Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

UNIVERSAL WISER PUBLISHER

Research Article

Existence Results for Multivalued Contractive Type Mappings Involving w_b-Distances

Abdul Latif[1]*, Ahad Hamoud Alotaibi[2], Maha Noorwali[3]

[1] Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
[2] Department of Mathematics, King Abdulaziz University, Rabigh Branch, Jeddah, Saudi Arabia
[3] Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia
E-mail: alatif@kau.edu.sa

Received: 28 March 2024; **Revised:** 28 April 2024; **Accepted:** 10 May 2024

Abstract: We present some new results on the existence of fixed points for multivalued contractive type mappings involving generalized distance on metric type spaces. In support of our main results, some examples are also given. Finally, we conclude that our new presented results either improve or generalize some interesting fixed point results of the existing literature.

***Keywords*:** metric type space, fixed point, w_b-distance, multivalued contraction

MSC: 47H09, 54H25

1. Introduction

Introducing the notion of multivalued contractions via Hausdorff metric, Nadler [1] has presented a multivalued version of the well-known Banach Contraction Principle (BCP). Since then, a number of extensions of this interesting result have been appeared. It is worth to mention that many of these results can be extended to various cases without relying on the Hausdorff metric; see [2–4], and others.

The classical metric space has been studied and extended by a number of authors via significant modifications to the metric axioms. Specifically, the concept of a metric type (or b-metric) space represents a valuable extension of the classical metric space. In fact, this idea initially explored by Bakhtin [5], and later refining the idea of b-metric, Czerwik [6] studied some basic fixed point results including the BCP. In this direction much work has been on the existence of fixed points for contraction type mappings; see [7–9] and some other related references [10–13].

In [14], Kada et al. introduced the idea of w-distance on metric spaces and then improve some known result. In [15], Suzuki and Takahashi proposed the concepts of singlevalued and multivalued weakly contractive mappings with respect to w-distance, and then extended a number of classical fixed point results in this context including BCP and Nadler fixed point result. Subsequently, much work of a significant quality has been done in this area; see [16–18] and references therein. Hussain et al. defined the wt-distance on metric type spaces and proved certain fixed point results for singlevalued mappings via wt-distance. For further research work in this area, see [9, 19–21] and others.

DOI: https://doi.org/10.37256/cm.5420244662

In this paper, using the concept of w_b-distance, we first establish key lemmas and then present some new results on the existence of fixed point for some multivalued contractive type mappings involving general conditions. Two nontrivial examples are included. Our results either generalize or improve a number of fixed point results including the corresponding results of Feng and Liu [3], Klim and Wardowski [4], Ciric [2, 22], Latif and Abdou [23, 24], Liu et al. [17, 25, 26] and Latif et al. [27, 28].

2. Materials and methods

Before presenting our main results, we recall some useful notations, concepts, and facts. In this section, we consider X is a metric space with the metric d, otherwise stated. We denote $2^X = \{E \subset X\colon E \neq \emptyset\}$, $C(X) = \{E \subset X\colon E$ is non-empty and closed$\}$, $CB(X) = \{E \subset X\colon E$ is non-empty closed and bounded$\}$. For any $L, Z \in CB(X)$, define

$$H(L, Z) = \max\{\sup_{s\in L} d(s, Z),\ \sup_{z\in Z} d(z, L)\},$$

where $d(s, Z) = \inf_{z\in Z} d(s, z)$. It is known that H is a metric on $CB(X)$, referred to as the Hausdorff metric.

A multivalued mapping $T\colon X \to 2^X$ is called multivalued contraction if for any s, z of X, $H(T(s), T(z)) \leq ad(s, z)$, for a fixed $a \in (0, 1)$. An element $s \in X$ is called a fixed point of T if s in $T(s)$. The set of all fixed points of T will be denoted by $\mathrm{Fix}(T)$. A sequence $\{s_n\}$ in X is called an orbit of T at $s_0 \in X$ if $s_n \in T(s_{n-1})$ for all $n \geq 1$. A map $f\colon X \to \mathbb{R}$ is said to be lower semi-continuous if, for any sequence $\{s_n\} \subset X$ with $s_n \to s \in X$, imply that $f(s) \leq \liminf_{n\to\infty} f(s_n)$. We denote $\mathbb{R}^+ = [0, \infty)$.

Using the concept of Hausdorff metric, Nadler [1] established the following multivalued version of the Banach Contraction Principle.

Theorem 1 [1] For a complete metric space X, each multivalued contraction mapping T from X into $CB(X)$ has a fixed point.

In [29], Mizoguchi and Takahashi generalized Theorem 1 as follows.

Theorem 2 [29] Let T be a closed and bounded valued mapping on X. Assume that X is complete and for all s, z of X, $H(T(s), T(z)) \leq \psi(d(s, z))d(s, z)$, where $\psi\colon \mathbb{R}^+ \to [0, 1)$ with $\limsup_{v\to t^+} \psi(v) < 1$ for every $t \in \mathbb{R}^+$. Then $\mathrm{Fix}(T) \neq \emptyset$.

Without using the Hausdorff metric, Feng and Liu [3] generalized Theorem 1 as follows.

Theorem 3 [3] Let T be a closed valued mapping on X and let h be a lower semi-continuous function on X with $h(s) = d(s, T(s))$. Assume that X is complete and for each s of X and for fixed constants $c, a \in (0, 1)$ with, $c < a$ there is z of $I_a^s = \{z \in T(s)\colon ad(s, z) \leq h(s)\}$ such that $h(z) \leq cd(s, z)$. Then $\mathrm{Fix}(T) \neq \emptyset$.

Later, Klim and Wardowski [4] obtained the following result which contains Theorem 3.

Theorem 4 [4] Let T be a closed valued mapping on X and let h be a lower semi-continuous function on X with $h(s) = d(s, T(s))$. Assume that X is complete and for each s of X and for a fixed constant $a \in (0, 1)$ there is $z \in I_a^s$ such that $h(z) \leq \psi(d(s, z))d(s, z)$, where $\psi\colon \mathbb{R}^+ \to [0, a)$ with $\limsup_{v\to t^+} \psi(v) < a$ for every $t \in \mathbb{R}^+$. Then $\mathrm{Fix}(T) \neq \emptyset$.

In [22], Ciric established a more general fixed point results as follows.

Theorem 5 [22] Let T be a closed valued mapping on X and let h be a lower semi-continuous function on X with $h(s) = d(s, T(s))$. Assume that X is complete and for each s of X there is z of $T(s)$ such that

$$\sqrt{\varphi(d(s, z))}\, d(s, z) \leq h(s) \quad \text{and} \quad h(z) \leq \varphi(d(s, z))\, d(s, z), \tag{1}$$

where φ is a function from $\mathbb{R}^+$ to $[c, 1)$, $c \in (0, 1)$ with $\limsup_{v\to t^+} \varphi(v) < 1$, $t \geq 0$. Then $\mathrm{Fix}(T) \neq \emptyset$.

