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Research Article

Derivation of Sawada-Kotera and Kaup-Kupershmidt Equations KdV Flow Equations from Modified Nonlinear Schrödinger Equation (MNLS)

Murat Koparan

Department of Mathematics and Science Education, Anadolu University Faculty of Education, Mathematics Education Division, Yunusemre Campus, 26470, Eskişehir, Turkey
E-mail: mkoparan@anadolu.edu.tr

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Abstract: Mathematical models of problems that arise in almost every branch of science are nonlinear evolution equations (NLEE). As a result, nonlinear evolution equations have served as a language for formulating many engineering and scientific problems. For this reason, many different and effective techniques have been developed regarding nonlinear evolution equations and solution methods. The main reason for this situation is that nonlinear evolution equations involve the problem of nonlinear wave propagation. In this study, (1 + 1) dimensional fifth-order nonlinear Korteweg-de Vries (fKdV) type equations were obtained by applying the multi-scale method known as the perturbation method for the modified nonlinear Schrödinger (MNLS) equation. Thus, we showed the relationship between KdV equations and MNLS-type equations.

***Keywords*:** multiple scales method, Sawada-Kotera equation, Kaup-Kupershmidt equation, MNLS equations

MSC: 34D10, 34E13, 35Q53, 35Q55, 37K10

1. Introduction

Although the word wave generally refers to the shapes formed on the water surface in daily life, there are many areas where the wave finds its place. Nonlinear waves appear as partial differential equations that characterize wave propagation in many areas of physics such as dispersive wave equations, fluid mechanics, elasticity theory, nonlinear optics, and plasma physics. Nonlinear evolution equations serve as potent mathematical instruments for modeling intricate phenomena across diverse disciplines encompassing physics, engineering, and biology [1]. By delving into the resolution of these equations, researchers can glean profound insights into the fundamental mechanisms and dynamics underpinning these phenomena. Since data on their exact solutions facilitates the confirmation of numerical solvers and supports in decisiveness analysis of solutions, the analytical study of these NLEE is significant. This not only helps to better understand the solutions and also helps us to understand the phenomenon they describe [2–4].

The following nonlinear Schrödinger-type equation:

$$iq_\tau + q_{\xi\xi} + i\delta(q|q|^2)_\xi + 2\alpha q|q|^2 = 0, \tag{1}$$

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the equation including the nonlinear dispersion; for temporal pulses in optical fibers, q is the field amplitude, t is the propagating distance, ξ is the time measured in a frame moving with the group velocity, and δ and α are constants representing the relative magnitudes of the nonlinear dispersion term and the nonlinear term, is suggested to define the short pulse propagation in long single-mode optical fibers, taking into account the connatural characteristic of the asymmetric output pulse spectrum [5–8]. Since Eq. (1) is reduced to the conventional nonlinear Schrödinger equation (NLS), this equation is known as the modified nonlinear Schrödinger (MNLS) equation (see, for example, [9]), or mixed NLS-DNLS equation (see, for example, [10]):

$$iq_\tau + q_{\xi\xi} + 2\alpha q|q|^2 = 0, \tag{2}$$

when $\delta = 0$ and the derivative is nonlinear Schrödinger equation (DNLS):

$$iq_\tau + q_{\xi\xi} + i\delta(q|q|^2)_\xi = 0, \tag{3}$$

when $\alpha = 0$ which occurs in different physical context, [11], represents well known integrable systems. The stability of the solutions, as well as the integrability of such systems that are unique to such equations, are thought to be due to a fine equilibrium between their linear dispersive and nonlinear collapse components. And so, in some physical conditions, demanding addition of higher nonlinear terms to (2) or (3) [12–17], this equilibrium is seemingly lost and the system not becomes integrable in a way that never allows for analytical solutions [18–20]. Otherwise, along with the well known NLS (2) and DNLS (3) equations, there has always been a tower of equations in which higher nonlinear terms appear in their integrable hierarchies. However, to restore balance and compensate for higher nonlinearities, higher-order linear dispersive terms appear in these integrable equations. It has been shown that the Eq. (1) is fully integrable [21]. Only the single soliton solution has been solved, which is obtained by integrating in a moving coordinate [6]. As a result, certain investigations must rely on numerical analyses that are appropriate for the application requirements [22]. The one-soliton and N-soliton solutions are provided in [6, 9].

