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Research Article

Exploring the Elzaki Transform: Unveiling Solutions to Reaction-Diffusion Equations with Generalized Composite Fractional Derivatives

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Abstract: This article investigates the use of the Elzaki transform on a generalized composite fractional derivative. To establish the framework for this inquiry, numerous essential lemmas about the Elzaki transform are presented. We successfully extract the solution to the reaction-diffusion problem using both the Elzaki and Fourier transforms, which include a generalized composite fractional derivative. We also look at special examples of the generalized equation, which helps us understand its applications and consequences better. The results show that the Elzaki transform is successful in dealing with complicated fractional differential equations, introducing new analytical approaches and solutions to the subject of fractional calculus and its applications in reaction-diffusion systems.

Keywords: elzaki transform, generalised composite fractional derivative, reaction-diffusion equation, fourier transform, fractional calculus applications

MSC: 34A08, 26A33, 35A22

1. Introduction

Fractional calculus provides a more flexible mathematical foundation and focuses on derivatives and integrals of non-integer order. Adding fractional derivatives to Reaction-diffusion equations (RDE) provides more flexibility when simulating complex systems with anomalous diffusion and memory effects. The fractional RDE has numerous applications, ranging from biology, and chemistry to physics and economics. They provide a valuable tool for modeling anomalous diffusion, subdiffusion, superdiffusion, and long-range interactions, as opposed to standard integer-order RDE [1–3]. Fractional derivatives are non-local and non-linear, making fractional RDE challenging to solve. A variety of numerical techniques, including finite difference, finite element, and spectral methods, can be used to approximate the solutions [4, 5]. In 1996, Grindrod, P. offers the conventional RDE [5].

$$\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial x^2} + \upsilon \cdot X(P), \tag{1}$$

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where $X(P)$ is a nonlinear function representing the kinetics of a reaction and D is the diffusion coefficient.

Manne et al. [6] presented a generalization of the equation (1) as

$$\frac{\partial^2 P}{\partial t^2} + \vartheta \frac{\partial P}{\partial t} = \vartheta^2 \frac{\partial^2 P}{\partial x^2} + \lambda^2 P(x, t). \tag{2}$$

Saxena et al. [7] further generalized the RDE by illustrating with fractional derivatives

$$_0\mathbb{D}_t^\nu P(x, t) + \vartheta \cdot {_0\mathbb{D}_t^\mu} P(x, t) = \vartheta^2_{-\infty}\mathbb{D}_x^\gamma P(x, t) + \lambda^2 P(x, t) + \Theta(x, t). \tag{3}$$

which describes the diffusion transport of the quantity $P(x, t)$ in space, λ indicates the strength of the nonlinearity of the system, ϑ^2 is the diffusive constant, Θ is a constant that describes the nonlinearity in the system, $\Theta(x, t)$ is a nonlinear function for reaction kinetics. The terms $_0\mathbb{D}_t^\nu$ and $_0\mathbb{D}_t^\mu$ represent the fractional time derivatives of orders ν and μ, respectively, which reflect the memory effect and non-local features of the process. The system is nonlinear if $\nu > \mu$ and both $\Theta(x, t)$ and ϑ are nonzero. Several writers have recently researched RDEs with fractional derivatives [8–13].

Simulations of complex phenomena continue to demonstrate the utility of fractional derivatives. As a result, fractional derivatives continue to make consistent progress. This fractional derivative, as defined by Hilfer [14], incorporates the characteristics of the Riemann-Liouville fractional derivative and the Caputo fractional derivative of the same order [15]. To obtain a closed-form solution to a generalized fractional FLE equation, Garg et al. [16] defined the composition of these derivatives, allowing for different fractional orders of Riemann-Liouville and Caputo fractional derivatives. In addition, using Fourier and Sumudu transforms, Alha et al. [17] found the solution to the nonlinear RDE with a generalized composite fractional (GCF) derivative. Numerous scientific and technological disciplines employ integral transforms such as Laplace, Mellin, Fourier, Sumudu, Hankel, Elzaki, etc. [18–20]. Tarig Elzaki [21] created the Elzaki Transform as an alternative to the traditional Fourier integral for solving ordinary and partial differential equations in the time domain. This work discusses the Elzaki transform of the GCF derivative and its application to solving the reaction-diffusion equation (RDE). The uniqueness comes in extending the Elzaki transform to handle complicated GCF derivatives, which is a difficult undertaking owing to their convoluted structure. This study introduces new analytical techniques and advances the practical applications of fractional calculus in a variety of scientific domains.