Remark 1 Note that Theorem 4 generalizes Theorem 1 and Theorem 3. In [4], Klim and Wardowski pointed out that their Theorem 4 do not generalize Theorem 2. However, Theorem 5 generalized all the above mentioned fixed point results.

Theorem 6 [22] Let T be a closed valued mapping on X and let h be a lower semi-continuous function on X with $h(s) = d(s, T(s))$. Assume that X is complete and for each s of X there is z of $T(s)$ such that

$$\sqrt{\varphi(h(s))}\, d(s, z) \leq h(s) \quad \text{and} \quad h(z) \leq \varphi(h(s))\, d(s, z), \tag{2}$$

where φ is a function from $\mathbb{R}^+$ to $[c, 1)$, $c \in (0, 1)$ with $\limsup_{v \to t^+} \varphi(v) < 1$, $t \geq 0$. Then $\mathrm{Fix}(T) \neq \emptyset$.

In [26], Liu et al. extended Theorem 6 and Theorem 3 as follows.

Theorem 7 [26] Let T be a closed valued mapping on X and let h be a lower semi-continuous function on X with $h(s) = d(s, T(s))$. Assume that X is complete and for each s of X there is z of $T(s)$ satisfying

$$\alpha(h(s))d(s, z) \leq h(s) \quad \text{and} \quad h(z) \leq \beta(h(s))\, d(s, z), \tag{3}$$

where

$$\alpha: B \to (0, 1],\ \beta: B \to [0, 1) \text{ with } B = \begin{cases} [0, \sup h(X)] & \text{if } \sup h(X) < \infty \\ [0, \infty) & \text{if } \sup h(X) = \infty \end{cases}$$

such that for all $t \in B$

$$\liminf_{v \to 0^+} \alpha(v) > 0 \quad \text{and} \quad \limsup_{v \to t^+} \frac{\beta(v)}{\alpha(v)} < 1.$$

Then $\mathrm{Fix}(T) \neq \emptyset$.

Kada et al. [14], introduced the concept of w-distance on metric spaces as follows.

A function p: $X \times X \to \mathbb{R}^+$ is called a w-distance on X if it satisfies the following conditions for each $s, z, u \in X$:

(i) $p(s, u) \leq p(s, z) + p(z, u)$;

(ii) the function $p(s, \cdot)$: $X \to \mathbb{R}^+$ is lower semi-continuous (that is, if a sequence $\{z_n\}$ in X with $z_n \to z \in X$, then $p(s, z) \leq \liminf_{n \to \infty} p(s, z_n)$);

(iii) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(u, s) \leq \delta$ and $p(u, z) \leq \delta$ imply $d(s, z) \leq \varepsilon$.

Clearly, any metric d is a w-distance on X. Let $(M, \|\cdot\|)$ be a normed space. Then, the functions p_1, p_2: $M \times M \to \mathbb{R}^+$ defined by $p_1(u, v) = \|v\|$ and $p_2(u, v) = \|u\| + \|v\|$ for all $u, v \in M$ are w-distances [14]. Let Z be a metric space, and let T: $Z \to Z$ be a continuous map. The function p: $Z \times Z \to \mathbb{R}^+$ defined by $p(u, v) = \max\{d(T(u), v), d(T(u), T(v))\}$ for all $u, v \in Z$ is a w-distance [14]. For further examples and properties of w-distance, we refer [14].

Kada et al. [14] improved certain standard conclusions in metric fixed point theory by using the notion of the w-distance. While, Susuki and Takahashi [15] presented fixed point results for singlevalued and multivalued contractive type mappings with respect to w-distance and consequently extended the Nadler fixed point result and BCP. In the existing literature, a number of known metric fixed point results have been generalized with respect to w-distance.

In [24], Latif and Abdou improved Theorem 6 [22, Theorem 2.1] as follows.

Theorem 8 [24] Let p be a w-distance on a complete metric space X and T be a closed valued mapping on X. Assume that h is lower semi-continuous function on X, defined by $h(s) = p(s, T(s))$ and for each s of X there is z of $T(s)$ with

$$\sqrt{\varphi(h(s))}\ p(s, z) \leq h(s) \quad \text{and} \quad h(z) \leq \varphi(h(s))\ p(s, z), \tag{4}$$

where φ is a function from $\mathbb{R}^+$ to $[c, 1)$, $c \in (0, 1)$ with $\limsup_{v \to t^+} \varphi(v) < 1$, $t \geq 0$. Then there exists $z_0 \in X$ such that $h(z_0) = 0$. Further, if $p(z_0, z_0) = 0$, then $z_0 \in T(z_0)$.

Further results in this direction can be founded in [16–18, 25].

In [6, 8] Czerwik introduced the following notion of metric type or (b-metric) space.

Let X be a nonempty set, $b \geq 1$ and D_b: $X \times X \to \mathbb{R}^+$ be a function satisfying the following conditions for all $s, z, u \in X$:

(i) $D_b(s, z) = 0$ if and only if $s = z$;

(ii) $D_b(s, z) = D_b(z, s)$;

(iii) $D_b(s, z) \leq b[D_b(s, u) + D_b(u, z)]$.

Then D_b is called a b-metric on X, and (X, D_b) is called a b-metric space (also known as a metric type space [9]). In the sequel, we also call it a metric type space. Note that, every metric space is a metric type space but the converse may not be true, see [5, 6, 30]. Thus, the family of metric type spaces contains the family of metric spaces.

Example 1 [5] If $X = [0, 1]$ and a function D_b: $X \times X \to \mathbb{R}^+$, defined by $D_b(s, z) = (s - z)^2$, for any $s, z \in X$. Then (X, D_b) is a metric type space with $b = 2$, but it is not a metric space.

For further details of metric type spaces, see [6]. Contrarily to the metric, metric type D_b may not be continuous in each variable, in general; see [31] (Examples 3.9 and 3.10). But, it has been observed that a topology can be defined with convergence on such spaces [31]. A set M in (X, D_b) is called open if and only if for any s of M, there is a positive number ς such that the open ball $B_o(s, \varsigma)$ is contained in M. We denote τ as a collection of all open subsets of X, which becomes a topology on (X, D_b). For metric type spaces, the notions of convergence sequence, Cauchy sequence, etc can be defined usual way as of metric spaces, see [7, 9, 30, 32, 33]. Further, any $M \neq \emptyset$ of X is closed provided any sequence $\{s_n\}$ in M converging to s, implies $s \in M$, see [9]. Also, recall that a real-valued function h on X is b-lower semi-continuous if for any sequence $\{s_n\}$ in X with $s_n \to s \in X$, then $h(s) \leq \liminf_{n \to \infty}(b\, h(s_n))$.

The following basic results for metric type spaces are useful.

Lemma 1 [8] If M is a closed set of (X, D_b) and $s \in X$. Then $D_b(s, M) = 0 \Leftrightarrow s \in \overline{M} = M$, where $D_b(s, M) = \inf\{D_b(s, z)$: $z \in M\}$, and $\overline{M}$ is the closure of the set M.

