mathbf{t}f\circledast_\gamma E_\gamma D_s(\check{\bar{\psi}}E_\gamma)\right)(\mathbf{x})\\
&= \frac{d_\gamma E_\gamma(\mathbf{x})}{|s|}\int_{\mathbb{R}^2}\overline{\psi\left(\frac{\mathbf{z}-\mathbf{x}}{s}\right)}E_\gamma\left(\frac{\mathbf{x}-\mathbf{z}}{s}\right)f(\mathbf{z}-\mathbf{t})E_\gamma^{-1}(\mathbf{z})d\mathbf{z}\\
&= \frac{d_\gamma E_\gamma(\mathbf{x})}{|s|}\int_{\mathbb{R}^2}\overline{\psi\left(\frac{(\mathbf{w}+\mathbf{t})-\mathbf{x}}{s}\right)}E_\gamma\left(\frac{\mathbf{x}-(\mathbf{w}+\mathbf{t})}{s}\right)f(\mathbf{w})E_\gamma^{-1}(\mathbf{w}+\mathbf{t})d\mathbf{w}\ \text{(Substituting } \mathbf{z}-\mathbf{t}=\mathbf{w})\\
&= \frac{d_\gamma E_\gamma^{-1}(\mathbf{t})E_\gamma(\mathbf{x}-\mathbf{t})\mathrm{e}^{i2\tilde{a}_\gamma\mathbf{x}\cdot\mathbf{t}}}{|s|}\int_{\mathbb{R}^2}\overline{\psi\left(\frac{\mathbf{w}-(\mathbf{x}-\mathbf{t})}{s}\right)}E_\gamma\left(\frac{(\mathbf{x}-\mathbf{t})-\mathbf{w}}{s}\right)f(\mathbf{w})E_\gamma^{-1}(\mathbf{w})E_\gamma^{-1}(\mathbf{t})\mathrm{e}^{i2\tilde{a}_\gamma\mathbf{w}\cdot\mathbf{t}}d\mathbf{w}\\
&= E_\gamma^{-2}(\mathbf{t})\mathrm{e}^{i2\tilde{a}_\gamma\mathbf{x}\cdot\mathbf{t}}\left(E_\gamma D_s(\check{\bar{\psi}}E_\gamma)\circledast_\gamma M_{-2\tilde{a}_\gamma\mathbf{t}}f\right)(\mathbf{x}-\mathbf{t})\\
&= E_\gamma^{-2}(\mathbf{t})\mathrm{e}^{i2\tilde{a}_\gamma\mathbf{x}\cdot\mathbf{t}}W_\psi^\gamma(M_{-2\tilde{a}_\gamma\mathbf{t}}f)(\mathbf{x}-\mathbf{t},\, s).
\end{aligned}
$$

4. For $\mathbf{z} \in \mathbb{R}^2$,

$$\begin{aligned} D_{\frac{1}{\lambda}}(E_\gamma D_s(\check{\bar{\psi}} E_\gamma))(\mathbf{z}) &= |\lambda| E_\gamma(\lambda \mathbf{z}) D_s(\check{\bar{\psi}} E_\gamma)(\lambda \mathbf{z}) \\ &= \frac{|\lambda|}{|s|} E_\gamma(\lambda \mathbf{z}) \check{\bar{\psi}} \left(\frac{\lambda \mathbf{z}}{s}\right) E_\gamma \left(\frac{\lambda \mathbf{z}}{s}\right) \\ &= \frac{1}{|s|} E_\eta(\mathbf{z}) \check{\bar{\phi}} \left(\frac{\mathbf{z}}{s}\right) E_\eta \left(\frac{\mathbf{z}}{s}\right) \text{ (where } \phi = D_{\frac{1}{\lambda}} \psi \text{ and } \eta = \cot^{-1}(\lambda^2 \cot \gamma)) \\ &= E_\eta(\mathbf{z}) D_s(\check{\bar{\phi}} E_\eta)(\mathbf{z}). \end{aligned} \tag{10}$$

Now using Corollary 1, we have

$$\begin{aligned} W_\psi^\gamma(D_\lambda f)(\mathbf{x}, s) &= (D_\lambda f \circledast_\gamma E_\gamma D_s(\check{\bar{\psi}} E_\gamma))(\mathbf{x}) \\ &= \frac{|\lambda| d_\gamma}{d_\eta} D_\lambda (f \circledast_\eta D_{\frac{1}{\lambda}}(E_\gamma D_s(\check{\bar{\psi}} E_\gamma)))(\mathbf{x}) \\ &= \frac{d_\gamma}{d_\eta} [f \circledast_\eta (E_\eta D_s(\check{\bar{\phi}} E_\eta))] \left(\frac{\mathbf{x}}{\lambda}\right) \quad \text{(Using (10))} \\ &= \frac{d_\gamma}{d_\eta} W_\phi^\eta f \left(\frac{\mathbf{x}}{\lambda}, s\right), \quad \text{by Lemma 6.} \end{aligned}$$

