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Research Article

Calculus for the Delta Distribution

Fred Brackx

Clifford Research Group, Foundations Lab, Department of Electronics and Information Systems, Faculty of Engineering and Architecture, Ghent University, Ghent, Belgium
E-mail: Freddy.Brackx@UGent.be

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Abstract: A compendium of interesting identities involving the delta distribution in higher dimensional Euclidean space is presented, ready to use as a reference work whenever modelling with the *delta function* is involved. The formulæ are expressed as well in vector, as in Cartesian and spherical variables, the latter case being especially important since distributions in spherical coordinates have to be treated with the utmost care. Special attention is paid to an *alter ego* of the delta distribution, the so-called delta signumdistribution, acting on test functions showing a singularity at the origin, which appears, mostly unnoticed, when radial functions and negative powers of the radial distance are used.

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1. Introduction

Without any doubt the *delta function* is a fundamental mathematical tool in a broad spectrum of theoretical physics and engineering sciences. However it often happens that the pure mathematical background of distribution theory in general and the delta distribution in particular, is circumvented, especially in three dimensional space, in favour of ad hoc constructions involving volume integrals, limit processes, special functions, and the like. Moreover such an approach sometimes leads to a disagreement about the correctness of one or another formula. Let us give an example.

Consider, in three dimensional Euclidean space $\mathbb{R}^3$, the Coulomb potential of a unit point charge, given, up to constants, by

$$\frac{1}{r} = \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}.$$

It is the fundamental solution of the Laplace operator

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$$\Delta_3 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

and its second order Cartesian derivatives with respect to the real variables $(x_1,\ x_2,\ x_3)$ are given by

$$\partial_{x_k}\partial_{x_j}\frac{1}{r} = 3x_k x_j\,\frac{1}{r^5}, \quad j \neq k, \quad j,\ k = 1,\ 2,\ 3$$

$$\partial^2_{x_j}\frac{1}{r} = 3x_j^2\frac{1}{r^5} - \frac{4}{3}\pi\delta(\underline{x}) - \frac{1}{r^3}, \quad j = 1,\ 2,\ 3$$

where $\delta(\underline{x})$ is the *delta distribution*, sometimes called the *delta function*, in $\mathbb{R}^3$. In [1] it is argued that, with respect to test functions that are not smooth at the origin, the above formulæ should be replaced by the alternative formulæ

$$\partial_{x_k}\partial_{x_j}\frac{1}{r} = 3x_k x_j\,\frac{1}{r^5} - 4\pi\frac{x_j x_k}{r^2}\delta(\underline{x}), \quad j \neq k, \quad j,\ k = 1,\ 2,\ 3$$

$$\partial^2_{x_j}\frac{1}{r} = 3x_j^2\,\frac{1}{r^5} - 4\pi\frac{x_j^2}{r^2}\delta(\underline{x}) - \frac{1}{r^3}, \quad j = 1,\ 2,\ 3.$$

Mathematically speaking it is not clear what in [1] is meant by “test functions that are not smooth at the origin”. Nevertheless, taking into account that distributions in general and the delta distribution and its derivatives in particular are only defined on spaces of differentiable test functions, it is readily observed that the above two sets of formulæ simply *coincide*, since it may be proved that, in general dimension m,

$$\frac{x_j^2}{r^2}\,\delta(\underline{x}) = \frac{1}{m}\,\delta(\underline{x}), \quad j = 1,\ \ldots,\ m$$

and

$$\frac{x_j x_k}{r^2}\,\delta(\underline{x}) = 0, \quad j \neq k, \quad j,\ k = 1,\ \ldots,\ m.$$

But there is more at stake than the correctness of certain formulæ. Let us introduce spherical coordinates $(r,\ \theta,\ \phi)$ in $\mathbb{R}^3$ by the transition formulæ

$$x_1 = r\sin\theta\cos\phi,\ \ x_2 = r\sin\theta\sin\phi,\ \ x_3 = r\cos\theta$$

or

$$x_1 = r\omega_1,\ \ x_2 = r\omega_2,\ \ x_3 = r\omega_3$$

where we have put

$$\omega_1 = \sin\theta\cos\phi, \quad \omega_2 = \sin\theta\sin\phi, \quad \omega_3 = \cos\theta$$

and

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

In [2] a formula is proved for the expression

$$\partial_{x_i}\left(\omega_{j_1}\cdots\,\omega_{j_n}\,\frac{1}{r^2}\right), \quad i,\ j_1,\ j_2,\ j_3 = 1,\ 2,\ 3$$

the inverse square field $\frac{1}{r^2} = \frac{1}{x_1^2 + x_2^2 + x_3^2}$ being at the heart of the discussion of idealised point sources in three dimensional space. An elegant proof based on more general formulæ is given in [3]. We focus on the special case ($n = 1$)

$$\partial_{x_i}\left(\omega_j\,\frac{1}{r^2}\right) = \delta_{ij}\,\frac{1}{r^3} - 3\,\omega_i\,\omega_j\,\frac{1}{r^3} + \frac{4\pi}{3}\,\delta_{ij}\,\delta(\underline{x}) \tag{1}$$

and note that if $\frac{1}{r^2}$ would be interpreted as a distribution, then, due to the non-differentiability of ω_j at the origin, the expression $\omega_j\frac{1}{r^2}$ becomes a *signumdistribution*, i.e. a continuous linear functional on a space of test functions showing a singularity at the origin, the theory of which was studied in [4, 5] and which is discussed in Section 12. In order to validate the proposed result (1), $\omega_j\frac{1}{r^2}$ should be a distribution whence $\frac{1}{r^2}$ has to be interpreted here as a signumdistribution, which is far more than a subtlety. For some more examples in the same vein we refer to [6, Section 12].

So it occurred to us that a self-contained compendium of identities involving the delta distribution and numerous differential operators acting on it, including their behaviour in the signumdistributional case, could be a very useful reference work, which lead to the underlying paper.

2. Basic notions and notations

The setting for this paper is m-dimensional Euclidean space $\mathbb{R}^m$ with orthonormal basis $(e_1,\ \ldots,\ e_m)$. The basis vectors $(e_j,\ j = 1,\ \ldots,\ m)$ are considered to be Clifford 1-vectors; they generate the Clifford algebra $\mathbb{R}_{0,\,m}$, where the non-commutative *geometric* or *Clifford product* is governed by the rules

$$e_j^2 = -1, \quad j = 1,\ \ldots,\ m$$

$$e_i\,e_j = -e_j\,e_i, \quad i \neq j.$$

The Clifford product splits into the commutative inner product, denoted by $\cdot$, and the anti-commutative outer product, denoted by $\wedge$. For the 1-vectors $\underline{x} = \sum_{j=1}^{m} e_j x_j$ and $\underline{y} = \sum_{j=1}^{m} e_j y_j$, it holds that

$$\underline{x}\,\underline{y} = \underline{x} \cdot \underline{y} + \underline{x} \wedge \underline{y}$$

and, in particular,

$$\underline{x}^2 = \underline{x} \cdot \underline{x} = -\,|\underline{x}|^2.$$

Note that the inner product is the negative of the standard scalar product of geometric vectors:

$$\underline{x} \cdot \underline{y} = -\vec{x} \circ \vec{y} = -\sum_{j=1}^{m} x_j y_j.$$

The derivatives with respect to Cartesian variables $x_j,\ j = 1, \ldots, m$ are incorporated into the so-called Dirac operator $\underline{\partial}$, which may be seen, see e.g. [7], as a Stein-Weiss projection of the well known gradient operator ∇:

$$\underline{\partial} = \sum_{j=1}^{m} e_j\, \partial_{x_j}.$$

The Dirac operator lies at the hart of the higher dimensional theory of monogenic functions (see e.g. [8]), also called Clifford analysis. It is a direct and elegant generalization to higher dimension of the theory of holomorphic functions in the complex plane. One of the important properties of the Dirac operator is that it linearizes the Laplace operator:

$$\underline{\partial}^2 = -\,\Delta.$$

For more on Clifford algebras we refer to e.g. [9].

For the sake of completeness, we recall some basic notions from the theory of distributions. Distributions, also known as *generalized functions*, generalize the traditional notion of a function. They act on *test functions*, which we consider here to be infinitely differentiable and with compact support in $\mathbb{R}^m$. This space of test functions is denoted by $\mathscr{D}(\mathbb{R}^m)$, and the action of a distribution T on a test function $\varphi \in \mathscr{D}(\mathbb{R}^m)$ is denoted by

$$\langle\, T,\ \varphi \,\rangle.$$

A distribution is a *continuous linear functional* on $\mathscr{D}(\mathbb{R}^m)$ according to the following definition. Note that this is only one of several possible equivalent definitions.

Definition 1 A real distribution T in $\mathbb{R}^m$ is a linear functional $T : \mathscr{D}(\mathbb{R}^m) \to \mathbb{R}$ such that for every compact set $K \subset \mathbb{R}^m$, for every multi-index α, and for every sequence of test functions $(\varphi_j)_{j=1}^{\infty}$ whose supports are contained in K, it holds that whenever the sequence $(\partial^\alpha \varphi_j)_{j=1}^{\infty}$ converges to 0 uniformly on K, then the numerical sequence $(\langle T,\ \partial^\alpha \varphi_j\rangle)_{j=1}^{\infty}$ converges to 0.

The space of distributions in $\mathbb{R}^m$ is denoted by $\mathscr{D}'(\mathbb{R}^m)$. A typical example of a distribution is the delta or Dirac distribution $\delta(\underline{x})$, defined in $\mathbb{R}^m$ by

$$\langle\, \delta(\underline{x}),\ \varphi(\underline{x})\,\rangle = \varphi(0), \quad \forall\, \varphi \in \mathscr{D}(\mathbb{R}^m).$$

And any *locally integrable function* $f(\underline{x})$ may be interpreted as a distribution T_f, called a *regular distribution*, by putting

$$\langle\, T_f,\ \varphi\,\rangle = \int_{\mathbb{R}^m} f(\underline{x})\,\varphi(\underline{x})\,d\underline{x}, \quad \forall\, \varphi \in \mathscr{D}(\mathbb{R}^m).$$

A fundamental property of a distribution $T \in \mathscr{D}'(\mathbb{R}^m)$ is its infinitely differentiability, its partial derivatives with respect to the Cartesian coordinates being given, for all test functions $\varphi(\underline{x}) \in \mathscr{D}(\mathbb{R}^m)$, by

$$\langle\, \partial_{x_j} T,\ \varphi(\underline{x})\,\rangle = -\langle\, T,\ \partial_{x_j}\varphi(\underline{x})\,\rangle, \quad j = 1,\ \ldots,\ m.$$

Another important operation on distributions is the multiplication of a distribution T by an analytic function $\alpha(\underline{x})$, i.e. a function which in the neighbourhood of any point $\underline{x}^* \in \mathbb{R}^m$ can be developed into a convergent multiple power series in $(x_j - x_j^*),\ j = 1,\ \ldots,\ m$. This multiplication is defined by

$$\langle\, \alpha(\underline{x})\,T,\ \varphi\,\rangle = \langle\, T,\ \alpha(\underline{x})\,\varphi\,\rangle, \quad \forall\, \varphi \in \mathscr{D}(\mathbb{R}^m).$$

We call an operator acting on distributions *Cartesian* if it involves partial derivation with respect to the Cartesian coordinates, and multiplication and division by analytic functions. However, this last operation, although being well-defined, is not uniquely determined, but results into an equivalence class of distributions involving (derivatives of) the translated delta distribution $\delta(\underline{x} - \underline{y})$, where $\underline{y}$ is a zero of the analytic function under consideration. More explicitly, if the analytic function $\beta(\underline{x})$ shows a zero at the single point $\underline{y}$ then

$$\frac{1}{\beta(\underline{x})}\,T = S + T^*$$

where S is any distribution for which $\beta(\underline{x})\,S = T$ and T^* is a distribution with support at the point $\underline{y}$, i.e. a finite linear combination of distributional derivatives of the translated delta distribution $\delta(\underline{x} - \underline{y})$:

$$T^* = \sum_{\alpha} c_\alpha\, \partial^\alpha \delta(\underline{x} - \underline{y}).$$

Two basic Cartesian operators on distributions are the multiplication operator $\underline{x}$ and the Dirac operator $\underline{\partial}$. Their actions are, quite naturally, well-defined and uniquely determined as

$$\langle\, \underline{x}T,\ \varphi(\underline{x})\,\rangle = \langle\, T,\ \underline{x}\,\varphi(\underline{x})\,\rangle$$

and

$$\langle\, \underline{\partial} T,\ \varphi(\underline{x})\,\rangle = -\langle\, T,\ \underline{\partial}\,\varphi(\underline{x})\,\rangle.$$

Also their squares: the multiplication operator $-\underline{x}^2 = |\underline{x}|^2$ and the Laplace operator $-\underline{\partial}^2 = \Delta$, are Cartesian operators with uniquely determined actions on distributions.

Division of a distribution by $\underline{x}$ may be approached in two ways. Either the function $\underline{x}$ is considered as a vector polynomial of the first degree, whence a vector analytic function with a single zero at the origin, or, as $\frac{1}{\underline{x}} = -\frac{\underline{x}}{|\underline{x}|^2}$, division by $\underline{x}$ is seen as the composition of two operations: first division by $r^2 = |\underline{x}|^2 = x_1^2 + \ldots + x_m^2$, followed by multiplication by $(-\underline{x})$. In [5] the following lemma was proven showing that both approaches are equivalent.

Lemma 1 For a scalar distribution T it holds that

$$\frac{1}{\underline{x}} T = \underline{S} + \delta(\underline{x})\,\underline{c}$$

for any distribution $\underline{S}$ such that $\underline{x}\,\underline{S} = T$, $\underline{c}$ being a vectorial constant.

Henceforth we use the notation $\left[\frac{1}{\underline{x}} T\right]$ for the equivalent class of distributions $\underline{S}$ such that $\underline{x}\,\underline{S} = T$.

The two fundamental formulæ in monogenic function theory, involving the (anti-)commutator of $\underline{x}$ and $\underline{\partial}$, viz.

$$\{\underline{x},\ \underline{\partial}\} = \underline{x}\,\underline{\partial} + \underline{\partial}\,\underline{x} = -2\mathbb{E} - m \quad \text{and} \quad [\underline{x},\ \underline{\partial}] = \underline{x}\,\underline{\partial} - \underline{\partial}\,\underline{x} = m - 2\Gamma$$

give rise to two other well known Cartesian operators: the scalar Euler operator

$$\mathbb{E} = \sum_{j=1}^{m} x_j\, \partial_{x_j}$$

and the bivector (orbital) angular momentum operator

$$\Gamma = -\sum_{j<k} e_j e_k L_{jk} = -\sum_{j<k} e_j e_k (x_j \partial x_k - x_k \partial_{x_j}).$$

It follows that

$$\underline{x}\,\underline{\partial} = -\mathbb{E} - \Gamma$$

or more precisely

$$\underline{x} \cdot \underline{\partial} = -\mathbb{E} \quad \text{and} \quad \underline{x} \wedge \underline{\partial} = -\Gamma. \tag{2}$$

On the other hand it holds that

$$\underline{\partial}\,\underline{x} = -m - \mathbb{E} + \Gamma$$

which splits into the scalar and bivector parts

$$\underline{\partial} \cdot \underline{x} = -m - \mathbb{E} \quad \text{and} \quad \underline{\partial} \wedge \underline{x} = \Gamma.$$

In the sequel we will also use of a special subclass of distributions, viz. *tempered distributions*. A distribution T, initially defined on the space $\mathscr{D}(\mathbb{R}^m)$ of compactly supported infinitely differentiable test functions, is said to be *tempered* if it can be extended to a continuous linear functional on the Schwartz space $\mathscr{S}(\mathbb{R}^m)$ of *rapidly decreasing* smooth test functions. Rapidly decreasing means decaying to 0 for $|\underline{x}| \to \infty$ faster than the inverse of any polynomial. An example of a rapidly decreasing function is $|\underline{x}|^k \exp(-|\underline{x}|^\ell)$, for any natural numbers k and ℓ. More explicitly, rapidly decreasing test functions are defined as follows.

Definition 2 A function $\psi(\underline{x})$ is a rapidly decreasing test function in $\mathbb{R}^m$ if it is infinitely differentiable in $\mathbb{R}^m$ and, together with all its derivatives, satisfies, for all multi-indices α and β,

$$p_{\alpha,\,\beta}(\psi) = \sup_{\mathbb{R}^m} |\underline{x}^\alpha\, \partial^\beta\, \psi(\underline{x})| < \infty.$$

Here we have used the short hand notations $\underline{x}^\alpha = x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ and $\partial^\beta = \partial_{x_1^{\beta_1}} \cdots \partial_{x_m^{\beta_m}}$. In a natural way, a tempered distribution is now defined as follows.

Definition 3 A real tempered distribution T in $\mathbb{R}^m$ is a linear functional $T : \mathscr{S}(\mathbb{R}^m) \to \mathbb{R}$ such that for every sequence of rapidly decreasing test functions $(\psi_j)_{j=1}^{\infty}$ for which, for all multi-indices α and β,

$$\lim_{j\to\infty} p_{\alpha,\,\beta}\, \psi_j(\underline{x}) = 0$$

the numerical sequence $(\langle T,\ \psi_j \rangle)_{j=1}^{\infty}$ converges to 0.

The space of tempered distributions in $\mathbb{R}^m$ is denoted by $\mathscr{S}'(\mathbb{R}^m)$, and it holds that

$$\mathscr{D}(\mathbb{R}^m) \subset \mathscr{S}(\mathbb{R}^m) \quad \text{and} \quad \mathscr{S}'(\mathbb{R}^m) \subset \mathscr{D}'(\mathbb{R}^m).$$

Tempered distributions are highly useful in practical applications, if only because the Fourier transform $\mathscr{F}$, given by

$$\mathscr{F}[\psi](\underline{y}) = \int_{\mathbb{R}^m} \psi(\underline{x}) \exp\left(-2\pi i \langle \underline{x},\ \underline{y} \rangle\right) d\underline{x}, \quad \psi \in \mathscr{S}(\mathbb{R}^m)$$

and

$$\langle\, \mathscr{F}[T],\ \psi\, \rangle = \langle\, T,\ \mathscr{F}[\psi]\, \rangle, \quad \forall \psi \in \mathscr{S}(\mathbb{R}^m), \quad T \in \mathscr{S}'(\mathbb{R}^m)$$

is an isomorphism of both spaces $\mathscr{S}(\mathbb{R}^m)$ and $\mathscr{S}'(\mathbb{R}^m)$. Typical examples of tempered distributions are constant functions and the delta distribution $\delta(\underline{x})$ defined by

$$\langle\, \delta(\underline{x}),\ \psi(\underline{x})\,\rangle = \psi(0), \quad \forall \psi \in \mathscr{S}(\mathbb{R}^m)$$

which are interrelated by the Fourier transform:

$$\mathscr{F}[\delta(\underline{x})] = 1_{\underline{y}}.$$

3. The one-dimensional delta distribution

The delta distribution $\delta(t)$ on the real line is defined, for all test functions $\varphi(t) \in \mathscr{D}(\mathbb{R})$, by

$$\langle\, \delta(t),\ \varphi(t)\,\rangle = \varphi(0).$$

It enjoys the following properties:
• it is pointly supported at the origin $t = 0$;
• it is an even distribution: $\delta(-t) = \delta(t)$;
• it is homogeneous of order (-1): $\delta(at) = \frac{1}{|a|}\,\delta(t)$;
• it is of finite order, more specifically of order zero, i.e. $\delta(t)$ is continuously extendable to $C_0(\mathbb{R})$.
As any other distribution it is infinitely differentiable and its kth order derivative, given by

$$\langle\, \delta^{(k)}(t),\ \varphi(t)\,\rangle = (-1)^k \langle\, \delta(t),\ \varphi^{(k)}(t)\,\rangle = (-1)^k\, \varphi^{(k)}(0),$$

is of finite order k.

