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Research Article

The Nonlinear Schrödinger Equation Derived from the Third Order Korteweg-de Vries Equation Using Multiple Scales Method

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Abstract: Nonlinear equations of evolution (NLEE) are mathematical models used in various branches of science. As a result, nonlinear equations of evolution have served as a language for formulating many engineering and scientific problems. For this reason, many different and effective techniques have been developed regarding nonlinear equations of evolution and solution methods. Although the origin of nonlinear equations of evolution dates back to ancient times, there have been significant developments regarding these equations from the past to the present. The main reason for this situation is that nonlinear equations of evolution involve the problem of nonlinear wave propagation. In recent years, equations of formation have become increasingly important in applied mathematics. This work focuses on the perturbation approach, often known as many scales, for nonlinear evolution equations. The article focuses on the analysis of the $(1+1)$ dimensional third-order nonlinear Korteweg-de Vries (KdV) equation using the multiple scales method, which resulted in obtaining nonlinear Schrödinger (NLS) type equations.

***Keywords*:** perturbation, multiple scales method, third-order Korteweg-de Vries (KdV3) equation

MSC: 34D10, 34E13, 35Q53, 35Q55, 37K10

1. Introduction

Although the word wave generally refers to the shapes formed on the water surface in daily life, there are many areas where the wave finds its place. The solutions of these waves constitute an important group of solutions of differential equations used in many fields such as physics, engineering, economics, and biology, as well as applied mathematics. Nonlinear waves appear as partial differential equations that characterize wave propagation in many areas of physics such as dispersive wave equations, fluid mechanics, elasticity theory, nonlinear optics, and plasma physics.

Very diverse of physical, chemical, and biological phenomena are depicted by nonlinear evolution equations (NLEE). Recently, NLEEs have become an important field of study in applied mathematics [1–3]. Also, nonlinear equations that model these scientific phenomena in other branches of science have long been a major concern for research studies [4]. Since data of their exact solutions facilitates the confirmation of numerical solvers and supports in decisiveness analysis of solutions, analytical study of these NLEE is significant.This not only helps to better understand the solutions and also helps us to understand the phenomenon they describe.

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The KdV equation is a general model for the study of weak nonlinear long waves, which includes the most important nonlinearity and dispersion [5, 6]. Also, it can be thought of as modeling the unilateral propagation of long wavelength gravitational waves of small amplitude in a shallow channel [7–10]. In the context of nonlinear third-order KdV type equations, studies are developing as these equations can describe real properties in various scientific applications and engineering fields and have very practical/physical meaning. The simplest form of the KdV equation is written as

$$u_t + auu_x + u_{xxx} = 0. \tag{1}$$

This equation includes two striving effects: linear dispersion represented by u_{xxx}, and nonlinearity represented by uu_x. Dispersion scatters the wave, nonlinearity tends to localize the wave [11, 12]. The stable equilibrium between these two mild nonlinearities and dispersion explains the formation of solitons composed of single-humped waves [13–21].

It is an example of a universal nonlinear model, as the nonlinear Schrödinger (NLS) equation explains a wide range of physical systems. As a result, the equation may be used to describe a wide range of nonlinear physical events [22]. It is renowned that a multiscale analysis of the KdV equation gives rise to the NLS equation for modulated amplitude [23–27]. In [23] Zakharov and Kuznetsov demonstrated a much deeper correspondence between these integrable equations, not only at the equation level but also at the linear spectral problem level, by showing that multiscale analysis of the Schrödinger spectral problem yields the Zakharov-Shabat problem for the NLS equation. Özer and Dağ demonstrated a similar link between the NLS and integrable fifth-order nonlinear evolution equations [28].

The nonlinear Schrödinger (NLS) equation is a nonlinear PDE that plays a key role in many aspects of scientific disciplines, including nonlinear optics, biophysics, Bose-Einstein condensation, fluid mechanics, microeconomic theory, electromagnetism spinning waves, image sensors, and so on. There are so many studies devoted to determining the exact optical soliton and rogue wave (RW) dynamics using the NLS equation [29–33].

