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# Fractional Frontier: Navigating Cauchy-Type Equations with Formable and Fourier Transformations

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**Abstract:** The Formable integral transform for the Hilfer-Prabhakar and Prabhakar fractional derivatives, as well as their regularised forms, are derived in this article. We solve several Cauchy-type fractional differential equations with Hilfer-Prabhakar fractional derivatives by applying the Formable integral and Fourier transformations in their entirety, including the three-parameter Mittag-Leffler function. With the help of analytical solutions and improved comprehension of complicated fractional differential equations, this work expands the use of integral transforms in fractional calculus across a range of scientific and engineering fields.

*Keywords*: Prabhakar integral, Hilfer-Prabhakar derivatives, formable transform, Fourier transform, Mittag-Leffler function

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## 1. Introduction

The subfield of fractional calculus deals with fractional integrals and fractional derivatives of real or complex orders. Due to its broad spectrum of applications and ability to bridge disciplines, fractional calculus is a rapidly expanding field of study. We employ a number of fractional integrals and derivatives in this study, such as the Caputo fractional (CF) derivative, the Hilfer fractional (HF) derivative, and the Riemann-Liouville fractional (RLF) integral and derivative [1, 2]. Almost every scientific discipline applies fractional calculus to real-world problems. Sound transmission, continuum mechanics, fluid flow, linear viscoelasticity, biological tissues, and other scientific and technological disciplines can all benefit from its application in modeling and engineering [3–13].

In solving differential equations, integral transforms such as the Laplace transform, Fourier transform, and Mellin transform are utilized by relocating the equation to a new domain. Integral transformation converts the equation to its algebraic form, which is significantly more manageable. Then, the inverse integral transform can be used to return the result to the original domain, where the differential equation can be solved.

The Prabhakar fractional (PF) integral [14] is the modification of the Riemann-Liouville integral by extending its kernel with the three-parameter Mittag-Leffler function. The Hilfer-Prabhakar derivative and its regularized version were first introduced in [15]. Many researchers used Hilfer-Prabhakar fractional derivatives in modeling and other fields due to their special properties, especially the combination of several integral transforms like Laplace, Sumudu, Elzaki, Shehu, and others [15–18].

The Formable integral (FMBI) transform is a modern integral transform that is introduced by Saadeh et al. [19] in 2021. In this study, we determine the Formable transformation for the Prabhakar integral, the Prabhakar derivative, the Hilfer-Prabhakar derivative, and their regularised forms. To further apply these findings, we used the Hilfer-Prabhakar fractional (HPF) derivatives expressed in terms of the generalized Mittag-Leffler (ML) function to solve a few Cauchy-type fractional differential equations.

## 2. Definitions and preliminaries

**Definition 1** [2] The RLF integral of order $\upsilon > 0$ of a function $\zeta(t)$ is

$$_0\beth_t^\upsilon \zeta(t) = \frac{1}{\Gamma(\upsilon)} \int_0^t (t-\ell)^{\upsilon-1} \zeta(\ell) d\ell, \ \ t > 0. \tag{1}$$

**Definition 2** [2] The RLF derivative of order $\upsilon$ of a function $\zeta(t)$ is

$$_0\mathbb{D}_t^\upsilon \zeta(t) = \frac{1}{\Gamma(n-\upsilon)} \frac{d^n}{dt^n} \int_0^t (t-\ell)^{n-\upsilon-1} \zeta(\ell) d\ell, \ \ n-1 < \upsilon < n, \ \ n \in \mathbb{N}. \tag{2}$$

**Definition 3** [2] CF derivative of order $\upsilon$ of a function $\zeta(t)$ is

$$_0^C\mathbb{D}_t^\upsilon \zeta(t) = \frac{1}{\Gamma(n-\upsilon)} \int_0^t (t-\ell)^{n-\upsilon-1} \zeta^{(n)}(\ell) d\ell, \ \ n-1 < \upsilon < n, \ \ n \in \mathbb{N}. \tag{3}$$

**Definition 4** [1] For $0 < \upsilon < 1$, and $0 \le \rho \le 1$, the HF derivative of order $\upsilon$ and $\rho$ of a function $\zeta(t)$ is

$$_0\mathbb{D}_t^{\upsilon, \rho} \zeta(t) = \left( _0\beth_t^{\rho(1-\upsilon)} \frac{d}{dt} \left( _0\beth_t^{(1-\upsilon)(1-\rho)} \zeta(t) \right) \right). \tag{4}$$

**Definition 5** [20] Let $\zeta(x)$ be a piecewise continuous function defined on $(-\infty, \infty)$ in each partial interval and absolutely integrable in $(-\infty, \infty)$. The Fourier integral (FRI) transform is defined by

$$F[\zeta(x), k] = \zeta^*(k) = \int_{-\infty}^{\infty} \zeta(x) exp(ikx) dx. \tag{5}$$

**Definition 6** [14] Three parameter Mittag-Leffler function is defined by

$$E_{\upsilon, \rho}^\gamma(\ell) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\upsilon k + \rho)} \frac{(\ell)^k}{k!}, \ \ \ell \in \mathbb{C}, \tag{6}$$

A generalized form of (6) given by Garra et al. [15] as

$$e_{\upsilon,\,\rho,\,\varpi}^{\gamma} = t^{\rho-1}E_{\upsilon,\,\rho}^{\gamma}(\varpi t^{\upsilon}), \;\; t > 0, \tag{7}$$

where $\upsilon,\,\rho,\,\gamma,\,\varpi \in \mathbb{C}$; $\upsilon > 0$.