Hence, the theorem follows. □

4. Uncertainty principle for 2D-coupled fractional wavelet transform

First we recall the uncertainty principle for CFRFT from [14]. If $f \in \mathscr{L}^1(\mathbb{R}^2) \cap \mathscr{L}^2(\mathbb{R}^2)$ and $P_k f, P_k \mathscr{F}_{\alpha, \beta} f \in \mathscr{L}^2(\mathbb{R}^2)$ for $k = 1, 2,$ then

$$\left(\frac{1}{2\pi} \int_{\mathbb{R}^2} \|\mathbf{x}\|^2 |f(\mathbf{x})|^2 d\mathbf{x}\right)^{\frac{1}{2}} \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} \|\mathbf{u}\|^2 |\mathscr{F}_{\alpha, \beta} f(\mathbf{u})|^2 d\mathbf{u}\right)^{\frac{1}{2}} \geq \frac{|\sin \gamma|}{2\pi} \int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x}. \tag{11}$$

Lemma 7 If $f, P_k \mathscr{F}_{\alpha, \beta} f \in \mathscr{L}^2(\mathbb{R}^2)$, for $k = 1, 2$ and $\psi \in \mathscr{L}^1(\mathbb{R}^2) \cap \mathscr{L}^2(\mathbb{R}^2)$ such that $0 \neq C_\psi < \infty$, then

$$\int_{\mathbb{R}^*} \int_{\mathbb{R}^2} \|\mathbf{u}\|^2 |\mathscr{F}_{\alpha, \beta}(W_\psi^\gamma f(\cdot, s))(\mathbf{u})|^2 \frac{d\mathbf{u} ds}{|s|^3} = C_\psi \int_{\mathbb{R}^2} \|\mathbf{u}\|^2 |\mathscr{F}_{\alpha, \beta} f(\mathbf{u})|^2 d\mathbf{u}.$$

Proof. By direct calculation, we get

$$\begin{aligned}
\int_{\mathbb{R}^*}\int_{\mathbb{R}^2}\|\mathbf{u}\|^2|\mathscr{F}_{\alpha,\beta}(W_\psi^\gamma f(\cdot,s))(\mathbf{u})|^2\frac{d\mathbf{u}ds}{|s|^3} &= \int_{\mathbb{R}^*}\int_{\mathbb{R}^2}\|\mathbf{u}\|^2|\mathscr{F}_{\alpha,\beta}(f\circledast_\gamma E_\gamma D_s(\check{\bar{\psi}}E_\gamma))(\mathbf{u})|^2\frac{d\mathbf{u}ds}{|s|^3}\\
&= \int_{\mathbb{R}^*}\int_{\mathbb{R}^2}\|\mathbf{u}\|^2|E_\gamma(\mathbf{u})\mathscr{F}_{\alpha,\beta}f(\mathbf{u})\mathscr{F}_{\alpha,\beta}(E_\gamma D_s(\check{\bar{\psi}}E_\gamma))(\mathbf{u})|^2\frac{d\mathbf{u}ds}{|s|^3}\\
&= \int_{\mathbb{R}^*}\int_{\mathbb{R}^2}\|\mathbf{u}\|^2|\mathscr{F}_{\alpha,\beta}f(\mathbf{u})|^2|\mathscr{F}_{\alpha,\beta}\psi(s\mathbf{u})|^2\frac{d\mathbf{u}ds}{s} \text{ (Using (1))}\\
&= \int_{\mathbb{R}^2}\|\mathbf{u}\|^2|\mathscr{F}_{\alpha,\beta}f(\mathbf{u})|^2\left(\int_{\mathbb{R}^*}|\mathscr{F}_{\alpha,\beta}\psi(s\mathbf{u})|^2\frac{ds}{s}\right)d\mathbf{u}\\
&= C_\psi\int_{\mathbb{R}^2}\|\mathbf{u}\|^2|\mathscr{F}_{\alpha,\beta}f(\mathbf{u})|^2d\mathbf{u}.
\end{aligned}$$

Thus the lemma follows. □

Theorem 10 (Uncertainty principle for CFRWT) Let $\psi\in\mathscr{L}^1(\mathbb{R}^2)\cap\mathscr{L}^2(\mathbb{R}^2)$ satisfy $0\neq C_\psi<\infty$. If $f, P_k\mathscr{F}_{\alpha,\beta}f\in\mathscr{L}^2(\mathbb{R}^2)$, $W_\psi^\gamma f(\cdot,s)\in\mathscr{L}^1(\mathbb{R}^2)$ and $P_k(\mathbf{x})W_\psi^\gamma f(\mathbf{x},s)\in\mathscr{L}^2(\mathbb{R}^2\times\mathbb{R}^*)$, for $k=1,2$, then

$$\left(\frac{1}{2\pi}\int_{\mathbb{R}^*}\int_{\mathbb{R}^2}\|\mathbf{x}\|^2|W_\psi^\gamma f(\mathbf{x},s)|^2\frac{d\mathbf{x}ds}{|s|^3}\right)\left(\frac{1}{2\pi}\int_{\mathbb{R}^2}\|\mathbf{u}\|^{\mathbf{2}}|\mathscr{F}_{\alpha,\beta}f(\mathbf{u})|^2d\mathbf{u}\right)\geq C_\psi\sin^2\gamma\|f\|_2^4.$$