Lemma 2 [34] Let (X, D_b) be a metric type space and let $\{z_n\}$ be a sequence in X. Assume that there exists $a \in [0, 1)$ satisfying $D_b(z_{n+1}, z_{n+2}) \leq a D_b(z_n, z_{n+1})$ for any $n \in \mathbb{N}$. Then $\{z_n\}$ is Cauchy.

Applying Lemma 2, Suzuki [34] established a general fixed point result for multivalued mappings of metric type spaces and then deduced classical fixed point results due to Nadler [1] and Mizoguchi and Takahashi [29].

Motivated by the work of Kada et al. [14], Hussain et al. [30] defined w-distance on metric type spaces, called it wt-distance (in the sequel, we call it w_b-distance).

Let (X, D_b) be a metric type space. A function p_b: $X \times X \to \mathbb{R}^+$ is called w_b-distance on X, if it satisfies the following conditions for any $s, z, u \in X$:

(i) the b-weighted triangle inequality holds (that is, $p_b(s, u) \leq b[p_b(s, z) + p_b(z, u)]$);

(ii) the function $p_b(s, \cdot)$: $X \to \mathbb{R}^+$ is b-lower semi-continuous (that is for any sequence $\{s_n\}$ in X with $s_n \to x \in X$, then $p_b(s, x) \leq \liminf_{n \to \infty}(b\, p_b(s, s_n))$);

(iii) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p_b(u, s) \leq \delta$ and $p_b(u, z) \leq \delta$ yield $D_b(s, z) \leq \varepsilon$.

Note that for $b = 1$, each w_b-distance reduces to the w-distance.

Example 2 [30] Let $X = \mathbb{R}$ (the set of reals) and $D_b(u, v) = (u - v)^2$, $u, v \in X$. Then, the functions p_{b_1}, p_{b_2}: $X \times X \to \mathbb{R}^+$ defined by $p_{b_1}(u, v) = |u|^2 + |v|^2$ and $p_{b_2}(u, v) = |v|^2$ for every $u, v \in X$ are w_b-distances on X.

Several examples of w_b-distances may be found in [30, 32, 35]. It has been observed that each metric type D_b is a w_b-distance but the converse may not be true, in general [35]. The w_b-function p_b induces via natural way a topology $\tau(p_b)$ on X which can be constructed, as metric type case; that is $\tau(p_b)$ is collection of all sets in X which contains some open ball $B_p(x, \eta)$ of X with respect to p_b, where $B_p(x, \eta) = \{y \in X\colon p_b(x, y) < \eta\}$. Finally, due to uniformity $\tau(p_b)$ turns out a metrizable topology. For further facts concerning w_b-function, see [21, 30, 32, 33, 35].

The following results concerning convergence and Cauchy sequences via w_b-distance, play important roles for the proof of our main results.

Lemma 3 [30] Let (X, D_b) be a metric type space, and let p_b be a w_{b}- distance on X. Let $\{s_n\}$ and $\{z_n\}$ be sequences in X. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $\mathbb{R}^+$ converging to zero. Then the following hold for any $s, z, u \in X$:

(a) if $p_b(s_n, z) \leq \alpha_n$ and $p_b(s_n, u) \leq \beta_n$ for any $n \in \mathbb{N}$, then $z = u$. In particular, if $p_b(s, z) = 0$ and $p_b(s, u) = 0$, then $z = u$;

(b) if $p_b(s_n, z_n) \leq \alpha_n$ and $p_b(s_n, u) \leq \beta_n$ for any $n \in \mathbb{N}$, then $D_b(z_n, u) \to 0$;

(c) if $p_b(s_n, s_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{s_n\}$ is a Cauchy sequence;

(d) if $p_b(z, s_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{s_n\}$ is a Cauchy sequence.

Lemma 4 [21] Let A be a closed subset of a metric type space (X, D_b), and let p_b be a w_b-distance on X. Suppose that there exists $u \in X$ such that $p_b(u, u) = 0$. Then $p_b(u, A) = 0 \Leftrightarrow u \in A$, where $p_b(u, A) = \inf\{p_b(u, v)\colon v \in A\}$.

Let (X, D_b) be a metric type space, $T\colon X \to C(X)$ and $h(u) = p_b(u, T(u)),\ u \in X$. We denote diameter of the space X with $\operatorname{diam}(X) = \sup\{p_b(s, z)\colon s, z \in X\}$, also we define

$$A_{p_b} = \begin{cases} [0, \operatorname{diam}(X)] & \text{if } \operatorname{diam}(X) < \infty \\ [0, \infty) & \text{if } \operatorname{diam}(X) = \infty \end{cases}$$

and

$$B_{p_b} = \begin{cases} [0, \sup h(X)] & \text{if } \sup h(X) < \infty \\ [0, \infty) & \text{if } \sup h(X) = \infty \end{cases}.$$

3. Results

Throughout this section, (X, D_b) is a metric type space and p_b is a w_b-distance on X. In this section, we present our results on the existence of fixed points and iterative approximations for nonlinear multivalued contractive type mappings with respect to w_b-distance on metric type spaces.

First, we prove key lemmas in the setting of metric type spaces.

Lemma 5 Consider a mapping $T\colon X \to C(X)$ with a non-negative real-valued function h on X defined by $h(s) = p_b(s, T(s))$. Assume that the following conditions hold: for any $s \in X$, there is $z \in T(s)$ satisfying

$$\alpha(h(s))\, \varphi(p_b(s, z)) \leq h(s) \quad \text{and} \quad h(z) \leq \beta(h(s))\, \psi(p_b(s, z)), \tag{5}$$

where α and β are functions from B_{p_b} into $(0, 1]$ and $[0, 1)$, respectively, with

$$\beta(0) < \alpha(0),\ \liminf_{v \to 0^+} \alpha(v) > 0,\ \limsup_{v \to t^+} \frac{\beta(v)}{\alpha(v)} < 1,\ \forall t \in B_{p_b} \text{ and } \psi(t) \leq \varphi(t),\ \forall t \in A_{p_b}, \tag{6}$$

where φ and ψ are functions from A_{p_b} into $\mathbb{R}^+$. Then, there exists an orbit $\{s_n\}$ of T in X such that the sequence of non-negative real numbers $\{h(s_n)\}$ is strictly decreasing to zero.