**Definition 7** [14] Let $\zeta \in L^1[0,\,b]$; $0 < t < b < \infty$; $\zeta * e_{\upsilon,\,\rho,\,\varpi}^{\gamma} \in W^{n,\,1}[0,\,b]$, $n = \lceil \rho \rceil$. The PF integral is defined as

$$\beth_{\upsilon,\,\rho,\,\varpi,\,0^+}^{\gamma}\zeta(t) = \int_0^t (t-\ell)^{\rho-1}E_{\upsilon,\,\rho}^{\gamma}(\varpi(t-\ell)^{\upsilon})\zeta(\ell)d\ell$$
$$= (\zeta * e_{\upsilon,\,\rho,\,\varpi}^{\gamma})(t), \tag{8}$$

where $\upsilon,\,\rho,\,\gamma,\,\varpi \in \mathbb{C}$; $\upsilon,\,\rho > 0$ and $W^{n,\,1}[0,\,b]$ is the Sobolev space.

**Definition 8** [14] Let $\zeta \in L^1[0,\,b]$; $0 < t < b < \infty$. The PF derivative is defined as

$$\mathbb{D}_{\upsilon,\,\rho,\,\varpi,\,0^+}^{\gamma}\zeta(t) = \frac{d^n}{dt^n}\beth_{\upsilon,\,n-\rho,\,\varpi,\,0^+}^{-\gamma}\zeta(t), \tag{9}$$

where $\upsilon,\,\rho,\,\gamma,\,\varpi \in \mathbb{C}$ and $\upsilon,\,\rho > 0$.

**Definition 9** [15] Let $\zeta \in L^1[0,\,b]$, $0 < t < b < \infty$, and $n = \lceil \rho \rceil$. The regularized PF derivative is defined as

$$^C\mathbb{D}_{\upsilon,\,\rho,\,\varpi,\,0^+}^{\gamma}\zeta(t) = \beth_{\upsilon,\,n-\rho,\,\varpi,\,0^+}^{-\gamma}\frac{d^n}{dt^n}\zeta(t), \tag{10}$$

where $\upsilon,\,\rho,\,\gamma,\,\varpi \in \mathbb{C}$ and $\upsilon,\,\rho > 0$.

**Definition 10** [15, 21] Let $\zeta \in L^1[0,\,b]$, $0 < \rho < 1$, $0 \le v \le 1$, $0 < b < t < \infty$, $\zeta * e_{\upsilon,\,(1-v)(1-\rho),\,\varpi}^{-\gamma(1-v)}(.) \in AC^1[0,\,b]$. The HPF derivative is defined as

$$\mathbb{D}_{\upsilon,\,\varpi,\,0^+}^{\gamma,\,\rho,\,v}\zeta(t) = \left(\beth_{\upsilon,\,v(1-\rho),\,\varpi,\,0^+}^{-\gamma v}\frac{d}{dt}\left(\beth_{\upsilon,\,(1-v)(1-\rho),\,\varpi,\,0^+}^{-\gamma(1-v)}\zeta\right)\right)(t), \tag{11}$$

where $\varpi,\,\gamma \in \mathbb{R}$ and $\upsilon > 0$.

**Definition 11** [21] Let $\zeta \in L^1[0,\,b]$, $0 < \rho < 1$, $0 \le v \le 1$, $0 < b < t < \infty$. The regularized HPF derivative of $\zeta(t)$ is given by

$$^C\mathbb{D}_{\upsilon,\,\varpi,\,0^+}^{\gamma,\,\rho,\,v}\zeta(t) = \left(\beth_{\upsilon,\,v(1-\rho),\,\varpi,\,0^+}^{-\gamma v}\left(\beth_{\upsilon,\,(1-v)(1-\rho),\,\varpi,\,0^+}^{-\gamma(1-v)}\frac{d}{dt}\zeta\right)\right)(t)$$
$$= \beth_{\upsilon,\,1-\rho,\,\varpi,\,0^+}^{-\gamma}\frac{d}{dt}\zeta(t), \tag{12}$$

where $\varpi,\,\gamma \in \mathbb{R}$ and $\upsilon > 0$.

**Definition 12** [19] Let $\mathbb{B}(\dot{r},\,\eta)$ be the FMBI transform of $\zeta(t)$ and is defined as

$$F[\zeta(t)] = \mathbb{B}(\dot{r},\ \eta) = \dot{r} \int_0^\infty \zeta(\eta t) exp(-\dot{r}t) dt$$

$$= \frac{\dot{r}}{\eta} \int_0^\infty exp\left(\frac{-\dot{r}t}{\eta}\right) \zeta(t) dt,\ \ \dot{r} \in (\lambda_1,\ \lambda_2), \tag{13}$$

over the set of functions

$$\mathscr{W} = \left\{ \zeta(t) : \exists\, N > 0,\ \lambda_1 > 0,\ \lambda_2 > 0,\ k > 0 \ \text{ such that } \ |\zeta(t)| \le Ne^{\left(\frac{t}{\lambda_j}\right)} \ \text{if } t \in (-1)^j \times [0,\ \infty) \right\}.$$

This means that the set $\mathscr{W}$ consists of functions $\zeta(t)$ that are bounded by an exponential function, with specific parameters $N$, $\lambda_1$, $\lambda_2$, and $k$, depending on the interval $(-1)^j \times [0,\ \infty)$ for $t$.

The integral transform (13) is defined for all values of $\zeta(t)$ that are greater than $k$.

**Proposition 1** [19] If $\mathbb{F}(\dot{r},\ \eta)$ and $\mathbb{G}(\dot{r},\ \eta)$ are the FMBI transforms of the functions $\zeta(t)$ and $\chi(t)$ respectively, then the Formable transform of their convolution is given as

$$F[\zeta(t) * \chi(t)),\ \dot{r}] = \frac{\eta}{\dot{r}} \mathbb{F}(\dot{r},\ \eta) \mathbb{G}(\dot{r},\ \eta), \tag{14}$$

where

$$\zeta(t) * \chi(t) = \int_0^\infty \zeta(\ell) \chi(t - \ell) d\ell. \tag{15}$$

• **Formable-Sumudu duality** [19] Let $G(\eta)$ be the Sumudu transform of $g(t)$, we have

$$\mathbb{B}(1,\ \eta) = G(\eta). \tag{16}$$

• **Formable-Shehu duality** [19] Let $V(\dot{r},\ \eta)$ be the Shehu transform of $g(t)$, we have

$$\mathbb{B}(\dot{r},\ \eta) = \frac{\dot{r}}{\eta} V(\dot{r},\ \eta). \tag{17}$$