2. Definition and preliminaries

Definition 1 [15] The Riemann Liouville fractional (RLF) integral of order $\nu > 0$ of a function $\Theta(t)$ is

$$_0\mathbb{I}_t^\nu \Theta(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-u)^{\nu-1} \Theta(u) du, \quad \nu \in \mathbb{C} \text{ and } t > 0. \tag{4}$$

Definition 2 [15] The RLF derivative of order $\nu > 0$ of a function $\Theta(t)$ is

$$_0\mathbb{D}_t^\nu \Theta(t) = \frac{1}{\Gamma(\kappa-\nu)} \frac{d^\kappa}{dt^\kappa} \int_0^t (t-u)^{\kappa-\nu-1} \Theta(u) du, \quad \kappa-1 < \nu < \kappa, \quad \kappa \in \mathbb{N}. \tag{5}$$

Definition 3 [15] Caputo fractional (CF) derivative of order $\nu > 0$ of a function $\Theta(t)$ is

$$ {}_0^C\mathbb{D}_t^{\nu}\Theta(t) = \frac{1}{\Gamma(\kappa-\nu)}\int_0^t (t-u)^{\kappa-\nu-1}\Theta^{(\kappa)}(u)du, \;\; \kappa-1<\nu<\kappa, \quad \kappa\in\mathbb{N}. \tag{6} $$

Definition 4 [7, 22] Weyl fractional (WF) derivative of order $\nu > 0$ is defined as

$$ {}_{-\infty}\mathbb{D}_t^{\nu}\Theta(t) = \frac{1}{\Gamma(\kappa-\nu)}\frac{d^{\kappa}}{dt^{\kappa}}\int_{-\infty}^t (t-u)^{\kappa-\nu-1}\Theta(u)du, \quad \kappa-1<\nu<\kappa, \; \kappa\in\mathbb{N}. \tag{7} $$

The modified Fourier integral transform (FIT) of the operator (7) given by Metzler and Klafter [23] is

$$ F\{{}_{-\infty}\mathbb{D}_t^{\nu}\Theta(t)\} = -\kappa^{\nu}\Theta^*(\kappa), \tag{8} $$

where the FIT is defined by means of the integral equation

$$ \Theta^*(\kappa) = \int_{-\infty}^{\infty}\Theta(\kappa)exp(i\kappa y). \tag{9} $$

Definition 5 [14] Hilfer fractional (HF) derivative of order ν and μ of a function $\Theta(t)$ is defined as

$$ {}_0\mathbb{D}_t^{\nu,\,\mu}\Theta(t) = \left({}_0\mathbb{I}_t^{\mu(1-\nu)}\frac{d}{dt}({}_0\mathbb{I}_t^{(1-\nu)(1-\mu)}\Theta(t))\right), \tag{10} $$

where $0<\nu\leq 1$, and $0\leq\mu\leq 1$.

Definition 6 [16] The GCF derivative of a function $\Theta(t)$ is defined as

$$ {}_0\mathbb{D}_t^{\nu,\,\mu;\,\alpha}\Theta(t) = \left({}_0\mathbb{I}_t^{\alpha(\kappa-\mu)}\frac{d^{\kappa}}{dt^{\kappa}}({}_0\mathbb{I}_t^{(1-\alpha)(\kappa-\nu)}\Theta(t))\right), \tag{11} $$

where $\kappa-1<\nu,\; \mu\leq\kappa;\; 0\leq\alpha\leq 1$ and $\kappa\in N$.

For $\alpha=0$ and $\alpha=1$, the GCF derivative (11) simplifies to the RLF derivative of order ν (5) and the CF derivative of order μ (6). For $\nu=\mu$, the GCF derivative is same as the ν-order, α-type derivative of HF derivative (10).

Definition 7 [24] A generalization of the Mittag-Leffler function proposed by Prabhakar is

$$ E_{\nu,\,\mu}^{\gamma}(t) = \sum_{j=0}^{\infty}\frac{(\gamma)_j}{\Gamma(\nu j+\mu)}\cdot\frac{t^j}{j!}, \tag{12} $$

where $\nu,\; \mu$ and γ are the complex numbers with $Re(\nu),\; Re(\mu)>0$.

Definition 8 [21] The Elzaki integral transform (EIT) denoted by $T(s)$ for the function $\Theta(t)$ is expressed as follows:

$$ E[\Theta(t),\; s] = s^2\int_0^{\infty}exp(-t)\Theta(st)dt = s\int_0^{\infty}exp(-\frac{t}{s})\Theta(t)dt, \quad s\in(\ell_1,\; \ell_2), $$

for the set of functions

$$\mathscr{A} = \left\{ \Theta(t) \mid \exists\, M,\ \ell_1,\ \ell_2 > 0,\ |\Theta(t)| < M\, exp\left(\frac{|t|}{\ell_j}\right),\ if\ t \in (-1)^j \times [0,\ \infty) \right\}.$$

In the upcoming sections, it will help to know about some of the EIT's most important features

Proposition 1 [25] If $M(s)$ and $N(s)$ are the EIT's for $\Theta(t)$ and $\varphi(t)$ respectively, then their convolution is defined as

$$E[\Theta(t) * \varphi(t)),\ s] = \frac{1}{s} M(s)N(s), \tag{13}$$

or equivalently

$$E^{-1}\left[\frac{1}{s} M(s)N(s),\ t\right] = (\Theta(t) * \varphi(t)), \tag{14}$$

where

$$(\Theta(t) * \varphi(t)) = \frac{1}{s}\int_0^t \Theta(u)\varphi(t-u)du.$$

Lemma 1 [26] The EIT of RLF integral of order ν is defined by

$$E[{}_0\mathbb{I}_t^{\nu}\Theta(t),\ s] = s^{\nu}T(s), \quad \mathbb{R}(\nu) > 0. \tag{15}$$