Proof. Replacing $f(\mathbf{x})$ by $W_\psi^\gamma f(\mathbf{x},s)$ in (11), we get

$$\left(\frac{1}{2\pi}\int_{\mathbb{R}^2}\|\mathbf{x}\|^2|W_\psi^\gamma f(\mathbf{x},s)|^2d\mathbf{x}\right)^{\frac{1}{2}}\left(\frac{1}{2\pi}\int_{\mathbb{R}^2}\|\mathbf{u}\|^2|\mathscr{F}_{\alpha,\beta}(W_\psi^\gamma f(\cdot,s))(\mathbf{u})|^2d\mathbf{u}\right)^{\frac{1}{2}}\geq\frac{|\sin\gamma|}{2\pi}\int_{\mathbb{R}^2}|W_\psi^\gamma f(\mathbf{x},s)|^2d\mathbf{x}. \quad (12)$$

Then,

$$\begin{aligned}
C_\psi\sin^2\gamma\|f\|_2^4 &= \frac{C_\psi\sin^2\gamma}{4\pi^2C_\psi^2}\left(\int_{\mathbb{R}^*}\int_{\mathbb{R}^2}|W_\psi^\gamma f(\mathbf{x},s)|^2\frac{d\mathbf{x}ds}{|s|^3}\right)^2 \text{ (Using Remark 1)}\\
&= \frac{1}{C_\psi}\left(\int_{\mathbb{R}^*}\left(\frac{|\sin\gamma|}{2\pi}\int_{\mathbb{R}^2}|W_\psi^\gamma f(\mathbf{x},s)|^2d\mathbf{x}\right)\frac{ds}{|s|^3}\right)^2\\
&\leq \frac{1}{C_\psi}\left(\int_{\mathbb{R}^*}\left(\frac{1}{2\pi}\int_{\mathbb{R}^2}\|\mathbf{x}\|^2|W_\psi^\gamma f(\mathbf{x},s)|^2d\mathbf{x}\right)^{\frac{1}{2}}\right.
\end{aligned}$$

$$\times \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} \|\mathbf{u}\|^2 |\mathscr{F}_{\alpha,\beta}(W_\psi^\gamma f(\cdot, s))(\mathbf{u})|^2 d\mathbf{u} \right)^{\frac{1}{2}} \frac{ds}{|s|^3} \Bigg)^2 \quad \text{(Using (12))}$$

$$\leq \frac{1}{C_\psi} \left(\frac{1}{2\pi} \int_{\mathbb{R}^*} \int_{\mathbb{R}^2} \|\mathbf{x}\|^2 |W_\psi^\gamma f(\mathbf{x}, s)|^2 \frac{d\mathbf{x}ds}{|s|^3} \right) \left(\frac{1}{2\pi} \int_{\mathbb{R}^*} \int_{\mathbb{R}^2} \|\mathbf{u}\|^2 |\mathscr{F}_{\alpha,\beta}(W_\psi^\gamma f(\cdot, s))(\mathbf{u})|^2 \frac{d\mathbf{u}ds}{|s|^3} \right)$$

(Using Cauchy-Schwartz inequality)

$$= \frac{1}{C_\psi} \left(\frac{1}{2\pi} \int_{\mathbb{R}^*} \int_{\mathbb{R}^2} \|\mathbf{x}\|^2 |W_\psi^\gamma f(\mathbf{x}, s)|^2 \frac{d\mathbf{x}ds}{|s|^3} \right) \left(\frac{C_\psi}{2\pi} \int_{\mathbb{R}^2} \|\mathbf{u}\|^2 |\mathscr{F}_{\alpha,\beta} f(\mathbf{u})|^2 d\mathbf{u} \right)$$

(Using Lemma 7)

$$= \left(\frac{1}{2\pi} \int_{\mathbb{R}^*} \int_{\mathbb{R}^2} \|\mathbf{x}\|^2 |W_\psi^\gamma f(\mathbf{x}, s)|^2 \frac{d\mathbf{x}ds}{|s|^3} \right) \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} \|\mathbf{u}\|^{\mathbf{2}} |\mathscr{F}_{\alpha,\beta} f(\mathbf{u})|^2 d\mathbf{u} \right).$$

Hence, the theorem follows. □

5. Conclusion

We defined the coupled fractional wavelet transform CFRWT using the fractional convolution $\circledast_\gamma$ so that the CFRWT satisfies the expected properties including the Parsevals' identity, inversion formula and the uncertainty principle. When $\gamma = \frac{\pi}{2}$, W_ψ^γ becomes the the classical two-dimensional wavelet transform. Introducing a higher dimensional fractional wavelet transform will be an interesting open problem.

Acknowledgement

The authors thank the referees for their valuable comments which help to improve the presentation of the paper.

Conflict of interest

The authors declare no competing financial interest.

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