Proof. Put $\gamma(t) = \frac{\beta(t)}{\alpha(t)}$ for all $t \in B_{p_b}$. Note that

$$0 \leq \gamma(t) < 1, \quad \forall t \in B_{p_b}. \tag{7}$$

For any fixed element s_0 of X, there is $s_1 \in T(s_0)$ satisfying

$$\alpha(h(s_0))\,\varphi(p_b(s_0, s_1)) \leq h(s_0) \quad \text{and} \quad h(s_1) \leq \beta(h(s_0))\,\psi(p_b(s_0, s_1)). \tag{8}$$

Note that

$$h(s_1) \leq \beta(h(s_0))\varphi(p_b(s_0, s_1)) \leq \beta(h(s_0))\frac{h(s_0)}{\alpha(h(s_0))}$$

and thus,

$$h(s_1) \leq \gamma(h(s_0))h(s_0).$$

By this way, we can get an orbit $\{s_n\}$ of T at $s_0 \in X$ with $s_{n+1} \in T(s_n)$ satisfying

$$\alpha(h(s_n))\,\varphi(p_b(s_n, s_{n+1})) \leq h(s_n) \quad \text{and} \quad h(s_{n+1}) \leq \beta(h(s_n))\,\psi(p_b(s_n, s_{n+1})). \tag{9}$$

Thus, we get

$$h(s_{n+1}) \leq \gamma(h(s_n))h(s_n). \tag{10}$$

From (7) we have for all $n \geq 0$, $h(s_{n+1}) < h(s_n)$. Thus, the sequence of non-negative real numbers $\{h(s_n)\}$ is strictly decreasing and bounded below, thus convergent. Therefore, there is some $\eta \geq 0$ such that $\lim_{n\to\infty} h(s_n) = \eta$. Suppose that $\eta > 0$. Using (6), (7) and (10) we get

$$\eta \leq \eta \limsup_{v\to\eta^+} \gamma(v) < \eta,$$

which is a contradiction. Hence $\eta = 0$, that is; $\lim_{n\to\infty} h(s_n) = 0$. $\square$

Lemma 6 Suppose that all the hypotheses of Lemma 5 hold. Further, assume that the function φ satisfying the following conditions:

$$\varphi \text{ is subadditive and strictly increasing on } A_{p_b} \text{ with } \lim_{t\to 0^+} \varphi^{-1}(t) = 0, \tag{11}$$

where φ^{-1} is the inverse of the function φ. Then, there exists an orbit $\{s_n\}$ of T in X which is a Cauchy sequence.

Proof. As in the proof of Lemma 5, we get an orbit $\{s_n\}$ of T at $s_0 \in X$ such that

$$\lim_{n\to\infty} h(s_n) = 0. \tag{12}$$

Now, put $l = \limsup_{n\to\infty} \gamma(h(s_n))$ and $c = \liminf_{n\to\infty} \alpha(h(s_n))$. Then we get

$$0 \leq l < 1 \quad \text{and} \quad c > 0. \tag{13}$$

Since $b \geqslant 1$, choose $q \in \left(0, \frac{1}{b}\right)$ with $l < q < 1$. Let $k \in (0, c)$. Then, there is $n_0 > 0$ and for all $n \geq n_0$, we have

$$\gamma(h(s_n)) < q \quad \text{and} \quad \alpha(h(s_n)) > k.$$

Using (9) and (10) we deduce that

$$\varphi(p_b(s_n, s_{n+1})) \leq \frac{h(s_n)}{k} \quad \text{and} \quad h(s_{n+1}) \leq qh(s_n).$$

By induction, for all $n \geq n_0$, we obtain

$$\varphi(p_b(s_n, s_{n+1})) \leq \frac{h(s_{n_0})}{k} q^{n-n_0} \quad \text{and} \quad h(s_{n+1}) \leq q^{n+1-n_0} h(s_{n_0}). \tag{14}$$

Since p_b is the w_b-distance, then for any $n, m \in \mathbb{N}$, $m > n$, we have

$$\begin{aligned}
p_b(s_n, s_m) &\leq b\left[p_b(s_n, s_{n+1}) + p_b(s_{n+1}, s_m)\right] \\
&\leq b\, p_b(s_n, s_{n+1}) + b\left(b\left[p_b(s_{n+1}, s_{n+2}) + p_b(s_{n+2}, s_m)\right]\right) \\
&\leq b\, p_b(s_n, s_{n+1}) + b^2 p_b(s_{n+1}, s_{n+2}) \\
&\quad + b^2\left(b\left[p_b(s_{n+2}, s_{n+3}) + p_b(s_{n+3}, s_m)\right]\right) \\
&\quad \vdots \\
&\leq b\, p_b(s_n, s_{n+1}) + b^2 p_b(s_{n+1}, s_{n+2}) + \ldots \\
&\quad + b^{m-n-1}\left(p_b(s_{m-2}, s_{m-1}) + p_b(s_{m-1}, s_m)\right).
\end{aligned} \tag{15}$$

Since φ is subadditive on A_{p_b} and from (14) we get

$$\varphi(p_b(s_n, s_m)) \leq \frac{b}{k} q^{n-n_0} \left[1 + bq + (bq)^2 + \ldots + (bq)^{m-n-2} + b^{m-n-2} q^{m-n-1}\right] h(s_{n_0}).$$

Since $bq < 1$, then for all $m, n \in \mathbb{N}$ with $m > n \geq n_0$, we have

$$\varphi(p_b(s_n, s_m)) \leq \frac{b q^{n-n_0}}{k(1-bq)} h(s_{n_0}). \tag{16}$$

Since φ is strictly increasing, so does φ^{-1}. Then we obtain

$$p_b(s_n, s_m) = \varphi^{-1}(\varphi(p_b(s_n, s_m))) \leq \varphi^{-1}\left(\frac{b h(s_{n_0})}{k(1-bq)} q^{n-n_0}\right), \quad \text{for all } m > n \geq n_0.$$

Since $\lim_{t \to 0^+} \varphi^{-1}(t) = 0$, then $\varphi^{-1}\left(\frac{bh(s_{n_0})}{k(1-bq)} q^{n-n_0}\right) \to 0$ as $n \to \infty$. We conclude from Lemma 3 that $\{s_n\}$ is a Cauchy sequence in X. □

Now, we present a general result on the existence of fixed points for multivalued mappings of metric type spaces, which improve/generalize a number of known fixed point results.

Theorem 9 Let (X, D_b) be a complete metric type space. Suppose that all the hypotheses of Lemma 6 hold. Assume that the function h is b-lower semi-continuous on X. Then there is some $v \in X$ such that $p_b(v, T(v)) = 0$. Further, if $p_b(v, v) = 0$, then $v \in T(v)$.

Proof. Note that there exists an orbit $\{s_n\}$ of T, which becomes a Cauchy sequence in X. Due to the completeness of the X, there is some $u_0 \in X$ such that $\{s_n\}$ converges to u_0. Now, using the properties of the function h and (12), we obtain

$$0 \leq h(u_0) \leq \liminf_{n \to \infty}(bh(s_n)) = 0,$$

and hence, $h(u_0) = p_b(u_0, T(u_0)) = 0$. If $p_b(u_0, u_0) = 0$, then it follows from Lemma 4 that $u_0 \in T(u_0)$. □

We observe that the conclusion of the Theorem 9 still holds, if we replace the b-lower semi-continuity of the function h with another suitable conditions.

Theorem 10 Suppose that all the hypotheses of Theorem 9 hold except the b-lower semi-continuity of the function h. Assume that one of the following hold for every $u \in X$ with $u \notin T(u)$:

$$\inf\{p_b(s_n, u) + \varphi(p_b(s_n, s_{n+1})) : n \geq 0\} > 0; \tag{17}$$

$$\inf\{p_b(s_n, u) + p_b(s_n, T(s_n)) : n \geq 0\} > 0. \tag{18}$$

Then $\mathrm{Fix}(T) \neq \emptyset$.