The modified nonlinear Schrödinger (MNLS) equation can be written in the following normalized form:

$$iq_\tau + q_{\xi\xi} + i(q|q|^2)_\xi + 2\alpha q|q|^2 = 0, \tag{4}$$

where α is a real constant.

It is an example of a universal nonlinear model, as the nonlinear Schrödinger (NLS) equation explains a wide range of physical systems. As a result, the equation may be used to describe a wide range of nonlinear physical events [23, 24]. It is generally understood that doing a multi-scale analysis on the Korteweg-de Vries (KdV) equation and many other equations yields the nonlinear Schrödinger (NLS) equation for modulated amplitudes [25]. It was shown by Zakharov and Kuznetsov that multiscale analysis of the Schrödinger spectral problem leads to the Zakharov-Shabat problem for the NLS equation. With this analysis, it has been shown that there is a deeper relationship between the integrable equations, not only at the level of the equations, but also at the level of the linear spectral problems. In this article, we derive the KdV flow equations from the MNLS equation by applying the multi-scale method. This is an important derivation as it is derived from the MNLS equation. When we compare our derivative with the KdV-MNLS derivative, the equations for coefficients of all orders in the epsilon do not contain secular terms. Therefore, there is no freedom in choosing the coefficient and the expansion is uniquely determined. The derivation of this hierarchy is not a simple case of algorithms, but basically relies on two facts. First, different time flows must commute; i.e.: $q_{t_it_j} = q_{t_jt_i}$. Second, In each order in epsilon, the coefficient equations contain secular terms. Eliminating the secular terms requires q_{t_i} to have a certain high symmetry (flow) of the hierarchy, and this way all coefficients of expansion are fixed and no arbitrariness is left.

The paper is organized as follows. In Section 2, we expressed shortly the fifth order Korteweg-de Vries Equation (KdV5) flow equations and in section 3, we derive (KK equation) and (SK equation) as well as KdV flow equations by applying the method to the MNLS equation. Last section is given conclusions. We extensively used Reduce to calculate our results in the paper.

2. Background materials

In this section, we present some background material on the best-known fifth-order KdV equations and the multi-scale method.

2.1 *The fifth order Korteweg-de Vries Equation (KdV5) flow equations*

The best-known fifth-order KdV equations look like this

$$u_t = \omega u_{xxxxx} + \alpha u_{xxx} + \beta u_x u_{xx} + \gamma u^2 u_x. \tag{5}$$

where α, β, γ and ω are arbitrary nonzero and real parameters, and $u = u(x, t)$ is a sufficiently smooth function. This will greatly change the properties of the fKdV equation (5). since the parameters α, β, γ and ω are arbitrary and take different values. Many forms of the fKdV equation can be created by changing the actual values of the parameters. The fKdV equations, which have wide applications in nonlinear optics and quantum mechanics, are an significant mathematical model. Characteristic examples are widely used in various fields such as plasma physics, quantum field theory, solid state physics and liquid physics [26, 27].

Some important particular cases of Eq. (5) are:

Kaup-Kupershmidt equation (KK equation) [28–32]

$$u_t = u_{xxxxx} + 10uu_{xxx} + 25u_x u_{xx} + 20u^2 u_x, \tag{6}$$

Sawada-Kotera equation (SK equation) [33, 34]

$$u_t = u_{xxxxx} + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x, \tag{7}$$

Caudrey-Dodd-Gibbon equation

$$u_t = u_{xxxxx} + 30u_{xxx} + 30u_x u_{xx} + 180u^2 u_x, \tag{8}$$

Lax equation [26]

$$u_t = u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x, \tag{9}$$

Ito equation [35, 36]

$$u_t = u_{xxxxx} + 3u_{xxx} + 6u_x u_{xx} + 2u^2 u_x, \tag{10}$$

Changing the variables α, β and γ change, significantly affects the equation's characteristics (5). For instance, the KK equation with ($\alpha = 10$, $\beta = 25$, and $\gamma = 20$) is integrable and has bilinear representations [30], and has bilinear representations [30, 32], but the exact form of the N-soliton solutions is unknown. These two equations have N-soliton solutions and an unlimited number of conserved densities. Another example is the SK equation where $\alpha = \beta = \gamma = 5$, and the Lax equation with $\alpha = 10$, $\beta = 20$, and $\gamma = 30$, are completely integrable. One last equation in this class is the Ito equation, with $\alpha = 3$, $\beta = 6$, and $\gamma = 2$, which cannot be fully integrated, but have a limited number of special conserved densities [36].

