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Research Article

Biharmonic Extensions on Infinite Trees

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Abstract: In the investigation of harmonic and potential functions on the Euclidean spaces, the Runge-type approximation theorem and Laurent decomposition theorem for harmonic functions are important. Their extensions to subharmonic functions are also crucial. In this note, we investigate various aspects of these results in the context of discrete potential theory on infinite trees. Given an infinite tree T with positive potentials, we prove that for a harmonic function h outside a finite set, there exists a harmonic function H on T such that $h-H$ is bounded outside a finite set. Developing other results based on this theorem, we investigate in detail biharmonic functions on T and study their properties. The thrust is to extend these results to the study of discrete biharmonic and bisuperharmonic functions on infinite trees. This is always true in $\mathbb{R}^n$, $n \geq 5$ because the fundamental solution of Δ^2 on this case tends to 0 at infinity. Based on this property we also define the notion of a tapered biharmonic space.

Keywords: Biharmonic functions, tapered space

MSC: 31A05, 31A10, 31A30

1. Introduction

The study of biharmonic functions on $\mathbb{R}^n$ by Nicolesco [1] with the help of the Almansi representation leads to Liouville-Picard-Hadamard theorem for biharmonic functions; and that of bisuperharmonic functions by Smyrnelis [2] in the framework of Brelot-Bauer axiomatic potential theory on locally compact spaces is carried out extending the known properties of harmonic functions and potentials on the Euclidean spaces. A discrete analogue of the biharmonic Green kernel is given in Yamasaki [3] when the transition functions of the infinite network are symmetric. A discrete biharmonic calculus for Δ^2 is presented in Ben-Artzi and Katriel [4]; and a recently published paper by Bajunaid [5] which deals with biharmonic functions on Schrödinger networks. Motivation for this work is provided by the work of Anandam in axiomatic potential theory where he used a local Riesz representation to give a meaning to an equation of the form $\Delta u = f$, where Δ has a minimal use as a differential operator (see [6]). Consequently, the biharmonic function and related notions can be defined not only in a Riemannian manifold and in $\mathbb{R}^n$, but also in any Riemann surface and more generally in any infinite tree.

Our aim in this work is to prove extension theorems for biharmonic functions and a consequence a very useful representation for biharmonic functions defined outside a finite set is obtained. Section 2 is the preliminaries giving definitions and notation on trees. The definition of harmonic functions and potentials are recalled (see [7]); some important

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properties of superharmonic functions are obtained. Riesz and Laurent decomposition are investigated. In Section 3 which entitled "Bisuperharmonic functions near infinity on $\mathbb{R}^n$", we introduce the biharmonic and the bisuperharmonic functions on the Euclidean spaces without making use of the derivatives. That would serve as a model in the development of discrete biharmonic and bisuperharmonic functions on infinite trees. Section 4, outlines the concepts of bisuperharmonic functions on infinite trees and prove the discrete version of the classical Riquier problem in a general form. Section 5 on "Biharmonic Extensions", a study of bipotentials which are discrete versions of the fundamental solutions of Δ^2 on $\mathbb{R}^n$, $n \geq 5$ is concerned. In Section 6, many properties of biharmonic functions are developed: Laurent decomposition for biharmonic functions defined on ring domains, the problem of finding biharmonic function B in a tree T such that for biharmonic function b given outside a finite set in T, $(b-B)$ is bounded near infinity. Finally, based on this last property, we define the notion of a tapered bipotential trees and prove that in such a space above-stated problem has a solution. A tapered tree is a discrete counterpart of the Euclidean spaces $\mathbb{R}^n$, $n \geq 5$, in which the fundamental solution of Δ^2 is proportional to r^{4-n}, hence tends to 0 at the point at infinity.

2. Preliminaries

In mathematical terms, a tree T is a special kind of drawing (graph) with vertices connected by edges. The key thing is that there are no circles or loops, and each dot only connects to a limited number of others. We can also think of the tree as just the collection of all its vertices. In a tree, vertices called v and w are considered neighbors if an edge directly connects them. We use $v \sim w$ to show this connection. Imagine following a sequence of connected vertices in a tree, like hopping from one to the next. This sequence, written as $[x_0, x_1, \ldots]$, such that $x_k \sim x_{k+1}$ is called a path. A special type of path, called *geodesic path* is a path $[x_0, x_1, \ldots]$ such that $x_{k-1} \neq x_{k+1}$ for all k. A *ray is* an infinite geodesic path. If x and y are any vertices, the unique geodesic path joining them is denoted by $[x, y]$. Fixing a nonterminal vertex e as a root of the tree, the predecessor u^- of a vertex u, with $u \neq e$, is the next to the last vertex of the path from e to u.

Imagine a tree where every vertex has the same number of neighbors. A homogeneous tree of degree $q+1$ (where q is a number greater than or equal to 2) is a tree such that each vertex has exactly $q+1$ neighbors.

Definition 2.1 For a finite subset S of T, the interior of S is the set $\overset{\circ}{S}$ only includes vertices $x \in S$ such that every vertex of T which is neighbor of x belongs to S. In contrast, the boundary of S in T is defined as the set $\partial S = S \backslash \overset{\circ}{S}$.

Let $B_n = \{x : |x| \leq n\}$, $n \geq 1$.

A tree T may be endowed with a metric d as follows: If x and y are vertices, then $d(x, y)$ is the number of edges you need to traverse along the geodesic path to get from x to y. Additionally, we can assign a *length* to each vertex x based on its distance from e. This length is written as $|x| = d(x, e)$. Given $x \in T$, let $N(x)$ denotes the set of neighbors of x.