**Theorem 1** [19] Suppose $\mathbb{B}(\dot{r},\ \eta)$ is the FMBI transform of $\zeta(t)$, then the Formable transform of $n^{th}$ derivative $\zeta^{(n)}(t)$ is defined as

$$F[\zeta^{(n)}(t)] = \left(\frac{\dot{r}}{\eta}\right)^n \mathbb{B}(\dot{r},\ \eta) - \sum_{k=0}^{n-1} \left(\frac{\dot{r}}{\eta}\right)^{n-k} \zeta^{(k)}(0),\ \ n \ge 0. \tag{18}$$

**Definition 13** [22] The Shehu transform of the generalized ML function (7) is given by

$$SH\left[t^{\rho-1}E_{\upsilon,\,\rho}^{\gamma}(\varpi t^{\upsilon})\right](\dot{r},\,\eta)=\left(\frac{\eta}{\dot{r}}\right)^{\rho}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{-\gamma},\quad\varpi\in\mathbb{C},\tag{19}$$

where $0<\upsilon<1;\ \rho,\,\gamma>0$.

## 3. Main results

**Lemma 1** The Formable transform of the generalized ML function (7) is defined as

$$F\left[t^{\rho-1}E_{\upsilon,\,\rho}^{\gamma}(\varpi t^{\upsilon})\right](\dot{r},\,\eta)=\left(\frac{\dot{r}}{\eta}\right)^{1-\rho}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{-\gamma},\quad\varpi\in\mathbb{C},\tag{20}$$

where $0<\upsilon<1;\ \rho,\,\gamma>0$.

**Proof.** Using the definition (19) and the duality of Formable-Shahu transform (17), we got the desired result (20). □

**Lemma 2** The FMBI transform of PF integral is defined by

$$F[\beth_{\upsilon,\,\rho,\,\varpi,\,0^{+}}^{\gamma}\zeta(t)]=\left(\frac{\eta}{\dot{r}}\right)^{\rho}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{-\gamma}\mathbb{B}(\dot{r},\,\eta).\tag{21}$$

**Proof.** Applying Formable transform (13) on definition (8), we have

$$F\left[\beth_{\upsilon,\,\rho,\,\varpi,\,0^{+}}^{\gamma}\zeta(t)\right](\dot{r},\,\eta)=F\left[\int_{0}^{t}(t-\ell)^{\rho-1}E_{\upsilon,\,\rho}^{\gamma}[\varpi(t-\ell)^{\upsilon}]\zeta(\ell)d\ell\right](\dot{r},\,\eta),$$

in view of (14) and (20), we get

$$F\left[\beth_{\upsilon,\,\rho,\,\varpi,\,0^{+}}^{\gamma}\zeta(t)\right](\dot{r},\,\eta)=\frac{\eta}{\dot{r}}\times\left(\frac{\eta}{\dot{r}}\right)^{\rho-1}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{-\gamma}\mathbb{B}(\dot{r},\,\eta).$$

We arrive at (21). □

**Theorem 4** The FMBI transform of the PF derivative is defined as

$$F\left[\mathbb{D}_{\upsilon,\,\rho,\,\varpi,\,0^{+}}^{\gamma}\zeta(t)\right]=\left(\frac{\dot{r}}{\eta}\right)^{\rho}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}\mathbb{B}(\dot{r},\,\eta)-\sum_{k=0}^{m-1}\left(\frac{\dot{r}}{\eta}\right)^{n-k}\mathbb{D}_{\upsilon,\,k-n+\rho,\,\varpi,\,0^{+}}^{\gamma}\zeta(0^{+}).\tag{22}$$

**Proof.** Applying the Formable transform (13) to the PF derivative (9), we have

$$F\left[\mathbb{D}_{\upsilon,\,\rho,\,\varpi,\,0^{+}}^{\gamma}\zeta(t)\right](\dot{r},\,\eta)=F\left[\frac{d^{n}}{dt^{n}}g(t)\right](\dot{r},\,\eta),\ \text{where}\ g(t)=\beth_{\upsilon,\,n-\rho,\,\varpi,\,0^{+}}^{-\gamma}\zeta(t),\tag{23}$$

on using (18), we can write (23) as

$$F\left[\mathbb{D}^{\gamma}_{\upsilon,\,\rho,\,\varpi,\,0^+}\zeta(t)\right](\dot{r},\,\eta)$$

$$= \left(\frac{\dot{r}}{\eta}\right)^n F[g(t)](\dot{r},\,\eta) - \sum_{k=0}^{n-1}\left(\frac{\dot{r}}{\eta}\right)^{n-k} g^{(k)}(0),\;\; g^{(k)}(0) = \frac{d^k}{dt^k}\mathtt{J}^{-\gamma}_{\upsilon,\,n-\rho,\,\varpi,\,0^+}\zeta(0). \tag{24}$$

Now, using result (21) in (24), we get

$$F\left[\mathbb{D}^{\gamma}_{\upsilon,\,\rho,\,\varpi,\,0^+}\zeta(t)\right](\dot{r},\,\eta)$$

$$= \left(\frac{\dot{r}}{\eta}\right)^n\left(\frac{\eta}{\dot{r}}\right)^{n-\rho}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma} F[\zeta(t)] - \sum_{k=0}^{n-1}\left(\frac{\dot{r}}{\eta}\right)^{n-k}\frac{d^k}{dt^k}\mathtt{J}^{-\gamma}_{\upsilon,\,n-\rho,\,\varpi,\,0^+}\zeta(t)\big|_{t=0},$$

on simplification, we arrive at (22). □

**Theorem 5** The FMBI of regularised PF derivative is defined as

$$F\left[{}^{C}\mathbb{D}^{\gamma}_{\upsilon,\,\rho,\,\varpi,\,0^+}\zeta(t)\right] = \left(\frac{\dot{r}}{\eta}\right)^{\rho}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}\mathbb{B}(\dot{r},\,\eta) - \sum_{k=0}^{n-1}\left(\frac{\dot{r}}{\eta}\right)^{\rho-k}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}\zeta^{(k)}(0^+). \tag{25}$$