Multiplication of $\delta(t)$ by natural powers of the variable t results into the following easily checked identities:

$$t\,\delta(t) = 0$$

and

$$t\,\delta'(t) = -\,\delta(t)$$

$$t^2\,\delta'(t) = 0$$

and also

$$t\,\delta''(t) = -2\,\delta'(t)$$

$$t^2\,\delta''(t) = 2\,\delta(t)$$

$$t^3\,\delta'''(t) = 0$$

etc. It holds that

$$t^j\,\delta^{(k)}(t) = (-1)^j\,k(k-1)\cdots(k-j+1)\,\delta^{(k-j)}(t), \quad j \le k$$

$$t^j\,\delta^{(k)}(t) = 0, \quad j > k$$

and in particular

$$t^k\,\delta^{(k)}(t) = (-1)^k\,k!\,\delta(t).$$

Division of $\delta(t)$ by the variable t leads, a priori, to an equivalence class of distributions, since t shows a single zero at the origin:

$$\frac{1}{t}\,\delta(t) = -\,\delta'(t) + c\,\delta(t),$$

but, as the left-hand side is a homogeneous distribution of order (-2), as is $\delta'(t)$ at the right-hand side, the arbitrary constant c must be zero, eventually leading to

$$\frac{1}{t}\,\delta(t) = -\,\delta'(t).$$

Along similar lines it is found that

$$\frac{1}{t^j}\,\delta^{(k-j)}(t) = (-1)^j\,\frac{1}{k(k-1)\cdots(k-j+1)}\,\delta^{(k)}(t)$$

and

$$\frac{1}{t^k}\,\delta(t) = (-1)^k\,\frac{1}{k!}\,\delta^{(k)}(t)$$

and also

$$\frac{1}{t}\,\delta^{(k-1)}(t) = -\frac{1}{k}\,\delta^{(k)}(t).$$

4. The higher dimensional delta distribution: the basics

The delta distribution $\delta(\underline{x}) \in \mathscr{D}'(\mathbb{R}^m)$ is a scalar distribution defined, for all test functions $\varphi \in \mathscr{D}(\mathbb{R}^m)$, by

$$\langle\ \delta(\underline{x}),\ \varphi(\underline{x})\ \rangle = \varphi(0).$$

It is a distribution which is:

• pointly supported at the origin $\underline{x} = 0$;

• rotation invariant: $\delta(A\,\underline{x}) = \delta(\underline{x})$, $\forall A \in \mathrm{SO}(m)$, where $\mathrm{SO}(m)$ is the group of $m\times m$ orthogonal matrices of determinant 1, called the *special orthogonal group* or *rotation group*;

• even: $\delta(-\underline{x}) = \delta(\underline{x})$;

• homogeneous of order $(-m)$: $\delta(a\underline{x}) = \dfrac{1}{|a|^m}\,\delta(\underline{x})$;

• of finite order zero.

Its Cartesian derivatives are given by

$$\langle\ \partial^s_{x_j}\delta(\underline{x}),\ \varphi(\underline{x})\ \rangle = (-1)^s\langle\ \delta(\underline{x}),\ \partial^s_{x_j}\varphi(\underline{x})\ \rangle = (-1)^s\{\partial^s_{x_j}\varphi(\underline{x})\}|_{\underline{x}=0}, \qquad j = 1,\ \ldots,\ m.$$

In particular the action of the Dirac operator $\underline{\partial}$ results into the vector-valued distribution $\underline{\partial}\delta(\underline{x})$ given by

$$\langle\ \underline{\partial}\,\delta(\underline{x}),\ \varphi(\underline{x})\ \rangle = -\langle\ \delta(\underline{x}),\ \underline{\partial}\,\varphi(\underline{x})\ \rangle = -\{\underline{\partial}\varphi(\underline{x})\}|_{\underline{x}=0}.$$

The action of the Euler operator $\mathbb{E}$ reveals its homogeneous character:

$$\mathbb{E}\,\delta(\underline{x}) = (-m)\,\delta(\underline{x})$$

whereas the action of the angular momentum operator $\Gamma = -\sum_{j<k} e_j e_k(x_j\partial x_k - x_k\partial_{x_j})$ leads to

$$\Gamma\,\delta(\underline{x}) = 0$$

which is in accordance with the rotation invariant or *radial* character of $\delta(\underline{x})$.

5. Multiplication by natural powers of the vector variable $\underline{x}$

A straightforward application of the definition of multiplication of a distribution by an analytic function, leads to the following identities for multiplication of the delta distribution and its Dirac derivatives by the vector variable $\underline{x}$ and natural powers thereof. It holds that

$$\underline{x}\,\delta(\underline{x}) = 0$$

and

$$\underline{x}\,\underline{\partial}\,\delta(\underline{x}) = m\,\delta(\underline{x})$$

$$\underline{x}^2\,\delta(\underline{x}) = 0$$

and also

$$\underline{x}\,\underline{\partial}^2\,\delta(\underline{x}) = 2\,\underline{\partial}\,\delta(\underline{x})$$

$$\underline{x}^2\,\underline{\partial}^2\,\delta(\underline{x}) = 2\,m\,\delta(\underline{x})$$

$$\underline{x}^3\,\underline{\partial}^2\,\delta(\underline{x}) = 0$$

etc. More generally one has

$$\underline{x}\,\underline{\partial}^{2\ell+1}\,\delta(\underline{x}) = (m+2\ell)\,\underline{\partial}^{2\ell}\,\delta(\underline{x})$$

$$\underline{x}\,\underline{\partial}^{2\ell}\,\delta(\underline{x}) = (2\ell)\,\underline{\partial}^{2\ell-1}\,\delta(\underline{x}).$$

By iteration we obtain the following identities involving natural powers of the vector variable $\underline{x}$:

$$\underline{x}^2\,\underline{\partial}^{2\ell+1}\,\delta(\underline{x}) = (m+2\ell)(2\ell)\,\underline{\partial}^{2\ell-1}\,\delta(\underline{x})$$

$$\underline{x}^3\,\underline{\partial}^{2\ell+1}\,\delta(\underline{x}) = (m+2\ell)(2\ell)(m+2\ell-2)\,\underline{\partial}^{2\ell-2}\,\delta(\underline{x})$$

$$\underline{x}^4\,\underline{\partial}^{2\ell+1}\,\delta(\underline{x}) = (m+2\ell)(m+2\ell-2)(2\ell)(2\ell-2)\,\underline{\partial}^{2\ell-3}\,\delta(\underline{x})$$

$$\vdots$$

$$\underline{x}^{2\ell+1}\,\underline{\partial}^{2\ell+1}\,\delta(\underline{x}) = C(m,\ \ell)\,\delta(\underline{x})$$

where we have introduced the constant

$$
\begin{aligned}
C(m,\ \ell) &= (m+2\ell)(m+2\ell-2)\cdots(m)(2\ell)(2\ell-2)\cdots(2)\\
&= 2^{\ell}\,\ell!\,m(m+2)\cdots(m+2\ell)\\
&= 2^{2\ell+1}\,\ell!\,\frac{\Gamma(m/2+\ell+1)}{\Gamma(m/2)}.
\end{aligned}
$$

Similarly we find

$$
\begin{aligned}
&\underline{x}^2\,\underline{\partial}^{2\ell}\delta(\underline{x}) = (m+2\ell-2)(2\ell)\,\underline{\partial}^{2\ell-2}\delta(\underline{x})\\
&\underline{x}^3\,\underline{\partial}^{2\ell}\delta(\underline{x}) = (m+2\ell-2)(2\ell)(2\ell-2)\,\underline{\partial}^{2\ell-3}\delta(\underline{x})\\
&\underline{x}^4\,\underline{\partial}^{2\ell}\delta(\underline{x}) = (m+2\ell-2)(m+2\ell-4)(2\ell)(2\ell-2)\,\underline{\partial}^{2\ell-4}\delta(\underline{x})\\
&\vdots\\
&\underline{x}^{2\ell}\,\underline{\partial}^{2\ell}\delta(\underline{x}) = \frac{1}{m+2\ell}C(m,\ \ell)\,\delta(\underline{x}).
\end{aligned}
$$

More generally it holds that, for $k \leq \ell$,

$$
\begin{aligned}
&\underline{x}^{2k}\,\underline{\partial}^{2\ell}\delta(\underline{x}) = (2\ell)(2\ell-2)\ \cdots\ (2\ell-2k+2)(m+2\ell-2)(m+2\ell-4)\ \cdots\ (m+2\ell-2k)\,\underline{\partial}^{2\ell-2k}\delta(\underline{x})\\
&\underline{x}^{2k+1}\,\underline{\partial}^{2\ell}\delta(\underline{x}) = (2\ell)(2\ell-2)\ \cdots\ (2\ell-2k)(m+2\ell-2)(m+2\ell-4)\ \cdots\ (m+2\ell-2k)\,\underline{\partial}^{2\ell-2k-1}\delta(\underline{x})\\
&\underline{x}^{2k}\,\underline{\partial}^{2\ell+1}\delta(\underline{x}) = (2\ell)(2\ell-2)\ \cdots\ (2\ell-2k+2)(m+2\ell)(m+2\ell-2)\ \cdots\ (m+2\ell-2k+2)\,\underline{\partial}^{2\ell-2k+1}\delta(\underline{x})\\
&\underline{x}^{2k+1}\,\underline{\partial}^{2\ell+1}\delta(\underline{x}) = (2\ell)(2\ell-2)\ \cdots\ (2\ell-2k+2)(m+2\ell)(m+2\ell-2)\ \cdots\ (m+2\ell-2k)\,\underline{\partial}^{2\ell-2k}\delta(\underline{x}).
\end{aligned}
$$

6. Multiplication by natural powers of the Cartesian variables x_j, $j = 1, ..., m$

In this section we will establish formulæ for products of partial derivatives of the delta distribution $\delta(\underline{x})$ by Cartesian variables involving expressions of the form

$$
x_{j_1}x_{j_2}\ \cdots\ x_{j_p}\,\partial_{k_1}\partial_{k_2}\ \cdots\ \partial_{k_q}\,\delta(\underline{x})
$$

where ∂_k is shorthand for ∂_{x_k}, $k = 1, \ldots, m$, more explicitly: $\partial_k \, \delta(\underline{x})$ denotes the partial derivative of the delta distribution $\delta(\underline{x})$ with respect to the Cartesian variable x_k, $k = 1, \ldots, m$. Such an expression can always been written in the alternative form

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m} \partial_1^{\beta_1} \partial_2^{\beta_2} \cdots \partial_m^{\beta_m} \delta(\underline{x})$$

with $|\alpha| = \alpha_1 + \cdots + \alpha_m = p$ and $|\beta| = \beta_1 + \cdots + \beta_m = q$.

We proceed stepwise via a sequence of lemmata. Note that, since the delta distribution $\delta(\underline{x})$ is tempered, see Section 2, the test functions under consideration can be taken either in $\mathscr{D}(\mathbb{R}^m)$ or in $\mathscr{S}(\mathbb{R}^m)$.

Lemma 2 For all $j = 1, \ldots, m$ it holds that

$$x_j \, \delta(\underline{x}) = 0,$$

the zero distribution.

Proof. For all test functions $\varphi(\underline{x})$, and any $j = 1, \ldots, m$ it holds that

$$\langle \, x_j \, \delta(\underline{x}), \; \varphi(\underline{x}) \, \rangle = \langle \, \delta(\underline{x}), \; x_j \, \varphi(\underline{x}) \, \rangle = \{x_j \, \varphi(\underline{x})\}_{\underline{x}=0} = 0.$$

$\square$

Lemma 3 For all $j, \; k = 1, \ldots, m$ it holds that the commutator of the operators x_j and ∂_j is given by

$$\{x_j, \; \partial_j\} = -1$$

whereas for $j \neq k$

$$\{x_j, \; \partial_k\} = 0.$$

Proof. For each distribution T it holds that

$$\partial_j \, (x_j \, T) = T + x_j \, \partial_j \, T$$

and thus

$$(x_j \, \partial_j - \partial_j \, x_j) \, T = -T.$$

For $j \neq k$ it holds trivially that

$$\partial_k (x_j T) = x_j \partial_k T.$$

$\square$

Note that the above result can be written as

$$\{x_j,\ \partial_k\} = -\delta_{jk}$$

where δ_{jk} is the Kronecker delta.

Corollary 1 For the delta distribution $\delta(\underline{x})$ it holds that, for every $j = 1,\ \ldots,\ m$,

$$x_j \partial_j \delta(\underline{x}) = -\delta(\underline{x}).$$

Lemma 4 For the delta distribution $\delta(\underline{x})$ it holds that, for every $j = 1,\ \ldots,\ m$ and every $n \in \mathbb{N}$,

$$x_j^n \partial_j^n \delta(\underline{x}) = (-1)^n n!\, \delta(\underline{x}).$$

Proof. The proof is by induction on n.

(i) Corollary 1 shows the result is true for $n = 1$.

(ii) We assume the result to be valid for n:

$$x_j^n \partial_j^n \delta(\underline{x}) = (-1)^n n!\, \delta(\underline{x})$$

and prove it to be true for $(n+1)$. We have consecutively, making use of the results in Lemma 3:

$$\begin{aligned}
x_j^{n+1} \partial_j^{n+1} \delta(\underline{x}) &= x_j^n \partial_j x_j \partial_j^n \delta(\underline{x}) - x_j^n \partial^n \delta(\underline{x}) \\
&= x_j^{n-1} \partial_j x_j^2 \partial_j^n \delta(\underline{x}) - 2 x_j^n \partial^n \delta(\underline{x}) \\
&= x_j^{n-2} \partial_j x_j^3 \partial_j^n \delta(\underline{x}) - 3 x_j^n \partial^n \delta(\underline{x}) \\
&= \ldots \\
&= x_j \partial_j x_j^n \partial_j^n \delta(\underline{x}) - n x_j^n \partial^n \delta(\underline{x}) \\
&= x_j \partial_j ((-1)^n n!\, \delta(\underline{x})) + (-1)^{n+1} n\, n!\, \delta(\underline{x})
\end{aligned}$$

$$= (-1)^{n+1}\, n!\, \delta(\underline{x}) + (-1)^{n+1}\, n\, n!\, \delta(\underline{x})$$

$$= (-1)^{n+1}\, (n+1)!\, \delta(\underline{x}).$$

□

Proposition 1 For the delta distribution $\delta(\underline{x})$ it holds that the distribution

$$x_1^{\alpha_1}\, x_2^{\alpha_2}\, \cdots x_m^{\alpha_m}\, \partial_1^{\beta_1}\, \partial_2^{\beta_2}\, \cdots \partial_m^{\beta_m}\, \delta(\underline{x})$$

is the zero distribution whenever at least one of the exponents, say α_j, is greater than the corresponding order of derivation β_j.

Proof. Assuming $\alpha_j > \beta_j$ we put $\alpha_j - \beta_j = s_j > 0$. Taking into account that the operators x_j and ∂_k commute whenever $j \neq k$ (see Lemma 3), the given distribution can be rewritten as:

$$x_1^{\alpha_1}\, x_2^{\alpha_2}\, \cdots\, [x_j^{\alpha_j}]\, \cdots\, x_m^{\alpha_m}\, \partial_1^{\beta_1}\, \cdots\, [\partial_j^{\beta_j}]\, \cdots\, \partial_m^{\beta_m}\, \delta(\underline{x})\, x_j^s\, x_j^{\beta_j}\, \partial_j^{\beta_j}\, \delta(\underline{x})$$

or, in view of Lemma 4,

$$x_1^{\alpha_1}\, x_2^{\alpha_2}\, \cdots\, [x_j^{\alpha_j}]\, \cdots\, x_m^{\alpha_m}\, \partial_1^{\beta_1}\, \cdots\, [\partial_j^{\beta_j}]\, \cdots\, \partial_m^{\beta_m}\, \delta(\underline{x})\, x_j^s\, (-1)^{\beta_j}\, \beta_j!\, \delta(\underline{x})$$

which clearly is the zero distribution seen the fact that $x_j\, \delta(\underline{x}) = 0$. □

Proposition 2 For the delta distribution $\delta(\underline{x})$ it holds that, for all multi-indices $(\alpha_1, \ldots, \alpha_m)$,

$$x_1^{\alpha_1}\, x_2^{\alpha_2}\, \cdots x_m^{\alpha_m}\, \partial_1^{\alpha_1}\, \partial_2^{\alpha_2}\, \cdots \partial_m^{\alpha_m}\, \delta(\underline{x}) = (-1)^{\alpha_1}\, \alpha_1!\, \cdots (-1)^{\alpha_m}\, \alpha_m!\, \delta(\underline{x}) = (-1)^{|\alpha|}\, \alpha!\, \delta(\underline{x}).$$

Proof. Again taking into account that the operators x_j and ∂_k commute whenever $j \neq k$, the given distribution may be rewritten as:

$$(x_1^{\alpha_1}\, \partial_1^{\alpha_1}) \cdots (x_m^{\alpha_m}\, \partial_m^{\alpha_m})\, \delta(\underline{x})$$

which, invoking the result in Lemma 4, equals

$$(-1)^{\alpha_1}\, \alpha_1! \cdots (-1)^{\alpha_m}\, \alpha_m!\, \delta(\underline{x}) = (-1)^{|\alpha|}\, \alpha!\, \delta(\underline{x}).$$

□

Proposition 3 If $p < q$, with $p + s = q$, then for the delta distribution $\delta(\underline{x})$ it holds that

$$x_{j_1} x_{j_2} \cdots x_{j_p} \partial_{k_1} \partial_{k_2} \cdots \partial_{k_q} \delta(\underline{x})$$

$$= (-1)^p \sum_{(k_{p+1} k_{p+2} \ldots k_{p+s})} \left(\sum_{\pi(j_1 j_2 \ldots j_p)} \delta_{j_1 k_1} \delta_{j_2 k_2} \cdots \delta_{j_p k_p} \right) \partial_{k_{p+1}} \partial_{k_{p+2}} \cdots \partial_{k_{p+s}} \delta(\underline{x}) \tag{3}$$

where the first summation runs over all possible combinations $(k_{p+1} k_{p+2} \ldots k_{p+s})$ out of $\{k_1, k_2, \ldots, k_q\}$ and the second summation runs over all possible permutations of $(j_1 \ldots j_p)$ repetitions being allowed.