Given a tree T, let p be a nearest-neighbor *transition probability* on the vertices of T, that is, $p(x, y) \geq 0$, $p(x, y) > 0$, if and only if x and y are neighbors, and for any fixed vertex x, $\sum_{y \sim x} p(x, y) = 1$. By a function on a tree, we mean a function on its vertices.

The *Laplacian* of a function $f\colon T \to \mathbb{R}$ is defined as

$$-\Delta f(x) = \sum_{y \sim x} p(x,y)\left[f(x) - f(y)\right] \quad \text{for all vertices } \; x \in T.$$

Definition 2.2 A real valued function u on T is considered *harmonic* at x if $\Delta u(x)$ equals zero. A real valued function u on T is said to be superharmonic at x if $\Delta u(x)$ is less than or equal to zero. Conversely, a real valued function u on T is said to be subharmonic if $\Delta u(x)$ is greater than or equal to zero. Finally, a potential is a special type of function. It has non-negative values and is superharmonic, but with an important twist: there's no positive harmonic function on the tree that's less than or equal to it.

Definition 2.3 The harmonic support of a superharmonic function s is the complement of the largest open set on which s is harmonic.

Definition 2.4 A P-tree is a tree on which there is a positive potential. An S-tree is a tree on which no positive potential exists.

Theorem 2.5 For some $n > 1$, let $u(x)$ be a real-valued function defined for $|x| = n$. Then $u(x)$ extends as a harmonic function on T.

Proof. First extend $u(x)$ harmonically on $|x| < n$ (take the Dirichlet solution on $|x| < n$ with boundary value u (see Theorem 2 [8] and [9]). Here's the next step: imagine expanding the area by one layer, including vertices with distance $|x| = n+1$. Note $\{x : |x| = n+1\} \subset \cup_{|y|=n} N(y)$. Assign a value α to all the new vertices on $N(y) \cap \{x : |x| = n+1\}$, for some y with $|y| = n$, such that u is harmonic at y. Since y is arbitrary except for the condition $|y| = n$, this method defines the function $u(x)$ for $|x| = n+1$ also. Thus $u(x)$ is defined on B_{n+1} and $\Delta u(x) = 0$ on $\overset{\circ}{B}_{n+1}$. By induction, $u(x)$ is defined harmonically on T. □

Corollary 2.6 Let u be harmonic for $|x| \leq n$, for some $n \geq 1$. Then we can expand u to encompass the entire tree while preserving its harmonic property and the resulting function $h(x) = u(x)$ for $|x| \leq n$.

Theorem 2.7 For a vertex e in T, there is a function $q_e(x)$ on T such that $-\Delta q_e(x) = \delta_e(x)$, the Dirac measure at e. This function acts as the Green potential $G_e(x)$ when T is P-tree.

Proof. See, [8, Theorem 1]. □

If u is a nonnegative superharmonic function which has a harmonic minorant, then u has a greatest harmonic minorant (see [10, Theorem 2.4]).

Riesz decomposition: Suppose s is a nonnegative real-valued superharmonic function on a set ω in a tree T. Then s can be expressed as a sum $s = p + h$ of a potential p on ω and a positive harmonic function h on ω, where h is the greatest harmonic minorant of s on ω. This way of breaking down a superharmonic function s is unique.

Let T be an S-tree. Fix an unbounded subharmonic function H which has the following properties on T :

$H \geq 0$, $\Delta H(e) = 1$, $H(e) = 0$ and H is harmonic on $T \backslash \{e\}$. For the existence of such an H see [10, Proposition 4.2].

Recalling Theorem 4.3 in Bajunaid *et al.* [10], it is shown that given any vertex e in an S-tree T, there are a unique function called the pseudo-potential, denoted by $q_e(x)$ which is superharmonic on T with harmonic point support $\{e\}$, and a uniquely determined positive constant α_e such that $(-\Delta) q_e(x) = \delta_e(x)$ for all x in T, and $q_e(x) - \alpha_e H(x)$ is bounded outside a finite set in T.

If A is a set of vertices, the function $q(x) = \sum_{x_i \in A} \beta_i q_{x_i}(x)$, is a superharmonic function on T, where for each i, q_{x_i} is pseudopotential with harmonic support $\{x_i\}$ and β_i is positive constants. $q(x)$ is referred as a pseudo-potential with harmonic support A.

Lemma 2.8 Suppose $u(x)$ is defined on $n \leq |x| \leq n+m$ and harmonic on $n < |x| < n+m$. Then $u(x)$ can be extended as a harmonic function on $|x| > n$.

Proof. Let $|x_0| = n+m+1$. Then $x_0 \in N(y)$ for some y with $|y| = n+m$. Choose the constant α such that if $u(x) = \alpha$ on $A = N(y) \cap \{x : |x| = n+m+1\}$, then $u(x)$ is harmonic at y. That is $u(y) = \alpha_0 p(y, y^-) + \alpha \sum_{x \in A} p(y, x)$, where $\alpha_0 = u(y^-)$.

Thus $u(x)$ is defined for $|x| = n+m+1$ and harmonic on $n < |x| < n+m+1$. By induction, $u(x)$ is defined harmonically on $|x| > n$. □

Proposition 2.9 If u is a subharmonic function defined outside a finite set in a P-tree (respectively. S-tree) T, then there exist a subharmonic function v and two potentials (respectively. pseudo-potentials) p_1 and p_2 with finite harmonic support on T such that $u = v + p_1 - p_2$ outside a finite set and $p_1 - p_2$ is bounded on T if it is a P-tree. Moreover, in case u is harmonic outside a finite set, v is harmonic on T; and in this case the harmonic function v is uniquely determined. If T is an S-tree, since the p's are pseudo-potentials with finite harmonic support, for some constant α, $v - v' - \alpha H$ is bounded outside a finite set. Since $v - v'$ is harmonic on T, $\alpha = 0$. This implies that $v - v'$ becomes constant everywhere on the tree T.