**Proof.** Applying the Formable transform (13) to the regularized PF derivative (10), we have

$$F\left[{}^{C}\mathbb{D}^{\gamma}_{\upsilon,\,\rho,\,\varpi,\,0^+}\zeta(t)\right](\dot{r},\,\eta)F\left[\mathtt{J}^{-\gamma}_{\upsilon,\,n-\rho,\,\varpi,\,0^+}h(t)\right](\dot{r},\,\eta),\;\text{where}\;\;h(t) = \frac{d^n}{dt^n}\zeta(t), \tag{26}$$

using result (21) in (26), we get

$$F\left[{}^{C}\mathbb{D}^{\gamma}_{\upsilon,\,\rho,\,\varpi,\,0^+}\zeta(t)\right](\dot{r},\,\eta) = \left(\frac{\eta}{\dot{r}}\right)^{n-\rho}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma} F[h(t)](\dot{r},\,\eta). \tag{27}$$

in view of (18), we get

$$F\left[{}^{C}\mathbb{D}^{\gamma}_{\upsilon,\,\rho,\,\varpi,\,0^+}\zeta(t)\right](\dot{r},\,\eta) = \left(\frac{\eta}{\dot{r}}\right)^{n-\rho}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}\left[\left(\frac{\eta}{\dot{r}}\right)^{-n} F[\zeta(t)] - \sum_{k=0}^{m-1}\left(\frac{\eta}{\dot{r}}\right)^{k-n}\zeta^{(k)}(0)\right].$$

On simplification, we arrive at (25). □

**Theorem 6** The FMBI transform of the HPF derivative is defined as

$$F\left[\mathbb{D}_{\upsilon,\,\varpi,\,0^+}^{\gamma,\,\rho,\,\nu}\zeta(t)\right] = \left(\frac{\dot{r}}{\eta}\right)^{\rho}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}\mathbb{B}(\dot{r},\,\eta)$$

$$-\left(\frac{\dot{r}}{\eta}\right)^{\nu(\rho-1)+1}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma\nu}\mathbb{J}_{\upsilon,\,(1-\nu)(1-\rho),\,\varpi,\,0^+}^{-\gamma(1-\nu)}\zeta(t)|_{t=0^+}. \tag{28}$$

**Proof.** Applying the Formable transform (13) to the HPF derivative (11), we have

$$F\left[\mathbb{D}_{\upsilon,\,\varpi,\,0^+}^{\gamma,\,\rho,\,\nu}\zeta(t)\right](\dot{r},\,\eta)$$

$$= F\left[\mathbb{J}_{\upsilon,\,\nu(1-\rho),\,\varpi,\,0^+}^{-\gamma\nu}k(t)\right](\dot{r},\,\eta),\ \ \text{where}\ \ k(t) = \frac{d}{dt}\mathbb{J}_{\upsilon,\,(1-\nu)(1-\rho),\,\varpi,\,0^+}^{-\gamma(1-\nu)}\zeta(t), \tag{29}$$

using result (21) and then (18) in (29), we get

$$F\left[\mathbb{D}_{\upsilon,\,\varpi,\,0^+}^{\gamma,\,\rho,\,\nu}\zeta(t)\right](\dot{r},\,\eta)$$

$$= \left(\frac{\eta}{\dot{r}}\right)^{\nu(1-\rho)}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma\nu} \tag{30}$$

$$\times\left[\left(\frac{\eta}{\dot{r}}\right)^{-1}F\left[\mathbb{J}_{\upsilon,\,(1-\nu)(1-\rho),\,\varpi,\,0^+}^{-\gamma(1-\nu)}\zeta(t)\right](\dot{r},\,\eta) - \left(\frac{\eta}{\dot{r}}\right)^{-1}\mathbb{J}_{\upsilon,\,(1-\nu)(1-\rho),\,\varpi,\,0^+}^{-\gamma(1-\nu)}\zeta(0^+)\right],$$

again using (21) in (30), we get

$$F[\mathbb{D}_{\upsilon,\,\varpi,\,0^+}^{\gamma,\,\rho,\,\nu}\zeta(t) = \left(\frac{\eta}{\dot{r}}\right)^{\nu(1-\rho)}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma\nu}$$

$$\times\left[\left(\frac{\eta}{\dot{r}}\right)^{(1-\nu)(1-\rho)-1}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma(1-\nu)}F[\zeta(t)] - \left(\frac{\eta}{\dot{r}}\right)^{-1}\mathbb{J}_{\upsilon,\,(1-\nu)(1-\rho),\,\varpi,\,0^+}^{-\gamma(1-\nu)}\zeta(0^+)\right].$$

On simplification, we arrive at (28). □

**Theorem 7** The FMBI transform of the regularized HPF derivative is defined as

$$F\left[{}^{C}\mathbb{D}_{\upsilon,\,\varpi,\,0^+}^{\gamma,\,\rho,\,\nu}\zeta(t)\right] = \left(\frac{\dot{r}}{\eta}\right)^{\rho}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}\mathbb{B}(\dot{r},\,\eta) - \left(\frac{\dot{r}}{\eta}\right)^{\rho}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}\zeta(0^+). \tag{31}$$