Proof. The proof is by induction on p and q.

First we compute directly a particular case for p and q, say $p = 2$ and $q = 4$ en we find:

$$x_j x_k \partial_a \partial_b \partial_c \partial_d \delta(\underline{x}) = (\delta_{jc} \delta_{kd} + \delta_{jd} \delta_{kc}) \partial_a \partial_b \delta(\underline{x})$$

$$+ (\delta_{jb} \delta_{kd} + \delta_{jd} \delta_{kb}) \partial_a \partial_c \delta(\underline{x})$$

$$+ (\delta_{jb} \delta_{kc} + \delta_{jc} \delta_{kb}) \partial_a \partial_d \delta(\underline{x})$$

$$+ (\delta_{ja} \delta_{kd} + \delta_{jd} \delta_{ka}) \partial_b \partial_c \delta(\underline{x})$$

$$+ (\delta_{ja} \delta_{kc} + \delta_{jc} \delta_{ka}) \partial_b \partial_d \delta(\underline{x})$$

$$+ (\delta_{ja} \delta_{kb} + \delta_{jb} \delta_{ka}) \partial_c \partial_d \delta(\underline{x})$$

which is in agreement with formula (3).

Next we increase p by 1 by multiplying the left-hand side of (3) by a Cartesian variable, say x_{j_0}. Doing so, we expect the order of derivation of $\delta(\underline{x})$ at the right-hand side of (3) to diminish by 1. We find, using the commutating relations of the operators x_j and ∂_j, consecutively:

$$x_{j_0} \partial_{k_{p+1}} \partial_{k_{p+2}} \ldots \partial_{k_{p+s}} \delta(\underline{x}) = \partial_{k_{p+1}} x_{j_0} \partial_{k_{p+2}} \ldots \partial_{k_{p+s}} \delta(\underline{x}) - \delta_{j_0,\, k_{p+1}} \partial_{k_{p+2}} \ldots \partial_{k_{p+s}} \delta(\underline{x})$$

$$= \partial_{k_{p+1}} \left(\partial_{k_{p+2}} x_{j_0} \ldots \partial_{k_{p+s}} \delta(\underline{x}) - \delta_{j_0,\, k_{p+2}} \partial_{k_{p+3}} \ldots \partial_{k_{p+s}} \delta(\underline{x}) \right)$$

$$- \delta_{j_0,\, k_{p+1}} \partial_{k_{p+2}} \ldots \partial_{k_{p+s}} \delta(\underline{x})$$

$$= \partial_{k_{p+1}} \partial_{k_{p+2}} \partial_{k_{p+3}} x_{j_0} \partial_{k_{p+4}} \ldots \partial_{k_{p+s}} \delta(\underline{x}) - \delta_{j_0,\, k_{p+1}} \partial_{k_{p+2}} \ldots \partial_{k_{p+s}} \delta(\underline{x})$$

$$- \delta_{j_0,\, k_{p+2}} \partial_{k_{p+1}} \partial_{k_{p+3}} \ldots \partial_{k_{p+s}} \delta(\underline{x}) - \delta_{j_0,\, k_{p+3}} \partial_{k_{p+1}} \partial_{k_{p+2}} \ldots \partial_{k_{p+s}} \delta(\underline{x})$$

$$= \ldots$$

$$= - \delta_{j_0,\, k_{p+1}} \partial_{k_{p+2}} \ldots \partial_{k_{p+s}} \delta(\underline{x})$$

$$- \delta_{j_0,\, k_{p+2}} \partial_{k_{p+1}} \partial_{k_{p+3}} \ldots \partial_{k_{p+s}} \delta(\underline{x})$$

$$- \ldots$$

$$- \delta_{j_0,\, k_{p+s}} \partial_{k_{p+1}} \partial_{k_{p+2}} \ldots \partial_{k_{p+s-1}} \delta(\underline{x})$$

which leads to

$$x_{j_0} x_{j_1} x_{j_2} \cdots x_{j_p} \partial_{k_1} \partial_{k_2} \cdots \partial_{k_q} \delta(\underline{x})$$

$$= (-1)^p \sum_{(k_{p+1} k_{p+2} \ldots k_{p+s})} \left(\sum_{\pi(j_1 j_2 \ldots j_p)} \delta_{j_1 k_1} \delta_{j_2 k_2} \cdots \delta_{j_p k_p} \right) x_{j_0} \partial_{k_{p+1}} \partial_{k_{p+2}} \ldots \partial_{k_{p+s}} \delta(\underline{x})$$

where now the right-hand side takes the form

$$(-1)^p \sum_{(k_{p+1} k_{p+2} \ldots k_{p+s})} \left(\sum_{\pi(j_1 j_2 \ldots j_p)} \delta_{j_1 k_1} \delta_{j_2 k_2} \cdots \delta_{j_p k_p} \right)$$

$$\times \left(- \delta_{j_0,\, k_{p+1}} \partial_{k_{p+2}} \ldots \partial_{k_{p+s}} \delta(\underline{x}) - \delta_{j_0,\, k_{p+2}} \partial_{k_{p+1}} \partial_{k_{p+3}} \ldots \partial_{k_{p+s}} \delta(\underline{x}) - \ldots - \delta_{j_0, k_{p+s}} \partial_{k_{p+1}} \partial_{k_{p+2}} \ldots \partial_{k_{p+s-1}} \delta(\underline{x}) \right)$$

which indeed equals

$$(-1)^{p+1} \sum_{(k_{p+2} \ldots k_{p+s})} \left(\sum_{\pi(j_0 j_1 j_2 \ldots j_p)} \delta_{j_1 k_1} \delta_{j_2 k_2} \cdots \delta_{j_p k_p} \delta_{j_0 k_{p+1}} \right) \partial_{k_{p+2}} \ldots \partial_{k_{p+s}} \delta(\underline{x}).$$

To prove the induction on q we increase q by 1 by acting with the operator $\partial_{k_{q+1}}$ on both sides of equation (3). The action on the right-hand side is simply

$$(-1)^p \sum_{(k_{p+1} k_{p+2} \ldots k_q)} \left(\sum_{\pi(j_1 j_2 \ldots j_p)} \delta_{j_1 k_1} \delta_{j_2 k_2} \cdots \delta_{j_p k_p} \right) \partial_{k_{p+1}} \partial_{k_{p+2}} \ldots \partial_{k_q} \partial_{k_{q+1}} \delta(\underline{x})$$

The action of $\partial_{k_{q+1}}$ on the left-hand side yields

$$x_{j_1} \cdots x_{j_p} \partial_{k_{q+1}} \partial_{k_1} \partial_{k_2} \cdots \partial_{k_q} \delta(\underline{x}) + V$$

where the distribution V is given by

$$\delta_{k_{q+1},\, j_1} x_{j_2} \ldots x_{j_p} \partial_{k_1} \partial_{k_2} \cdots \partial_{k_q} \delta(\underline{x}) + \delta_{k_{q+1},\, j_2} x_{j_1} x_{j_3} \ldots x_{j_p} \partial_{k_1} \partial_{k_2} \cdots \partial_{k_q} \delta(\underline{x})$$

$$+ \cdots + \delta_{k_{q+1},\, j_p} x_{j_1} x_{j_2} \ldots x_{j_{p-1}} \partial_{k_1} \partial_{k_2} \cdots \partial_{k_q} \delta(\underline{x}).$$

On each of the terms of V, formula (3) can be applied, yielding the following expression for V:

$$\delta_{k_{q+1},\, j_1} (-1)^{p-1} \sum_{(k_1 k_{p+1} k_{p+2} \ldots k_q)} \left(\sum_{\pi(j_2 \ldots j_p)} \delta_{j_2 k_2} \cdots \delta_{j_p k_p} \right) \partial_{k_1} \partial_{k_{p+1}} \partial_{k_{p+2}} \ldots \partial_{k_q} \delta(\underline{x}) +$$

$$\ldots$$

$$+ \delta_{k_{q+1},\, j_1} (-1)^{p-1} \sum_{(k_p k_{p+1} k_{p+2} \ldots k_q)} \left(\sum_{\pi(j_1 \ldots j_{p-1})} \delta_{j_1 k_1} \cdots \delta_{j_{p-1} k_{p-1}} \right) \partial_{k_p} \partial_{k_{p+1}} \partial_{k_{p+2}} \ldots \partial_{k_q} \delta(\underline{x}).$$

This leads to

$$x_{j_1} \cdots x_{j_p} \partial_{k_1} \partial_{k_2} \cdots \partial_{k_q} \partial_{k_{q+1}} \delta(\underline{x})$$

$$= (-1)^p \sum_{(k_{p+1} k_{p+2} \ldots k_q)} \left(\sum_{\pi(j_1 j_2 \ldots j_p)} \delta_{j_1 k_1} \delta_{j_2 k_2} \cdots \delta_{j_p k_p} \right) \partial_{k_{p+1}} \partial_{k_{p+2}} \ldots \partial_{k_q} \partial_{k_{q+1}} \delta(\underline{x}) - V$$

which, by substituting the above obtained expression for V, takes the desired form

$$x_{j_1} \cdots x_{j_p} \partial_{k_1} \partial_{k_2} \cdots \partial_{k_q} \partial_{k_{q+1}} \delta(\underline{x})$$

$$= (-1)^p \sum_{(k_{p+1} k_{p+2} \ldots k_{q+1})} \left(\sum_{\pi(j_1 j_2 \ldots j_p)} \delta_{j_1 k_1} \delta_{j_2 k_2} \cdots \delta_{j_p k_p} \right) \partial_{k_{p+1}} \partial_{k_{p+2}} \ldots \partial_{k_q} \partial_{k_{q+1}} \delta(\underline{x}).$$

□

Now we illustrate the results of the Propositions 1, 2 and 3 by some straightforward examples, meanwhile recovering former formulæ.

Example 1 For all $j = 1, \ldots, m$, one has $x_j\, \delta(\underline{x}) = 0$, whence $\underline{x}\, \delta(\underline{x}) = 0$. Similarly it holds that $x_j x_k\, \partial_a \delta(\underline{x}) = 0$, whence $\underline{x}^2\, \underline{\partial}\, \delta(\underline{x}) = 0$, etc.

Example 2 For all $j = 1, \ldots, m$, one has

$$x_j\, \partial_j \delta(\underline{x}) = -\, \delta(\underline{x})$$

and also

$$x_j\, \partial_k\, \delta(\underline{x}) = 0, \quad j \neq k.$$

It follows at once that

$$\mathbb{E}\, \delta(\underline{x}) = \sum_{j=1}^{m} x_j\, \partial_j\, \delta(\underline{x}) = (-m)\, \delta(\underline{x}),$$

that

$$\Gamma\, \delta(\underline{x}) = -\sum_{j<k} e_j e_k (x_j \partial_k - x_k \partial_j)\, \delta(\underline{x}) = 0,$$

and

$$\underline{x}\, \underline{\partial}\, \delta(\underline{x}) = \sum_j \sum_k e_j e_k\, x_j\, \partial_k\, \delta(\underline{x}) = -\sum_j x_j\, \partial_j\, \delta(\underline{x}) = m\, \delta(\underline{x}).$$

It also holds that

$$x_j\, \underline{\partial} \delta(\underline{x}) = -\, e_j\, \delta(\underline{x})$$

and

$$\underline{x}\, \partial_j \delta(\underline{x}) = -\, e_j\, \delta(\underline{x}).$$

Example 3 For $j \neq k$ one has

$$x_j\, \partial_k^2\, \delta(\underline{x}) = 0$$

whereas

$$x_j\, \partial_j^2\, \delta(\underline{x}) = -2\, \partial_j\, \delta(\underline{x})$$

whence

$$x_j\, \underline{\partial}^2\, \delta(\underline{x}) = -x_j \sum_{k=1}^{m} \partial_k^2\, \delta(\underline{x}) = 2\, \partial_j\, \delta(\underline{x})$$

and

$$\underline{x}\, \underline{\partial}^2\, \delta(\underline{x}) = 2\, \underline{\partial}\, \delta(\underline{x}).$$

Example 4 For $j \neq k$ one has

$$x_j\, x_k\, \partial_a^2\, \delta(\underline{x}) = \left(\sum_{\pi(jk)} \delta_{ja}\, \delta_{ka} \right) \delta(\underline{x}) = 0$$

whence

$$x_j\, x_k\, \underline{\partial}^2\, \delta(\underline{x}) = 0.$$

For $j \neq a$ one has

$$x_j^2\, \partial_a^2\, \delta(\underline{x}) = \left(\sum_{\pi(jj)} \delta_{ja}\, \delta_{ja} \right) \delta(\underline{x}) = 0$$

whereas

$$x_j^2\, \partial_j^2\, \delta(\underline{x}) = \left(\sum_{\pi(jj)} \delta_{jj}\, \delta_{jj} \right) \delta(\underline{x}) = 2\, \delta(\underline{x})$$

whence

$$x_j^2\, \underline{\partial}^2\, \delta(\underline{x}) = -2\, \delta(\underline{x})$$

and

$$\underline{x}^2\,\underline{\partial}^2\,\delta(\underline{x}) = -\sum_{j=1}^{m} x_j^2\,\underline{\partial}^2\,\delta(\underline{x}) = 2m\;\delta(\underline{x}).$$

Example 5 For $j \neq k$ one has

$$x_j\,\partial_k\,\partial_a^2\,\delta(\underline{x}) = -2\,\delta_{ja}\,\partial_k\,\partial_a\,\delta(\underline{x})$$

whence

$$x_j\,\partial_k\,\partial_a^2\,\delta(\underline{x}) = 0, \quad j \neq a$$

and

$$x_j\,\partial_k\,\partial_j^2\,\delta(\underline{x}) = -2\,\partial_k\,\partial_j\,\delta(\underline{x})$$

yielding

$$x_j\,\partial_k\,\underline{\partial}^2\,\delta(\underline{x}) = 2\,\partial_j\,\partial_k\,\delta(\underline{x}).$$

On the other hand one has

$$x_j\,\partial_j\,\partial_a^2\,\delta(\underline{x}) = -2\,\delta_{ja}\,\partial_j\,\partial_a\,\delta(\underline{x}) - \partial_a^2\,\delta(\underline{x})$$

whence

$$x_j\,\partial_j\,\partial_a^2\,\delta(\underline{x}) = -\,\partial_a^2\,\delta(\underline{x}), \quad j \neq a$$

and

$$x_j\,\partial_j\,\partial_j^2\,\delta(\underline{x}) = -3\,\partial_j\,\partial_j\,\delta(\underline{x})$$

yielding

$$x_j\,\partial_j\,\underline{\partial}^2\,\delta(\underline{x}) = -\,\underline{\partial}^2\,\delta(\underline{x}) + 2\,\partial_j^2\,\delta(\underline{x}).$$

It follows that

$$
\begin{aligned}
\underline{x}\,\underline{\partial}^3\,\delta(\underline{x}) &= \sum_j \sum_k e_j e_k\, x_j\, \partial_k\, \underline{\partial}^2\,\delta(\underline{x}) \\
&= -\sum_j x_j\,\partial_j\,\underline{\partial}^2\,\delta(\underline{x}) + \sum_{j\neq k} e_j e_k\, x_j\,\partial_k\,\underline{\partial}^2\,\delta(\underline{x}) \\
&= m\,\underline{\partial}^2\,\delta(\underline{x}) + 2\,\underline{\partial}^2\,\delta(\underline{x}) + \sum_{j\neq k} e_j e_k\, 2\,\partial_j\,\partial_k\,\delta(\underline{x}) \\
&= (m+2)\,\underline{\partial}^2\,\delta(\underline{x})
\end{aligned}
$$

and also

$$
\mathbb{E}\,\underline{\partial}^2\,\delta(\underline{x}) = \sum_j x_j\,\partial_j\,\underline{\partial}^2\,\delta(\underline{x}) = \sum_j \left(-\,\underline{\partial}^2\,\delta(\underline{x}) + 2\,\partial_j^2\,\delta(\underline{x})\right) = -\,(m+2)\,\underline{\partial}^2\,\delta(\underline{x})
$$

and

$$
\Gamma\,\underline{\partial}^2\,\delta(\underline{x}) = -\sum_{j<k} e_j e_k\,(x_j\partial_k - x_k\partial_j)\,\underline{\partial}^2\,\delta(\underline{x}) = 0.
$$

In the same order of ideas the following results may be proven. Note however that a proof by induction is much simpler.

Proposition 4 For $j = 1, \ldots, m$ one has

$$
\begin{aligned}
&x_j\,\underline{\partial}^{2\ell+1}\,\delta(\underline{x}) = -\,e_j\,\underline{\partial}^{2\ell}\,\delta(\underline{x}) + (2\ell)\,\partial_j\,\underline{\partial}^{2\ell-1}\,\delta(\underline{x}) \\
&x_j\,\underline{\partial}^{2\ell}\,\delta(\underline{x}) = (2\ell)\,\partial_j\,\underline{\partial}^{2\ell-2}\,\delta(\underline{x}).
\end{aligned}
$$

Proposition 5 For $j,\ k = 1, \ldots, m$ one has

$$
\begin{aligned}
&x_j\,\partial_k\,\underline{\partial}^{2\ell}\,\delta(\underline{x}) = (2\ell)\,\partial_j\,\partial_k\,\underline{\partial}^{2\ell-2}\,\delta(\underline{x}), \quad j\neq k \\
&x_j\,\partial_j\,\underline{\partial}^{2\ell}\,\delta(\underline{x}) = -\,\underline{\partial}^{2\ell}\,\delta(\underline{x}) + (2\ell)\,\partial_j^2\,\underline{\partial}^{2\ell-2}\,\delta(\underline{x}) \\
&x_j\,\partial_k\,\underline{\partial}^{2\ell+1}\,\delta(\underline{x}) = -\,e_j\,\partial_k\,\underline{\partial}^{2\ell}\,\delta(\underline{x}) + (2\ell)\,\partial_j\,\partial_k\,\underline{\partial}^{2\ell-1}\,\delta(\underline{x}), \quad j\neq k \\
&x_j\,\partial_j\,\underline{\partial}^{2\ell+1}\,\delta(\underline{x}) = -\,\underline{\partial}^{2\ell+1}\,\delta(\underline{x}) - e_j\,\partial_j\,\underline{\partial}^{2\ell}\,\delta(\underline{x}) + (2\ell)\,\partial_j^2\,\underline{\partial}^{2\ell-1}\,\delta(\underline{x}).
\end{aligned}
$$

Making use of the results in Propositions 4 and 5 the following well known formulæ, may be proven straightforwardly.
Corollary 2 One has
(i) $\mathbb{E}\,\underline{\partial}^{2\ell}\,\delta(\underline{x}) = -\,(m+2\ell)\,\underline{\partial}^{2\ell}\,\delta(\underline{x})$;
(ii) $\mathbb{E}\,\underline{\partial}^{2\ell+1}\,\delta(\underline{x}) = -\,(m+2\ell+1)\,\underline{\partial}^{2\ell+1}\,\delta(\underline{x})$;
(iii) $\Gamma\,\underline{\partial}^{2\ell}\,\delta(\underline{x}) = 0$;
(iv) $\Gamma\,\underline{\partial}^{2\ell+1}\,\delta(\underline{x}) = (m-1)\,\underline{\partial}^{2\ell+1}\,\delta(\underline{x})$.