Proof. As in the proof of Theorem 9, we get an orbit $\{s_n\}$ of T, which becomes a Cauchy sequence in X. Due to the completeness of the X, there is some $u_0 \in X$ such that $\{s_n\}$ converges to u_0. From (14) we conclude that,

$$\lim_{n\to\infty} \varphi\left(p_b\left(s_n,\, s_{n+1}\right)\right) = 0. \tag{19}$$

Now we show that $\lim_{n\to\infty} p_b\left(s_n,\, u_0\right) = 0$. From Lemma 6 we observe that for all $m > n \geq n_0$

$$p_b\left(s_n,\, s_m\right) = \varphi^{-1}\left(\varphi\left(p_b\left(s_n,\, s_m\right)\right)\right) \leq \varphi^{-1}\left(\frac{b\,h\left(s_{n_0}\right)}{k(1-bq)} q^{n-n_0}\right).$$

Thus by the b-lower semi-continuity of p_b and $\lim_{t\to 0^+} \varphi^{-1}(t) = 0$, we have

$$p_b\left(s_n,\, u_0\right) \leq \liminf_{m\to\infty}(b\, p_b\left(s_n,\, s_m\right)) \leq \varphi^{-1}\left(\frac{b\,h\left(s_{n_0}\right)}{k(1-bq)} q^{n-n_0}\right) \to 0 \quad \text{as } n\to\infty.$$

Suppose that $u_0 \notin T\left(u_0\right)$. If the condition (17) holds. Then we obtain that

$$0 < \inf\left\{p_b\left(s_n,\, u_0\right) + \varphi\left(p_b\left(s_n,\, s_{n+1}\right)\right) : n \geqslant 0\right\} = 0,$$

which is a contradiction. Now if (18) holds, then we conclude that

$$0 < \inf\left\{p_b\left(s_n,\, u_0\right) + p_b\left(s_n,\, T\left(s_n\right)\right) : n \geqslant 0\right\} = 0,$$

which is also contradiction. Thus, $u_0 \in T\left(u_0\right)$. □

Lemma 7 Suppose that all the hypotheses of Lemma 5 with α and β are functions from A_{p_b} into (0, 1] and [0, 1), respectively such that either α or β is non-decreasing on A_{p_b}. Assume that for any $s \in X$, there is $z \in T(s)$ satisfying

$$\alpha(p_b(s,\, z))\ \varphi(p_b(s,\, z)) \leq h(s) \quad \text{and} \quad h(z) \leq \beta(p_b(s,\, z))\ \psi(p_b(s,\, z)). \tag{20}$$

Further, assume that φ is strictly increasing on A_{p_b}. Then, there exists an orbit $\{s_n\}$ of T in X such that the sequence $\{h(s_n)\}$ is decreasing to zero.

Proof. Putting $\gamma(t) = \frac{\beta(t)}{\alpha(t)}$. Note that $0 \leq \gamma(t) < 1$, for all $t \in A_{p_b}$. Following similar arguments as in the proof of Lemma 5, one can construct an iterative sequence $\{s_n\}$ in X such that $s_{n+1} \in T\left(s_n\right)$ and satisfying

$$\alpha\left(p_b(s_n,\, s_{n+1})\right) \varphi\left(p_b\left(s_n,\, s_{n+1}\right)\right) \leq h\left(s_n\right), \tag{21}$$

and

$$h(s_{n+1}) \leq \beta\left(p_b(s_n,\, s_{n+1})\right) \psi(p_b\left(s_n,\, s_{n+1}\right)). \tag{22}$$

For each $n \geq 0$ put $\tau_n = p_b(s_n, s_{n+1})$. Then we obtain that

$$h(s_{n+1}) \leq \gamma(\tau_n)h(s_n). \tag{23}$$

Using (21) and (22) we get

$$\varphi(\tau_{n+1}) \leq \frac{\beta(\tau_n)\psi(\tau_n)}{\alpha(\tau_{n+1})}. \tag{24}$$

Now we claim that $\tau_{n+1} \leq \tau_n$, for all $n \geq 0$. Suppose that there is a positive integer n_0 satisfying $\tau_{n_0+1} > \tau_{n_0}$, it follows from (24) that

$$\varphi\left(\tau_{n_0+1}\right) \leq \frac{\beta(\tau_{n_0})\psi(\tau_{n_0})}{\alpha\left(\tau_{n_0+1}\right)},$$

as either α or β is non-decreasing, we have $\beta(\tau_{n_0+1}) > \beta(\tau_{n_0})$. Hence,

$$\begin{aligned}
\varphi\left(\tau_{n_0+1}\right) &\leq \frac{\beta(\tau_{n_0})\psi(\tau_{n_0})}{\alpha\left(\tau_{n_0+1}\right)} \\
&\leq \frac{\beta(\tau_{n_0+1})\psi(\tau_{n_0})}{\alpha\left(\tau_{n_0+1}\right)} \\
&= \gamma(\tau_{n_0+1})\psi(\tau_{n_0}) \\
&\leq \max\left\{\gamma\left(\tau_{n_0+1}\right), \gamma\left(\tau_{n_0}\right)\right\}\psi(\tau_{n_0}).
\end{aligned}$$

Since $\gamma(t) < 1$ for each $t \in A_{p_b}$ and from (6) we have $\varphi\left(\tau_{n_0+1}\right) \leq \varphi(\tau_{n_0})$. Since φ is strictly increasing, so we get that

$$\varphi\left(\tau_{n_0+1}\right) \leq \varphi(\tau_{n_0}) < \varphi\left(\tau_{n_0+1}\right),$$

which is impossible. Thus, $\tau_{n+1} \leq \tau_n$, for all $n \geq 0$, that is the sequence $\{\tau_n\}$ is non-negative and decreasing. Hence there is some $\theta \geq 0$ such that $\lim_{n\to\infty} \tau_n = \theta$. Now we show that $\lim_{n\to\infty} h(s_n) = 0$. Since $\gamma(t) < 1$ for all $t \in A_{p_b}$, we conclude that the sequence $\{h(s_n)\}$ is strictly decreasing and bounded below, thus convergent. Therefore, there is some $\eta \geq 0$ such that $\lim_{n\to\infty} h(s_n) = \eta$. Suppose that $\eta > 0$. Using (23), we get

$$\eta \leq \eta \limsup_{t\to\theta^+} \gamma(t) < \eta,$$

which is a contradiction. Thus, $\lim_{n\to\infty} h(s_n) = 0$. □

Lemma 8 Suppose that all the hypotheses of Lemma 7 hold with condition (11). Then, there exists an orbit $\{s_n\}$ of T in X which is a Cauchy sequence.