**Proof.** Applying the Formable transform (13) to the definition (12), we have

$$F\left[{}^{C}\mathbb{D}_{\upsilon,\,\varpi,\,0^{+}}^{\gamma,\,\rho,\,\nu}\zeta(t)\right](\dot{r},\,\eta)=F\left[\beth_{\upsilon,\,1-\rho,\,\varpi,\,0^{+}}^{-\gamma}z(t)\right](\dot{r},\,\eta),\;\text{where}\;z(t)=\frac{d}{dt}\zeta(t),\tag{32}$$

using result (21) in (32), we get

$$F\left[{}^{C}\mathbb{D}_{\upsilon,\,\varpi,\,0^{+}}^{\gamma,\,\rho,\,\nu}\zeta(t)\right](\dot{r},\,\eta)=\left(\frac{\eta}{\dot{r}}\right)^{1-\rho}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}F[z(t)](\dot{r},\,\eta),\tag{33}$$

in view of (18), we can write (33) as

$$F\left[{}^{C}\mathbb{D}_{\upsilon,\,\varpi,\,0^{+}}^{\gamma,\,\rho,\,\nu}\zeta(t)\right](\dot{r},\,\eta)=\left(\frac{\eta}{\dot{r}}\right)^{1-\rho}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}\left[\left(\frac{\eta}{\dot{r}}\right)^{-1}F[\zeta(t)]-\left(\frac{\eta}{\dot{r}}\right)^{-1}\zeta(0^{+})\right],$$

on simplification, we got the desired result (31). $\qquad\square$

# 4. Applications

In this section, we will use the FRI and FMBI transformations of HP and regularized HP fractional derivative to find solutions to some Cauchy-type fractional differential equations.

**Theorem 8** The solution of the generalized Cauchy-type problem for the fractional advection-dispersion equation

$$\mathbb{D}_{\upsilon,\,\varpi,\,0^{+}}^{\gamma,\,\rho,\,\nu}\zeta(x,\,t)=-w\mathbb{D}_{x}\zeta(x,\,t)+\vartheta\,\Delta^{\frac{\curlywedge}{2}}\zeta(x,\,t),\tag{34}$$

subjected to

$$\beth_{\upsilon,\,(1-\nu)(1-\rho),\,\varpi,\,0^{+}}^{-\gamma(1-\nu)}\zeta(x,\,0^{+})=g(x)\tag{35}$$

$$\lim_{|x|\to\infty}\zeta(x,\,t)=0,\;\;t>0,$$

is given by

$$\zeta(x,\,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}e^{(-ikx)}g(k)(iwk-\vartheta|k|^{\curlywedge})^{n}t^{\nu(1-\rho)+n\rho+\rho-1}E_{\upsilon,\,\nu(1-\rho)+\rho(n+1)}^{\gamma(1+n)-\gamma\nu}(\varpi t^{\upsilon})dk,\tag{36}$$

where $\Delta^{\frac{\curlywedge}{2}}$ is the fractional generalized Laplace operator of order $\curlywedge$, with $\curlywedge\in(0,\,2]$; $0<\rho<1$, $0\le\nu\le1$; $x,\,\varpi,\,\gamma\in\mathbb{R}$; $t,\,\upsilon>0$, $\gamma\ge0$; $w$ and $\vartheta$ represent the fluid velocity and dispersion coefficient, respectively; and $\mathbb{D}_{x}$ is the partial derivative of $\zeta$ with respect to $x$. The Fourier transform of $\Delta^{\frac{\curlywedge}{2}}$ is $-|k|^{\curlywedge}$, as discussed in [23].

**Proof.** Applying the FRI transform (5) on (34), we have

$$\mathbb{D}_{\upsilon, \varpi, 0^+}^{\gamma, \rho, \nu} \zeta^*(x, t) = iwk\zeta^*(k, t) - \vartheta|k|^\wedge \zeta^*(k, t),\tag{37}$$

where $\zeta^*(k, t)$ is the Fourier transform of $\zeta(x, t)$ with respect to variable $x$. Now, applying the FMBI transform (13) on (37) and then using result (28) and (35), we have

$$\left(\frac{\eta}{\dot{r}}\right)^{-\rho} \left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^\upsilon\right)^\gamma \overline{\zeta}^*(k, \dot{r}, \eta) - \left(\frac{\eta}{\dot{r}}\right)^{\nu(1-\rho)-1} \left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^\upsilon\right)^{\gamma\nu} g^*(k)$$

$$= iwk\overline{\zeta}^*(k, \dot{r}, \eta) - \vartheta|k|^\wedge \overline{\zeta}^*(k, \dot{r}, \eta),$$

where $\overline{\zeta}^*(k, \dot{r}, \eta)$ is the FMBI transform of $\zeta^*(k, t)$ with respect to variable $t$. Consider

$$\overline{\zeta}^*(k, \dot{r}, \eta)\left[\left(\frac{\eta}{\dot{r}}\right)^{-\rho} \left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^\upsilon\right)^\gamma + \vartheta|k|^\wedge - iwk\right] = \left(\frac{\eta}{\dot{r}}\right)^{\nu(1-\rho)-1} \left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^\upsilon\right)^{\gamma\nu} g^*(k).$$

On simplification,

$$\overline{\zeta}^*(k, \dot{r}, \eta) = \frac{\left(\frac{\eta}{\dot{r}}\right)^{\nu(1-\rho)-1} \left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^\upsilon\right)^{\gamma\nu} g^*(k)}{\left(\frac{\eta}{\dot{r}}\right)^{-\rho} \left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^\upsilon\right)^\gamma \left[1 + \frac{\vartheta|k|^\wedge - iwk}{\left(\frac{\eta}{\dot{r}}\right)^{-\rho} \left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^\upsilon\right)^\gamma}\right]}, \quad \text{if } \frac{\vartheta|k|^\wedge - iwk}{\left(\frac{\eta}{\dot{r}}\right)^{-\rho} \left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^\upsilon\right)^\gamma} < 1$$

$$\overline{\zeta}^*(k, \dot{r}, \eta) = \left(\frac{\eta}{\dot{r}}\right)^{\nu(1-\rho)+\rho-1} \left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^\upsilon\right)^{\gamma\nu-\gamma} g^*(k) \sum_{n=0}^{\infty} \left[\frac{-\vartheta|k|^\wedge + iwk}{\left(\frac{\eta}{\dot{r}}\right)^{-\rho} \left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^\upsilon\right)^\gamma}\right]^n$$

$$\overline{\zeta}^*(k, \dot{r}, \eta) = \sum_{n=0}^{\infty} (iwk - \vartheta|k|^\wedge)^n \left(\frac{\eta}{\dot{r}}\right)^{\nu(1-\rho)+\rho+\rho n-1} \left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^\upsilon\right)^{\gamma\nu-\gamma n-\gamma} g^*(k).$$