Proof. Let $y \in \overset{\circ}{T}$, and let $Q_y(x)$ be the potential (respectively pseudo-potential) with harmonic support at y if T is a $P-$tree (respectively an $S-$tree). In all cases, we have $(-\Delta) Q_y(y) = 1$. For a fixed nonterminal vertex e, and for large n, let $B_n u$ denote the Dirichlet solution on $|x| < n$ with boundary values $u(x)$ on $|x| = n$.

Define

$$s(x)=\begin{cases} u(x) & \text{if} \quad n<|x| \\ \\ B_n u & \text{if} \quad n \geq |x| \end{cases}.$$

Then $\Delta s(x) \geq 0$ if $n<|x|$ and $\Delta s(x)=0$ if $n>|x|$. Let $v(x)=s(x)+\sum_{|y|=n} \Delta s(y) Q_y(x)$. Clearly $\Delta v(x) \geq 0$ if $|x| \neq n$ and if $|x|=n$, $x=y$, $\Delta v(y)=0$. Thus $\Delta v(x) \geq 0$ for every $x \in \overset{\circ}{T}$. Therefore v is subharmonic on the entire tree T and when $|x|>n$, $u(x)=v(x)+p_1(x)-p_2(x)$ where $p_1(x)=\sum_{|y|=n} [\Delta s(y)]^- Q_y(x)$ and $p_2(x)=\sum_{|y|=n} [\Delta s(y)]^+ Q_y(x)$ so that p_1 and p_2 are potentials (respectively pseudo-potentials) with finite harmonic support if T is a $P-$tree (respectively. an $S-$tree) [10, Theorem 2.11], if T is a $P-$tree, p_1-p_2 is bounded.

Finally, suppose u is harmonic outside a finite set. Then $\Delta v(x)=\Delta s(x)=0$ if $|x| \neq n$, so that $\Delta v=0$ on $\overset{\circ}{T}$. Hence v is harmonic on T. Now, suppose $u=v'+p'_1-p'_2$ is another such representation outside a finite set. Then, if T is a P-tree, the subharmonic function $|v-v'|$ is majorized by the sum of all potentials $p_1+p_2+p'_1+p'_2$ outside a finite set A. Select a potential $L>0$ on T. Because A is finite, we can find a positive constant λ such that $|v-v'| \leq \lambda L$ on A. This implies that $|v-v'| \leq \lambda L+(p_1+p_2+p'_1+p'_2)$ on T. Since the subharmonic function $|v-v'|$ is majorized by a potential on T, $|v-v'| \leq 0$ and hence v is unique.

If T is an S-tree, since the p's are pseudo -potentials with finite harmonic support, for some constant α, $v-v'-\alpha H$ is bounded outside a finite set. Since $v-v'$ is harmonic on T, $\alpha=0$. This implies that $v-v'$ becomes constant everywhere on the tree. □

Corollary 2.10 (Laurent decomposition) Suppose $u(x)$ is defined on $n \leq |x| \leq n+m$, for an integer $m \geq 1$ and harmonic on $n<|x|<n+m$. Then there exist a harmonic function $t(x)$ on $|x| \leq n+m$ and a harmonic function $s(x)$ on $|x| \geq n$ such that $u(x)=s(x)-t(x)$ on $n \leq |x| \leq n+m$. Furthermore, $s(x)$ can be chosen as follows:

(1) if T is a P-tree, then there exists a potential $p(x)$ on T such that $|s(x)| \leq p(x)$ everywhere except for a small finite set of the tree. Hence the decomposition is unique.

(2) if T is an S-tree, then there exists a unique α such that $s(x)-\alpha H(x)$ is bounded outside a finite set. Hence the decomposition is unique up to an additive constant.

Proof. First apply Lemma 2.8 to extend u as a harmonic function on all of $|x|>n$. Then by Proposition 2.9, there exists a harmonic function v on T such that $u=v+p_1-p_2$ on $|x|>n$, where if T is a P-tree, p_1 and p_2 are potentials with finite harmonic support and if T is an S-tree, p_1 and p_2 are pseudo-potentials with finite harmonic support; the harmonic supports of p_1 and p_2 are in $|x|=n$.

Define $s(x)=u(x)-v(x)$ on $|x| \geq n$ and $t(x)=-v(x)$ on $|x| \leq n+m$. This decomposition allows us to confirm the claims made in the corollary. □

A harmonic function defined outside a finite set of vertices does not necessarily extend to a harmonic function on the whole tree. An example will be shown next to illustrate this point:

Example 2.11 Let x_0 be a vertex of a tree $T, C_1, C_2, ..., C_m$ be the connected components of $T \backslash \{x_0\}$, and $\beta_1, \beta_2, ..., \beta_m$ be distinct real constant. Let h be defined on $T \backslash \{x_0\}$ by $h(x)=\beta_i$ if $x \in C_i$, $i=1, ..., m$. h is harmonic outside $\{x_0\}$ and its neighbors since it is locally constant, but h can not be extended to a harmonic function on the whole tree T.

3. Bisuperharmonic functions near infinity on $\mathbb{R}^n$

Terminology: The concept "near infinity" describes a set which is the complement of a compact set in $\mathbb{R}^n$, $n \geq 2$.