After the introduction, in section 2, we briefly expressed the third-order Korteweg-de Vries Equation (KdV3) flow equations and the multiple scales method, respectively. In section 3, we applied the method given in section 2 to the $(1+1)$ dimensional KdV3. The last part consists of the conclusion part.

2. Background materials

In this section, we present some background material on the best-known third-order KdV equations and the multi-scale method.

2.1 *Third-order KdV equations*

The third-order KdV family is as follows

$$u_t + P(u)u_x + u_{xxx} = 0, \tag{2}$$

where $u(x,\ t)$ is a function of space x and time variable t. Constants can be used as coefficients of $P(u)u_x$ and u_{xxx}, but these constants are generally scalable. The nonlinear term $P(u)$ is shown in the figures below

$$P(u)=\begin{cases} au \\ au^2 \\ au^n \\ u_x \\ au^n-bu^{2n} \end{cases} \tag{3}$$

For $P(u)=\pm 6u$ we get the standard KdV equation

$$u_t \pm 6u_x + u_{xxx} = 0 \tag{4}$$

where the factor ± 6 suitable for full integrability. That is, KdV equation has N-soliton solutions [34].

2.2 *The multiple scales method for KdV equation*

The multiple scales method is a perturbation method. In the multiple scales method first recommended by Zakharov and Kuznetzov [23], Zakharov and Kuznetzov used this method to decrease the KdV equation to the NLS equation and apply it to a class of nonlinear evolution equations. Using this method, they showed that integrable systems can be decreased to integrable systems. If the system we have taken at the beginning is not an integrable system, it has been seen that the reduced system as a result of the application of the method is either integrable or non-integrable. However, if the method is applied to a suitable integrable system, it is seen that the system obtained as a result of the analysis is always an integrable system. This is the master purpose of applying the multi-scale expansion method to integrable systems.

In this section, multiple scales method of nonlinear evolution equations is discussed. By applying the Zakharov and Kuznetzuv [23] technique, the steps of the multi-scale method in obtaining NLS type equations from KdV equations are shown in order.

Let's consider the general evolution equation in the following form

$$u_t = K[u]. \tag{5}$$

General nonlinear evolution equations $K[u]$ is a function of u and its derivatives in the x-spatial variables. The most well-known of these equations is KdV equation. $L[\partial_x,\ \partial_y]u$ represents the linear component of $K[u]$. So, using $K[u]$ we can get the dispersion relation for the Eq. (5). Substituting the plane wave solution

$$\begin{aligned} u_k &= Ae^{i(kx+ry-\omega(k)t)} \\ &\equiv Ae^{i\theta} \end{aligned} \tag{6}$$

into the linear part of Eq. (5)

$$u_t = L[\partial_x,\ \partial_y]u \tag{7}$$

we obtain dispersion relation

$$\omega(k) = iL[ik, ir] \tag{8}$$

Then, dispersion relation (8) is substituted in Eq. (5). We suppose the following series expansions for the solution of the Eq. (5):

$$u(x, y, t) = \sum_{n=1}^{\infty} \varepsilon^n U_n(x, y, t, \xi, \tau)$$

Using this method, we define slow spaces ξ and multiple time variable τ for scaling parameters $\varepsilon > 0$ as shown below. A nonlinear equation modulates the amplitude of this plane wave solution in such a way that it may be considered dependent on the slow variables:

$$\begin{aligned} \xi &= \varepsilon \left(x - \frac{d\omega(k)}{dk} t \right) \\ \tau &= -\frac{1}{2}\varepsilon^2 \frac{d^2\omega(k)}{dk^2} t \end{aligned} \tag{9}$$

By selecting different forms for the slow variables, we can derive higher-order NLS equations. The multiple scales analysis starts with the assumption that:

$$u(x, t) = U(x, y, t, \xi, \tau) \tag{10}$$

and solution of U is in the form

$$U(x, y, t, \xi, \tau) = (\varepsilon U_1 + \varepsilon^2 U_2 + \varepsilon^3 U_3 + \ldots) \tag{11}$$

In this case, considering the transformation (10) and solution (11), using (8) and (9), the terms included derivative in Eq. (5) are obtained. Substituting these terms with (10) and (11) into the Eq. (5), we get a polynomial in ε. We obtain a series of algebraic equations by equalizing each coefficient of this polynomial to zero. Using wave solution space (6) and dispersion relation (8), these equations may be solved by iteration and Reduce. Thus, we can obtain NLS type equations from Eq. (5). Furthermore, this approach allows us to obtain numerical solutions to KdV-type problems.