Now, we can arrive at (36) by using the inverse form of result (20) and (5). $\qquad\square$

**Remark 1** If we take $w = 0$ and $\vartheta = \dfrac{ih}{2m}$ in equation (34), the result will reach to one-dimensional space-time Schrodinger fractional equation for mass $m$ and plank constant $h$ with solution

$$\zeta(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(-ikx)} g(k) \sum_{n=0}^{\infty} \left(-\frac{ih}{2m}|k|^\wedge\right) t^{\nu(1-\rho)+n\rho+\rho-1} E_{\upsilon, \nu(1-\rho)+\rho(n+1)}^{\gamma(1+n)-\gamma\nu}(\varpi t^\upsilon) dk.\tag{38}$$

**Theorem 9** The solution of the generalized Cauchy-type problem for the fractional advection-dispersion equation

$$^C\mathbb{D}_{\upsilon,\,\varpi,\,0^+}^{\gamma,\,\rho,\,\nu}\zeta(x,\,t) = -w\mathbb{D}_x\zeta(x,\,t) + \vartheta\,\Delta^{\frac{\lambda}{2}}\zeta(x,\,t), \tag{39}$$

subjected to

$$\zeta(x,\,0^+) = g(x),\ \ x \in \mathbb{R}$$

$$\tag{40}$$

$$\lim_{|x|\to\infty}\zeta(x,\,t) = 0,\ \ t > 0,$$

is given by

$$\zeta(x,\,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{(-ikx)}g(k)\sum_{n=0}^{\infty}(iwk - \vartheta|k|^{\lambda})t^{\rho n}E_{\upsilon,\,\rho n+1}^{\gamma n}(\varpi t^{\upsilon})dk, \tag{41}$$

where $0 < \rho < 1$, $0 \le \nu \le 1$; $x,\ \varpi,\ \gamma \in \mathbb{R}$; $t,\ \upsilon > 0$; $\gamma \ge 0$. FRI transform of $\Delta^{\frac{\lambda}{2}}$ is $-|k|^{\lambda}$ and discussed in [23].

**Proof.** Applying the FRI transforms (5) on (39), we have

$$^C\mathbb{D}_{\upsilon,\,\varpi,\,0^+}^{\gamma,\,\rho,\,\nu}\zeta^*(k,\,t) = iwk\zeta^*(k,\,t) - \vartheta|k|^{\lambda}\zeta^*(k,\,t), \tag{42}$$

where $\zeta^*(k,\,t)$ is the FRI transform of $\zeta(x,\,t)$ with respect to variable $x$. Now, applying the FMBI transform (13) on (42) and then using result (31) and (40), we have

$$\left(\frac{\eta}{\dot{r}}\right)^{-\rho}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}\overline{\zeta}^*(k,\,\dot{r},\,\eta) - \left(\frac{\eta}{\dot{r}}\right)^{-\rho}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}\zeta^*(k,\,0)$$

$$= iwk\overline{\zeta}^*(k,\,\dot{r},\,\eta) - \vartheta|k|^{\lambda}\overline{\zeta}^*(k,\,\dot{r},\,\eta),$$

where $\overline{\zeta}^*(k,\,\dot{r},\,\eta)$ is the FMBI transform of $\zeta^*(k,\,t)$ with respect to variable $t$. Consider

$$\overline{\zeta}^*(k,\,\dot{r},\,\eta)\left[\left(\frac{\eta}{\dot{r}}\right)^{-\rho}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma} + \vartheta|k|^{\lambda} - iwk\right] = \left(\frac{\eta}{\dot{r}}\right)^{-\rho}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}g^*(k),$$

on simplification,

$$\overline{\zeta}^*(k, \dot{r}, \eta) = \frac{\left(\frac{\eta}{\dot{r}}\right)^{-\rho}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma} g^*(k)}{\left(\frac{\eta}{\dot{r}}\right)^{-\rho}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}\left[1 + \frac{\vartheta|k|^{\wedge} - iwk}{\left(\frac{\eta}{\dot{r}}\right)^{-\rho}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}}\right]}, \quad \text{if} \quad \left[\frac{\vartheta|k|^{\wedge} - iwk}{\left(\frac{\eta}{\dot{r}}\right)^{-\rho}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}}\right] < 1$$

$$\overline{\zeta}^*(k, \dot{r}, \eta) = \frac{\left(\frac{\eta}{\dot{r}}\right)^{-\rho}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma} g^*(k)}{\left(\frac{\eta}{\dot{r}}\right)^{-\rho}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}}\left[1 + \frac{\vartheta|k|^{\wedge} - iwk}{\left(\frac{\eta}{\dot{r}}\right)^{-\rho}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}}\right]^{-1}$$

$$\overline{\zeta}^*(k, \dot{r}, \eta) = g^*(k)\sum_{n=0}^{\infty}(iwk - \vartheta|k|^{\wedge})^n\left(\frac{\eta}{\dot{r}}\right)^{\rho n}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{-\gamma n}.$$

Now, we can arrive at (41) by using the inverse form of result (20) and (5). □

**Theorem 10** The solution of the generalized Cauchy-type problem for the fractional differential equation

$$\mathbb{D}^{\gamma, \rho, \nu}_{\upsilon, \varpi, 0^+}\zeta(x, t) = M\frac{\partial^2}{\partial x^2}\zeta(x, t), \tag{43}$$

subjected to

$$\beth^{-\gamma(1-\nu)}_{\upsilon, (1-\nu)(1-\rho), \varpi, 0^+}\zeta(x, t)|_{t=0} = g(x), \tag{44}$$

$$\lim_{|x|\to\infty}\zeta(x, t) = 0,$$

is given by

$$\zeta(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ikx}g(k)\sum_{n=0}^{\infty}(-Mk^2)^n t^{\rho(n+1)-\nu(\rho-1)-1}E^{\gamma(n+1-\nu)}_{\upsilon, \rho(n+1)+\nu(1-\rho)}(\varpi t^{\upsilon})dk. \tag{45}$$

where $0 < \rho < 1$, $0 \le \nu \le 1$; $\varpi, \gamma, x \in \mathbb{R}$; $M, t, \upsilon > 0$; $\gamma \ge 0$.