Recall:

(1) If h is a harmonic function defined outside a compact in $\mathbb{R}^n$, then there exists a harmonic function H in $\mathbb{R}^n$ such that $h(x)-H(x)$ is bounded near infinity if $n \geq 3$ and $h(x)-H(x)-\alpha \log|x|$ is bounded near infinity if $n=2$.

(2) If $n \geq 3$, $h(x) = H(x) + b(x)$ near infinity where $b(x)$ tends to 0 at infinity and $|b(x)| \leq p(x)$ for a potential $p(x)$ in $\mathbb{R}^n$.

(3) [11] If u is a superharmonic function defined outside a compact set in $\mathbb{R}^n$, then there exists a nonconstant superharmonic function v in $\mathbb{R}^n$ such that outside a compact set $u(x) = v(x) - \alpha \log|x|$ if $n = 2$, for some constant $\alpha \leq 0$, and $u(x) = v(x) - \beta |x|^{2-n}$ if $n \geq 3$, for some constant $\beta \geq 0$.

In particular, near infinity $u(x)$ is the difference of two superharmonic functions defined on $\mathbb{R}^n$.

(4) [12] If $f(x)$ is a non-negative locally Lebesgue integrable function on $\mathbb{R}^n$, then there exists a superharmonic function $s(x)$ on $\mathbb{R}^n$ such that $-\Delta s(x) = f(x)$ on $\mathbb{R}^n$. If $f(x)$ is a locally Lebesgue integrable function on $\mathbb{R}^n$, then there exist two superharmonic functions $s_1(x)$ and $s_2(x)$ on $\mathbb{R}^n$ such that $s(x) = s_1(x) - s_2(x)$ and $-\Delta s(x) = f(x)$.

Definition 3.1 A function u defined on an open set ω in $\mathbb{R}^n$ is said to be bisuperharmonic (respectively, a bisubharmonic) if $-\Delta u(x)$ is superharmonic (respectively, subharmonic) on ω.

Definition 3.2 Let p and q be two potentials defined on an open set ω in $\mathbb{R}^n$, such that $-\Delta q(x) = p(x)$ on ω. Then q is called a bipotential on ω and ω is called a bipotential domain.

Remark 3.3

(a) A bisuperharmonic function u defined on an open set ω in $\mathbb{R}^n$ is the difference of two superharmonic functions in ω. For $\Delta u = s$ can be split into two equations $\Delta u_1 = s^+$ and $\Delta u_2 = s^-$.

(b) A biharmonic function $b(x)$ in $|x| < r$ in $\mathbb{R}^n$ can be represented as $b(x) = |x|^2 u(x) + h(x)$ where $u(x)$ and $h(x)$ are harmonic in $|x| < r$.

(c) If $b(x)$ is biharmonic and $s(x)$ is subharmonic in $\mathbb{R}^n$, $n \geq 1$ such that $b(x) \geq s(x)$, then $b(x) = c|x|^2 + h(x)$ where $c \geq 0$ is constant and h is harmonic.

(d) Any bounded biharmonic function in $\mathbb{R}^n$, $n \geq 1$ is constant.

(e) $|x|^{4-n}$ is a positive bisuperharmonic function that is superharmonic in $\mathbb{R}^n$, $n \geq 5$. But there are no nonconstant positive bisuperharmonic functions that are superharmonic in $\mathbb{R}^n$, $2 \leq n \leq 4$.

We see that a superharmonic function defined outside a compact set is the difference of two superharmonic functions on $\mathbb{R}^n$. As an extension, we remark that a bisuperharmonic function near infinity in $\mathbb{R}^n$ is the difference of two bisuperharmonic functions on $\mathbb{R}^n$ up to an additive harmonic function near infinity. Precisely:

Proposition 3.4 Let $s(x)$ be a bisuperharmonic function near infinity. Then there exist bisuperharmonic functions $s_1(x)$ and $s_2(x)$ on $\mathbb{R}^n$ such that $s(x) = s_1(x) - s_2(x) + h(x)$ near infinity, where $h(x)$ is harmonic near infinity.

Proof. Outside a compact set K in $\mathbb{R}^n$, $-\Delta s(x) = u(x)$ is a superharmonic function.

Hence, if $n = 2$, $u(x) = v(x) - \alpha \log|x|$ where $\alpha \leq 0$ and $v(x)$ is a superharmonic function defined on $\mathbb{R}^n$. Take $s_1(x)$ such that $-\Delta s_1(x) = v(x)$ on $\mathbb{R}^n$. Then $s_1(x)$ is a bisuperharmonic function on $\mathbb{R}^n$. Consequently, $-\Delta s(x) = u(x) = -\Delta s_1(x) - \Delta\left[\gamma |x|^2 (-1 + \log|x|)\right]$ outside the compact set K and $\gamma \leq 0$. Hence $s(x)$ can be expressed as $s(x) = s_1(x) - s_2(x) + h(x)$ outside K where $s_2(x)$ is a bisuperharmonic function on $\mathbb{R}^n$ and $h(x)$ is a harmonic function near infinity.

If $n \geq 3$, $u(x) = v(x) - \beta |x|^{2-n}$ where $\beta \geq 0$. Take $s_1(x)$ such that $-\Delta s_1(x) = v(x)$ on $\mathbb{R}^n$. Consequently, outside a compact set, $-\Delta s(x) = u(x) = -\Delta s_1(x) - \Delta\left[\delta |x|^{4-n}\right]$, where $\delta > 0$. Hence, $s(x) = s_1(x) - s_2(x) + h(x)$ outside a compact set, where $s_2(x)$ is a bisuperharmonic function on $\mathbb{R}^n$ (incidentally, it is actually a bipotential in $\mathbb{R}^n$ if $n \geq 5$) and $h(x)$ is a harmonic function near infinity. □

4. Discrete bisuperharmonic functions on infinite trees

Theorem 4.1 If f is a real-valued function defined on a tree T, then there exists a function u on T such that $(-\Delta)u(x) = f(x)$ on T.