7. Division by natural powers of the vector variable $\underline{x}$

Division of a distribution by the vector variable $\underline{x}$ is a Cartesian operation, whence always defined, albeit, in general, not uniquely determined. However, from [5] we know that division of a distribution by $\underline{x}$ is uniquely determined when this distribution is either radial or homogeneous with homogeneity degree different from $(-m+1)$, where m is the dimension of the Euclidean space considered. It follows that the division of the delta distribution and its Dirac derivatives by natural powers of the vector variable $\underline{x}$ will always be uniquely determined. Let us illustrate this as follows.

According to the general theory we would have

$$\frac{1}{\underline{x}}\,\delta(\underline{x}) = \frac{1}{m}\,\underline{\partial}\,\delta(\underline{x}) + \delta(\underline{x})\,\underline{c}_0$$

with $\underline{c}_0$ an arbitrary vector constant, since $\underline{x}\,\underline{\partial}\,\delta(\underline{x}) = m\,\delta(\underline{x})$, $\underline{x}\,\delta(\underline{x}) = 0$ and $\underline{x}$ shows a simple zero at the origin. But the left-hand side is a homogeneous distribution of order $(-m-1)$ whence the right-hand side should also be homogeneous of order $(-m-1)$, forcing the arbitrary constant $\underline{c}_0$ to be zero, eventually leading to

$$\frac{1}{\underline{x}}\,\delta(\underline{x}) = \frac{1}{m}\,\underline{\partial}\,\delta(\underline{x}).$$

Similarly, based on the results of Section 5, we find:

$$\frac{1}{\underline{x}}\,\underline{\partial}\delta(\underline{x}) = \frac{1}{2}\,\underline{\partial}^2\delta(\underline{x})$$

$$\frac{1}{\underline{x}}\,\underline{\partial}^2\delta(\underline{x}) = \frac{1}{m+2}\,\underline{\partial}^3\delta(\underline{x})$$

$$\frac{1}{\underline{x}}\,\underline{\partial}^3\delta(\underline{x}) = \frac{1}{4}\,\underline{\partial}^4\delta(\underline{x})$$

$$\frac{1}{\underline{x}}\,\underline{\partial}^4\delta(\underline{x}) = \frac{1}{m+4}\,\underline{\partial}^5\delta(\underline{x})$$

etc. More generally, one has

$$\frac{1}{\underline{x}}\,\underline{\partial}^{2\ell}\delta(\underline{x}) = \frac{1}{m+2\ell}\,\underline{\partial}^{2\ell+1}\delta(\underline{x})$$

$$\frac{1}{\underline{x}}\,\underline{\partial}^{2\ell+1}\delta(\underline{x}) = \frac{1}{2\ell+2}\,\underline{\partial}^{2\ell+2}\delta(\underline{x}).$$

By iteration we find

$$\frac{1}{\underline{x}^2}\,\delta(\underline{x}) = \frac{1}{2m}\,\underline{\partial}^{2}\delta(\underline{x})$$

$$\frac{1}{\underline{x}^2}\,\underline{\partial}\delta(\underline{x}) = \frac{1}{2(m+2)}\,\underline{\partial}^{3}\delta(\underline{x})$$

$$\frac{1}{\underline{x}^2}\,\underline{\partial}^2\delta(\underline{x}) = \frac{1}{4(m+2)}\,\underline{\partial}^{4}\delta(\underline{x})$$

$$\frac{1}{\underline{x}^2}\,\underline{\partial}^3\delta(\underline{x}) = \frac{1}{4(m+4)}\,\underline{\partial}^{5}\delta(\underline{x})$$

etc. More generally, one has

$$\frac{1}{\underline{x}^2}\,\underline{\partial}^{2\ell}\delta(\underline{x}) = \frac{1}{(m+2\ell)(2\ell+2)}\,\underline{\partial}^{2\ell+2}\delta(\underline{x})$$

$$\frac{1}{\underline{x}^2}\,\underline{\partial}^{2\ell+1}\delta(\underline{x}) = \frac{1}{(2\ell+2)(m+2\ell+2)}\,\underline{\partial}^{2\ell+3}\delta(\underline{x}).$$

Still more generally, it holds that

$$\frac{1}{\underline{x}^{2k}}\,\underline{\partial}^{2\ell}\delta(\underline{x}) = \frac{1}{2^k\,(\ell+1)(\ell+2)\cdots(l+k)(m+2\ell)(m+2\ell+2)\cdots(m+2\ell+2k-2)}\,\underline{\partial}^{2\ell+2k}\delta(\underline{x})$$

$$\frac{1}{\underline{x}^{2k+1}}\,\underline{\partial}^{2\ell}\delta(\underline{x}) = \frac{1}{2^k\,(\ell+1)(\ell+2)\cdots(l+k)(m+2\ell)(m+2\ell+2)\cdots(m+2\ell+2k)}\,\underline{\partial}^{2\ell+2k+1}\delta(\underline{x})$$

$$\frac{1}{\underline{x}^{2k}}\,\underline{\partial}^{2\ell+1}\delta(\underline{x}) = \frac{1}{2^k\,(\ell+1)(\ell+2)\cdots(l+k)(m+2\ell+2)(m+2\ell+4)\cdots(m+2\ell+2k)}\,\underline{\partial}^{2\ell+2k+1}\delta(\underline{x})$$

$$\frac{1}{\underline{x}^{2k+1}}\,\underline{\partial}^{2\ell+1}\delta(\underline{x}) = \frac{1}{2^{k+1}\,(\ell+1)(\ell+2)\cdots(l+k+1)(m+2\ell+2)(m+2\ell+4)\cdots(m+2\ell+2k)}\,\underline{\partial}^{2\ell+2k+2}\delta(\underline{x}).$$

8. Spherical coordinates

We introduce in Euclidean space $\mathbb{R}^m$ spherical coordinates $\underline{x} = r\underline{\omega}$, $r = |\underline{x}|$, $\underline{\omega} = \sum_{j=1}^{m} e_j \omega_j \in S^{m-1}$. Note that in physics texts $\underline{\omega}$ is usually denoted by $\vec{e}_r$ or $\widehat{\underline{r}}$.

The Dirac operator takes the form

$$\underline{\partial} = \underline{\partial}_{rad} + \underline{\partial}_{ang}$$

with

$$\underline{\partial}_{rad} = \underline{\omega}\, \partial_r \quad \text{and} \quad \underline{\partial}_{ang} = \frac{1}{r}\, \partial_{\underline{\omega}}.$$

We have indeed

$$\underline{\partial} = \sum_{j=1}^{m} e_j \left(\partial_r\, \omega_j + \sum_{k=1}^{m} \partial_{\omega_k} \frac{1}{r}\, \delta_{jk} \right)$$

$$= \sum_{j=1}^{m} e_j\, \omega_j\, \partial_r + e_j\, \partial_{\omega_j} \frac{1}{r}$$

$$= \underline{\omega}\, \partial_r + \frac{1}{r}\, \partial_{\underline{\omega}}$$

where we have put

$$\partial_{\underline{\omega}} = \sum_{j=1}^{m} e_j\, \partial_{\omega_j}.$$

This angular differential operator $\partial_{\underline{\omega}}$ may be seen as the Clifford vector version of the so-called *spherical gradient* $\vec{\nabla}_0$ for which it holds that

$$\vec{\nabla} = \frac{1}{r}\, \vec{\nabla}_0 + \vec{e}_r\, \partial_r.$$

To illustrate the meaning of this *spherical Dirac operator* $\partial_{\underline{\omega}}$ we consider the traditional cases of low dimension: $m = 2$ and $m = 3$.

In dimension $m = 2$, where $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ and $\omega_1 = \cos\theta$, $\omega_2 = \sin\theta$, it holds that

$$\begin{bmatrix} \partial_r \\ \frac{1}{r}\partial_\theta \end{bmatrix} = A_2 \begin{bmatrix} \partial_{x_1} \\ \partial_{x_2} \end{bmatrix}$$

where the SO(2)-matrix A_2 is given by

$$A_2 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

It follows that

$$\underline{\partial} = e_1 \left(\cos\theta \, \partial_r - \frac{1}{r} \sin\theta \, \partial_\theta \right) + e_2 \left(\sin\theta \, \partial_r + \frac{1}{r} \cos\theta \, \partial_\theta \right)$$

whence

$$\partial_{\underline{\omega}} = e_\theta \, \partial_\theta$$

with $e_\theta = -e_1 \sin\theta + e_2 \cos\theta$ a unit vector tangent to the unit circle. Note that

$$\partial_{\omega_1} = -\sin\theta \; \partial_\theta \quad \text{and} \quad \partial_{\omega_2} = \cos\theta \; \partial_\theta.$$

In dimension $m = 3$, where $x_1 = r\sin\theta\cos\phi$, $x_2 = r\sin\theta\sin\phi$, $x_3 = r\cos\theta$ and $\omega_1 = \sin\theta\cos\phi$, $\omega_2 = \sin\theta\sin\phi$, $\omega_3 = \cos\theta$, it holds that

$$\begin{bmatrix} \partial_r \\ \frac{1}{r}\partial_\theta \\ \frac{1}{r}\frac{1}{\sin\theta}\partial_\phi \end{bmatrix} = A_3 \begin{bmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{bmatrix}$$

where the SO(3)-matrix A_3 is given by

$$A_3 = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}.$$

It follows that

$$\underline{\partial} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \times \begin{bmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \times A_3^{-1} \times \begin{bmatrix} \partial_r \\ \frac{1}{r}\partial_\theta \\ \frac{1}{r}\frac{1}{\sin\theta}\,\partial_\phi \end{bmatrix}$$

whence

$$\partial_{\underline{\omega}} = e_\theta\,\partial_\theta + e_\varphi\,\frac{1}{\sin\theta}\,\partial_\varphi$$

with $e_\theta = e_1\cos\theta\cos\phi + e_2\cos\theta\sin\phi - e_3\sin\theta$ and $e_\phi = -e_1\sin\phi + e_2\cos\phi$ two orthogonal unit vectors tangent to the unit sphere. Note that

$$\partial_{\omega_1} = \cos\theta\,\cos\phi\,\partial_\theta - \sin\phi\,\frac{1}{\sin\theta}\,\partial_\varphi$$

$$\partial_{\omega_2} = \cos\theta\,\sin\phi\,\partial_\theta + \cos\phi\,\frac{1}{\sin\theta}\,\partial_\varphi$$

$$\partial_{\omega_3} = -\sin\theta\,\partial_\theta.$$

The angular operators $\partial_{\omega_j},\ j=1,\ \ldots,\ m$ should be manipulated with great care, moreover taking into account that the $\omega_j,\ j=1,\ \ldots,\ m$ are not independent since $\sum_j \omega_j^2 = 1$. Note the following formulæ:

- $\partial_{\omega_i}[\omega_i] = 1-\omega_i^2$,
- $\partial_{\omega_i}[\omega_j] = \partial_{\omega_j}[\omega_i] = -\,\omega_i\omega_j,\ i\neq j$,
- $\partial_{\omega_j}[\underline{\omega}] = \partial_{\underline{\omega}}[\omega_j] = e_j - \omega_j\,\underline{\omega}$,
- $\partial_{\underline{\omega}}[\underline{\omega}] = 1-m$.

Taking into account that $\partial_{\underline{\omega}}$ is orthogonal to $\underline{\omega}$, the Euler operator takes the well known form

$$\mathbb{E} = -\underline{x}\cdot\underline{\partial} = -\,r\underline{\omega}\cdot\underline{\partial}_{rad} = -\,r\underline{\omega}\cdot\underline{\omega}\,\partial_r = r\,\partial_r$$

whereas the angular momentum operator Γ takes the form

$$\Gamma = -\underline{x}\wedge\underline{\partial} = -\,r\underline{\omega}\wedge\underline{\partial}_{ang} = -\,r\underline{\omega}\wedge\frac{1}{r}\,\partial_{\underline{\omega}} = -\,\underline{\omega}\wedge\partial_{\underline{\omega}} = -\underline{\omega}\,\partial_{\underline{\omega}} = -\sum_{j<k} e_je_k(\omega_j\partial\omega_k - \omega_k\partial_{\omega_j}).$$

In dimension $m=2$ the operator Γ is given by

$$\Gamma = -\,e_1\,e_2\,\partial_\theta$$

whereas in dimension $m = 3$ it holds that

$$\Gamma = e_2\, e_3\, (\sin\phi\, \partial_\theta + \cot\theta\, \cos\phi\, \partial_\phi) + e_3\, e_1\, (\cot\theta\, \sin\phi\, \partial_\phi - \cos\phi\, \partial_\theta) + e_1\, e_2\, (-\partial_\phi).$$

There is still an interesting observation to be made about the angular momentum operators and the spherical Dirac operator. Introducing in each $(x_j,\, x_k)$-plane (with $j < k$) polar coordinates:

$$x_j = r_{jk} \cos\theta_{jk}, \quad x_k = r_{jk} \sin\theta_{jk}$$

with $x_j^2 + x_k^2 = r_{jk}^2$, it is easily shown that

$$L_{jk} = \partial_{\theta_{jk}}$$

whence

$$\Gamma = -\sum_{j<k} e_j\, e_k\, \partial_{\theta_{jk}}.$$

For the spherical Dirac operator we then obtain

$$\begin{aligned}
\partial_{\underline{\omega}} &= \underline{\omega} \cdot \Gamma \\
&= -\sum_{i=1}^{m} \sum_{j<k} \omega_i\, e_i \cdot e_j\, e_k\, L_{jk} \\
&= \sum_{j<k} \omega_j\, e_k\, L_{jk} - \omega_k\, e_j\, L_{jk} \\
&= \sum_{j=1}^{m} \sum_{k=1}^{m} \omega_j\, e_k\, L_{jk}
\end{aligned}$$

and also, componentwise,

$$\partial_{\omega_k} = \sum_{j=1}^{m} \omega_j\, L_{jk} = \sum_{j=1}^{m} \omega_j\, \partial_{\theta_{jk}}.$$

The question now is how to define, if possible, the separate action of the operators $\underline{\partial}_{rad}$ and $\underline{\partial}_{ang}$ on a distribution. To that end both operators should first be shown to be Cartesian, which is achieved by putting

$$\underline{\partial}_{rad} = \underline{\omega}\,\partial_r = -\frac{1}{\underline{x}}\,\mathbb{E} \quad \text{and} \quad \underline{\partial}_{ang} = \frac{1}{r}\,\partial_{\underline{\omega}} = -\frac{1}{\underline{x}}\,\Gamma.$$

Due to the division by the vector variable $\underline{x}$, see Lemma 1, the result by the action of $\underline{\partial}_{rad}$ and $\underline{\partial}_{ang}$ on a distribution clearly will be an equivalence class of distributions modulo a multiple of the delta distribution. This leads to the following definition.

Definition 4 Let $T(\underline{x}) \in \mathscr{D}'(\mathbb{R}^m)$ be a distribution. Then we put

$$\underline{\partial}_{rad}\,T(\underline{x}) = (\underline{\omega}\,\partial_r)\,T(r\underline{\omega}) = -\left[\frac{1}{\underline{x}}\,\mathbb{E}\,T(\underline{x})\right] \tag{4}$$

and

$$\underline{\partial}_{ang}\,T(\underline{x}) = \left(\frac{1}{r}\,\partial_{\underline{\omega}}\right)\,T(r\underline{\omega}) = -\left[\frac{1}{\underline{x}}\,\Gamma\,T(\underline{x})\right]. \tag{5}$$

So it becomes clear that the actions of $\underline{\partial}_{rad}$ and $\underline{\partial}_{ang}$ on the distribution $T(\underline{x})$ are well-defined but not uniquely determined. However, if $\underline{S}_1$ and $\underline{S}_2$ are distributions arbitrarily chosen in the equivalent classes (4) and (5) respectively, i.e.

$$\underline{x}\,\underline{S}_1 = -\,\mathbb{E}\,T(\underline{x}) \quad \text{and} \quad \underline{x}\,\underline{S}_2 = -\,\Gamma T(\underline{x}),$$

then

$$\underline{\partial}_{rad}\,T(\underline{x}) = \underline{S}_1 + \underline{c}_1\,\delta(\underline{x})$$

$$\underline{\partial}_{ang}\,T(\underline{x}) = \underline{S}_2 + \underline{c}_2\,\delta(\underline{x})$$

and it must hold that

$$\underline{S}_1 + \underline{c}_1\,\delta(\underline{x}) + \underline{S}_2 + \underline{c}_2\,\delta(\underline{x}) = \underline{\partial}_{rad}\,T(\underline{x}) + \underline{\partial}_{ang}\,T(\underline{x}) = \underline{\partial}\,T(\underline{x}) \tag{6}$$

where the distribution at the utmost right-hand side is, quite naturally, a known distribution once the distribution T is given. One could say that the differential operators $\underline{\partial}_{rad}$ and $\underline{\partial}_{ang}$ are *entangled* in the sense that the results of their actions on a distribution are subject to (6) which becomes a condition on the arbitrary vector constants $\underline{c}_1$ and $\underline{c}_2$. Henceforth we call (6) the *entanglement condition* for the operators $\underline{\partial}_{rad}$ and $\underline{\partial}_{ang}$.