Proof. Put $l = \limsup_{n\to\infty} \gamma(p_b(s_n, s_{n+1}))$ and $c = \liminf_{n\to\infty} \alpha(p_b(s_n, s_{n+1}))$. It follows from (6) that $0 \leq l < 1$ and $c > 0$. Since $b \geqslant 1$, choose $q \in \left(0, \frac{1}{b}\right)$ with $l < q < 1$. Let $k \in (0, c)$. Then there is a positive integer n_0 such that for all $n \geq n_0$ we have

$$\gamma(p_b(s_n, s_{n+1})) < q \quad \text{and} \quad \alpha(p_b(s_n, s_{n+1})) > k,$$

using (21) and (23) we obtain

$$\varphi(p_b(s_n, s_{n+1})) \leq \frac{h(s_n)}{k} \quad \text{and} \quad h(s_{n+1}) \leq qh(s_n),$$

then the following inequalities have been observed in the proof of Lemma 6, for all $n \geq n_0$

$$\varphi(p_b(s_n, s_{n+1})) \leq \frac{h(s_{n_0})}{k} q^{n-n_0} \quad \text{and} \quad h(s_{n+1}) \leq q^{n+1-n_0} h(s_{n_0})$$

$$p_b(s_n, s_m) \leq b\, p_b(s_n, s_{n+1}) + b^2 p_b(s_{n+1}, s_{n+2}) + \ldots + b^{m-n-1}\left(p_b(s_{m-2}, s_{m-1}) + p_b(s_{m-1}, s_m)\right).$$

Proceeding as in the proof of Lemma 6, we can get an orbit $\{s_n\}$ of T in X which is a Cauchy sequence. □

Following the similar method as in the proof of Theorem 9, we can obtain the following fixed point result.

Theorem 11 Let (X, D_b) be a complete metric type space. Suppose that all the hypotheses of Lemma 8 hold. Assume that the function h is b-lower semi-continuous on X. Then, there is some $u_0 \in X$ such that $p_b(u_0, T(u_0)) = 0$. Further, if $p_b(u_0, u_0) = 0$, then $u_0 \in T(u_0)$.

Following the proof of Theorem 9, and techniques of Theorem 10, we have the following result which extend the results [17, Theorem 2.2] and [25, Theorem 3.4].

Theorem 12 Suppose that all the hypotheses of Theorem 11 hold except the b-lower semi-continuity of the function h. Assume that either the condition (17) or the condition (18) hold. Then $\text{Fix}(T) \neq \emptyset$.

Now we present the following example in support of Theorem 9.

Example 3 Let $X = [0, 1] \cup \{\frac{13}{10}\}$. For each $s, z \in X$, we define $D_b(s, z) = (s - z)^2$ and $p_b(s, z) = z^2$. Then X is a metric type space with $b = 2$ and p_b is a w_b-distance on X. Let $T\colon X \to C(X)$ be a multivalued mapping defined by

$$T(s) = \begin{cases} \left\{\frac{s^2}{2}\right\}, & s \in \left[0, \frac{7}{10}\right) \cup \left(\frac{7}{10}, 1\right] \\ \left\{\frac{7}{40}, \frac{9}{40}\right\}, & s \in \left\{\frac{7}{10}, \frac{13}{10}\right\} \end{cases}$$

and define the functions α, β from $[0, \frac{1}{4}]$ to $(0, 1]$, $[0, 1)$ respectively and φ, ψ: $\left[0, \frac{169}{100}\right] \to \mathbb{R}^+$ by

$$\alpha(t) = \frac{4+\sqrt{t}}{5}, \quad \beta(t) = \frac{3+\sqrt{t}}{5}, \quad \forall t \in \left[0, \frac{1}{4}\right]$$

$$\varphi(t) = t, \quad \forall t \in \left[0, \frac{169}{100}\right], \quad \psi(t) = \begin{cases} \frac{t}{2}, & t \in [0, \frac{169}{100}) \\ 0, & t = \frac{169}{100} \end{cases}$$

it is easy to see that $A_{p_b} = \left[0, \frac{169}{100}\right]$, $B_{p_b} = \left[0, \frac{1}{4}\right]$, $\psi(t) \leq \varphi(t)$ for all $t \in \left[0, \frac{169}{100}\right]$ and φ is subadditive and strictly increasing on A_{p_b} with $\lim_{t \to 0^+} \varphi^{-1}(t) = 0$. Note that

$$h(s) = p_b(s, T(s)) = \begin{cases} \frac{s^4}{4}, & s \in \left[0, \frac{7}{10}\right) \cup \left(\frac{7}{10}, 1\right] \\ \left(\frac{7}{40}\right)^2, & s \in \{\frac{7}{10}, \frac{13}{10}\} \end{cases}$$

is b-lower semi-continuous. Moreover, for each $t \in B_{p_b}$

$$\beta(0) = \frac{3}{5} < \frac{4}{5} = \alpha(0), \ \liminf_{\nu \to 0^+} \alpha(\nu) = \frac{4}{5} > 0$$

and

$$\limsup_{\nu \to t^+} \frac{\beta(\nu)}{\alpha(\nu)} = \frac{3+\sqrt{t}}{4+\sqrt{t}} < 1.$$

For each $s \in \left[0, \frac{7}{10}\right) \cup \left(\frac{7}{10}, 1\right]$, there exists $z = \frac{s^2}{2} \in T(s) = \left\{\frac{s^2}{2}\right\}$ satisfying

$$\alpha\left(h(s)\right) \varphi(p_b(s, z)) = \left(\frac{4+\frac{s^2}{2}}{5}\right)\left(\frac{s^4}{4}\right) \leq \frac{s^4}{4} = h(s)$$

and

$$h(z) = \frac{(\frac{s^2}{2})^4}{4} = \left(\frac{s^2}{16}\right)\left(\frac{s^4}{4}\right) \leq \left(\frac{3+\frac{s^2}{2}}{10}\right)\left(\frac{s^4}{4}\right) = \beta\left(h(s)\right) \psi(p_b(s, z)).$$

Letting, $s \in \left\{\frac{7}{10}, \frac{13}{10}\right\}$, we have $T(s) = \left\{\frac{7}{40}, \frac{9}{40}\right\}$. Clearly, there exists $z = \frac{7}{40} \in T(s)$ such that

$$\alpha\left(h(s)\right) \varphi(p_b(s, z)) = \left(\frac{4+\frac{7}{40}}{5}\right)\left(\frac{7}{40}\right)^2 \leq \left(\frac{7}{40}\right)^2 = h(s)$$

and

$$h(z) = \frac{(\frac{7}{40})^4}{4} \leq \left(\frac{3+\frac{7}{40}}{10}\right)\left(\frac{7}{40}\right)^2 = \beta\,(h(s))\,\psi(p_b(s,\,z)).$$

Thus, all the assumptions of Theorem 9 are satisfied. Hence, $\text{Fix}(T) \neq \emptyset$ and $\text{Fix}(T) = \{0\}$. Note that p_b is not a metric on X, consequently the results of [26, Theorem 2.1], [22, Theorem 2.1] and [3, Theorem 3.1] are unapplicable. In addition, note that p_b is not a w-distance on X, so the results of [17, Theorem 2.1] and [25, Theorem 3.1] are not applicable.