Lemma 2 [27] The EIT of m^{th} derivative $\Theta^{(m)}(t)$ is defined by

$$E[\Theta^m(t),\ s] = s^{-m}T(s) - \sum_{j=0}^{m-1} s^{j-m+2}\Theta^{(j)}(0), \quad m \geq 0. \tag{16}$$

Lemma 3 [28] The EIT of the RLF and CF derivatives of order ν are defined as

$$E[{}_0\mathbb{D}_t^{\nu}\Theta(t),\ s] = s^{-\nu}T(s) - \sum_{j=0}^{\kappa-1} s^{-(j-2)}[\mathbb{D}^{\nu-j}\Theta(t)|_{t=0}]$$

and

$$E[{}_0^C\mathbb{D}_t^{\nu}\Theta(t),\ s] = s^{-\nu}T(s) - \sum_{j=0}^{\kappa-1} s^{2-\nu+j}[\mathbb{D}^{\kappa}\Theta(t)|_{t=0}],$$

where $\kappa-1 \leq \nu < \kappa$, $\kappa \in \mathbb{N}$.

Lemma 4 [29] In the complex plane $\mathbb{C}$, the following equality holds for the inverse of EIT:

$$E^{-1}\left[u^{\gamma+1}(1-\rho u^{\mu})^{-\delta}\right] = t^{\gamma-1}E^{\delta}_{\mu,\,\gamma}(\rho u^{\mu}), \tag{17}$$

where μ, γ, δ and $\rho \in \mathbb{C}$ with $Re(\mu) > 0$, $Re(\gamma) > 0$.

3. Main results

Lemma 5 Let $0 \leq \alpha \leq 1$, $\nu-\alpha(\nu-\mu) > \omega-\alpha(\omega-\delta)$ and $\kappa \in \mathbb{N}$, therefore, the succeeding equivalence holds true

$$\begin{aligned} & E^{-1}\left[\frac{1}{s^{-1}(s^{\alpha(\nu-\mu)-\nu}+\vartheta\cdot s^{\alpha(\omega-\delta)-\omega}+\eta)}\right] \\ &= \sum_{r=0}^{\infty}(-\eta)^r t^{\{\nu-\alpha(\nu-\mu)\}(r+1)-1}\times E^{(r+1)}_{\nu-\alpha(\nu-\mu)-\omega+\alpha(\omega-\delta),\,\{\nu-\alpha(\nu-\mu)\}(r+1)}\left(-\vartheta\cdot t^{\nu-\alpha(\nu-\mu)-\omega+(\omega-\delta)}\right), \end{aligned} \tag{18}$$

where ν, $\mu > \kappa-1$; $\kappa > \omega$, δ.

Proof. For LHS of (18), we consider

$$\begin{aligned} & \frac{1}{s^{-1}(s^{\alpha(\nu-\mu)-\nu}+\vartheta\cdot s^{\alpha(\omega-\delta)-\omega}+\eta)} \\ &= \left[\frac{1}{s^{-1}(s^{\alpha(\nu-\mu)-\nu}+\vartheta\cdot s^{\alpha(\omega-\delta)-\omega})}\right]\times\left[1+\frac{\eta}{s^{\alpha(\nu-\mu)-\nu}+\vartheta\cdot s^{\alpha(\omega-\delta)-\omega}}\right]^{-1} \\ &= \left[\frac{s}{s^{\alpha(\nu-\mu)-\nu}+\vartheta\cdot s^{\alpha(\omega-\delta)-\omega}}\right]\times\sum_{r=0}^{\infty}\frac{(-\eta)^r}{\left[s^{\alpha(\nu-\mu)-\nu}+\vartheta\cdot s^{\alpha(\omega-\delta)-\omega}\right]^r} \\ &= \sum_{r=0}^{\infty}(-\eta)^r\cdot s^{\{\nu-\alpha(\nu-\mu)\}(r+1)+1}\times\left[1+\vartheta\cdot s^{\nu-\alpha(\nu-\mu)-\omega+\alpha(\omega-\delta)}\right]^{-(r+1)}. \end{aligned} \tag{19}$$

Now, taking the inverse of EIT on (19) and using (17), we arrive at (18). □

Lemma 6 Let $0 \leq \alpha \leq 1$, $\nu-\alpha(\nu-\mu) > \omega-\alpha(\omega-\delta)$ and $\kappa \in \mathbb{N}$, therefore, the succeeding equivalence holds true

$$E^{-1}\left[\frac{s^{\alpha(\kappa-\mu)-\kappa}}{(s^{\alpha(\nu-\mu)-\nu}+\vartheta\cdot s^{\alpha(\omega-\delta)-\omega}+\eta)}\right]$$

$$=\sum_{r=0}^{\infty}(-\eta)^r t^{\alpha(\kappa+r\mu)+\nu(r+1)(1-\alpha)-\kappa-2}\times E^{(r+1)}_{\nu-\alpha(\nu-\mu)-\omega+\alpha(\omega-\delta),\alpha(\kappa+r\mu)+\nu(r+1)(1-\alpha)-\kappa-1} \tag{20}$$

$$\left(-\vartheta\cdot t^{\nu-\alpha(\nu-\mu)-\omega+\alpha(\omega-\delta)}\right),$$

where $\nu,\,\mu>\kappa-1;\ \kappa>\omega,\,\delta$.