**Proof.** Applying the FRI transform (5) on (43), we have

$$\mathbb{D}^{\gamma, \rho, \nu}_{\upsilon, \varpi, 0^+}\zeta^*(x, t) = -Mk^2\zeta^*(k, t) \tag{46}$$

where $\zeta^*(k, t)$ is the FRI transform of $\zeta(x, t)$ with respect to variable $x$. Now, applying the FMBI transform (13) on (46) and then using result (28) and (44), we have

$$\left(\frac{\eta}{\dot{r}}\right)^{-\rho}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}\overline{\zeta}^{*}(k,\,\dot{r},\,\eta)-\left(\frac{\eta}{\dot{r}}\right)^{\upsilon(1-\rho)-1}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma\upsilon}g^{*}(k)=-Mk^{2}\overline{\zeta}^{*}(k,\,\dot{r},\,\eta),$$

where $\overline{\zeta}^{*}(k,\,\dot{r},\,\eta)$ is the FMBI transform of $\zeta^{*}(k,\,t)$ with respect to variable $t$. Consider

$$\overline{\zeta}^{*}(k,\,\dot{r},\,\eta)\left[\left(\frac{\eta}{\dot{r}}\right)^{-\rho}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}+Mk^{2}\right]=\left(\frac{\eta}{\dot{r}}\right)^{\upsilon(1-\rho)-1}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma\upsilon}g^{*}(k),$$

on simplification,

$$\overline{\zeta}^{*}(k,\,\dot{r},\,\eta)=\frac{\left(\frac{\eta}{\dot{r}}\right)^{\upsilon(1-\rho)-1}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma\upsilon}g^{*}(k)}{\left(\frac{\eta}{\dot{r}}\right)^{-\rho}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}+Mk^{2}}$$

$$\overline{\zeta}^{*}(k,\,\dot{r},\,\eta)=\frac{\left(\frac{\eta}{\dot{r}}\right)^{\upsilon(1-\rho)-1}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma\upsilon}g^{*}(k)}{\left(\frac{\eta}{\dot{r}}\right)^{-\rho}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}\left[1+\dfrac{Mk^{2}}{\left(\frac{\eta}{\dot{r}}\right)^{-\rho}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}}\right]},\quad \text{if}\ \left(\dfrac{Mk^{2}}{\left(\frac{\eta}{\dot{r}}\right)^{-\rho}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}}\right)<1$$

$$\overline{\zeta}^{*}(k,\,\dot{r},\,\eta)=\left(\frac{\eta}{\dot{r}}\right)^{\upsilon(1-\rho)+\rho-1}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma\upsilon-\gamma}g^{*}(k)\sum_{n=0}^{\infty}\left[\dfrac{-Mk^{2}}{\left(\frac{\eta}{\dot{r}}\right)^{-\rho}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}}\right]^{n}$$

$$\overline{\zeta}^{*}(k,\,\dot{r},\,\eta)=\left(\frac{\eta}{\dot{r}}\right)^{\upsilon(1-\rho)+\rho-1}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma\upsilon-\gamma}g^{*}(k)\sum_{n=0}^{\infty}(-Mk^{2})^{n}\left(\frac{\eta}{\dot{r}}\right)^{\rho n}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{-\gamma n}$$

$$\overline{\zeta}^{*}(k,\,\dot{r},\,\eta)=g^{*}(k)\sum_{n=0}^{\infty}(-Mk^{2})^{n}\left(\frac{\eta}{\dot{r}}\right)^{\rho n+\upsilon(1-\rho)+\rho-1}\left(1-\varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma\upsilon-\gamma n-\gamma}.$$

Now, we can arrive at (45) by using the inverse form of both (20) and (5). □

**Theorem 11** The solution of the generalized Cauchy-type fractional differential equation

$$\mathbb{D}_{\upsilon,\,\varpi,\,0^{+}}^{\gamma,\,\rho,\,\upsilon}\zeta(t)=\lambda\mathbb{J}_{\upsilon,\,\rho,\,\varpi,\,0^{+}}^{\delta}\zeta(t)+y(t),\qquad(47)$$

subjected to

$$\left( \beth^{-\gamma(1-\nu)}_{\upsilon,\ (1-\nu)(1-\rho),\ \varpi,\ 0^+} \zeta(t) \right) |_{t=0} = M \tag{48}$$

is given by

$$\zeta(t) = \sum_{n=0}^{\infty} \curlywedge^n \beth^{\gamma+n(\delta+\gamma)}_{\upsilon,\ \rho(2n+1),\ \varpi,\ 0^+} y(t) + M \sum_{n=0}^{\infty} \curlywedge^n t^{\rho(2n+1)+\nu(1-\rho)-1}$$

$$\times E^{\delta n+\gamma n+\gamma-\gamma\nu}_{\upsilon,\ \nu(1-\rho)+\rho(2n+1)}(\varpi t^\upsilon). \tag{49}$$

where $y(t) \in L^1[0,\ \infty);\ 0 < \rho < 1,\ 0 \le \nu \le 1;\ \varpi,\ \gamma,\ \curlywedge \in \mathbb{R};\ t,\ M,\ \upsilon > 0;\ \gamma,\ \delta \ge 0$.