It is a very well known fact that the Dirac operator linearizes the Laplace operator:

$$-\,\underline{\partial}^2 = \Delta = -\,|\underline{\partial}|^2.$$

In terms of spherical coordinates we obtain, in view of

$$\underline{\partial}_{rad}\,\underline{\partial}_{rad} = -\,\partial_r^2$$

$$\underline{\partial}_{rad}\,\underline{\partial}_{ang} = -\frac{1}{r^2}\,\underline{\omega}\,\partial_{\underline{\omega}} + \frac{1}{r}\,\underline{\omega}\,\partial_{\underline{\omega}}\,\partial_r$$

$$\underline{\partial}_{ang}\,\underline{\partial}_{rad} = -\,(m-1)\frac{1}{r}\,\partial_r - \frac{1}{r}\,\partial_r\,\underline{\omega}\,\partial_{\underline{\omega}}$$

$$\underline{\partial}_{ang}\,\underline{\partial}_{ang} = \frac{1}{r^2}\,\partial_{\underline{\omega}}^2$$

that

$$\Delta = \partial_r^2 + (m-1)\,\frac{1}{r}\,\partial_r + \frac{1}{r^2}\,\Delta^*$$

where $\Delta^* = \underline{\omega}\,\partial_{\underline{\omega}} - \partial_{\underline{\omega}}^2$ is the so-called Laplace-Beltrami operator. In view of the orthogonality of $\underline{\omega}$ and $\partial_{\underline{\omega}}$ we have

$$\Delta^* = \underline{\omega}\wedge\partial_{\underline{\omega}} - \partial_{\underline{\omega}}\cdot\partial_{\underline{\omega}} - \partial_{\underline{\omega}}\wedge\partial_{\underline{\omega}}$$

which, as Δ^* is a scalar operator, implies that

$$\Delta^* = -\,\partial_{\underline{\omega}}\cdot\partial_{\underline{\omega}} = |\partial_{\underline{\omega}}|^2$$

and

$$\underline{\omega}\wedge\partial_{\underline{\omega}} = \partial_{\underline{\omega}}\wedge\partial_{\underline{\omega}} = -\,\Gamma.$$

So, whereas the Laplace operator Δ is the normsquared of the Dirac operator, the spherical Laplace operator or Laplace-Beltrami operator is the normsquared of the spherical Dirac operator.

Contrary to its appearance, the Laplace-Beltrami operator Δ^* is a Cartesian operator, and so is the operator $\partial_{\underline{\omega}}^2$. The following result indeed holds.

Proposition 6 [10] The angular differential operators $\partial_{\underline{\omega}}^2$ and Δ^* may be written in terms of Cartesian derivatives as

$$\partial_{\underline{\omega}}^2 = \Gamma^2 - (m-1)\,\Gamma$$

and

$$\Delta^* = (m-2)\,\Gamma - \Gamma^2.$$

The actions of the Laplace operator and the Laplace-Beltrami operator on a distribution being well-defined and uniquely determined, the question arises how to define the separate actions on a distribution of the three parts of the Laplace operator expressed in spherical coordinates. It turns out that these operators are Cartesian, their actions on a distribution being well-defined, though not uniquely determined, through equivalent classes of distributions.

Proposition 7 The operators ∂_r^2, $\frac{1}{r}\,\partial_r$ and $\frac{1}{r^2}\Delta^*$ are Cartesian, and it holds, for a distribution T, that

$$\partial_r^2\, T = \left[\frac{1}{r^2}\,\mathbb{E}\,(\mathbb{E}-1)\,T\right]$$

$$\frac{1}{r}\,\partial_r\, T = \left[\frac{1}{r^2}\,\mathbb{E}\,T\right]$$

$$\frac{1}{r^2}\,\Delta^*\, T = \left[\frac{1}{r^2}\,((m-2)\Gamma - \Gamma^2)\,T\right]$$

leading to

(i) $\partial_r^2\, T = S_2 + \delta(\underline{x})\,c_2 - \sum_{j=1}^m c_{1,\,j}\,\partial_{x_j}\delta(\underline{x})$ for arbitrary constants c_2 and $c_{1,\,j},\ j=1,\,\ldots,\,m$ and any distribution S_2 such that $\underline{x}\,S_2 = \mathbb{E}\,\underline{S}_1$ with $\underline{x}\,\underline{S}_1 = -\,\mathbb{E}\,T$;

(ii) $\frac{1}{r}\,\partial_r\, T = S_3 + \frac{1}{m}\sum_{j=1}^m c_{1,\,j}\,\partial_{x_j}\delta(\underline{x}) + c_3\,\delta(\underline{x})$ for arbitrarily constant c_3 and any distribution S_3 such that $\underline{x}\,S_3 = \underline{S}_1$;

(iii) $\frac{1}{r^2}\,\Delta^*\, T = S_4 + c_4\,\delta(\underline{x}) + \sum_{j=1}^m c_{5,\,j}\,\partial_{x_j}\delta(\underline{x})$ for arbitrary constants c_4 and $c_{5,\,j},\ j=1,\,\ldots,\,m$ and any distribution S_4 such that $r^2\,S_4 = \Delta^*\,T$.

Proof. (i) A direct computation shows that $r^2\,\partial_r^2 = \mathbb{E}(\mathbb{E}-1)$, whence

$$\partial_r^2 = \frac{1}{r^2}\,\mathbb{E}\,(\mathbb{E}-1).$$

Further we have

$$(\underline{\omega}\,\partial_r)\,T = -\left[\frac{1}{\underline{x}}\,\mathbb{E}\,T\right] = \underline{S}_1 + \delta(\underline{x})\,\underline{c}_1$$

with $\underline{x}\,\underline{S}_1 = -\,\mathbb{E}\,T$. It follows that

$$\begin{aligned}
\partial_r^2 T &= -(\underline{\omega}\,\partial_r)^2 T \\
&= -(\underline{\omega}\,\partial_r)\,(\underline{S}_1 + \delta(\underline{x})\,\underline{c}_1) \\
&= \left[\frac{1}{\underline{x}}\,\mathbb{E}\,\underline{S}_1\right] - \underline{\partial}\,\delta(\underline{x})\,\underline{c}_1 \\
&= S_2 + \delta(\underline{x})\,c_2 - \underline{\partial}\,\delta(\underline{x})\,\underline{c}_1
\end{aligned}$$

with $\underline{x}\,S_2 = \mathbb{E}\,\underline{S}_1$.

(ii) As $r\,\partial_r = \mathbb{E}$, it follows that $\frac{1}{r}\partial_r = \frac{1}{r^2}\mathbb{E}$ and also

$$\begin{aligned}
\frac{1}{r}\,\partial_r T &= \frac{1}{\underline{x}}\,(\underline{\omega}\,\partial_r)\,T \\
&= \frac{1}{\underline{x}}\,(\underline{S}_1 + \delta(\underline{x})\,\underline{c}_1) \\
&= S_3 + \frac{1}{\underline{x}}\,\delta(\underline{x})\,\underline{c}_1 \\
&= S_3 + \frac{1}{m}\,\underline{\partial}\,\delta(\underline{x})\,\underline{c}_1 + \delta(\underline{x})\,c_3
\end{aligned}$$

with $\underline{x}\,S_3 = \underline{S}_1$.

(iii) The distribution $\Delta^*\,T$ being uniquely well-defined and r^2 being an analytic function with a second order zero at the origin, the result follows immediately. □

Remark 1 The operators ∂_r^2, $\frac{1}{r}\partial_r$ and $\frac{1}{r^2}\,\Delta^*$ are *entangled* in the sense that, given a distribution T and having chosen appropriately the distributions $\underline{S}_1$, S_2, S_3 and S_4, all arbitrary constants appearing in the expressions of Proposition 7 should satisfy the entanglement condition generated by

$$\partial_r^2\,T + (m-1)\,\frac{1}{r}\,\partial_r\,T + \frac{1}{r^2}\,\Delta^*\,T = \Delta T$$

the distribution at the right-hand side being uniquely determined.

9. The action of the operators $\underline{\omega}\,\partial_r$, ∂_r^2 and $\frac{1}{r}\partial_r$ on the delta distribution

We already mentioned in Section 4 that the delta distribution $\delta(\underline{x})$ is *spherically symmetric* or *rotation invariant* or *radial* for short; we may think of $\delta(\underline{x})$ as being only depending on the radial distance r, but we will keep the notation $\delta(\underline{x})$

to avoid confusion with the one-dimensional delta distribution $\delta(r)$. Due to this radial character the actions of the Dirac, Euler, Gamma and Laplace operators take a simpler form.

Recall that Euler operator $\mathbb{E} = r\,\partial_r$ is a purely radial operator and its action on $\delta(\underline{x})$ results, quite naturally, into a radial distribution, viz. the delta distribution itself, up to the homogeneity factor $(-m)$:

$$\mathbb{E}\,\delta(\underline{x}) = r\,\partial_r\,\delta(\underline{x}) = -\,m\,\delta(\underline{x}).$$

Also recall that angular momentum operator $\Gamma = -\underline{\omega}\,\partial_{\underline{\omega}} = -\,\underline{\omega} \wedge \partial_{\underline{\omega}}$ is a purely angular operator which, quite naturally, annihilates the radial delta distribution:

$$\Gamma\,\delta(\underline{x}) = -\underline{\omega}\,\partial_{\underline{\omega}}\,\delta(\underline{x}) = 0.$$

For the same reason, when acting with Dirac operator $\underline{\partial} = \underline{\omega}\,\partial_r + \frac{1}{r}\partial_{\underline{\omega}}$ on the delta distribution only the radial part $\underline{\partial}_{rad} = \underline{\omega}\,\partial_r$ will play an active role, which leads to the following specific result about the delta distribution.

Proposition 8 The actions of the radial and angular parts of the Dirac operator on the delta distribution are uniquely determined and it holds that

$$\underline{\partial}_{rad}\,\delta(\underline{x}) = \underline{\omega}\,\partial_r\,\delta(\underline{x}) = \underline{\partial}\,\delta(\underline{x}) \quad \text{and} \quad \underline{\partial}_{ang}\,\delta(\underline{x}) = \frac{1}{r}\,\partial_{\underline{\omega}}\,\delta(\underline{x}) = 0.$$

Proof. By Definition 4 one has

$$(\underline{\omega}\,\partial_r)\,\delta(\underline{x}) = -\,\frac{1}{\underline{x}}\mathbb{E}\,\delta(\underline{x}) = m\,\frac{1}{\underline{x}}\,\delta(\underline{x}) = \underline{\partial}\delta(\underline{x})$$

and

$$\left(\frac{1}{r}\,\partial_{\underline{\omega}}\right)\,\delta(\underline{x}) = -\,\frac{1}{\underline{x}}\,\Gamma\,\delta(\underline{x}) = 0.$$

□

As the Laplace-Beltrami operator is a purely angular operator, which implies that $\Delta^*\,\delta(\underline{x}) = 0$, the action of the Laplace operator on the delta distribution takes the form

$$\Delta\,\delta(\underline{x}) = \partial_r^2\,\delta(\underline{x}) + (m-1)\,\frac{1}{r}\,\partial_r\,\delta(\underline{x}).$$

We will now show that the two operator parts of the Laplace operator, viz. (∂_r^2) and $\left(\frac{1}{r}\partial_r\right)$ have well-determined actions on the delta distribution. Note that the latter operator is, up to a constant factor, nothing else but the derivative with respect to r^2 since it holds indeed that $\left(\frac{1}{r}\partial_r\right) = 2\,\partial_{r^2}$.

Proposition 9 The actions of operators (∂_r^2) and $\left(\frac{1}{r}\partial_r\right)$ on delta distribution $\delta(\underline{x})$ are uniquely determined, and it holds that

$$(\partial_r^2)\,\delta(\underline{x}) = \frac{1}{2}(m+1)\,\Delta\,\delta(\underline{x}) \quad \text{and} \quad \left(\frac{1}{r}\,\partial_r\right)\delta(\underline{x}) = -\frac{1}{2}\,\Delta\,\delta(\underline{x}).$$

Proof. By Proposition 7 we find:

$$\partial_r^2\,\delta(\underline{x}) = \frac{1}{r^2}\,\mathbb{E}\,(\mathbb{E}-1)\,\delta(\underline{x}) = m(m+1)\,\frac{1}{r^2}\,\delta(\underline{x}) = \frac{1}{2}\,(m+1)\,\Delta\,\delta(\underline{x})$$

and

$$\frac{1}{r}\,\partial_r\,\delta(\underline{x}) = \frac{1}{r^2}\,\mathbb{E}\,\delta(\underline{x}) = (-m)\,\frac{1}{r^2}\,\delta(\underline{x}) = -\frac{1}{2}\,\Delta\,\delta(\underline{x}).$$

$\square$

Remark 2 Note that the results of Proposition 9 are consistent with the action of the Laplace operator since

$$(\partial_r^2)\,\delta(\underline{x}) + (m-1)\left(\frac{1}{r}\,\partial_r\right)\delta(\underline{x}) = \frac{1}{2}(m+1)\,\Delta\,\delta(\underline{x}) + (m-1)\left(-\frac{1}{2}\right)\Delta\,\delta(\underline{x}) = \Delta\,\delta(\underline{x}).$$

Combining the results concerning the actions on the delta distribution of the operators $\underline{\omega}\,\partial_r$ and $\partial_r^2 = -(\underline{\omega}\,\partial_r)^2$, the following identities are obtained.

Corollary 3 For all $k \in \mathbb{N}$ one has

$$(\underline{\omega}\,\partial_r)^{2k}\,\delta(\underline{x}) = (-1)^k\,\partial_r^{2k}\,\delta(\underline{x}) = \frac{1}{2^k k!}\,(m+1)(m+3)\cdots(m+2k-1)\,\underline{\partial}^{2k}\,\delta(\underline{x})$$

$$(\underline{\omega}\,\partial_r)^{2k+1}\,\delta(\underline{x}) = (-1)^k\,\underline{\omega}\,\partial_r^{2k+1}\,\delta(\underline{x}) = \frac{1}{2^k k!}\,(m+1)(m+3)\cdots(m+2k-1)\,\underline{\partial}^{2k+1}\,\delta(\underline{x})$$

$$(\underline{\omega}\,\partial_r)\,\underline{\partial}^{2\ell}\,\delta(\underline{x}) = \underline{\partial}^{2\ell+1}\,\delta(\underline{x}) = \underline{\partial}^{2\ell}\,(\underline{\omega}\,\partial_r)\,\delta(\underline{x})$$

$$(\underline{\omega}\,\partial_r)\,\underline{\partial}^{2\ell+1}\,\delta(\underline{x}) = \frac{m+2\ell+1}{2(\ell+1)}\,\underline{\partial}^{2\ell+2}\,\delta(\underline{x}) = \frac{m+2\ell+1}{2(\ell+1)}\,\underline{\partial}^{2\ell+1}\,(\underline{\omega}\,\partial_r)\,\delta(\underline{x}).$$

Iteration of the action on the delta distribution of the operator $\left(\frac{1}{r}\partial_r\right)$ leads to the following result.

Corollary 4 For all $k \in \mathbb{N}$ one has

$$\left(\frac{1}{r}\,\partial_r\right)^k\delta(\underline{x}) = \frac{1}{2^k k!}\,\underline{\partial}^{2k}\,\delta(\underline{x}) = (-1)^k\,\frac{1}{2^k k!}\,\Delta^k\,\delta(\underline{x}).$$

The formulæ obtained in Corollary 3 may be generalized by considering products of:
(i) natural powers of the radial distance squared: $r^2 = -\underline{x}^2$;
(ii) natural powers of the radial derivative squared: $\partial_r^2 = -(\underline{\omega}\,\partial_r)^2$;
(iii) the vector variable $\underline{x} = r\,\underline{\omega}$, to obtain the following identities.

Proposition 10 One has, with $k \geq \ell$,

(i)

$$r^{2\ell}\,\partial_r^{2k}\,\delta(\underline{x}) = (-1)^{k+\ell}\,\frac{1}{2^{k-\ell}(k-\ell)!}\,(m+1)(m+3)\cdots(m+2k-1)(m+2k-2)$$

$$\times\,(m+2k-4)\cdots(m+2k-2\ell)\,\underline{\partial}^{2k-2\ell}\,\delta(\underline{x})\,;$$

(ii)

$$\underline{\omega}\,r^{2\ell+1}\,\partial_r^{2k}\,\delta(\underline{x}) = (-1)^{k+\ell}\,\frac{1}{2^{k-\ell-1}(k-\ell-1)!}\,(m+1)(m+3)\cdots(m+2k-1)(m+2k-2)$$

$$\times\,(m+2k-4)\cdots(m+2k-2\ell)\,\underline{\partial}^{2k-2\ell-1}\,\delta(\underline{x})\,;$$

(iii)

$$\underline{\omega}\,r^{2\ell}\,\partial_r^{2k+1}\,\delta(\underline{x}) = (-1)^{k+\ell}\,\frac{1}{2^{k-\ell}(k-\ell)!}\,(m+1)(m+3)\cdots(m+2k-1)(m+2k)$$

$$\times\,(m+2k-2)\cdots(m+2k-2\ell+2)\,\underline{\partial}^{2k-2\ell+1}\,\delta(\underline{x})\,;$$

(iv)

$$r^{2\ell+1}\,\partial_r^{2k+1}\,\delta(\underline{x}) = (-1)^{k+\ell+1}\,\frac{1}{2^{k-\ell}(k-\ell)!}\,(m+1)(m+3)\cdots(m+2k-1)(m+2k)$$

$$\times\,(m+2k-2)\cdots(m+2k-2\ell)\,\underline{\partial}^{2k-2\ell}\,\delta(\underline{x}).$$

Proposition 11 One has

(i)

$$(\underline{\omega}\,\partial_r)^{2k}\,\underline{\partial}^{2\ell}\,\delta(\underline{x}) = \frac{1}{2^k}\,\frac{(m+2\ell+1)(m+2\ell+3)\cdots(m+2\ell+2k-1)}{(\ell+1)(\ell+2)\cdots(\ell+k)}\,\underline{\partial}^{2\ell+2k}\,\delta(\underline{x});$$

(ii)

$$(\underline{\omega}\,\partial_r)^{2k}\,\underline{\partial}^{2\ell+1}\,\delta(\underline{x}) = \frac{1}{2^k}\,\frac{(m+2\ell+1)(m+2\ell+3)\cdots(m+2\ell+2k-1)}{(\ell+1)(\ell+2)\cdots(\ell+k)}\,\underline{\partial}^{2\ell+2k+1}\,\delta(\underline{x});$$

(iii)

$$(\underline{\omega}\,\partial_r)^{2k+1}\,\underline{\partial}^{2\ell}\,\delta(\underline{x}) = \frac{1}{2^k}\,\frac{(m+2\ell+1)(m+2\ell+3)\cdots(m+2\ell+2k-1)}{(\ell+1)(\ell+2)\cdots(\ell+k)}\,\underline{\partial}^{2\ell+2k+1}\,\delta(\underline{x});$$

(iv)

$$(\underline{\omega}\,\partial_r)^{2k+1}\,\underline{\partial}^{2\ell+1}\,\delta(\underline{x}) = \frac{1}{2^{k+1}}\,\frac{(m+2\ell+1)(m+2\ell+3)\cdots(m+2\ell+2k+1)}{(\ell+1)(\ell+2)\cdots(\ell+k+1)}\,\underline{\partial}^{2\ell+2k+2}\,\delta(\underline{x}).$$

10. The action of the operator $\frac{1}{r}\partial_{\underline{\omega}}$ on the delta distribution

Because the delta distribution $\delta(\underline{x})$ and its Dirac-derivatives of even order $\underline{\partial}^{2\ell}\,\delta(\underline{x})$ are radial distributions, the action of the operator $\frac{1}{r}\partial_{\underline{\omega}}$ annihilates them all:

$$\left(\frac{1}{r}\,\partial_{\underline{\omega}}\right)\underline{\partial}^{2\ell}\,\delta(\underline{x}) = 0, \quad \ell = 0,\ 1,\ 2,\ \ldots .$$

For the Dirac-derivatives of odd order we have e.g.