Proof. To illustrate (20), we consider

$$\frac{s^{\alpha(\kappa-\mu)-\kappa}}{\left(s^{\alpha(\nu-\mu)-\nu}+\vartheta\cdot s^{\alpha(\omega-\delta)-\omega}+\eta\right)}=\frac{s^{\alpha(\kappa-\mu)-\kappa}}{s^{\alpha(\nu-\mu)-\nu}+\vartheta\cdot s^{\alpha(\omega-\delta)-\omega}}\left(1+\frac{\eta}{s^{\alpha(\nu-\mu)-\nu}+\vartheta\cdot s^{\alpha(\omega-\delta)-\omega}}\right)^{-1}$$
$$=\sum_{r=0}^{\infty}(-\eta)^r s^{\alpha(\kappa-\mu)-\kappa-\{\alpha(\nu-\mu)-\nu\}(r+1)}\left(1+\vartheta\cdot s^{\nu-\alpha(\nu-\mu)-\omega+\alpha(\omega-\delta)}\right)^{-(r+1)}. \tag{21}$$

Now, taking the inverse of EIT on (21) and using (17), we arrive at (20). □

Similarly, we can write the following lemma:

Lemma 7 Let $0\le\alpha\le 1$, $\nu-\alpha(\nu-\mu)>\omega-\alpha(\omega-\delta)$ and $\kappa\in\mathbb{N}$, therefore, the succeeding equivalence holds true.

$$E^{-1}\left[\frac{s^{\alpha(\kappa-\delta)-\kappa}}{(s^{\alpha(\nu-\mu)-\nu}+\vartheta\cdot s^{\alpha(\omega-\delta)-\omega}+\eta)}\right]$$

$$=\sum_{r=0}^{\infty}(-\eta)^r t^{\kappa(\alpha-1)+(r+1)\{\nu(1-\alpha)+\alpha\mu\}-\alpha\delta-2} \tag{22}$$

$$\times E^{(r+1)}_{\nu-\alpha(\nu-\mu)-\omega+\alpha(\omega-\delta),\kappa(\alpha-1)+(r+1)\{\nu(1-\alpha)+\alpha\mu\}-\alpha\delta-1}\left(-\vartheta\cdot t^{\nu-\alpha(\nu-\mu)-\omega+\alpha(\omega-\delta)}\right),$$

where $\nu,\,\mu>\kappa-1;\ \kappa>\omega,\,\delta$.

3.1 *EIT of GCF derivative*

Theorem 8 Let the EIT of the function $\Theta(t)$ denoted by $T(s)$, then the EIT of the GCF derivative ${}_0\mathbb{D}_t^{\nu,\,\mu;\,\alpha}\Theta(t)$ is defined by

$$E[{}_0\mathbb{D}_t^{\nu,\,\mu;\,\gamma}\Theta(t),\,s]=s^{\alpha(\nu-\mu)-\nu}T(s)-\sum_{j=0}^{\kappa-1}s^{\alpha(\kappa-\mu)-\kappa+j+2}\left[\left(\mathbb{D}^j\left({}_0\mathbb{I}_t^{(1-\alpha)(\kappa-\nu)}\right)\Theta(t)\right)\Big|_{t=0}\right], \tag{23}$$

where $\kappa - 1 < \nu,\ \mu \leq \kappa$ with $\kappa \in \mathbb{N}$.

Proof. Suppose $\varphi(t) = \mathbb{D}^{\kappa}\ {}_0\mathbb{I}_t^{(1-\alpha)(\kappa-\nu)}\Theta(t) = \mathbb{D}^{\kappa}\varphi(t)$ for (11) and using (15), we get

$$E\left[{}_0\mathbb{D}_t^{\nu,\,\mu;\,\alpha}\Theta t,\ s\right] = E\left[{}_0\mathbb{I}_t^{\alpha(\kappa-\mu)}\varphi(t),\ s\right] = s^{\alpha(\kappa-\mu)}E[\varphi(t),\ s] = s^{\alpha(\kappa-\mu)}E[\mathbb{D}^{\kappa}\varphi(t),\ s], \tag{24}$$

where $\varphi(t) ={}_0 \mathbb{I}_t^{(1-\alpha)(\kappa-\nu)}$, using (16) in (24), we get

$$\begin{aligned} E\left[{}_0\mathbb{D}_t^{\nu,\,\mu;\,\alpha}\Theta(t),\ s\right] &= s^{\alpha(\kappa-\mu)-\kappa}E\left[{}_0\mathbb{I}_t^{(1-\alpha)(\kappa-\nu)}\Theta(t),\ s\right] \\ &- \sum_{j=0}^{\kappa-1} s^{\alpha(\kappa-\mu)-\kappa+j+2}\left[\left(\mathbb{D}^j\left({}_0\mathbb{I}_t^{(1-\alpha)(\kappa-\nu)}\Theta\right)(t)\right)\Big|_{t=0}\right], \end{aligned} \tag{25}$$