2.2 *The multiple scales method*

We derive the KdV flow equations from the MNLS equation (4) using the multi-scale method developed by Zakharov and Kuznetsov [25]. We also apply the method of deriving Hamilton functions for the KdV flow equations from that of the MNLS equation (4).

Now let's consider the MNLS equation (4) and look for a solution by separating phase and amplitude as follows:

$$q(\xi,\, \tau) = e^{i\theta(\xi,\, \tau)}\sqrt{N(\xi,\, \tau)},\;\; q^*(\xi,\, \tau) = e^{-i\theta(\xi,\, \tau)}\sqrt{N(\xi,\, \tau)} \tag{11}$$

Adding this default solution to the MNLS equation (4) and grouping the real and imaginary parts, respectively, we obtain the following partial differential equations:

$$\begin{aligned} N_\tau &= -2(N\theta)_\xi + 3N_\xi N, \\ \theta_\tau &= \theta_\xi^2 + \frac{N}{\theta_\xi} - 2\alpha N - \frac{N_{\xi\xi}}{2N} - \frac{N_\xi^2}{4N^2} \end{aligned} \tag{12}$$

If we take $\theta(\xi,\, \tau)_\xi = V(\xi,\, \tau)$ then we find a particular case of (12) as a system:

$$\begin{aligned} N_\tau &= -2(NV)_\xi + 3N_\xi N, \\ V_\tau &= \left(V^2 + \frac{N}{V} - 2\alpha N - \frac{N_{\xi\xi}}{2N} + \frac{N_\xi^2}{4N^2}\right)_\xi. \end{aligned} \tag{13}$$

Then we suppose the following series expansions for solutions:

$$\theta = 2\tau + \sum_{n=1}^{\infty} \varepsilon^{2n-1}\theta_n(x,\, t_1,\, t_2,\, \ldots,\, t_n),$$

$$N = 1 + \sum_{n=1}^{\infty} \varepsilon^{2n} N_n(x, t_1, t_2, \ldots, t_n),$$

$$V = \sum_{n=1}^{\infty} \varepsilon^{2n} V_n(x, t_1, t_2, \ldots, t_n). \tag{14}$$

We also define slow variables with respect to the scaling parameter $\varepsilon > 0$ respectively as follows:

$$x = \varepsilon(\xi + 2\tau), \quad t_n = \varepsilon^{2n+1}\tau \ , \ n = 1, 2, \ldots \tag{15}$$

Now we insert series expansions (14) with equations (15) into the system (13). Then we equate the coefficients in the ε powers individually to zero. Thus we get an infinite set of equations for N_n in the powers of ε for each n. If we allow $\varepsilon \to 0$ and zero the terms in the minimum powers of ε, considering the case $n \geq 1$ we get:

(i) For the coefficients of ε^3, we find

$$3N_{1x} + 2V_{1x} = 0,$$

$$(4\alpha - 2)N_{1x} + 6V_{1x} = 0, \tag{16}$$

(ii) For the coefficient of ε^5, we find

$$3N_{2x} + 3N_{1x}N_1 + 2N_{1x}V_1 + N_{1t_1} + 2V_{2x} + 2V_{1x}N_1 = 0,$$

$$4\alpha N_{2x} - 2N_{2x} + N_{1xxx} + (12\alpha - 2)N_{1x}N_1 + 2N_{1x}V_1$$

$$+ 6V_{2x} + 20V_{1x}N_1 - 4V_{1x}V_1 + 2V_{1t_1} = 0, \tag{17}$$

(iii) For the coefficients of ε^7, we find

$$3N_{3x} + 3N_{2x}N_1 + 2N_{2x}V_1 + N_{2t_1} + 3N_{1x}N_2 + 2N_{1x}V_2$$

$$+ N_{1t_2} + 2V_{3x} + 2V_{2x}N_1 + 2V_{1x}N_2 = 0,$$

$$(4\alpha - 2)N_{3x} + N_{2xxx} + (12\alpha - 2)N_{2x}N_1 + 2N_{2x}V_1 + 2N_{1xxx}N_1$$

$$- 2N_{1xx}N_{1x} + (12\alpha - 2)N_{1x}N_2 + 12\alpha N_{1x}N_1^2 + 2N_{2x}N_1V_1$$

$$+2N_{1x}V_2+6V_{3x}+20V_{2x}N_1-4N_{2x}V_1+2V_{2t_1}+20V_{1x}N_2$$

$$+22\alpha V_{1x}N_1^2-12V_{1x}N_1V_1-4V_{1_x}V_2+6V_{1t_1}N_1+2V_{1t_2}=0. \tag{18}$$