We present the following example in support of Theorem 11.

Example 4 Let $X = \mathbb{R}^+$ with D_b and p_b as in Example 3. Let $T\colon X \to C(X)$ be defined by

$$T(s) = \begin{cases} \left\{\frac{s}{4}\right\}, & s \in [0,\,1] \\ \left\{0,\, s-\frac{1}{3}\right\}, & s \in (1,\,\infty) \end{cases}$$

Define the functions $\alpha\colon \mathbb{R}^+ \to (0,\,1]$, $\beta\colon \mathbb{R}^+ \to [0,\,1)$ and $\varphi,\ \psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\alpha(t) = \begin{cases} \frac{40+t}{500}, & t \in [0,\,1] \\ \frac{95+t^5}{100+t^5}, & t \in (1,\,\infty) \end{cases}, \quad \beta(t) = \begin{cases} \frac{30+t}{500}, & t \in [0,\,1] \\ \frac{130}{170+t^5}, & t \in (1,\,\infty) \end{cases}$$

$$\varphi(t) = 2t, \quad \forall t \in [0,\,\infty), \quad \psi(t) = \begin{cases} 2t, & t \in [0,\,1) \\ 0, & t \in [1,\,\infty) \end{cases}$$

it is easy to see that $A_{p_b} = [0,\,\infty)$, $\psi(t) \leq \varphi(t)$ for all $t \in A_{p_b}$ and φ is subadditive and strictly increasing on A_{p_b} with $\lim\limits_{t\to 0^+} \varphi^{-1}(t) = 0$. Note that

$$h(s) = p_b(s,\,T(s)) = \begin{cases} \frac{s^2}{16}, & s \in [0,\,1] \\ 0, & s \in (1,\,\infty) \end{cases}$$

is b-lower semi-continuous and α is nondecreasing. If $t \in [0,\,1]$, then

$$\beta(0) = \frac{30}{500} < \frac{40}{500} = \alpha(0),\ \ \liminf_{v\to 0^+} \alpha(v) = \frac{4}{50} > 0,$$

and

$$\limsup_{v\to t^+}\frac{\beta(v)}{\alpha(v)}=\frac{30+t}{40+t}<1.$$

If $t\in(1,\infty)$, then

$$\liminf_{v\to 0^+}\alpha(v)=\frac{95}{100}>0$$

and

$$\limsup_{v\to t^+}\frac{\beta(v)}{\alpha(v)}=\limsup_{v\to t^+}\frac{130(100+v^5)}{(170+v^5)(95+v^5)}=\frac{13000+130t^5}{16150+265t^5+t^{10}}<1.$$

For each $s\in[0,1]$, there exists $z=\frac{s}{4}\in T(s)=\left\{\frac{s}{4}\right\}$ satisfying

$$\alpha\left(p_b(s,z)\right)\varphi(p_b(s,z))=\left(\frac{80+\frac{s^2}{8}}{500}\right)\left(\frac{s^2}{16}\right)\le\frac{s^2}{16}=h(s)$$

and

$$h(z)=\frac{(\frac{s}{4})^2}{16}=\frac{s^2}{256}\le\left(\frac{30+\frac{s^2}{16}}{500}\right)\left(\frac{2x^2}{16}\right)=\beta\left(p_b(s,z)\right)\psi(p_b(s,z))$$

if $s\in(1,\infty)$, there exists $z=0\in T(s)=\left\{0,\, s-\frac{1}{3}\right\}$ satisfying

$$\alpha\left(p_b(s,z)\right)\varphi(p_b(s,z))=0=h(s)$$

and

$$h(z)=0=\beta\left(p_b(s,z)\right)\psi(p_b(s,z)).$$

Thus, for each $s\in\mathbb{R}^+$, all the conditions of Theorem 3.3 are satisfied. Hence, $\mathrm{Fix}(T)\neq\emptyset$ and $\mathrm{Fix}(T)=\{0\}$. Note that p_b is a w_b-distance but not a metric on X, so T does not satisfy the hypotheses of [26, Theorem 2.3], [22, Theorem 2.2], [3, Theorem 3.1], [4, Theorem 2.1] and [2, Theorem 6]. Further, note that p_b is a w_b-distance but not a w-distance on X. Therefore, the results of [17, Theorem 2.2] and [25, Theorem 3.3] are not applicable.

4. Conclusion

(1) Theorem 9 generalizes the corresponding fixed point results of Liu et al. [17, Theorem 2.1] and [25, Theorem 3.1]. Further, Theorem 9 contains [26, Theorem 2.1] of Liu et al. as a special case.

(2) Theorem 9 extends fixed point results of Feng and Liu [3, Theorem 3.1], Ciric [22, Theorem 2.1], Latif and Albar [24, Theorem 2.1], [23, Theorem 2.2], Latif et al. [27, Theorem 2.1] and [28, Theorem 3.1].

(3) Theorem 10 generalizes fixed point results of Liu et al. [17, Theorem 2.1] and [25, Theorem 3.2].

(4) Theorem 10 extends fixed point results of Latif and Albar [24, Theorem 2.2], [23, Theorem 2.4], Latif et al. [27, Theorem 2.2] and [28, Theorem 3.2].

(5) Theorem 11 generalizes fixed point results of Liu et al. [17, Theorem 2.2] and [25, Theorem 3.3]. Further, Theorem 11 contains [26, Theorem 2.3] of Liu et al. as a special case.

(6) Theorem 11 extends and unifies the fixed point results of Feng and Liu [3, Theorem 3.1], Klim and Wardowski [4, Theorem 2.1], Ciric [22, Theorem 2.2], Latif and Albar [24, Theorem 2.3], Ciric [2, Theorem 6], Latif et al. [27, Theorem 2.3] and [28, Theorem 3.3].

(7) Theorem 12 generalizes the fixed point results of Latif and Albar [24, Theorem 2.5], Latif et al. [27, Theorem 2.4] and [28, Theorem 3.4].

Acknowledgments

The authors thank the learned referees for their valuable comments and suggestions.

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] Nadler SB. Multi-valued contraction mappings. *Pacific Journal of Mathematics*. 1969; 30(2): 475-488.

[2] Ciric LB. Fixed point theorems for multi-valued contractions in complete metric spaces. *Journal of Mathematical Analysis and Applications*. 2008; 348(1): 499-507. Available from: https://doi.org/10.1016/j.jmaa.2008.07.062.

[3] Feng Y, Liu S. Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings. *Journal of Mathematical Analysis and Applications*. 2006; 317(1): 103-112. Available from: https://doi.org/10.1016/j.jmaa.2005.12.004.

[4] Klim D, Wardowski D. Fixed point theorems for set-valued contractions in complete metric spaces. *Journal of Mathematical Analysis and Applications*. 2007; 334(1): 132-139. Available from: https://doi.org/10.1016/j.jmaa.2006.12.012.

[5] Bakhtin IA. The contraction mapping principle in almost metric spaces. *Functional Analysis and Its Applications*. 1989; 30: 26-37.