•

$$\begin{aligned}\left(\frac{1}{r}\,\partial_{\underline{\omega}}\right)\underline{\partial}\,\delta(\underline{x}) &= \left(\frac{1}{r}\,\partial_{\underline{\omega}}\right)(\underline{\omega}\,\partial_r)\,\delta(\underline{x})\\ &= (1-m)\left(\frac{1}{r}\,\partial_r\right)\delta(\underline{x})\\ &= -\frac{1}{2}\,(m-1)\,\underline{\partial}^2\,\delta(\underline{x});\end{aligned}$$

•

$$\left(\frac{1}{r}\,\partial_{\underline{\omega}}\right)\underline{\partial}^{3}\,\delta(\underline{x}) = \left(\frac{1}{r}\,\partial_{\underline{\omega}}\right)(\underline{\omega}\,\partial_r)\,\underline{\partial}^{2}\,\delta(\underline{x})$$

$$= (1-m)\left(\frac{1}{r}\,\partial_r\right)\underline{\partial}^{2}\,\delta(\underline{x})$$

$$= -2\,(m-1)\left(\frac{1}{r}\,\partial_r\right)^{2}\delta(\underline{x})$$

$$= -\frac{1}{4}\,(m-1)\,\underline{\partial}^{4}\,\delta(\underline{x})\,.$$

More generally, it holds that

$$\left(\frac{1}{r}\,\partial_{\underline{\omega}}\right)\underline{\partial}^{2\ell+1}\,\delta(\underline{x}) = -\frac{1}{2\ell+2}\,(m-1)\,\underline{\partial}^{2\ell+2}\,\delta(\underline{x})\,.$$

Indeed, we have consecutively

$$\left(\frac{1}{r}\,\partial_{\underline{\omega}}\right)\underline{\partial}^{2\ell+1}\,\delta(\underline{x}) = \left(\frac{1}{r}\,\partial_{\underline{\omega}}\right)(\underline{\omega}\,\partial_r)\,\underline{\partial}^{2\ell}\,\delta(\underline{x})$$

$$= (1-m)\left(\frac{1}{r}\,\partial_r\right)\underline{\partial}^{2\ell}\,\delta(\underline{x})$$

$$= -2^{\ell}\,\ell!\,(m-1)\left(\frac{1}{r}\,\partial_r\right)^{\ell+1}\delta(\underline{x})$$

$$= -\frac{1}{2\ell+2}\,(m-1)\,\underline{\partial}^{2\ell+2}\,\delta(\underline{x})\,.$$

11. The delta distribution in spherical coordinates

In physics texts one often encounters the following expression for the delta distribution in spherical coordinates:

$$\delta(\underline{x}) = \frac{1}{a_m}\,\frac{\delta(r)}{r^{m-1}} \tag{7}$$

where $a_m = \dfrac{2\pi^{m/2}}{\Gamma(m/2)}$ is the area of the unit sphere $\mathbb{S}^{m-1}$ in $\mathbb{R}^m$, and $\delta(r)$ is the one dimensional delta distribution on the r-axis.

Apparently this can be mathematically explained in the following way. Write the action of the delta distribution as an integral:

$$
\begin{aligned}
\varphi(0) &= \langle\ \delta(\underline{x}),\ \varphi(\underline{x})\ \rangle \\
&= \int_{\mathbb{R}^m} \delta(\underline{x})\, \varphi(\underline{x})\, dV(\underline{x}) \\
&= \int_0^\infty r^{m-1} \delta(\underline{x})\, dr \int_{\mathbb{S}^{m-1}} \varphi(r\,\underline{\omega})\, dS_{\underline{\omega}} \\
&= a_m \int_0^\infty r^{m-1}\, \delta(\underline{x})\, \Sigma^0[\varphi](r)\, dr
\end{aligned}
$$

using the so-called *spherical mean* (see e.g. [11]) of the test function φ given by

$$
\Sigma^0[\varphi](r) = \frac{1}{a_m} \int_{\mathbb{S}^{m-1}} \varphi(r\,\underline{\omega})\, dS_{\underline{\omega}}\,.
$$

As it is easily seen that $\Sigma^0[\varphi](0) = \varphi(0)$, it follows that

$$
a_m \int_0^\infty r^{m-1}\, \delta(\underline{x})\, \Sigma^0[\varphi](r)\, dr = \langle\ \delta(r),\ \Sigma^0[\varphi](r)\ \rangle = \int_0^\infty \delta(r)\, \Sigma^0[\varphi](r)\, dr
$$

which explains (7). However we prefer to interpret this expression as

$$
\varphi(0) = \langle\ \delta(\underline{x}),\ \varphi(\underline{x})\ \rangle = \langle\ \delta(r),\ \Sigma^0[\varphi](r)\ \rangle = \Sigma^0[\varphi](0) \tag{8}
$$

which can be generalized to higher even order Dirac-derivatives of the delta distribution:

$$
\begin{aligned}
\{\underline{\partial}^{2\ell} \varphi(\underline{x})\}|_{\underline{x}=0} &= \langle\ \underline{\partial}^{2\ell}\, \delta(\underline{x}),\ \varphi(\underline{x})\ \rangle \\
&= (-1)^\ell\, \frac{C(m,\ \ell)}{(2\ell)!(m+2\ell)}\, \langle\ \partial_r^{2\ell}\, \delta(r),\ \Sigma^0[\varphi](r)\ \rangle \\
&= (-1)^\ell\, \frac{C(m,\ \ell)}{(2\ell)!(m+2\ell)}\, \{\partial_r^{2\ell}\, \Sigma^0[\varphi](r)\}|_{r=0}\,.
\end{aligned}
$$

Recall that the constant $C(m,\ \ell)$ is given by

$$
C(m,\ \ell) = 2^\ell\, \ell!\, m(m+2)\cdots(m+2\ell)\,.
$$

In physics language of integrals this formula would be written as

$$\int_0^{+\infty} r^{m-1}\,\underline{\partial}^{2\ell}\,\delta(\underline{x})\,dr \int_{S^{m-1}} \varphi(r\underline{\omega})\,dS_{\underline{\omega}} = (-1)^\ell \frac{C(m,\,\ell)}{(2\ell)!(m+2\ell)} \int_0^{+\infty} \delta^{(2\ell)}(r)\,dr\,\frac{1}{a_m}\int_{S^{m-1}} \varphi(r\underline{\omega})\,dS_{\underline{\omega}}$$

leading to

$$\underline{\partial}^{2\ell}\,\delta(\underline{x}) = (-1)^\ell \frac{C(m,\,\ell)}{(2\ell)!(m+2\ell)}\,\frac{1}{a_m}\,\frac{\delta^{(2\ell)}(r)}{r^{m-1}} \tag{9}$$

or, by means of the results of Corollary 3,

$$\partial_r^{2\ell}\,\delta(\underline{x}) = \frac{1}{(2\ell)!}(m)(m+1)\cdots(m+2\ell-1)\,\frac{1}{a_m}\,\frac{\delta^{(2\ell)}(r)}{r^{m-1}}\,.$$

Note that the spherical mean $\Sigma^0[\varphi](r)$ is an even function of r, tacitly assuming that $\Sigma^0[\varphi](r)$ is extended to the whole of the real r-axis. Moreover its odd order derivatives vanish at the origin:

$$\langle\, -\partial_r^{2\ell+1}\,\delta(r),\,\Sigma^0[\varphi](r)\,\rangle = \{\partial_r^{2\ell+1}\,\Sigma^0[\varphi](r)\}|_{r=0} = 0\,.$$

For expressing, in a similar way, the higher odd order Dirac-derivatives of the delta distribution, we have to invoke the so-called *spherical mean of the second kind* $\Sigma^1[\varphi]$, which was introduced in [11]:

$$\Sigma^1[\varphi](r) = \frac{1}{a_m}\int_{\mathbb{S}^{m-1}} \underline{\omega}\,\varphi(r\,\underline{\omega})\,dS_{\underline{\omega}}\,.$$

The spherical mean of the second kind $\Sigma^1[\varphi](r)$ is a vector-valued odd function of r, whose even order derivatives vanish at the origin:

$$\langle\, \partial_r^{2\ell}\,\delta(r),\,\Sigma^1[\varphi](r)\,\rangle = \{\partial_r^{2\ell}\,\Sigma^1[\varphi](r)\}|_{r=0} = 0\,.$$

It holds that

$$\langle\, \underline{\partial}^{2\ell+1}\,\delta(\underline{x}),\,\varphi(\underline{x})\,\rangle = (-1)^\ell \frac{C(m,\,\ell)}{(2\ell+1)!}\,\langle\, \partial_r^{2\ell+1}\,\delta(r),\,\Sigma^1[\varphi](r)\,\rangle \tag{10}$$

or

$$\{\underline{\partial}^{2\ell+1}\varphi(\underline{x})\}|_{\underline{x}=0} = (-1)^\ell \frac{C(m,\,\ell)}{(2\ell+1)!}\,\{\partial_r^{2\ell+1}\,\Sigma^1[\varphi](r)\}|_{r=0}$$

which in physics language would then be written as

$$\underline{\partial}^{2\ell+1}\,\delta(\underline{x}) = (-1)^\ell\,\frac{C(m,\,\ell)}{(2\ell+1)!}\,\frac{1}{a_m}\,\frac{\delta^{(2\ell+1)}(r)}{r^{m-1}}\,\underline{\omega}$$

or, by means of the results of Corollary 3,

$$\underline{\omega}\,\partial_r^{2\ell+1}\,\delta(\underline{x}) = \frac{1}{(2\ell+1)!}(m)(m+1)\cdots(m+2\ell)\,\frac{1}{a_m}\,\frac{\delta^{(2\ell+1)}(r)}{r^{m-1}}\,\underline{\omega} \tag{11}$$

and, in particular for $\ell = 0$,

$$\underline{\partial}\,\delta(\underline{x}) = (\underline{\omega}\,\partial_r)\,\delta(\underline{x}) = \frac{1}{a_m}\,m\,\underline{\omega}\,\frac{\delta'(r)}{r^{m-1}}\,.$$

However the mathematics interpretation of (11) is not straightforward; the best we can think of is to see the right-hand side as a continuous linear functional on the space of test functions $\mathscr{D}(\mathbb{R})\times\mathscr{D}(S^{m-1})$. Moreover, if one would give in to temptation to multiply both members of (11) by $\underline{\omega}$, which is not allowed since $\underline{\omega}$ is not differentiable in the whole of $\mathbb{R}^m$, thus obtaining

$$\partial_r^{2\ell+1}\,\delta(\underline{x}) = \frac{1}{(2\ell+1)!}(m)(m+1)\cdots(m+2\ell)\,\frac{1}{a_m}\,\frac{\delta^{(2\ell+1)}(r)}{r^{m-1}}$$

one would have to give meaning to the radial derivative of a distribution, which is far from trivial as was already observed by Schwartz in his famous and seminal book [12], where he writes on page 51: *Using coordinate systems other than the Cartesian ones should be done with the utmost care* [our translation]. Derivation with respect to the spherical coordinates of a distribution, and of the delta distribution in particular, will be treated in detail in Section 13.

Nevertheless, expression (7) may be used when computing the action of some operators. Let us illustrate this phenomenon by obtaining, via this alternative way, already known results.

-

$$\begin{aligned}(\underline{\omega}\,\partial_r)\,\delta(\underline{x}) &= (\underline{\omega}\,\partial_r)\left(\frac{1}{a_m}\,\frac{\delta(r)}{r^{m-1}}\right)\\ &= \frac{1}{a_m}\,\underline{\omega}\left(\frac{\delta'(r)}{r^{m-1}} - (m-1)\,\frac{\delta(r)}{r^m}\right)\\ &= m\,\frac{1}{a_m}\,\underline{\omega}\,\frac{\delta'(r)}{r^{m-1}}\\ &= \underline{\partial}\,\delta(\underline{x})\end{aligned}$$

- $$\left(\frac{1}{r}\,\partial_{\underline{\omega}}\right)\,\delta(\underline{x}) = \left(\frac{1}{r}\,\partial_{\underline{\omega}}\right)\,\left(\frac{1}{a_m}\,\frac{\delta(r)}{r^{m-1}}\right) = 0$$

- $$\begin{aligned}\mathbb{E}\,\delta(\underline{x}) &= (r\,\partial_r)\,\left(\frac{1}{a_m}\,\frac{\delta(r)}{r^{m-1}}\right)\\ &= \frac{1}{a_m}\,\left(\frac{r\,\delta'(r)}{r^{m-1}} - (m-1)\,\frac{\delta(r)}{r^{m-1}}\right)\\ &= -\,\frac{1}{a_m}\,m\,\frac{\delta(r)}{r^{m-1}}\\ &= (-m)\,\delta(\underline{x})\end{aligned}$$

- $$\Gamma\,\delta(\underline{x}) = -\,\underline{\omega}\,\partial_{\underline{\omega}}\,\left(\frac{1}{a_m}\,\frac{\delta(r)}{r^{m-1}}\right) = 0$$

- $$\begin{aligned}\partial_r^2\,\delta(\underline{x}) &= \partial_r^2\,\left(\frac{1}{a_m}\,\frac{\delta(r)}{r^{m-1}}\right)\\ &= \frac{1}{a_m}\,\left(\frac{\delta''(r)}{r^{m-1}} - 2(m-1)\,\frac{\delta'(r)}{r^m} + m(m-1)\,\frac{\delta(r)}{r^{m+1}}\right)\\ &= \frac{1}{a_m}\,\left(\frac{\delta''(r)}{r^{m-1}} + (m-1)\,\frac{\delta''(r)}{r^{m-1}} + \frac{1}{2}\,m(m-1)\,\frac{\delta''(r)}{r^{m-1}}\right)\\ &= \frac{1}{a_m}\,\frac{1}{2}\,m(m+1)\,\frac{\delta''(r)}{r^{m-1}}\\ &= \frac{1}{2}\,(m+1)\,\Delta\,\delta(\underline{x})\end{aligned}$$

-

$$\frac{1}{r}\,\partial_r\,\delta(\underline{x}) = \frac{1}{a_m}\left(\frac{\delta'(r)}{r^m} - (m-1)\,\frac{\delta(r)}{r^{m+1}}\right)$$

$$= \frac{1}{a_m}\left(-\frac{1}{2}\,\frac{\delta''(r)}{r^{m-1}} - \frac{1}{2}\,(m-1)\,\frac{\delta''(r)}{r^{m-1}}\right)$$

$$= -\frac{1}{a_m}\,\frac{1}{2}\,m\,\frac{\delta''(r)}{r^{m-1}}$$

$$= -\frac{1}{2}\,\Delta\,\delta(\underline{x})$$

-

$$\underline{x}\,\underline{\partial}^{2\ell+1}\,\delta(\underline{x}) = r\,\underline{\omega}\,(-1)^{\ell}\,\frac{C(m,\ \ell)}{(2\ell+1)!}\,\frac{1}{a_m}\,\frac{\delta^{(2\ell+1)}(r)}{r^{m-1}}$$

$$= (-1)^{\ell+1}\,\frac{C(m,\ \ell)}{(2\ell+1)!}\,\frac{1}{a_m}\,(-1)\,(2\ell+1)\,\frac{\delta^{2\ell}(r)}{r^{m-1}}$$

$$= (m+2\ell)\,\underline{\partial}^{2\ell}\,\delta(\underline{x})$$

-

$$\underline{x}\,\underline{\partial}^{2\ell}\,\delta(\underline{x}) = r\,\underline{\omega}\,(-1)^{\ell}\,\frac{C(m,\ \ell)}{(2\ell)!(m+2\ell)}\,\frac{1}{a_m}\,\frac{\delta^{(2\ell)}(r)}{r^{m-1}}$$

$$= \underline{\omega}\,(-1)^{\ell-1}\,\frac{C(m,\ \ell-1)}{(2\ell-1)!}\,\frac{1}{a_m}\,(2\ell)\,\frac{\delta^{(2\ell-1)}(r)}{r^{m-1}}$$

$$= (2\ell)\,\underline{\partial}^{2\ell-1}\,\delta(\underline{x})$$

•

$$\underline{x}^{2k}\,\underline{\partial}^{2\ell}\,\delta(\underline{x}) = (-1)^{\ell}\,\frac{C(m,\,\ell)}{(2\ell)!(m+2\ell)}\,\frac{1}{a_m}\,(-1)^k\,\frac{r^{2k}\,\delta^{(2\ell)}(r)}{r^{m-1}}$$

$$= (-1)^{\ell+k}\,\frac{C(m,\,\ell)}{(2\ell)!(m+2\ell)}\,(2\ell)(2\ell-1)\cdots(2\ell-2k+1)\,\frac{1}{a_m}\,\frac{\delta^{(2\ell-2k)}(r)}{r^{m-1}}$$

$$= (-1)^{\ell+k}\,\frac{C(m,\,\ell)}{(2\ell-2k)!(m+2\ell)}\,(-1)^{\ell-k}\,\frac{(2\ell-2k)!\,(m+2\ell-2k)}{C(m,\,\ell-k)}\,\underline{\partial}^{2\ell-2k}\,\delta(\underline{x})$$

$$= 2^k\,(\ell)(\ell-1)\cdots(\ell-k+1)(m+2\ell-2)(m+2\ell-4)\cdots(m+2\ell-2k)\,\underline{\partial}^{2\ell-2k}\,\delta(\underline{x})$$

and similarly for the other general multiplication formulæ of Section 5.