Proof. For every vertex y on the tree T, there exists a unique superharmonic function $q_y(x)$ on T such that for every x in T, $(-\Delta)q_y(x) = \delta_y(x)$ (Theorem 2.7 and Remark 3.3).

Let

$$f_n(x) = \begin{cases} f(x) & \text{if} \quad |x| = n \\ \\ 0 & \text{if} \quad |x| \neq n \end{cases}.$$

If $s_n(x) = \sum_{y \in \partial B_n} f(y) q_y(x)$, then if $x \notin \partial B_n$, $(-\Delta)s_n(x) = 0$ and if $x \in \partial B_n$, $(-\Delta)s_n(x) = f(x)$ that is

$$(-\Delta)s_n(x) = f_n(x), \quad \forall x \in T.$$

Using Corollary 2.6, we can fined a harmonic function $h_n(x)$ on T such that $h_n(x) = s_n(x)$ for $|x| \leq n$.
Let

$$s(x) = \sum_{n=1}^{\infty} [s_n(x) - h_n(x)].$$

Note that in $\sum_{n=1}^{\infty} [s_n(x) - h_n(x)]$ all the terms except a finite number of them are 0 on any B_m, and hence the infinite sum is well-defined on B_m and consequently on T.

Hence $s(x)$ is a real-valued function such that

$$(-\Delta)s(x) = \sum_{n=1}^{\infty} (-\Delta)s_n(x) = \begin{cases} f(x) & \text{if} \quad |x| \geq 1 \\ \\ 0 & \text{if} \quad x = e \end{cases}.$$

Let $u(x) = f(e)q_e(x) + s(x)$. Then $(-\Delta)u(x) = f(x)$ on T. □

Corollary 4.2 Let f be a positive real-valued function defined on a tree T. Then there is a superharmonic function u on T such that $(-\Delta)u(x) = f(x)$ on T.

If f is an arbitrary real-valued function on T, then we have a δ-superharmonic function u on T such that $(-\Delta)u(x) = f(x)$ on T. We say that u is generated by f.

Definition 4.3 A function B on T is called biharmonic if there exists a harmonic function h on T such that $(-\Delta)B(x) = h(x)$ on T.

u is said to be a bisuperharmonic (respectively. a bisubharmonic) function on T if there exists a superharmonic (respectively. subharmonic) function v on T such that $(-\Delta)u(x) = v(x)$ on T.

We can now prove the discrete version of the classical Require problem in a general form.

Theorem 4.4 (Riquier Problem) Let T be a tree. Let g_1 and g_2 be two functions defined on $A = \{x : |x| = n\}$, $n \geq 1$. There exist a biharmonic function f on T such that $f|_A = g_1, -\Delta f|_A = g_2$ and f is unique determined in $|x| \leq n$.

Proof. First use Theorem 2.5 to extend g_2 as a harmonic function h_2 on T. Let u be a biharmonic function on T generated by h_2, that is $(-\Delta)u = h_2$.

Use Theorem 2.5 again to extend $u - g_1$ on A as a harmonic function h_1 on T. Let $f = u - h_1$ on T. Then $f|_A = u - (u - g_1) = g_1$.

f is biharmonic on T, and $-\Delta f|_A = h_2|_A = g_2$.

Suppose F is another such biharmonic solution. Let $\varphi = f - F$. Then $\Delta\varphi$ is harmonic on $|x| \le n$ and equals 0 on $|x| = n$ which implies that $\Delta\varphi = 0$ on $|x| \le n$. Hence φ is harmonic on $|x| \le n$. Note $\varphi = 0$ on $|x| = n$. Therefore $\varphi = 0$ on $|x| \le n$ and f is unique. □

5. Biharmonic extensions

Theorem 5.1 If b is a biharmonic function defined on $|x| \le n$ in T, then there is a biharmonic function B on T such that $b(x) = B(x)$ for $|x| \le n$.

Proof. Let h_1 be a harmonic function on $|x| \le n$ in T such that $(-\Delta)b(x) = h_1(x)$. By Corollary 2.6, there is a harmonic function H on T such that $h_1 = H$ on $|x| \le n$. Let $(-\Delta)B_1(x) = H(x)$. Then B_1 is a biharmonic function on T. Write $r(x) = B_1(x) - b(x)$ if $|x| \le n$. Then if $x \in \overset{\circ}{B}_n$,

$$
\begin{aligned}
(-\Delta)\, r(x) &= (-\Delta)\, B_1(x) - (-\Delta)\, b(x) \\
&= H(x) - h(x) \\
&= 0.
\end{aligned}
$$

Hence $r(x)$ is harmonic on $|x| \le n$, which implies that there exists a harmonic function h_2 on T such that $h_2 = r$ on $|x| \le n$. Take $B(x) = B_1(x) - h_2(x)$. Then $B(x)$ is biharmonic function defined on T such that $b(x) = B(x)$ for $|x| \le n$. □

Definition 5.2 In a P-tree T, a biharmonic function b defined outside a finite set is said to be regular at infinity if there exists a potential p, such that $|\Delta b| \le p$ outside a finite set.

Remark 5.3

(1) In a tree T, a biharmonic function B is regular at infinity if and only if B is harmonic in T. (2) If b_1 and b_2 are two biharmonic functions defined outside a finite set in T and regular at infinity, then the biharmonic function $\beta_1 b_1 + \beta_2 b_2$ is regular at infinity for any real numbers β_1 and β_2.