(iv) For the coefficients of ε^9, we find:

$$3N_{4x}+3N_{3x}N_1+2N_{3x}V_1+N_{3t_1}+3N_{2x}N_2+2N_{2x}V_2$$

$$+N_{2t_2}+3N_{1x}N_3+2N_{1x}V_3+N_{1t_3}+2V_{4x}+2V_{3x}N_1$$

$$+2V_{2x}N_2+2V_{1x}N_3=0,$$

$$(4\alpha-2)N_{4x}+N_{3xxx}+(12\alpha-2)N_{3x}N_1+2N_{3x}V_1+2N_{2xxx}N_1$$

$$-2N_{2xx}N_{1x}-2N_{2x}N_{1xx}+(12\alpha-2)N_{2x}N_2+12\alpha N_{2x}N_1^2$$

$$+2N_{2x}N_1V_1+2N_{2x}V_2+2N_{1xxx}N_2+N_{1xxx}N_1^2-2N_{1xx}N_{1x}N_1$$

$$+N_{1x}^3+(12\alpha-2)N_{1x}N_3+(24\alpha+2)N_{1x}N_2N_1+2N_{1x}N_2V_1$$

$$+4\alpha N_{1x}N_1^3+2N_{1x}N_1V_2+2N_{1x}V_3+6V_{4x}+20V_{3x}N_1$$

$$-4V_{3x}V_1+2V_{3t_1}+20V_{2x}N_2+22V_{2x}N_1^2-12V_{2x}N_1V_1$$

$$-4V_{2x}V_2+6V_{2t_1}N_1+2V_{2t_2}+20V_{1x}N_3+44V_{1x}N_2N_1$$

$$-12V_{1x}N_2V_1+8V_{1x}N_1^3-12V_{1x}N_1^2V_1-12V_{1x}N_1V_2$$

$$-4V_{1x}V_3+6V_{1t_1}N_2+6V_{1t_1}N_1^2+6V_{1t_1}N_1+2V_{1t_3}=0.$$

$$\vdots \tag{19}$$

and so on. Now taking $\alpha=\frac{11}{4}$ in (16), we find

$$N_1=-\frac{2}{3}V_1, \tag{20}$$

by assuming integration constants as zero.

2.3 *The derivation of KdV flow equations*

We now use (20) in the system (17) and take

$$N_2 = -\frac{2}{3}V_2 + \frac{1}{27}V_{1xx} + \frac{20}{81}V_1^2, \tag{21}$$

so that we find the following equation

$$V_{1t_1} = \frac{1}{18}(3V_{1xxx} + 4V_1V_{1x}), \tag{22}$$

or making the transformation

$$t_1 \to \frac{3}{18}t_1, \quad V_1 \to -\frac{9}{2}u$$

we derive the well known KdV equation

$$u_{t_1} = u_{xxx} + 6uu_x. \tag{23}$$

If we take in the equation (21) for V_2 as

$$V_2 = k_1V_1^2 + k_2V_{1xx}, \tag{24}$$

then insert this into the equation (18), we get the equation

$$V_{1t_2} = \frac{1}{972}\begin{pmatrix} 4374N_{3x} + 2916V_{3x} - (324k_1 + 2160k_2\% - 132)\,V_{1xxx}V_1 - \\ (162k_2 - 9)\,V_{1xxxxx} + (2592k_2 + 36)\,V_{1xx}V_{1x} \\ -(6264k_1 - 160)\,V_{1x}V^2 \end{pmatrix}. \tag{25}$$

Now chosing

$$N_3 = -\frac{2}{3}V_3 + \frac{1}{26244}(-7263V_{1xxxx} - 102096V_{1xx}V_1 - 6156V_{1x}^2 - 38848V_1^3), \tag{26}$$

with

$$k_1 = -\frac{82}{27}, \quad k_2 = -\frac{133}{18},$$

as a result from the equation (25),

$$V_{1t_2} = -\frac{1}{216}(V_{1xxxxx} - \frac{40}{3}V_{1xxx}V_1 - \frac{80}{3}V_{1xx}V_{1x} + \frac{160}{3}V_{1x}V^2)$$

the equation is obtained and using an appropriate transformation for $t_2 \to -\frac{1}{216}t_2,$ and $V_1 \to -\frac{3}{4}u$ we derive the Lax's fifth order KdV equation as t_2 KdV flow equation:

$$u_{t_2} = \left(u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3\right)_x. \tag{27}$$