3. Applications

Based on Zakharov and Kuznetsov [23] we apply the multi-scale method to derive NLS type equations from the KdV equation (12). To find dispersion relation for from KdV equation (12), we take notice the linear part of (12) in the form

$$u_t = u_{xxx} + 6uu_x \tag{12}$$

and linear differential equation (12) satisfies the solution.

$$u(x,\ t) = e^{i\theta},\ \ \theta = kx - \omega(k)t \tag{13}$$

Substituting the solution (13) into the linear differential equation (12), we get from this we get at the

$$\omega(k) = k^3 \tag{14}$$

dispersion relation. Thence the solution of linear differential equation (12) is as follows:

$$u(x,\ t) = e^{i(kx - w(k)t)} \tag{15}$$

ε scale parameter and slow variables

$$\xi_i = \xi_i(x,\ t,\ \varepsilon),\quad \tau_i = \tau_i(x,\ t,\ \varepsilon).$$

Then we suppose the following series expansions for solutions:

$$u(x,\ t) = \sum_{n=1}^{\infty} \varepsilon^n U_n(\xi_0,\ \xi_1,\ \cdots,\ \xi_n,\ \tau_1,\ \tau_2,\ \cdots,\ \tau_n) \tag{16}$$

We also define slow variables with respect to the scaling parameter $\varepsilon > 0$ respectively as follows:

$$\begin{aligned}
&\xi_0 = x\\
&\xi_1 = \varepsilon\left(x - \frac{d\omega}{dk}t\right)\\
&\tau = t\\
&\tau_1 = -\frac{\varepsilon^2}{2!}\frac{dw}{dk}t\\
&\tau_2 = -\frac{\varepsilon^3}{3!}\frac{dw}{dk^2}t\\
&\tau_3 = -\frac{\varepsilon^4}{4!}\frac{dw}{dk^3}t
\end{aligned} \tag{17}$$

Then, taking into account the transformation (12) and the solution (16), using the dispersion relation (14) and the slow variables (17), we obtain u and its derivatives with respect to ε in Eq. (12). We replace these terms with (15) and (16) in the Eq. (12). By adding together all terms with the same ε-order, the left hand side of Eq. (12) is turned into a polynomial in ε. After setting each coefficient of this polynomial equal to zero, we obtain a set of algebraic equations. If we let $\varepsilon \to 0$ and vanish the terms at minimal powers of ε, by considering the case $n \geq 1$ we obtain the following:

$$u_{1\tau_0} - u_{1\xi_0\xi_0\xi_0} = 0 \tag{18}$$

$$u_{2\tau_0} - u_{2\xi_0\xi_0\xi_0} = 3k^2 u_{1\xi_1} + 6u_1 u_{1\xi_0} + 3u_{1\xi_0\xi_0\xi_1} \tag{19}$$

$$\begin{aligned} u_{3\tau_0} - u_{3\xi_0\xi_0\xi_0} = {} & 3k^2 u_{1\tau_1} + 6u_1 u_{1\xi_1} + 3k^2 u_{2\xi_1} + 6u_2 u_{1\xi_0} \\ & + 6u_1 u_{2\xi_0} + 3u_{1\xi_0\xi_1\xi_1} + 3u_{2\xi_0\xi_0\xi_1} \end{aligned} \tag{20}$$