Theorem 5.4 If T is a P-tree, and b is a function defined on $n \le |x| \le N$ and biharmonic on $n < |x| < N$, then we can find a biharmonic function b_1 in $|x| > n$ regular at infinity, and a biharmonic function b_2 in $|x| < N$ such that $b = b_1 - b_2$ in $n < |x| < N$. Up to an additive harmonic function, this decomposition is unique in T.

Proof. Let $(-\Delta b) = h$, h is harmonic in $n < |x| < N$. Using Corollary 2.10, $h = s - t$ in $n \le |x| \le N$, where s is harmonic in $|x| > N$ and t is harmonic in $|x| < N$. From the proof of Corollary 2.10, it is clear that s can be chosen so that outside a finite set, $|s| \le p$ where p is a potential in T.

Now let u_1 be a function in $|x| > n$ such that $(-\Delta)\ u_1 = s$ and u_2 be a function in $|x| < N$ such that $(-\Delta)u_2 = t$. Consequently $(-\Delta)(u_1 - u_2) = s - t = h$ in $n < |x| < N$, and therefore $b - (u_1 - u_2)$ is harmonic in $n < |x| < N$. Hence there are functions s_1 and t_1 such that s_1 is harmonic in $|x| > n$ and t_1 is harmonic in $|x| < N$ and $b - (u_1 - u_2) = s_1 - t_1$ in $n < |x| < N$. Setting $b_1 = u_1 + s_1$ and $b_2 = u_2 + t_1$, we see that b_1 is biharmonic in $|x| > n$ which is regular at infinity, b_2 is biharmonic in $|x| < N$ and $b = b_1 - b_2$ in $n < |x| < N$.

To prove the uniqueness of this decomposition, consider another representation $b = b_1' - b_2'$. Then the function

$$
B(x) = \begin{cases} b_2 - b_2' & \text{in} \quad |x| < N \\ b_1 - b_1' & \text{in} \quad |x| > n, \end{cases}
$$

is biharmonic in T which is regular at infinity. Hence B is harmonic in the tree T. □

Proposition 5.5 Let b be a biharmonic function outside a finite set $|x| \leq n$ in a tree T. Then there is a biharmonic function B in T, unique up to an additive harmonic function, such that $B - b$ is regular at infinity.

Proof. Let N be a positive integer such that $N > n$. Then, by Theorem 5.4, there exist a biharmonic function b_1 in $|x| > n$, regular at infinity and a biharmonic function b_2 in $|x| < N$ such that $b = b_1 - b_2$ in $n < |x| < N$.

Define

$$B(x) = \begin{cases} b - b_1 & \text{in} \quad |x| > n \\ \\ -b_2 & \text{in} \quad |x| < N \end{cases}.$$

Then B is biharmonic in T. Moreover $B - b$ is biharmonic in $|x| > n$ and regular at infinity.

To prove the uniqueness of B, let B' be another such biharmonic extension of b, then $B - B'$ is biharmonic in T, regular at infinity. Consequently $B - B'$ is harmonic in T. □

Remark 5.6 In Proposition 5.5, $(b - B)$ may not be bounded outside a finite set. For example, if in a homogeneous tree T of order q, $b(x) = d(e,x)$ for $|x| \geq 1$ and $b(e) = 1$, then b is biharmonic in $|x| > 1$. For if $|x| = n$, $n > 1$

$$\Delta b(x) = \frac{q(n+1) + (n-1)}{q+1} - n$$

$$= \frac{q-1}{q+1}.$$

Hence $\Delta^2 b(x) = 0$ and b is biharmonic in $|x| > 1$. Take $B \equiv 0$, then B is biharmonic in T and $b - B$ is not bounded outside a finite set.

Definition 5.7 A potential $q > 0$ on a P-tree is called bipotential if $(-\Delta)q$ is a potential on T.

A P-tree is said to be a bipotential tree if there exist potentials p and q on T such that $\Delta q = -p$ on T.

6. Tapered bipotential trees

Definition 6.1 A bipotential tree T is said to be tapered if there exist potentials $p > 0$ and $q > 0$ in T, q being bounded outside a finite set such that $(-\Delta)q = p$ in T.

Proposition 6.2 A bipotential tree T is tapered if and only if there exist two bounded potentials p and q on T such that $\Delta q = -p$.

Proof. Let T be tapered. That is by Definition 6.1 there are two positive potentials p and q on T such that q is bounded outside a finite set and $\Delta q = -p$.

Since any superharmonic function is bounded on a finite set, by the definition, q should be bounded on T. Suppose $q \leq M$.

Then at vertex x,

$$
\begin{aligned}
p(x) &= (-\Delta)\, q(x) \\
&= q(x) - \sum_{z\sim x} p(x,z)\, q(z) \\
&\leq q(x) + \sum_{z\sim x} p(x,z)\, q(z) \\
&\leq M + M\left(\sum_{z\sim x} p(x,z)\right) \\
&= 2M.
\end{aligned}
$$

Hence p and q should be bounded potentials on T.

The converse is evident. □

Proposition 6.3 A bipotential tree is tapered if and only if there exists a bounded bipotential on T.

Proof. Let q be a bounded bipotential on T. Since q is bounded, $p = (-\Delta)\, q$ is bounded as shown above. Hence p and q are bounded potentials on T such that $\Delta q = -p$. That is, T is tapered.

Conversely, if T is tapered, there exist by the above proposition bounded potentials p and q such that $\Delta q = -p$. That is, q is a bounded bipotential on T. □

In a $P-$tree, it is not true that given a potential p we can find a potential q such that $\Delta q = -p$ on T. It is not even true in bipotential tree. In this context, the following theorem is useful.