2.4 *The derivation of Kaup-Kupershmidt equation*

If we take in the equation (21) for V_2 as (24) then add this to the equation (18), we get the equation (25). Now chosing equation (26) with

$$k_1 = -\frac{8}{3}, \quad k_2 = -\frac{53}{8},$$

as a result from the equation (25),

$$V_{1t_2} = -\frac{1}{216}(V_{1xxxxx} - \frac{40}{3}V_{1xxx}V_1 - \frac{100}{3}V_{1xx}V_{1x} + \frac{320}{9}V_{1x}V^2).$$

the equation is obtained and using an appropriate transformation for $t_2 \to -\frac{1}{216}t_2$, $V_1 \to -\frac{4}{3}u$ we derive the Kaup-Kupershmidt equation

$$u_t = u_{xxxxx} + 10uu_{xxx} + 25u_xu_{xx} + 20u^2u_x, \tag{28}$$

as t_2 KdV flow equation.

2.5 *The derivation of Sawada-Kotera equation*

If we take in the equation (21) for V_2 as (24) then add this to the equation (18), we get the equation (25). Now chosing equation (26) with

$$k_1 = -\frac{8}{3}, \quad k_2 = -\frac{77}{12},$$

as a result from the equation (25),

$$V_{1t_2} = -\frac{1}{216}(V_{1xxxxx} - \frac{40}{3}V_{1xxx}V_1 - \frac{40}{3}V_{1xx}V_{1x} + \frac{320}{9}V_{1x}V^2).$$

the equation is obtained and using an appropriate transformation for $t_2 \to -\frac{1}{216}t_2$, $V_1 \to -\frac{3}{8}u$ we derive the Sawada-Kotera equation

$$u_t = u_{xxxxx} + 5uu_{xxx} + 5u_xu_{xx} + 5u^2u_x, \tag{29}$$

as t_2 KdV flow equation.

We now insert (24) and (26) into the system (19) and choose

$$V_3 = k_3V_{1xxxx} + k_4V_{1xx}V_{1x} + k_5V_{1x}^2 + k_6V_1^3, \tag{30}$$

we finally obtain from the coefficients of ε^9, the seventh order KdV flow equation

$$\begin{aligned} u_{t_3} = {} & u_{xxxxxxx} + 14uu_{xxxxx} + 42u_xu_{xxxx} + 70u_{xx}u_{xxx} \\ & + 70u^2u_{xxx} + 280uu_xu_{xx} + 70u_x^3 + 140u^3u_x \end{aligned} \tag{31}$$

where we take

$$k_1 = \frac{323}{378}, \qquad k_2 = \frac{-i\sqrt{12548407} - 5769}{2016},$$

$$k_3 = \frac{817i\sqrt{12548407} - 132195829}{4064256}, \; k_4 = \frac{182i\sqrt{12548407} - 5402217}{92256},$$

$$k_5 = \frac{1051i\sqrt{12548407} - 17213059}{381024}, \; k_6 = -\frac{659513}{107163}$$

and make an appropriate transformation $t_3 \to \frac{1}{3888}t_3$ and $V_1 \to -\frac{7}{10}u$. In general, extending this proceeding the calculation as before, we obtain all equations of KdV flow equations

$$R[u] = \partial^2 + 4u + 2u_x\partial^{-1} \tag{32}$$

the KdV equation the hierarchy recursion operator and

$$u_{\tau_{2n+1}} = R^n[u]\,u_\xi = K_{2n+1}[u], \; n = 1, 2, 3, \ldots \tag{33}$$

the infinite hierarchy of mutually commuting flows satisfies relations.

3. Conclusion

We have used a multiple scales method to provide a new derivation of the KdV flow equations from the MNLS equation. The equations for the coefficients at each order in epsilon, contain no secular terms in our derivation of KdV flow equations. Therefore no freedom is left in choosing coefficients at each order in epsilon and the expansion is uniquely determined. Thus, a relation exists between the MNLS and the KdV flow equations.

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Data availability

All data generated or analysed during this study are included in this published article.

Conflict of interest

The author declares that they have no conflict of interests regarding the publication of this paper.

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