$$\begin{aligned} u_{4\tau_0} - u_{4\xi_0\xi_0\xi_0} = {} & -3ku_{1\tau_2} + 3k^2 u_{2\tau_1} + 6u_2 u_{1\xi_1} + 6u_1 u_{2\xi_1} + 3k^2 u_{3\xi_1} \\ & + u_{1\xi_1\xi_1\xi_1} + 6u_3 u_{1\xi_0} + 6u_2 u_{2\xi_0} + 6u_1 u_{3\xi_0} + 3u_{2\xi_0\xi_1\xi_1} + 3u_{3\xi_0\xi_0\xi_1} \end{aligned} \tag{21}$$

$$\begin{aligned} u_{5\tau_0} - u_{5\xi_0\xi_0\xi_0} = {} & u_{1\tau_3} - 3ku_{2\tau_2} + 3k^2 u_{3\tau_1} + 6u_3 u_{1\xi_1} + 6u_2 u_{2\xi_1} \\ & + 6u_1 u_{3\xi_1} + 3k^2 u_{4\xi_1} + u_{2\xi_1\xi_1\xi_1} + 6u_4 u_{1\xi_0} \\ & + 6u_3 u_{2\xi_0} \quad + 6u_2 u_{3\xi_0} + 6u_1 u_{4\xi_0} + 3u_{3\xi_0\xi_1\xi_1} + 3u_{4\xi_0\xi_0\xi_1} \end{aligned} \tag{22}$$

$$\vdots$$

Then, we can find the solution of (18) as follows

$$u_1(\xi_0,\ \xi_1,\ \tau_0,\ \tau_1) = v_1(\xi_1,\ \tau_1)\, e^{i(kx-wt)} + c.c. \tag{23}$$

where $c.c.$ is complex conjugate of v_1. Substituting the solution (23) into (19), the solution of (19) is in the form

$$\begin{aligned} u_2(\xi_0,\ \xi_1,\ \tau_0,\ \tau_1) = {} & v_2(\xi_1,\ \tau_1)\, e^{2i(kx-wt)} + c.c. \\ & + f_0(\xi,\ \tau) \end{aligned} \tag{24}$$

where f_0 is integration constant. Thus we take

$$v_2 = \frac{1}{k^2} v_1^2, \;\; v_{-2} = \frac{1}{k^2} v_{-1}^2 \tag{25}$$

where v_{-1} is the complex conjugate of v_1 and v_{-2} is the complex conjugate of v_2. Substituting solutions (23), (24) and (25) into the (22), we find the solution of (22) in the form

$$\begin{gathered} u_3(\xi_0, \; \xi_1, \; \tau_0, \; \tau_1) = v_3(\xi_1, \; \tau_1) e^{3i(kx - wt)} + c.c. \\ + f_1(\xi, \; \tau) \end{gathered} \tag{26}$$

where f_1 is integration constant and v_{-3} is the complex conjugate of v_3. Then we take

$$v_3 = \frac{3}{4k^4} v_1^3, \;\; v_{-3} = \frac{3}{4k^4} v_{-1}^3, \;\; f_0 = \frac{-2}{k^2} v_1 v_{-1} \tag{27}$$

and

$$i v_{1\tau_1} = v_{1\xi_1\xi_1} - \frac{2}{k^2} v_1^2 v_{-1}, \quad i v_{-1\tau_1} = -v_{-1\xi_1\xi_1} - \frac{2}{k^2} v_{-1}^2 v_1 \tag{28}$$

Describing as $v_1 = kq$ and $v_{-1} = kp$, from Eqs. (28) we get the NLS type equations

$$i q_{\tau_1} = q_{\xi_1\xi_1} - 2q^2 p, \;\; i p_{\tau_1} = -p_{\xi_1\xi_1} - 2p^2 q. \tag{29}$$