Theorem 6.4 Let q be a bipotential on T generated by a potential q_1. Let p_1 be a potential on T such that $p_1 \leq q_1$. Then there is a unique bipotential p such that $p \leq q$ on T and $\Delta p = -p_1$.

Proof. As in Theorem 4.1, choose a superharmonic function s on T such that $(-\Delta)s = p_1$ on T. By hypothesis, $(-\Delta)q = q_1 \geq p_1$ on T. Choose a superharmonic function t on T, such that $(-\Delta)t = q_1 - p_1$ on T. Then $q = s + t +$ (a harmonic function h_1). s has a subharmonic minorant on T since $q \geq 0$. So that $s = p + h_2$ and $t = p' + h'_2$ on T, where p and p' are potentials and h_2 and h'_2 are harmonic on T. Thus $q = p + p' + (h_1 + h_2 + h'_2)$. Equating the potential parts, we have $q = p + p'$. Therefore $p \leq q$; it's important to remember that $(-\Delta)p = (-\Delta)s = p_1$ on T.

For the uniqueness, suppose p_2 is another bipotential generated by p_1. Since $(-\Delta)p = p_1 = (-\Delta)p_2$, $p_2 = p+$ (a harmonic function h) on T; and since p and p_2 are potentials, $h \equiv 0$. Hence the bipotential generated by p_1 is unique. □

Corollary 6.5 In a tapered bipotential tree T, let p_1 be a potential with finite harmonic support. Then p_1 generates a bounded bipotential p on T.

Proof. Since T is tapered, there exist bounded potentials q and q_1 on T such that $\Delta q = -q_1$ on T. Then we can choose $\lambda > 0$ so that $p_1 \leq \lambda q_1$ on T, since p_1 has finite harmonic support. Hence p_1 generates a bipotential on T by Theorem 6.4, since λq_1 is bipotential on T. Moreover, if $\Delta p = -p_1$, then by construction $p \leq \lambda q$ on T. Hence p is bounded on T. □

Let $\widehat{R}_1^E = \inf\{s : s \text{ is superharmonic function} > 0 \text{ on } T \text{ such that } s \geq 1 \text{ on } E\}$. Then $u = \widehat{R}_1^E$ is a potential on T such that $u \leq 1$ on T, u is harmonic on $T \backslash E$ and $u \equiv 1$ on E. If T is a tapered bipotential tree, the by Corollary 6.5, $\widehat{R}_1^E$ generates a bipotential on T.

Theorem 6.6 A bipotential tree T is tapered if and only if for one (and hence any) finite set E in T, there is a bounded potential u in T such that $(-\Delta)\, u = \widehat{R}_1^E$.

Proof. Let E be a finite set in a tapered tree T. Since T is tapered, there are potentials p and q in T, q being bounded outside a finite set, such that $(-\Delta)q = p$. Now, p being a potential, $\widehat{R}_1^E \leq (\inf_E p)^{-1}\, p$. Hence, if u is the potential in T

such that $(-\Delta)u = \widehat{R}_1^E, u \leq (\inf_E p)^{-1} q$. For, if $q_1 = (\inf_E p)^{-1} q$, there exists a subharmonic function s in T such that $u = q_1 + s$ in T. This implies that $s \leq 0$ since the potential $u \geq s$; hence $u \leq q_1$. u is bounded outside a finite set since q_1 is so. Moreover, since u is bounded outside a finite set in T, u is bounded on T.

The converse is evident. □

Theorem 6.7 Let T be a tapered tree and s be a function generated by a potential with finite harmonic support in T. Then $s = v + h$ where h is harmonic in T and v is a potential bounded outside a finite set.

Proof. Let $(-\Delta)s = p$ where p is a potential with finite harmonic support E in T. If K is a finite set such that $\overset{\circ}{K} \supset E$, then there is a constant λ such that $p \leq \lambda \widehat{R}_1^K$ in T. Since by Corollary 6.5, there exists a bounded potential u in T such that $(-\Delta)u = \widehat{R}_1^K$, there also exists a potential v in T, bounded outside a finite set such that $(-\Delta)v = p$, consequently, $s = v + h$, where h is harmonic in T. Also, since the potential v is bounded outside a finite set in T, v is bounded in T. □

Theorem 6.8 In a bipotential tree T, the following are equivalent (1) T is tapered. (2) If b is a biharmonic function defined outside a finite set in T, there is a biharmonic function B in T such that $B - b$ is regular at infinity and bounded outside a finite set. (3) Any bipotential with finite harmonic support in T is bounded outside a finite set.

Proof. $(1) \Rightarrow (2)$: Let b be a biharmonic function defined outside a finite set in T and $(-\Delta)b = h$. Since h is harmonic outside a finite set, there exist two potentials p_1 and p_2 with finite harmonic support and a harmonic function H on T such that $u = p_1 - p_2 + H$ outside a finite set. If $(-\Delta)s_1 = p_1$, $(-\Delta)s_2 = p_2$ and $(-\Delta)B_1 = H$, then $b = -s_1 - s_2 + B_1 + v$ outside a finite set, where v is harmonic function. Writing $v = q_1 - q_2 + v_1$ where q_1 and q_2 are potentials with finite harmonic support and v_1 is harmonic in T. Let $B = B_1 + v_1$. Then B is biharmonic in T and $b = s_1 - s_2 + q_1 - q_2 + B$. Since by assumption T is tapered, s_i $(i = 1, 2)$ being superharmonic function with finite biharmonic support, can be considered as potential bounded outside a finite set as in Theorem 6.7. Since q_i is a potential with harmonic support, it is also bounded outside a finite set. Hence $(b - B)$ is bounded outside a finite set and $|\Delta(b - B)| \leq p_1 + p_2$ outside a finite set.