again using (15) in (25), we get

$$\begin{aligned} E\left[{}_0\mathbb{D}_{\nu,\,\mu;\,\alpha}\Theta(t),\ s\right] &= s^{\alpha(\nu-\mu)-\nu} \times s^{(1-\alpha)(\kappa-\nu)}E[\Theta(t),\ s] \\ &- \sum_{t=0}^{\kappa-1} s^{\alpha(\kappa-\mu)-\kappa+j+2}\left[\left(\mathbb{D}^j\left({}_0\mathbb{I}_t^{(1-\alpha)(\kappa-\nu)}\Theta\right)(t)\right)\Big|_{t=0}\right], \end{aligned} \tag{26}$$

we arrive at (23). □

3.2 *The GCF derivative and its solution to the RDE*

Theorem 9 For $0 \leq \alpha \leq 1;\ \kappa - 1 < \nu,\ \mu;\ \omega,\ \delta \leq \kappa,\ \kappa \in \mathbb{N}$ such that $\nu > \omega,\ \delta < \mu$. Consider the fractional RDE

$${}_0\mathbb{D}_t^{\nu,\,\mu;\,\alpha}P(x,\ t) + \vartheta \cdot {}_0\mathbb{D}_t^{\omega,\,\delta;\,\alpha}P(x,\ t) = \vartheta^2_{-\infty}\mathbb{D}_x^{\gamma}P(x,\ t) + \lambda^2 P(x,\ t) + \Theta(x,\ t), \tag{27}$$

with the initial conditions

$$\begin{cases} \mathbb{D}^j{}_0\mathbb{I}_t^{(1-\alpha)(\kappa-\nu)}P(x,\ t)\Big|_{t=0} = \varphi_1(x) \\ \mathbb{D}^j{}_0\mathbb{I}_t^{(1-\alpha)(\kappa-\omega)}P(x,\ t)\Big|_{t=0} = \varphi_2(x) \end{cases} ; \quad j = 0,\ 1,\ \cdots,\ \kappa-1, \tag{28}$$

where ϑ is a diffusion coefficient, α is a constant that represents the non-linearity of the system, and Θ is a non-linear function for reaction kinetics than the solution of (27) corresponding to $P(x,\ t)$ is as follows

$$
\begin{aligned}
P(x,\,t) = &\sum_{j=0}^{\kappa-1}\sum_{r=0}^{\infty}\frac{(-\eta)^r}{\sqrt{2\pi}}\int_{-\infty}^{\infty}t^{\alpha(\kappa+r\mu)+\nu(r+1)(1-\alpha)-\kappa+j}\varphi_1^*(N)exp(-iNx)\\
&\times E^{r+1}_{\nu-\alpha(\nu-\mu)-\omega+\alpha(\omega-\delta),\alpha(\kappa+r\mu)+\nu(r+1)(1-\alpha)-\kappa+j+1}\left(-\vartheta\cdot t^{\nu-\alpha(\nu-\mu)-\omega+\alpha(\omega-\delta)}\right)dN\\
&+\sum_{j=0}^{\kappa-1}\sum_{r=0}^{\infty}\vartheta\frac{(-\eta)^r}{\sqrt{2\pi}}\int_{-\infty}^{\infty}t^{\kappa(\alpha-1)+(r+1)(\nu-\alpha(\nu-\mu))-\alpha\delta+j}\varphi_2^*(N)exp(-iNx)\\
&\times E^{r+1}_{\nu-\alpha(\nu-\mu)-\omega+\alpha(\omega-\delta),\kappa(\alpha-1)+(r+1)(\nu-(\nu-\mu))-\alpha\delta+j+1}\left(-\vartheta\cdot t^{\nu-\alpha(\nu-\mu)-\omega+\alpha(\omega-\delta)}\right)dN\\
&+\sum_{r=0}^{\infty}\frac{(-\eta)^r}{\sqrt{2\pi}}\int_0^t u^{\{\nu-\alpha(\nu-\mu)\}(r+1)-1}\int_{-\infty}^{\infty}\Theta^*(N,\,t-u)exp(-iNx)\\
&\times E^{r+1}_{\nu-\alpha(\nu-\mu)-\omega+\alpha(\omega-\delta),\{\nu-\alpha(\nu-\mu)\}(r+1)}\left(-\vartheta\cdot t^{\nu-\alpha(\nu-\mu)-\omega+\alpha(\omega-\delta)}\right)dudN.
\end{aligned}
\tag{29}
$$