•

$$\frac{1}{\underline{x}}\,\underline{\partial}^{2\ell+1}\,\delta(\underline{x}) = -\frac{1}{r}\,\underline{\omega}\,(-1)^{\ell}\,\frac{C(m,\,\ell)}{(2\ell+1)!}\,\frac{1}{a_m}\,\frac{\delta^{(2\ell+1)}(r)}{r^{m-1}}\,\underline{\omega}$$

$$= (-1)^{\ell+1}\,\frac{C(m,\,\ell+1)}{(2\ell+1)!(m+2\ell+2)(2\ell+2)}\,\frac{1}{a_m}\,\frac{\delta^{2\ell+2}(r)}{r^{m-1}}\,\frac{1}{2\ell+2}$$

$$= \frac{1}{2\ell+2}\,\underline{\partial}^{2\ell+2}\,\delta(\underline{x})$$

•

$$\frac{1}{\underline{x}}\,\underline{\partial}^{2\ell}\,\delta(\underline{x}) = -\frac{1}{r}\,\underline{\omega}\,(-1)^{\ell}\,\frac{C(m,\,\ell)}{(2\ell)!(m+2\ell)}\,\frac{1}{a_m}\,\frac{\delta^{(2\ell)}(r)}{r^{m-1}}$$

$$= (-1)^{\ell}\,\underline{\omega}\,\frac{C(m,\,\ell)}{(2\ell)!(m+2\ell)}\,\frac{1}{a_m}\,\frac{\delta^{2\ell+1}(r)}{r^{m-1}}\,\frac{1}{2\ell+1}$$

$$= \frac{1}{m+2\ell}\,\underline{\partial}^{2\ell+1}\,\delta(\underline{x})$$

•

$$\frac{1}{\underline{x}^{2k}}\,\underline{\partial}^{2\ell}\,\delta(\underline{x}) = (-1)^{\ell}\,\frac{C(m,\,\ell)}{(2\ell)!(m+2\ell)}\,\frac{1}{a_m}\,(-1)^k\,\frac{1}{r^{2k}}\,\frac{\delta^{(2\ell)}(r)}{r^{m-1}}$$

$$= (-1)^{\ell+k}\,\frac{C(m,\,\ell)}{(2\ell)!(m+2\ell)}\,\frac{1}{(2\ell+2k)(2\ell+2k-1)\cdots(2\ell+1)}\,\frac{1}{a_m}\,\frac{\delta^{(2\ell+2k)}(r)}{r^{m-1}}$$

$$= \frac{C(m,\,\ell)}{(m+2\ell)}\,\frac{(m+2\ell+2k)}{C(m,\,\ell+k)}\,\underline{\partial}^{2\ell+2k}\,\delta(\underline{x})$$

$$= \frac{1}{2^k\,(\ell+1)(\ell+2)\cdots(\ell+k)(m+2\ell)(m+2\ell+2)\cdots(m+2\ell+2k-2)}\,\underline{\partial}^{2\ell+2k}\,\delta(\underline{x})$$

and similarly for the other general division formulæ of Section 7.

More generally it can be proved by induction that

$$(\underline{\omega}\,\partial_r)^{2k}\,\delta(\underline{x}) = (-1)^k\,\frac{1}{(2k)!}\,m(m+1)\cdots(m+2k-1)\,\frac{1}{a_m}\,\frac{\delta^{(2k)}(r)}{r^{m-1}}$$

$$(\underline{\omega}\,\partial_r)^{2k+1}\,\delta(\underline{x}) = (-1)^k\,\frac{1}{(2k+1)!}\,m(m+1)\cdots(m+2k)\,\frac{1}{a_m}\,\frac{\delta^{(2k+1)}(r)}{r^{m-1}}\,\underline{\omega}\,.$$

12. Signumdistributions

In [4] it was shown that derivation of the delta distribution with respect to spherical coordinates necessitates the introduction of a new concept: *signumdistribution*. The general theory of signumdistributions was developed in [5] and applied, in [6], on two specific families of distributions appearing in harmonic and Clifford analysis. Here we confine ourselves to a concise introduction of the concept of a signumdistribution.

We consider two spaces of test functions: traditional space $\mathscr{D}(\mathbb{R}^m)$ of compactly supported infinitely differentiable functions $\varphi(\underline{x})$ and space $\Omega(\mathbb{R}^m;\,\mathbb{R}^m) = \{\underline{\omega}\,\varphi(\underline{x}) : \varphi(\underline{x}) \in \mathscr{D}(\mathbb{R}^m)\}$. Clearly the test functions in $\Omega(\mathbb{R}^m;\,\mathbb{R}^m)$ are no longer differentiable in the whole of $\mathbb{R}^m$, since they are not defined at the origin, due to the function $\underline{\omega} = \dfrac{\underline{x}}{|\underline{x}|}$ which can be seen as the higher-dimensional counterpart of the one-dimensional signum function $sign(t) = \dfrac{t}{|t|}$, $t \in \mathbb{R}$. Obviously there is a one-to-one correspondence between spaces $\mathscr{D}(\mathbb{R}^m)$ and $\Omega(\mathbb{R}^m;\,\mathbb{R}^m)$. The continuous linear functionals on those spaces of test functions, both equipped with an appropriate topology, are the standard distributions and the signumdistributions respectively.

Given a standard distribution $T(\underline{x}) \in \mathscr{D}'(\mathbb{R}^m)$, signumdistribution $T^{\vee}(\underline{x}) \in \Omega'(\mathbb{R}^m;\,\mathbb{R}^m)$ is defined in such a way that for all test functions $\underline{\omega}\,\varphi \in \Omega(\mathbb{R}^m;\,\mathbb{R}^m)$ it holds that

$$\langle\, T^{\vee}(\underline{x}),\ \underline{\omega}\,\varphi(\underline{x})\,\rangle = -\,\langle\, T(\underline{x}),\ \varphi(\underline{x})\,\rangle\,. \tag{12}$$

Then $T^{\vee}(\underline{x})$ is called the signumdistribution associated to $T(\underline{x})$. In [5] it was proven that this associated signumdistribution is *unique*.

Conversely, for a given signumdistribution ${}^{s}U \in \Omega'(\mathbb{R}^m;\, \mathbb{R}^m)$, we define the associated distribution ${}^{s}U^{\wedge}$ by

$$\langle\, {}^{s}U^{\wedge}(\underline{x}),\ \varphi(\underline{x})\, \rangle = -\langle\, {}^{s}U(\underline{x}),\ \underline{\omega}\,\varphi(\underline{x})\, \rangle \quad \forall\varphi(\underline{x}) \in \mathscr{D}(\mathbb{R}^m)\,.$$

Clearly it holds that

$$T^{\vee\wedge} = T \quad \text{and} \quad {}^{s}U^{\wedge\vee} = {}^{s}U\,.$$

With each well-defined operator P acting between distributions, there corresponds an operator

$$P^{\vee} = \underline{\omega}\, P\, (-\underline{\omega})$$

acting between signumdistributions according to the following commutative diagram:

$$\begin{array}{ccc} T & \overset{P}{\longrightarrow} & P\,T \\ {\scriptstyle -\underline{\omega}}\uparrow & & \downarrow{\scriptstyle \underline{\omega}} \\ T^{\vee} = \underline{\omega}\, T & \underset{P^{\vee}}{\longrightarrow} & P^{\vee}\, T^{\vee} = \underline{\omega}\, P\, T \end{array}$$

in this way giving rise to a pair of operators P and $P^{\vee}$ which we call a *signum-pair of operators*. If the action result of the operator P is uniquely determined, then the action result of $P^{\vee}$ is also uniquely determined, in which case we use the notation $(P,\ P^{\vee})$ for this signum-pair of operators. If, on the contrary, the action result of P is an equivalence class of distributions, then the action result of $P^{\vee}$ will be an equivalence class of signumdistributions, in which case we use the notation $[P,\ P^{\vee}]$. In Table 1 a number of signum-pairs of operators are listed.

The above commutative diagram induces two more operators: the operator Q mapping a distribution to a signumdistribution, and the corresponding operator $Q^{c} = (-\underline{\omega})\, Q\, (-\underline{\omega}) = \underline{\omega}\, Q\, \underline{\omega}$ mapping a signumdistribution to a distribution according to the following commutative diagram:

$$\begin{array}{ccc} {}^{s}U^{\wedge} = -\underline{\omega}\, {}^{s}U & \overset{P}{\longrightarrow} & Q^{c}\, {}^{s}U \\ {\scriptstyle -\underline{\omega}}\uparrow & \overset{\nearrow\, Q^{c}}{\underset{\searrow\, Q}{}} & \uparrow{\scriptstyle -\underline{\omega}} \\ {}^{s}U & \underset{P^{\vee}}{\longrightarrow} & Q\,(-\underline{\omega})\, {}^{s}U \end{array}$$

Clearly the operators Q and Q^c cannot be Cartesian since they map distributions to signumdistributions and vice versa. We call the pair of operators Q and Q^c a *cross-pair of operators* denoted by either (Q, Q^c) or $[Q, Q^c]$ depending on the nature of their action result, similarly as in the case of a signum-pair of operators. In Table 2 a number of cross-pairs of operators are listed.

Table 1. Signum-pairs of operators

Signum-pairs of operators	
$(\underline{x}, \underline{x})$	
(r^2, r^2)	
$(\mathbb{E}, \mathbb{E})$	
$(\Gamma, -\partial_{\underline{\omega}}\underline{\omega})$	$(-\partial_{\underline{\omega}}\underline{\omega}, \Gamma)$
$(\Gamma^2, \Gamma^2-2(m-1)\Gamma+(m-1)^2)$	$(\Gamma^2-2(m-1)\Gamma+(m-1)^2, \Gamma^2)$
$(\underline{\partial}, \underline{D})$	$[\underline{D}, \underline{\partial}]$
$[\underline{\omega}\partial_r, \underline{\omega}\partial_r]$	
$\left[\frac{1}{r}\partial_{\underline{\omega}}, -\frac{1}{r}\partial_{\underline{\omega}}+(m-1)\frac{1}{r}\underline{\omega}\right]$	$\left[\frac{1}{r}\partial_{\underline{\omega}}+(m-1)\frac{1}{r}\underline{\omega}, \frac{1}{r}\partial_{\underline{\omega}}\right]$
$(\partial^2_{\underline{\omega}}, \partial^2_{\underline{\omega}})$	
$(\Delta^*, \mathbf{Z}^*)$	$(\mathbf{Z}^*, \Delta^*)$
$(\Delta, \mathbf{Z})$	$[\mathbf{Z}, \Delta]$
$[\partial^2_r, \partial^2_r]$	
$\left[\frac{1}{\underline{x}}, \frac{1}{\underline{x}}\right]$	
$\left[\frac{1}{r}\partial_r, \frac{1}{r}\partial_r\right]$	
$\left[\frac{1}{r^2}, \frac{1}{r^2}\right]$	
(∂_{x_j}, d_j)	$[d_j, \partial_{x_j}]$

Table 2. Cross-pairs of operators

Cross-pairs of operators	
$(\underline{\omega}, \underline{\omega})$	
$(r, -r)$	$(-r, r)$
$[\partial_r, -\partial_r]$	$[-\partial_r, \partial_r]$
$(\partial_{\underline{\omega}}, \underline{\omega}\partial_{\underline{\omega}}\underline{\omega})$	$(\underline{\omega}\partial_{\underline{\omega}}\underline{\omega}, \partial_{\underline{\omega}})$
$\left[\frac{1}{r}, -\frac{1}{r}\right]$	$\left[-\frac{1}{r}, \frac{1}{r}\right]$
$\left[\frac{1}{r}\partial_{\underline{\omega}}\underline{\omega}, \frac{1}{r}\underline{\omega}\partial_{\underline{\omega}}\right]$	$\left[-\frac{1}{r}\partial_{\underline{\omega}}\underline{\omega}, \frac{1}{r}\underline{\omega}\partial_{\underline{\omega}}\right]$

13. Spherical operators

An operator involving spherical coordinates is said to be *spherical* when it is not Cartesian. Clearly multiplication operators r and $\underline{\omega}$ are spherical operators, as are differential operators ∂_r and $\partial_{\underline{\omega}}$. The concepts of signumdistribution, signum-pair of operators and cross-pair of operators, introduced in Section 12, allow for a definition of the action of spherical operators on (signum)distributions.

Definition 5 The product of a scalar distribution T by the function $\underline{\omega}$ is the signumdistribution $T^\vee$ associated to T, and it holds for all test functions $\underline{\omega}\,\varphi \in \Omega(\mathbb{R}^m;\, \mathbb{R}^m)$ that

$$\langle\, \underline{\omega}T,\ \underline{\omega}\,\varphi \,\rangle = \langle\, T^\vee,\ \underline{\omega}\,\varphi \,\rangle = -\,\langle\, T,\ \varphi \,\rangle\,.$$

Similarly the product of the signumdistribution ${}^sU$ by the function $(-\,\underline{\omega})$ is its associated distribution ${}^sU^\wedge$.

Definition 6 The product of a scalar distribution T^{scal} by the function $\omega_j,\ j = 1,\ \ldots,\ m$ is the signumdistribution $\omega_j\, T^{scal}$ given by the uniquely determined expression

$$\omega_j\, T^{scal} = \{\underline{\omega}\, T^{scal}\}_j\,.$$

Similarly the product of the scalar signumdistribution ${}^sU^{scal}$ by the function ω_j is the distribution $\omega_j\, {}^sU^{scal}$ given by

$$\omega_j\, {}^sU^{scal} = \{\underline{\omega}\, {}^sU^{scal}\}_j\,.$$

Remark 3 For a general Clifford algebra valued distribution or signumdistribution, the action of the multiplicative operator ω_j is defined through linearity with respect to the scalar components.

Definition 7 The product of a scalar distribution T by the function r is the signumdistribution $rT = (-\,\underline{x}\,T)^\vee$ given for all test functions $\underline{\omega}\,\varphi \in \Omega(\mathbb{R}^m;\, \mathbb{R}^m)$ by

$$\langle\, rT,\ \underline{\omega}\,\varphi \,\rangle = \langle\, \underline{x}T,\ \varphi \,\rangle = \langle\, T,\ \underline{x}\,\varphi \,\rangle\,.$$

according to (the boldface part of) the commutative diagram

$$\begin{array}{ccc} \mathbf{T} & \xrightarrow{-\underline{x}} & -\,\underline{\mathbf{x}}\mathbf{T} \\ {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \nearrow^{-r}\ \searrow_{r} & \uparrow{\scriptstyle -\underline{\omega}}\ \downarrow{\scriptstyle \underline{\omega}} \\ T^\vee = \underline{\omega}\,T & \xrightarrow[-\underline{x}]{} & \mathbf{rT} \end{array}$$

involving the signum-pair of operators $(\underline{x},\, \underline{x})$, which induces the product of the signum-distribution $\underline{\omega}\,T$ by $\underline{x}$ to be $\underline{x}(\underline{\omega}\,T) = -\,rT$.

Definition 8 The derivative with respect to the radial distance r of a scalar distribution T is the equivalent class of signumdistributions

$$[\partial_r T] = [-\,\underline{\omega}\,\partial_r T]^{\vee} = \left[\frac{1}{\underline{x}}\,\mathbb{E}T\right]^{\vee} = (\underline{S} + \underline{c}\,\delta(\underline{x}))^{\vee} = \underline{\omega}\,\underline{S} + \underline{\omega}\,\delta(\underline{x})\,\underline{c}$$

for any vector distribution $\underline{S}$ satisfying $\underline{x}\,\underline{S} = \mathbb{E}T$, according to (the boldface part of) the commutative diagram

$$\begin{array}{ccc}
\mathbf{T} & \overset{-\underline{\omega}\partial_r}{\longrightarrow} & \left[\dfrac{\mathbf{1}}{\underline{\mathbf{x}}}\,\mathbb{E}\mathbf{T}\right] \\
{\scriptstyle -\underline{\omega}}\uparrow\;\downarrow{\scriptstyle \underline{\omega}} & \begin{array}{c}\nearrow {\scriptstyle -\partial_r} \\ \searrow {\scriptstyle \partial_r}\end{array} & \uparrow{\scriptstyle -\underline{\omega}}\;\downarrow{\scriptstyle \underline{\omega}} \\
T^{\vee} = \underline{\omega}\,T & \underset{-\underline{\omega}\partial_r}{\longrightarrow} & [\partial_{\mathbf{r}}\mathbf{T}]
\end{array}$$

involving the signum-pair of operators $[\underline{\omega}\,\partial_r,\ \underline{\omega}\,\partial_r]$, which induces the action of $(\underline{\omega}\,\partial_r)$ on the signumdistribution $\underline{\omega}\,T$ to be be $(\underline{\omega}\,\partial_r)\,(\underline{\omega}\,T) = [-\,\partial_r T]$.

Definition 9 The angular $\partial_{\underline{\omega}}$-derivative of a scalar distribution T is the unique signumdistribution $\partial_{\underline{\omega}} T = (\Gamma T)^{\vee}$ given for all test functions $\underline{\omega}\,\varphi \in \Omega(\mathbb{R}^m;\ \mathbb{R}^m)$ by

$$\langle\ \underline{\omega}\,\varphi,\ \partial_{\underline{\omega}} T\ \rangle = \langle\ \varphi,\ \underline{\omega}\,\partial_{\underline{\omega}} T\ \rangle = \langle\ \varphi,\ -\Gamma T\ \rangle$$

according to (the boldface part of) the commutative diagram

$$\begin{array}{ccc}
\mathbf{T} & \overset{-\underline{\omega}\partial_{\underline{\omega}}}{\longrightarrow} & \Gamma\mathbf{T} \\
{\scriptstyle -\underline{\omega}}\uparrow\;\downarrow{\scriptstyle \underline{\omega}} & \begin{array}{c}\nearrow {\scriptstyle \underline{\omega}\partial_{\underline{\omega}}\underline{\omega}} \\ \searrow {\scriptstyle \partial_{\underline{\omega}}}\end{array} & \uparrow{\scriptstyle -\underline{\omega}}\;\downarrow{\scriptstyle \underline{\omega}} \\
T^{\vee} = \underline{\omega}\,T & \underset{-\partial_{\underline{\omega}}\underline{\omega}}{\longrightarrow} & \partial_{\underline{\omega}}\mathbf{T}
\end{array}$$

involving the signumpair of operators $(\,\underline{\omega}\partial_{\underline{\omega}},\ \partial_{\underline{\omega}}\underline{\omega}\,)$ or $(\Gamma,\ -\,\partial_{\underline{\omega}}\,\underline{\omega}) = (\Gamma,\ (m-1)\mathbf{1} - \Gamma)$, which induces the action of the operator $\partial_{\underline{\omega}}\,\underline{\omega}$ on the signumdistribution $\underline{\omega}\,T$ to be $\partial_{\underline{\omega}}\,\underline{\omega}\,(\underline{\omega}\,T) = -\,\partial_{\underline{\omega}}\,T$.