**Proof.** Let $\mathbb{B}(\dot{r},\ \eta)$ be the FMBI transform of $\zeta(t)$, applying the FMBI transform (13) on both side of (47) and then using (28) and (48), we have

$$\left(\frac{\eta}{\dot{r}}\right)^{-\rho} \left(1 - \varpi \left(\frac{\eta}{\dot{r}}\right)^\upsilon\right)^\gamma \mathbb{B}(\dot{r},\ \eta) - \left(\frac{\eta}{\dot{r}}\right)^{\nu(1-\rho)-1} \left(1 - \varpi \left(\frac{\eta}{\dot{r}}\right)^\upsilon\right)^{\gamma\nu} \beth^{-\gamma(1-\nu)}_{\upsilon,\ (1-\nu)(1-\rho),\ \varpi,\ 0^+} \zeta(t)|_{t=0}$$

$$= \curlywedge \left(\frac{\eta}{\dot{r}}\right) \mathbb{B}(\dot{r},\ \eta) \left(\frac{\eta}{\dot{r}}\right)^{\rho-1} \left(1 - \varpi \left(\frac{\eta}{\dot{r}}\right)^\upsilon\right)^{-\delta} + \mathcal{F}[y(t)](\dot{r},\ \eta)$$

$$\left(\frac{\eta}{\dot{r}}\right)^{-\rho} \left(1 - \varpi \left(\frac{\eta}{\dot{r}}\right)^\upsilon\right)^\gamma \mathbb{B}(\dot{r},\ \eta) - \left(\frac{\eta}{\dot{r}}\right)^{\nu(1-\rho)-1} \left(1 - \varpi \left(\frac{\eta}{\dot{r}}\right)^\upsilon\right)^{\gamma\nu} M$$

$$= \curlywedge \left(\frac{\eta}{\dot{r}}\right)^\rho \left(1 - \varpi \left(\frac{\eta}{\dot{r}}\right)^\upsilon\right)^{-\delta} \mathbb{B}(\dot{r},\ \eta) + \mathcal{F}[y(t)](\dot{r},\ \eta),$$

on simplification,

$$\mathbb{B}(\dot{r}, \eta) = \frac{F[y(t)](\dot{r}, \eta) + \left(\frac{\eta}{\dot{r}}\right)^{\nu(1-\rho)-1}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma\nu} M}{\left(\frac{\eta}{\dot{r}}\right)^{-\rho}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}\left[1 - \frac{\lambda\left(\frac{\eta}{\dot{r}}\right)^{\rho}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{-\delta}}{\left(\frac{\eta}{\dot{r}}\right)^{-\rho}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}}\right]}, \quad \text{if} \quad \left[\frac{\lambda\left(\frac{\eta}{\dot{r}}\right)^{\rho}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{-\delta}}{\left(\frac{\eta}{\dot{r}}\right)^{-\rho}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}}\right] < 1$$

$$= \frac{F[y(t)](\dot{r}, \eta) + \left(\frac{\eta}{\dot{r}}\right)^{\nu(1-\rho)-1}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma\nu} M}{\left(\frac{\eta}{\dot{r}}\right)^{-\rho}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}} \sum_{n=0}^{\infty}\left[\frac{\lambda\left(\frac{\eta}{\dot{r}}\right)^{\rho}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{-\delta}}{\left(\frac{\eta}{\dot{r}}\right)^{-\rho}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}}\right]^{n}$$

$$= \frac{F[y(t)](\dot{r}, \eta) + \left(\frac{\eta}{\dot{r}}\right)^{\nu(1-\rho)-1}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma\nu} M}{\left(\frac{\eta}{\dot{r}}\right)^{-\rho}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma}} \sum_{n=0}^{\infty}\lambda^{n}\left(\frac{\eta}{\dot{r}}\right)^{2\rho n}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{-\delta n - \gamma n}$$

$$= \left(F[y(t)](\dot{r}, \eta) + \left(\frac{\eta}{\dot{r}}\right)^{\nu(1-\rho)-1}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{\gamma\nu} M\right)$$

$$\times \sum_{n=0}^{\infty}\lambda^{n}\left(\frac{\eta}{\dot{r}}\right)^{2\rho n + \rho}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{-\delta n - \gamma n - \gamma}$$

$$= F[y(t)](\dot{r}, \eta) \sum_{n=0}^{\infty}\lambda^{n}\left(\frac{\eta}{\dot{r}}\right)^{2\rho n + \rho}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{-\delta n - \gamma n - \gamma}$$

$$+ M \sum_{n=0}^{\infty}\lambda^{n}\left(\frac{\eta}{\dot{r}}\right)^{2\rho n + \rho + \nu(1-\rho)-1}\left(1 - \varpi\left(\frac{\eta}{\dot{r}}\right)^{\upsilon}\right)^{-\delta n - \gamma n - \gamma + \gamma\nu}.$$

Now, we can arrive at (49) by using the inverse form of both (20) and (21). □

## 5. Discussion

This work mainly deals with Hilfer-Prabhakar fractional derivatives of fractional differential equations of the Cauchy type, demonstrating the effectiveness of Formable and Fourier transformations. The strategies offered, however, are adaptable and may be used with many kinds of fractional differential equations. Our method may be used, for example, to fractional integro-differential equations, systems of fractional differential equations, time- and space-fractional diffusion equations, and Caputo and Riemann-Liouville derivatives. These expansions demonstrate how widely applicable our techniques are across several scientific and technical domains. These applications will be thoroughly investigated in future study.

# 6. Conclusion

This article analyses the Prabhakar fractional derivative, the Hilfer-Prabhakar fractional derivative, and their regularised forms using the Formable integral transform. Numerous Cauchy-type fractional differential equations involving Hilfer-Prabhakar fractional derivatives are solved in this paper using the Formable and Fourier transformations and the Mittag-Leffler function with three parameters. These results contribute to the expanding field of fractional calculus and aid in the solution of complex fractional differential equations in a variety of scientific and technical domains. By presenting derived transformations and solutions, this work provides the groundwork for further investigation and application of fractional calculus tools in various subject areas.

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## Conflict of interest

The authors declare no competing financial interest.

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