Proof. If we use the EIT on both sides of (27) with respect to t, we get

$$
\begin{aligned}
&E[{}_0\mathbb{D}_t^{\nu,\,\mu;\,\alpha}P(x,\,t),\,s]+E[\vartheta\cdot{}_0\mathbb{D}_t^{\omega,\,\delta;\,\alpha}P(x,\,t),\,s]\\
&=\vartheta^2E[{}_{-\infty}\mathbb{D}_x^{\gamma}P(x,\,t),\,s]+\lambda^2E[P(x,\,t),\,s]+E[\Theta(x,\,t)],
\end{aligned}
\tag{30}
$$

using (23) and (28) in (30), we have

$$
\begin{aligned}
&s^{\alpha(\nu-\mu)-\nu}\overline{P}(x,\,s)-\sum_{j=0}^{\kappa-1}s^{\alpha(\kappa-\mu)-\kappa+j+2}\left[\mathbb{D}^j\left({}_0\mathbb{I}_t^{(1-\alpha)(\kappa-\nu)}P(x,\,t)\right)\right]\Big|_{t=0}\\
&+\vartheta\cdot s^{\alpha(\omega-\delta)-\omega}\overline{P}(x,\,s)-\vartheta\cdot\sum_{j=0}^{\kappa-1}s^{\alpha(\kappa-\delta)-\kappa+j+2}\left[\mathbb{D}^j\left({}_0\mathbb{I}_t^{(1-\alpha)(\kappa-\omega)}P(x,\,t)\right)\right]\Big|_{t=0}\\
&=\vartheta^2_{-\infty}\mathbb{D}_x^{\gamma}\overline{P}(x,\,s)+\lambda^2\overline{P}(x,\,s)+\overline{\Theta}(x,\,s)\\
&s^{\alpha(\nu-\mu)-\nu}\overline{P}(x,\,s)-\sum_{j=0}^{\kappa-1}s^{\alpha(\kappa-\mu)-\kappa+j+2}\varphi_1(x)+\vartheta\cdot s^{\alpha(\omega-\delta)-\omega}\overline{P}(x,\,s)\\
&-\vartheta\cdot\sum_{j=0}^{\kappa-1}s^{\alpha(\kappa-\delta)-\kappa+j+2}\varphi_2(x)
\end{aligned}
$$

$$= \vartheta^2_{-\infty}\mathbb{D}^{\gamma}_{x}\overline{P}(x,\ s) + \lambda^2\overline{P}(x,\ s) + \overline{\Theta}(x,\ s). \tag{31}$$

Now, by applying FIT to both ends of (31) with respect to x and employing (7), we obtain

$$\begin{aligned} & s^{\alpha(\nu-\mu)-\nu}\overline{P}^*(N,\ s) - \sum_{j=0}^{\kappa-1} s^{\alpha(\kappa-\mu)-\kappa+j+2}\varphi_1^*(N) + \vartheta\cdot s^{\alpha(\omega-\delta)-\omega}\overline{P}^*(N,\ s) - \vartheta\cdot\sum_{j=0}^{\kappa-1} s^{\alpha(\kappa-\delta)-\kappa+j+2}\varphi_2^*(N) \\ & = -\vartheta^2|\kappa|^{\gamma}\overline{P}^*(N,\ s) + \lambda^2\overline{P}^*(N,\ s) + \overline{\Theta}^*(N,\ s), \end{aligned} \tag{32}$$

to solve $\overline{P}^*(N,\ s)$, above corresponds to

$$\begin{aligned} \overline{P}^*(N,\ s) = & \sum_{j=0}^{\kappa-1}\varphi_1^*(N)\frac{s^{\alpha(\kappa-\mu)-\kappa+j+2}}{s^{\alpha(\nu-\mu)-\nu}+\vartheta\cdot s^{\alpha(\omega-\delta)-\omega}+\eta} + \sum_{j=0}^{\kappa-1}\vartheta\cdot\varphi_2^*(N)\frac{s^{\alpha(\kappa-\delta)-\kappa+j+2}}{s^{\alpha(\nu-\mu)-\nu}+\vartheta\cdot s^{\alpha(\omega-\delta)-\omega}+\eta} \\ & + \frac{\overline{\Theta}^*(N,\ s)}{s^{\alpha(\nu-\mu)-\nu}+\vartheta\cdot s^{\alpha(\omega-\delta)-\omega}+\eta}, \end{aligned} \tag{33}$$

where $\eta = \vartheta^2|K|^{\gamma} - \lambda^2$. Now, using (18), (22), and the EIT convolution (14), we obtained by applying the inverse EIT on both ends of (33)