[6] Czerwik S. Nonlinear set-valued contraction mappings in b-metric spaces. *Proceedings of the Seminar on Mathematics and Physics of the University of Modena*. 1998; 46: 263-276.

[7] Agarwal RP, Karapinar E, O'Regan D, Roldán-López-de Hierro AF. *Fixed point theory in metric type spaces*. Switzerland: Springer International Publishing; 2015.

[8] Czerwik S, Dlutek K, Singh SL. Round-off stability of iteration procedures for set-valued operators in b-metric spaces. *Journal of Natural and Physical Sciences*. 2001; 11: 87-94. Available from: https://api.semanticscholar.org/CorpusID:145916382.

[9] Khamsi MA, Hussain N. KKM mappings in metric type spaces. *Nonlinear Analysis: Theory, Methods and Applications*. 2010; 73(9): 3123-3129. Available from: https://doi.org/10.1016/j.na.2010.06.084.

[10] Chuensupantharat N, Gopal D. On Caristi fixed point theorem in metric spaces with a graph. *Carpathian Journal of Mathematics*. 2020; 36(2): 259-268. Available from: https://doi.org/10.37193/CJM.2020.02.09.

[11] Gopal D, Agarwal P, Kumam P. *Metric structures and fixed point theory*. CRC Press; 2021.

[12] Lakzian H, Gopal D, Sintunavarat W. New fixed point results for mappings of contractive type with an application to nonlinear fractional differential equations. *Journal of Fixed Point Theory and Applications*. 2016; 18: 251-266. Available from: https://doi.org/10.1007/s11784-015-0275-7.

[13] Rouzkard F, Imdad M, Gopal D. Some existence and uniqueness theorems on ordered metric spaces via generalized distances. *Journal of Fixed Point Theory and Applications*. 2013; 2013: 1-20. Available from: https://doi.org/10.1186/1687-1812-2013-45.

[14] Kada O, Suzuki T, Takahashi W. Nonconvex minimization theorems and fixed point theorems in complete metric spaces. *Mathematica Japonicae*. 1996; 44: 381-391. Available from: https://api.semanticscholar.org/CorpusID:120511018.

[15] Suzuki T, Takahashi W. Fixed point theorems and characterizations of metric completeness. *Topological Methods in Nonlinear Analysis*. 1996; 8: 371-382.

[16] Kaneko S, Takahashi W, Wen CF, Yao JC. Existence theorems for single-valued and multivalued mappings with *w*-distances in metric spaces. *Fixed Point Theory and Applications*. 2016; 2016: 1-15. Available from: https://doi.org/10.1186/s13663-016-0527-2.

[17] Liu Z, Wang X, Kang SM, Cho SY. Fixed points for mappings satisfying some multi-valued contractions with *w*-distance. *Fixed Point Theory and Applications*. 2014; 2014: 1-17. Available from: https://doi.org/10.1186/1687-1812-2014-246.

[18] Rakoćević V. *Fixed point results in w-distance spaces*. Chapman and Hall/CRC; 2021.

[19] Demma M, Saadati R, Vetro P. Multi-valued operators with respect to *wt*-distance on metric type spaces. *Bulletin of the Iranian Mathematical Society*. 2016; 42(6): 1571-1582.

[20] Fallahi K, Savic D, Soleimani Rad G. The existence theorem for contractive mappings on *wt*-distance in *b*-metric spaces endowed with a graph and its application. *Sahand Communications in Mathematical Analysis*. 2019; 13(1): 1-15. Available from: https://doi.org/10.22130/scma.2018.89571.471.

[21] Latif A, Al Subaie R, Alansari M. Metric fixed points for contractive type mappings and applications. *Journal of Nonlinear Convex Analysis*. 2022; 23: 501-511.

[22] Ciric LB. Multivalued nonlinear contraction mappings. *Nonlinear Analysis: Theory, Methods and Applications*. 2009; 71(7-8): 2716-2723. Available from: https://doi.org/10.1016/j.na.2009.01.116.

[23] Latif A, Abdou A. Fixed points of generalized contractive maps. *Journal of Fixed Point Theory and Applications*. 2009; 487161. Available from: https://doi.org/10.1155/2009/487161.

[24] Latif A, Abdou A. Multivalued generalized nonlinear contractive maps and fixed points. *Nonlinear Analysis*. 2011; 74(4): 1436-1444. Available from: https://doi.org/10.1016/j.na.2010.10.017.

[25] Liu Z, Lu Y, Kang SM. Fixed point theorems for multi-valued contractions with *w*-distance. *Applied Mathematics and Computation*. 2013; 224: 535-552. Available from: https://doi.org/10.1016/j.amc.2013.08.061.

[26] Liu Z, Sun W, Kang SM, Ume JS. On fixed point theorems for multi-valued contractions. *Fixed Point Theory and Applications*. 2010; 2010: 1-18. Available from: https://doi.org/10.1155/2010/870980.

[27] Latif A, Alotaibi A, Noorwali M. Multivalued nonlinear contractive type mappings and fixed points. *Journal of Nonlinear Convex Analysis*. 2023. Available from: https://doi.org/10.1016/j.na.2010.10.017.

[28] Latif A, Alotaibi A, Noorwali M. Fixed point results via multivalued contractive type mappings involving generalized distance on metric type spaces. *Journal of Nonlinear Variational Analysis*. 2024; 8(5): 787-798. Available from: https://doi.org/10.23952/jnva.8.2024.5.06.

[29] Mizoguchi N, Takahashi W. Fixed point theorems for multivalued mappings on complete metric spaces. *Journal of Mathematical Analysis and Applications*. 1989; 141: 177-188.

[30] Hussain N, Saadati R, Agrawal RP. On the topology and *wt*-distance on metric type spaces. *Fixed Point Theory and Applications*. 2014; 88: 2014. Available from: https://doi.org/10.1186/1687-1812-2014-88.

[31] An TV, Tuyen LQ, Dungć NV. Stone-type theorem on *b*-metric spaces and applications. *Topology and its Applications*. 2015; 185-186: 50-64. Available from: https://doi.org/10.1016/j.topol.2015.02.005.

[32] Karapinar E, Chifu C. Results in *wt*-distance over *b*-metric spaces. *Mathematics*. 2020; 8(2): 220. Available from: https://doi.org/10.3390/math8020220.

[33] Romaguera S. An application of *wt*-distances to characterize complete *b*-metric spaces. *Axioms*. 2023; 12(2): 121. Available from: https://doi.org/10.3390/axioms12020121.

[34] Suzuki T. Basic inequality on a *b*-metric space and its applications. *Journal of Inequalities and Applications*. 2017; 2017(1): 256. Available from: https://doi.org/10.1186/s13660-017-1528-3.

[35] Ghosh SK, Nahak C. An extension of Lakzian-Rhoades results in the structure of ordered b-metric spaces via wt-distance with an application. *Applied Mathematics and Computation*. 2020; 378: 125197. Available from: https://doi.org/10.1016/j.amc.2020.125197.