$(2) \Rightarrow (3)$: Let q be bipotential with finite biharmonic support in T. Then, by the assumption, there exists a biharmonic function B in T such that $(q - B)$ is regular at infinity and bounded outside a finite set. Since $|\Delta(q - B)| \leq p$ outside a finite set, where p is a potential in T and since $(-\Delta)q$ is also a potential in T, the harmonic function ΔB is bounded by a potential outside a finite set in T and hence $\Delta B \equiv 0$; that is B harmonic in T. Since $|(q - B)| \leq M$ outside a finite set E and since q is also a potential in T, $|B| \leq q + M$ in $T \backslash E$. Hence the subharmonic function $|B|$ in T is bounded by M. Consequently, q is bounded outside the finite set E.

$(3) \Rightarrow (1)$: Let E be a finite set in T. Since T is a bipotential space, there is a bipotential q in T generated by $\widehat{R}_1^K$. Since q has finite biharmonic support E, by the assumption, q is bounded outside a finite set. Therefore T is tapered. □

Theorem 6.9 In a bipotential tree T, the following are equivalent (1) T is tapered. (2) For any biharmonic function b defined outside a finite set in T, and regular at infinity, there exists a harmonic function h in T such that $b - h$ is bounded outside a finite set.

Proof. $(1) \Rightarrow (2)$: Let b be a biharmonic function regular at infinity. Then using Theorem 6.8, there is a biharmonic function B in T such that $b - B$ is regular at infinity and bounded outside a finite set. Since b and $b - B$ are regular at infinity, so is the biharmonic function B defined on T. This implies that B is harmonic. Set $B = b$ to obtain (2).

$(2) \Rightarrow (1)$: Let E be a finite set in T and $(-\Delta)b = \widehat{R}_1^E$ in T. Then b is biharmonic in E^c and regular at infinity. Hence, by the assumption, there exists a harmonic function h in T such that $b - h$ is bounded outside a finite set.

Set $s = b - h$. Then s is a bounded function outside a finite set such that $(-\Delta)s = \widehat{R}_1^E$ is a positive superharmonic. Hence, s is superharmonic function with harmonic minorant outside a finite set and consequently is the sum of a potential in T and a harmonic function. This means that in the equation $(-\Delta)s = \widehat{R}_1^E$, s can be taken as a potential.

Since the superharmonic function s is bounded outside a finite set (and since is always bounced on a finite set), s is bounded on T. Hence its greatest harmonic minorant also is bounded on T, so that the potential part of s is bounded on T. □

Corollary 6.10 In a tapered tree T, let b be a function defined on $n \leq |x| \leq N$ and biharmonic on $n < |x| < N$. Then there exist a biharmonic function b_2 in $|x| < N$ and a biharmonic function b_1 in $|x| > n$ which is bounded outside a finite set such that $b = b_1 - b_2$ in $n < |x| < N$.

Proof. We have proved this corollary in a general P-tree (Theorem 5.4), excepting that b_1 is bounded outside a finite set. But in that theorem, b_1 was shown to be regular at infinity. Hence by the above theorem, there exists a harmonic function h in T such that $b_1 - h$ is bounded outside a finite set. Consequently, replacing b_1 (respectively, b_2) by $b_1 - h$ (respectively, $b_2 - h$) in the decomposition $b = b_1 - b_2$, we have proved the corollary. □

Proposition 6.11 In a tapered tree T, every bounded biharmonic function in T is harmonic if and only if every bounded biharmonic function outside a finite set is regular at infinity.

Proof. $\Rightarrow$) Let b be a bounded biharmonic function outside a finite set. Since T is tapered, there exists a biharmonic function B in T such that $b - B$ is bounded outside a finite set, and $b - B$ is regular at infinity.

This implies that B is bounded in T and hence by the assumption, B is harmonic. Since $b - B$ is regular at infinity, b is regular at infinity.

$\Leftarrow$) Let B be bounded biharmonic function in T. Then by the assumption, B is regular at infinity, that is $|\Delta B| \leq$ a potential p outside a finite set. Since ΔB is harmonic in T, $\Delta B \equiv 0$. Hence B is harmonic. □

In $\mathbb{R}^n$, $n \geq 5$, $s_n(x) = |x|^{4-n}$ is a bipotential tending to 0 at infinity. Hence these spaces are tapered and the biharmonic extensions mentioned in Theorem 2.5 and Proposition 6.11 are valid here. But the spaces $\mathbb{R}^n$, $2 \leq n \leq 4$, are not tapered. For, $\mathbb{R}^2$ is not hyperbolic; and $\mathbb{R}^3$ and $\mathbb{R}^4$ though hyperbolic, are not even bipotential spaces.

7. Conclusion

For an infinite tree T, suppose b is biharmonic function defined on a domain $B_n = \{x : |x| \leq n\}$, $n \geq 1$, there is always a biharmonic function B on the whole tree T such that $b = B$. If b is a biharmonic function defined outside B_n, then there is a biharmonic function B in T unique up to an additive harmonic function, such that $(b - B)$ is regular at infinity, that is $|\Delta(b - B)| \leq p$ outside a finite set where p is a potential on T. In Remark 5.6, we showed that $(b - B)$ may not be bounded outside a finite set and in Theorem 6.8, we give a necessary and sufficient conditions for boundness of $(b - B)$.

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Ethics approval

The conducted research is not related to either human or animal use.

Conflict of interest

Author declares there is no conflict of interest at any point with reference to research findings.

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