$$\begin{aligned} P^*(N,\ t) = & \sum_{j=0}^{\kappa-1}\varphi_1^*(N)\sum_{r=0}^{\infty}(-\eta)^r t^{\alpha(\kappa+r\mu)+\nu(r+1)(1-\alpha)-\kappa+j} \\ & \times E^{r+1}_{\nu-\alpha(\nu-\mu)-\omega+\alpha(\omega-\delta),\ \alpha(\kappa+r\mu)+\nu(r+1)(1-\alpha)-\kappa+j+1}\left(-\vartheta\cdot t^{\nu-\alpha(\nu-\mu)-\omega+\alpha(\omega-\delta)}\right) \\ & + \sum_{j=0}^{\kappa-1}\vartheta\cdot\varphi_2^*(N)\sum_{r=0}^{\infty}(-\eta)^r t^{\kappa(\alpha-1)+(r+1)\{\nu(1-\alpha)+\alpha\mu\}-\alpha\delta+j} \\ & \times E^{r+1}_{\nu-\alpha(\nu-\mu)-\omega+\alpha(\omega-\delta),\kappa(\alpha-1)+(r+1)\{\nu(1-\alpha)+\alpha\mu\}-\alpha\delta+j+1}\left(-\vartheta\cdot t^{\nu-\alpha(\nu-\mu)-\omega+\alpha(\omega-\delta)}\right) \\ & + \sum_{r=0}^{\infty}(-\eta)^r\int_0^t\Theta^*(N,\ t-u)u^{\{\nu-\alpha(\nu-\mu)\}(r+1)-1} \\ & \times E^{r+1}_{\nu-\alpha(\nu-\mu)-\omega+\alpha(\omega-\delta),\ \{\nu-\alpha(\nu-\mu)\}(r+1)}\left(-\vartheta\cdot t^{\nu-\alpha(\nu-\mu)-\omega+\alpha(\omega-\delta)}\right)du, \end{aligned} \tag{34}$$

when we do an inverse FIT on both ends of (34), we get (29). □

3.3 *Special cases*

Corollary 1 For $\alpha = 0$, the GCF-RDE (27) reduces into the RDE with RLF's derivative, that is

$$ {}_0\mathbb{D}_t^{\nu} P(x,\, t) + \vartheta \cdot {}_0\mathbb{D}_t^{\omega} P(x,\, t) = \vartheta^2_{-\infty}\mathbb{D}_x^{\gamma} P(x,\, t) + \lambda^2 P(x,\, t) + \Theta(x,\, t), \tag{35} $$

where $\kappa - 1 < \nu$; $\omega \leq \kappa$; $\nu > \omega$; $\kappa \in \mathbb{N}$; ${}_0\mathbb{D}_t^{\nu}$ and ${}_0\mathbb{D}_t^{\omega}$ are fractional derivatives in RLF's sense with the initial conditions

$$ \begin{cases} \mathbb{D}^j\ {}_0\mathbb{I}_t^{(1-\alpha)(\kappa-\nu)} P(x,\, t)|_{t=0} = \varphi_1(x) \\ \mathbb{D}^j\ {}_0\mathbb{I}_t^{(1-\alpha)(\kappa-\omega)} P(x,\, t)|_{t=0} = \varphi_2(x) \end{cases};\quad j = 0,\, 1,\, \cdots \kappa - 1. \tag{36} $$

Given initial conditions (36), the solution of (35) is given by

$$ \begin{aligned} P(x,\, t) = & \sum_{j=0}^{\kappa-1} \sum_{r=0}^{\infty} \frac{(-\eta)^r}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^{\nu(r+1)-\kappa+j} \varphi_1^*(N) exp(-iNx) \\ & \times E^{r+1}_{\nu-\omega,\, \nu(r+1)-\kappa+j+1} \left(-\vartheta \cdot t^{\nu-\omega}\right) dN \\ & + \sum_{j=0}^{\kappa-1} \sum_{r=0}^{\infty} \vartheta \frac{(-\eta)^r}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^{\nu(r+1)-\kappa+j} \varphi_2^*(N) exp(-iNx) \\ & \times E^{r+1}_{\nu-\omega,\, \nu(r+1)-\kappa+j+1} \left(-\vartheta \cdot t^{\nu-\omega}\right) dN \\ & + \sum_{r=0}^{\infty} \frac{(-\eta)^r}{\sqrt{2\pi}} \int_0^t u^{\{\nu(r+1)-1} \int_{-\infty}^{\infty} \Theta^*(N,\, t-u) exp(-iNx) \\ & \times E^{r+1}_{\nu-\omega,\, \nu(r+1)} \left(-\vartheta \cdot t^{\nu-\alpha(\nu-\mu)-\omega+\alpha(\omega-\delta)}\right) dudN. \end{aligned} \tag{37} $$

Corollary 2 For $\alpha = 1$, the GCF-RDE (27) reduces into a RDE with CF's derivative, that is

$$ {}_0\mathbb{D}_t^{\mu} P(x,\, t) + \vartheta \cdot {}_0\mathbb{D}_t^{\delta} P(x,\, t) = \vartheta^2_{-\infty}\mathbb{D}_x^{\gamma} P(x,\, t) + \lambda^2 P(x,\, t) + \Theta(x,\, t), \tag{38} $$

where $\kappa - 1 < \mu,\ \delta \leq \kappa$; $\mu > \delta,\ \kappa\ \in \mathbb{R}$; ${}_0\mathbb{D}_t^{\mu}$ and ${}_0\mathbb{D}_t^{\delta}$ are fractional derivatives in CF sense with the initial conditions

$$ \mathbb{D}_0^j P(x,\, t)\big|_{t=0} = \varphi_(x), \quad j = 0,\, 1,\, \cdots\ \kappa - 1. \tag{39} $$