Substituting solutions into (21) are obtained as

$$\begin{gathered} u_4(\xi_0, \; \xi_1, \; \tau_0, \; \tau_1) = v_4(\xi_1, \; \tau_1) e^{4i(kx - wt)} + c.c. \\ + f_2(\xi, \; \tau) \end{gathered} \tag{30}$$

where f_2 is integration constant and v_{-4} is the complex conjugate of v_4. Thus we take

$$v_4 = \frac{1}{2k^6} v_1^4, \;\; v_{-4} = \frac{1}{2k^6} v_{-1}^4, \quad f_1 = \frac{2i}{k^3} \left(-v_{1\xi_1} v_{-1} + v_1 v_{-1\xi_1} \right) \tag{31}$$

and

$$v_{1\tau_2} = \frac{1}{3k} v_{1\xi_1\xi_1\xi_1} - \frac{4}{k^3} v_1 v_{-1} v_{1\xi_1} - \frac{6}{k^3} v_1^2 v_{-1\xi_1}$$
$$v_{-1\tau_2} = \frac{1}{3k} v_{-1\xi_1\xi_1\xi_1} - \frac{4}{k^3} v_{-1} v_1 v_{-1\xi_1} - \frac{6}{k^3} v_{-1}^2 v_{1\xi_1} \tag{32}$$

Describing as $v_1 = kq$ and $v_{-1} = kp$, from Eqs. (32) we get equations

$$q_{\tau_2} = q_{\xi_1\xi_1\xi_1} - 12qpq_{\xi_1} - 18q^2 p_{\xi_1}$$
$$p_{\tau_2} = p_{\xi_1\xi_1\xi_1} - 12pqp_{\xi_1} - 18p^2 q_{\xi_1} \tag{33}$$

Similarly, substituting the solutions into (22) are obtained as

$$u_5(\xi_0,\ \xi_1,\ \tau_0, \tau_1) = v_5(\xi_1,\ \tau_1)\, e^{5i(kx-wt)} + c.c.$$
$$+ f_3(\xi,\ \tau) \tag{34}$$

where f_2 is integration constant and v_{-5} is the complex conjugate of v_5. Thus we take

$$v_5 = \frac{5}{16k^8} v_1^5,\ \ v_{-5} = \frac{5}{16k^8} v_{-1}^5,\ \ f_2 = \frac{2}{k^3}\left(v_1 v_{-1\xi_1\xi_1} + v_{-1} v_{1\xi_1\xi_1} - v_{1\xi_1} v_{-1\xi_1}\right) \tag{35}$$

and

$$v_{1\tau_3} = -\frac{24}{k^3} i v_1^2 v_{-1\xi_1\xi_1} - \frac{12}{k^3} i v_1 v_{1\xi_1} v_{-1\xi_1} + \frac{6}{k^3} i v_{-1} v_{1\xi_1}^2 - \frac{3}{2k^5} i v_1^3 v_{-1}^2$$
$$v_{-1\tau_3} = \frac{24}{k^3} i v_{-1}^2 v_{1\xi_1\xi_1} + \frac{12}{k^3} i v_{-1} v_{-1\xi_1} v_{1\xi_1} - \frac{6}{k^3} i v_1 v_{-1\xi_1}^2 + \frac{3}{2k^5} i v_{-1}^3 v_1^2 \tag{36}$$

Describing as $v_1 = kq$ and $v_{-1} = kp$, from Eqs. (36) we get equations

$$q_{\tau_3} = -16iq^2 p_{\xi_1\xi_1} - 8iqq_{\xi_1} p_{\xi_1} + 4ipq_{\xi_1}^2 - iq^3 p^2$$
$$p_{\tau_3} = 16ip^2 q_{\xi_1\xi_1} + 8ipp_{\xi_1} q_{\xi_1} - 4iqp_{\xi_1}^2 + ip^3 q^2 \tag{37}$$

4. Conclusion

We developed a powerful multiple scales method to solve nonlinear formation equations. We also investigated the solutions of the third-order Korteweg-de Vries Equation (KdV3) equation and the relationship between the third-order fKdV equation and the NLS equation. In the article, we merely examined how NLS-type equations are derived from KdV-type equations and how their solutions are using the multiple scales method. Finally, it is interesting to note that the implementation of these proposed methods is much more basic and uncomplicated. At the same time, the solutions obtained will be the basis for numerical calculations. Also, NLEE can be applied to many other equations such as differential-difference equations and fractional differential equations.

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Data availability

All data generated or analysed during this study are included in this published article.

Conflict of interest

The author declare that they have no conflict of interests regarding the publication of this paper.

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