The expression provides the solution of (38) with initial conditions (39)

$$P(x,\, t)=\sum_{j=0}^{\kappa-1}\sum_{r=0}^{\infty}\frac{(-\eta)^r}{\sqrt{2\pi}}\int_{-\infty}^{\infty}t^{\mu r+j}\varphi_1^*(N)exp(-iNx)\times E_{\mu-\delta,\,\mu r+j+1}^{r+1}\left(-\vartheta\cdot t^{\mu-\delta}\right)dN$$

$$+\sum_{j=0}^{\kappa-1}\sum_{r=0}^{\infty}\vartheta\frac{(-\eta)^r}{\sqrt{2\pi}}\int_{-\infty}^{\infty}t^{\mu(r+1)-\delta+j}\varphi_2^*(N)exp(-iNx)$$

$$\times E_{\mu-\delta,\,\mu(r+1)-\delta+j+1}^{r+1}\left(-\vartheta\cdot t^{\mu-\delta}\right)dN \tag{40}$$

$$+\sum_{r=0}^{\infty}\frac{(-\eta)^r}{\sqrt{2\pi}}\int_0^t u^{\mu(r+1)-1}\int_{-\infty}^{\infty}\Theta^*(N,\, t-u)exp(-iNx)\times E_{\mu-\delta,\,\mu(r+1)}^{r+1}\left(-\vartheta\cdot t^{\mu-\delta}\right)dudN.$$

In particular, if we take $\kappa=1$ in (38)-(40), we get a form of fractional RDE studied by Saxena et al. [7] and Gupta and Sharma [30].

Corollary 3 For $\nu=\mu;\ \omega=\delta$, the GCF-RDE (27) reduces into a RDE with HF's derivative, that is

$${}_0\mathbb{D}_t^{\nu,\,\alpha}P(x,\, t)+\vartheta\cdot{}_0\mathbb{D}_t^{\omega,\,\alpha}P(x,\, t)=\vartheta^2{}_{-\infty}\mathbb{D}_x^{\gamma}P(x,\, t)+\lambda^2P(x,\, t)+\Theta(x,\, t), \tag{41}$$

with the initial conditions

$$\begin{cases}\mathbb{D}^j\ {}_0\mathscr{I}_t^{(1-\alpha)(\kappa-\nu)}P(x,\, t)|_{t=0}=\varphi_1(x)\\ \mathbb{D}^j\ {}_0\mathscr{I}_t^{(1-\alpha)(\kappa-\omega)}P(x,\, t)|_{t=0}=\varphi_2(x)\end{cases};\ \ j=0,\,1,\,\cdots\,\kappa-1, \tag{42}$$

where $\kappa-1<\nu,\ \omega\le\kappa$ and $\kappa\in\mathbb{N}$ such that $0\le\alpha\le1$. The answer that corresponds to this problem can be found by

$$P(x,\, t)=\sum_{j=0}^{\kappa-1}\sum_{r=0}^{\infty}\frac{(-\eta)^r}{\sqrt{2\pi}}\int_{-\infty}^{\infty}t^{\nu(r+1-\alpha)+\kappa(\alpha-1)+j}\varphi_1^*(N)exp(-iNx)$$

$$\times E_{\nu-\omega,\,\nu(r+1-\alpha)+\kappa(\alpha-1)+j+1}^{r+1}\left(-\vartheta\cdot t^{\nu-\omega}\right)dN$$

$$+\sum_{j=0}^{\kappa-1}\sum_{r=0}^{\infty}\vartheta\frac{(-\eta)^r}{\sqrt{2\pi}}\int_{-\infty}^{\infty}t^{\kappa(\alpha-1)+\nu(r+1)-\alpha\omega+j}\varphi_2^*(N)exp(-iNx) \tag{43}$$

$$\times E_{\nu-\omega,\,\kappa(\alpha-1)+\nu(r+1)-\alpha\omega+j+1}^{r+1}\left(-\vartheta\cdot t^{\nu-\omega}\right)dN$$

$$+\sum_{r=0}^{\infty}\frac{(-\eta)^r}{\sqrt{2\pi}}\int_0^t u^{\nu(r+1)-1}\int_{-\infty}^{\infty}\Theta^*(N,\, t-u)exp(-iNx)$$

$$\times E_{\nu-\omega,\ \nu(r+1)}^{r+1}\left(-\vartheta\cdot t^{\nu-\omega}\right)dudN$$

Specifically, the Alkahtani et al. [8] studied FDE obtained in this scenario for $\kappa = 1$.

4. Conclusion

By deriving the inverse Elzaki transform and introducing multiple lemmas related to the generalized composite fractional derivative, this study has made significant contributions to the discipline. Researchers have effectively solved a nonlinear reaction-diffusion equation incorporating the generalized composite fractional derivative by using the Elzaki and Fourier transforms. In addition, some special cases of the general equation have been discussed. Furthermore, the provided approach is potentially applicable to different forms of fractional derivatives, implying larger value in a variety of scenarios. Future research might include extending the Elzaki transform to additional fractional derivatives and complicated systems, creating numerical methods to supplement analytical answers, and investigating multidisciplinary applications in physics, biology, and engineering.

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Conflict of interest

The authors declare no competing financial interest.

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