Example 6 Because the delta distribution is radial, it holds, trivially, that

$$\partial_{\underline{\omega}}\,\delta(\underline{x}) = 0\,.$$

Definition 10 The quotient of a scalar distribution T by the radial distance r is the equivalence class of signumdistributions

$$\left[\frac{1}{r}\,T\right] = \underline{\omega}\left[\frac{1}{\underline{x}}\,T\right] = \underline{\omega}\,(\underline{S} + \delta(\underline{x})\,\underline{c}) = \underline{\omega}\,\underline{S} + \underline{\omega}\,\delta(\underline{x})\,\underline{c} = \underline{S}^{\vee} + \delta(\underline{x})^{\vee}\,\underline{c}$$

for any vector-valued distribution $\underline{S}$ for which $\underline{x}\,\underline{S} = T$, according to (the boldface part of) the commutative diagram

$$\begin{array}{ccc}
\mathbf{T} & \xrightarrow{\frac{1}{\underline{x}}} & \left[\dfrac{\mathbf{1}}{\underline{\mathbf{x}}}\,\mathbf{T}\right] \\
{\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \begin{array}{c}\nearrow{\scriptstyle -\frac{1}{r}} \\ \searrow{\scriptstyle \frac{1}{r}}\end{array} & {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} \\
T^{\vee} = \underline{\omega}\,T & \xrightarrow[\frac{1}{\underline{x}}]{} & \left[\dfrac{\mathbf{1}}{\mathbf{r}}\,\mathbf{T}\right]
\end{array}$$

involving the signum-pair of operators $\left[\frac{1}{\underline{x}},\, \frac{1}{\underline{x}}\right]$, which induces the quotient of the signumdistribution $\underline{\omega}\,T$ by $\underline{x}$ to be $\frac{1}{\underline{x}}(\underline{\omega}\,T) = \left[\frac{1}{r}\,T\right]$.

Definition 11 The angular ∂_{ω_j}-derivative of a scalar distribution T^{scal} is the unique signumdistribution given by

$$\partial_{\omega_j} T^{scal} = \{\partial_{\underline{\omega}}\, T^{scal}\}_j, \quad j = 1, \ldots, m.$$

Remark 4 (i) An alternative expression for $\partial_{\omega_j} T^{scal}$ is:

$$\partial_{\omega_j} T^{scal} = r\,\partial_{x_j}\, T^{scal} - \omega_j\,\mathbb{E}\,T^{scal}.$$

Indeed, it follows from

$$-\Gamma\, T^{scal} = \underline{x}\,\underline{\partial}\, T^{scal} + \mathbb{E}\,T^{scal}$$

that

$$-\partial_{\underline{\omega}}\, T^{scal} = -\,\underline{\omega}\,\underline{x}\,\underline{\partial}\, T^{scal} + \underline{\omega}\,\mathbb{E}\,T^{scal} = -\,r\,\underline{\partial}\, T^{scal} + \underline{\omega}\,\mathbb{E}\,T^{scal}$$

whence the desired formula for each of the components.

(ii) For a general Clifford algebra valued distribution, the action of the spherical derivative operator ∂_{ω_j} is defined through linearity with respect to the scalar components.

Remark 5 Once the action of the spherical derivative operators $\partial_{\omega_j}, \ j = 1, \ldots, m$ is defined, see Definition 11, we are able to define the action of the corresponding Cartesian operators $-\underline{\omega}\,\partial_{\omega_j}, \ j = 1, \ldots, m$, through the commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{\ -\underline{\omega}\,\partial_{\omega_j}\ } & -\underline{\omega}\,\partial_{\omega_j} T \\ {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \nearrow^{\underline{\omega}\,\partial_{\omega_j}\underline{\omega}} \ \searrow_{\partial_{\omega_j}} & {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} \\ \underline{\omega}\,T & \xrightarrow[\ -\partial_{\omega_j}\underline{\omega}\ ]{} & \partial_{\omega_j} T \end{array}$$

In particular, for a scalar distribution T^{scal} it holds that

$$-\underline{\omega}\,\partial_{\omega_j} T^{scal} = -\underline{x}\,\partial_{x_j} T^{scal} + \omega_j\,\underline{\omega}\,\mathbb{E}\,T^{scal}, \quad j = 1, \ldots, m.$$

Example 7 Because $\delta(\underline{x})$ is a radial distribution we expect $\partial_{\omega_j}\delta(\underline{x})$, and thus also $-\underline{\omega}\,\partial_{\omega_j}\delta(\underline{x})$, to be zero. And indeed it holds that

$$\begin{aligned} \partial_{\omega_j}\,\delta(\underline{x}) &= r\,\partial_{x_j}\,\delta(\underline{x}) - \omega_j\,\mathbb{E}\,\delta(\underline{x}) \\ &= r\,\partial_{x_j}\,\delta(\underline{x}) + m\,\omega_j\,\delta(\underline{x}) \\ &= \{r\,\underline{\partial}\,\delta(\underline{x}) + m\,\underline{\omega}\,\delta(\underline{x})\}_j \\ &= \{\underline{\omega}\,\mathbb{E}\,\delta(\underline{x}) + m\,\underline{\omega}\,\delta(\underline{x})\}_j \\ &= 0. \end{aligned}$$

14. Signum-partners of Cartesian operators

In Section 8 we saw that Laplace-Beltrami operator Δ^* and the square of angular derivative $\partial^2_{\underline{\omega}}$ are Cartesian operators. Their signum-partners are straightforwardly computed to be

$$(\partial^2_{\underline{\omega}})^{\vee} = \underline{\omega}\,\partial^2_{\underline{\omega}}\,(-\underline{\omega}) = \partial^2_{\underline{\omega}}$$

and

$$\Delta^{*\vee} = -\Gamma^2 + m\Gamma - (m-1)\,\mathbf{1}.$$

Clearly also operator $\Delta^{*\vee}$ is Cartesian. Introducing the notation $\mathbf{Z}^* = \Delta^{*\vee}$, the signum-pairs of operators $(\partial^2_{\underline{\omega}},\ \partial^2_{\underline{\omega}})$, $(\Delta^*,\ \mathbf{Z}^*)$ and $(\mathbf{Z}^*,\ \Delta^*)$ follow, inducing the definition of the actions of operators $\partial^2_{\underline{\omega}}$, $\mathbf{Z}^*$ and Δ^* on a signumdistribution.

For signum-partner $\underline{D}$ of Dirac operator $\underline{\partial}$ we obtain the following expressions:

$$\underline{D} = \underline{\omega}\,\underline{\partial}\,(-\underline{\omega}) = \underline{\omega}\left(\underline{\omega}\,\partial_r + \frac{1}{r}\,\partial_{\underline{\omega}}\right)(-\underline{\omega})$$

$$= \underline{\omega}\,\partial_r + \frac{1}{r}\,\underline{\omega}\,\partial_{\underline{\omega}}\,(-\underline{\omega})$$

$$= \underline{\omega}\,\partial_r - \frac{1}{r}\,\partial_{\underline{\omega}} + (m-1)\,\frac{1}{r}\,\underline{\omega}$$

$$= \underline{\partial} - 2\frac{1}{r}\,\partial_{\underline{\omega}} + (m-1)\,\frac{1}{r}\,\underline{\omega}$$

$$= -\,\underline{\partial} + 2\,\underline{\omega}\,\partial_r + (m-1)\,\frac{1}{r}\,\underline{\omega}$$

giving rise to the signum-pairs of operators $(\underline{\partial},\ \underline{D})$ and $\left[\frac{1}{r}\partial_{\underline{\omega}},\ -\frac{1}{r}\partial_{\underline{\omega}} + (m-1)\frac{1}{r}\underline{\omega}\right]$, which induce the actions of the operators $\underline{D}$ and $-\frac{1}{r}\partial_{\underline{\omega}} + (m-1)\frac{1}{r}\underline{\omega}$ on a signumdistribution.

Note that whereas the actions of the operator $\underline{\partial}$ on distributions and of its signum-partner $\underline{D}$ on signumdistributions are uniquely determined, the action results of the operator $\frac{1}{r}\partial_{\underline{\omega}}$ on distributions and of its signum-partner $-\frac{1}{r}\partial_{\underline{\omega}} + (m-1)\frac{1}{r}\underline{\omega}$ on signumdistributions are, in general, equivalent classes of (signum)distributions.

Operator $\underline{D}$ is called the *signum-Dirac operator*. At first sight it is not clear if $\underline{D}$ is Cartesian. But this is indeed the case, and we have the following result.

Proposition 12 The operators $\underline{D}$ and $-\frac{1}{r}\partial_{\underline{\omega}} + (m-1)\frac{1}{r}\underline{\omega}$ are Cartesian operators.

Proof. In view of Definition 4 it holds that

$$-\frac{1}{r}\partial_{\underline{\omega}} + (m-1)\frac{1}{r}\underline{\omega} = \frac{1}{ux}(\Gamma - (m-1)\mathbf{1})$$

and also

$$\underline{D} = \frac{1}{\underline{x}}(-\mathbb{E} + \Gamma - (m-1)\mathbf{1})$$

$$= \underline{\partial} + \frac{1}{\underline{x}}(2\Gamma - (m-1)\mathbf{1})$$

$$= -\,\underline{\partial} + \frac{1}{\underline{x}}(-2\mathbb{E} - (m-1)\mathbf{1})\,.$$

□

This leads to the signum-pair of operators $[\underline{D},\ \underline{\partial}]$, which induces the action of the Dirac operator $\underline{\partial}$ on a signumdistribution. However, due to division by $\underline{x}$, the latter action results into an equivalence class of signumdistributions.

It is interesting to note that, in the same way as the Dirac operator factorizes the Laplace operator: $\underline{\partial}^2 = -\Delta$, signum-Dirac operator $\underline{D}$ factorizes the *signum-Laplace operator*, i.e. the signum-partner of the Laplace operator:

$$\underline{D}^2 = (\underline{\omega}\,\underline{\partial}\,(-\underline{\omega}))^2 = \underline{\omega}\,\underline{\partial}^2\,(-\underline{\omega}) = -\,\underline{\omega}\,\Delta\,(-\underline{\omega}) = -\Delta^{\vee}\,.$$

Introducing the notation $\mathbf{Z} = \Delta^{\vee}$, it follows that $(\Delta,\ \mathbf{Z})$ is a signum-pair of operators, with

$$\mathbf{Z} = -\,\underline{D}^2$$

$$= \partial_r^2 + (m-1)\frac{1}{r}\partial_r + \frac{1}{r^2}\mathbf{Z}^*\,.$$

Clearly also the operator $\mathbf{Z}$ is Cartesian; the signum-pair of operators $[\mathbf{Z},\ \Delta]$ follows, inducing the action of Laplace operator Δ on a signumdistribution.

Also from Section 8 we know that operators ∂_r^2, $\frac{1}{r}\partial_r$ and $\frac{1}{r^2}\Delta^*$, which are the constituents of Laplace operator Δ, are Cartesian operators. Their signum-partners are easily seen to be

$$(\partial_r^2)^{\vee} = \partial_r^2$$

$$\left(\frac{1}{r}\partial_r\right)^{\vee} = \frac{1}{r}\partial_r$$

$$\left(\frac{1}{r^2}\Delta^*\right)^{\vee} = \frac{1}{r^2}\mathbf{Z}^*$$

and the signum-pairs of operators $[\partial_r^2\,,\ \partial_r^2]$, $\left[\frac{1}{r}\partial_r,\ \frac{1}{r}\partial_r\right]$, $\left[\frac{1}{r^2}\Delta^*\,,\ \frac{1}{r^2}\mathbf{Z}^*\right]$ and $\left[\frac{1}{r^2}\mathbf{Z}^*,\ \frac{1}{r^2}\Delta^*\right]$ follow, inducing the action of the operators ∂_r^2, $\frac{1}{r}\partial_r$, $\frac{1}{r^2}\mathbf{Z}^*$ and $\frac{1}{r^2}\Delta^*$ on a signumdistribution.

15. Action uniqueness of some operators

In the preceding sections we encountered operators acting on (signum)distributions with a uniquely determined result and other ones whose actions are not uniquely determined but lead to equivalence classes of (signum)distributions instead. Nevertheless in [10] sufficient conditions were found guaranteeing the uniqueness of the latter operators' actions, involving homogeneous, radial and signum-radial (signum)distributions.

Definition 12 (i) A distribution T or a signumdistibution ${}^sU$ respectively, is said to be *radial* if it is $\mathrm{SO}(m)$-invariant and so only depends on $r = |\underline{x}|$: $T(\underline{x}) = T(r)$ or ${}^sU(\underline{x}) = {}^sU(r)$ respectively.

(ii) A distribution, signumdistribution respectively, is said to be signum-radial if its associated signumdistribution, distribution respectively, is radial.

Let us state these sufficient conditions obtained in [10].

(i) If the distribution T^{rad} is radial, then the following actions are uniquely determined:

$$(\underline{\omega}\,\partial_r)\,T^{rad} \qquad \left(-\frac{1}{r}\,\partial_{\underline{\omega}} + (m-1)\,\frac{1}{r}\,\underline{\omega}\right)T^{rad} \qquad \frac{1}{\underline{x}}\,T^{rad} \qquad \underline{D}\,T^{rad} \qquad \partial_r\,T^{rad} \qquad \frac{1}{r}\,T^{rad}$$

and the actions of the corresponding signum-partner operators on the signum-radial signumdistribution ${}^sU^{srad}$, viz.

$$(\underline{\omega}\,\partial_r)\,{}^sU^{srad} \qquad \left(\frac{1}{r}\,\partial_{\underline{\omega}}\right){}^sU^{srad} \qquad \frac{1}{\underline{x}}\,{}^sU^{srad} \qquad \underline{\partial}\,{}^sU^{srad} \qquad \partial_r\,{}^sU^{srad} \qquad \frac{1}{r}\,{}^sU^{srad}$$

are also uniquely determined.

(ii) If the distribution $T^{(k)}$ is homogeneous with homogeneity degree $k \neq -m+1$, then the following actions are uniquely determined:

$$(\underline{\omega}\,\partial_r)\,T^{(k)} \qquad \left(\frac{1}{r}\,\partial_{\underline{\omega}}\right)T^{(k)} \qquad \left(-\frac{1}{r}\,\partial_{\underline{\omega}} + (m-1)\,\frac{1}{r}\,\underline{\omega}\right)T^{(k)} \qquad \frac{1}{\underline{x}}\,T^{(k)} \qquad \underline{D}\,T^{(k)} \qquad \partial_r\,T^{(k)} \qquad \frac{1}{r}\,T^{(k)}$$

and the actions of the corresponding signum-partner operators on the homogeneous signumdistribution ${}^sU^{(k)}$, viz.

$$(\underline{\omega}\,\partial_r)\,{}^sU^{(k)} \qquad \left(-\frac{1}{r}\,\partial_{\underline{\omega}} + (m-1)\,\frac{1}{r}\,\underline{\omega}\right){}^sU^{(k)} \qquad \left(\frac{1}{r}\,\partial_{\underline{\omega}}\right){}^sU^{(k)} \qquad \frac{1}{\underline{x}}\,{}^sU^{(k)} \qquad \underline{\partial}\,{}^sU^{(k)} \qquad \partial_r\,{}^sU^{(k)} \qquad \frac{1}{r}\,{}^sU^{(k)}$$

are also uniquely determined.

(iii) If the distribution $T^{(k)}$ is homogeneous with homogeneity degree $k \neq -m+1,\ -m+2$, then the following actions are uniquely determined:

$$\partial_r^2\,T^{(k)} \qquad \frac{1}{r}\,\partial_r\,T^{(k)} \qquad \frac{1}{r^2}\,\Delta^*\,T^{(k)} \qquad \frac{1}{r^2}\,\mathbf{Z}^*\,T^{(k)} \qquad \mathbf{Z}\,T^{(k)} \qquad \frac{1}{r^2}\,T^{(k)}$$

and the actions of the corresponding signum-partner operators on the homogeneous signumdistribution ${}^sU^{(k)}$, viz.

$$\partial_r^2\,{}^sU^{(k)} \qquad \frac{1}{r}\,\partial_r\,{}^sU^{(k)} \qquad \frac{1}{r^2}\,\mathbf{Z}^*\,{}^sU^{(k)} \qquad \frac{1}{r^2}\,\Delta^*\,{}^sU^{(k)} \qquad \Delta\,{}^sU^{(k)} \qquad \frac{1}{r^2}\,{}^sU^{(k)}$$

are also uniquely determined.

(iv) The conclusions contained in (iii) remain valid if the distribution, signumdistribution respectively, under consideration is both radial, signum-radial respectively, and homogeneous with homogeneity degree $k \neq -m+2$.

(v) The conclusions contained in (iii) remain valid if the distribution, signumdistribution respectively, under consideration is both signum-radial, radial respectively, and homogeneous with homogeneity degree $k \neq -m+1$.

16. Signumdistributions associated to the delta distribution

Recall that to each distribution T there may be associated a signumdistribution $T^{\vee} = \underline{\omega}\, T$ which acts on test functions showing a point-wise singularity at the origin of $\mathbb{R}^m$. In this section we recall the definitions of signumdistributions associated to the deltadistribution $\delta(\underline{x})$, viz. $\underline{\omega}\,\delta(\underline{x})$, $\partial_r\,\delta(\underline{x})$ and $r\,\delta(\underline{x})$, which were stated with great detail in [4]. We will also define signumdistributions associated to Dirac-derivatives of $\delta(\underline{x})$. The basic definition is that of the signumdistribution $\delta(\underline{x})^{\vee} = \underline{\omega}\,\delta(\underline{x})$.

Definition 13 The signumdistribution $\delta(\underline{x})^{\vee} = \underline{\omega}\,\delta(\underline{x})$, associated to the delta distribution $\delta(\underline{x})$, is defined by

$$\langle\, \underline{\omega}\,\delta(\underline{x}),\ \underline{\omega}\,\varphi(\underline{x})\,\rangle = -\,\langle\, \delta(\underline{x}),\ \varphi(\underline{x})\,\rangle = -\,\varphi(0) \tag{13}$$

for all test functions $\underline{\omega}\,\varphi(\underline{x}) \in \Omega(\mathbb{R}^m;\ \mathbb{R})$.

Consider the distribution $(\underline{\omega}\,\partial_r)\,\delta(\underline{x}) = \underline{\partial}\,\delta(\underline{x})$ (see Proposition 8). Its associated signumdistribution is now defined in terms of the radial derivative of $\delta(\underline{x})$.

Definition 14 The signumdistribution $((\underline{\omega}\,\partial_r)\,\delta(\underline{x}))^{\vee}$, associated to $(\underline{\omega}\,\partial_r)\,\delta(\underline{x})$, is defined to be $-\,\partial_r\,\delta(\underline{x})$. It thus holds, for all test functions $\underline{\omega}\,\varphi(\underline{x}) \in \Omega(\mathbb{R}^m;\ \mathbb{R})$, that

$$\langle\, \partial_r\,\delta(\underline{x}),\ \underline{\omega}\,\varphi(\underline{x})\,\rangle = \langle\, (\underline{\omega}\,\partial_r)\,\delta(\underline{x}),\ \varphi(\underline{x})\,\rangle = \langle\, \underline{\partial}\,\delta(\underline{x}),\ \varphi(\underline{x})\,\rangle = -\,\underline{\partial}\,\varphi(0) \tag{14}$$

according to the commutative diagram

$$\begin{array}{ccc}
\delta(\underline{x}) & \overset{-\underline{\omega}\partial_r}{\longrightarrow} & -\,\underline{\partial}\,\delta(\underline{x}) \\
{\scriptstyle -\underline{\omega}}\uparrow\ \downarrow{\scriptstyle \underline{\omega}} & \begin{array}{c}\nearrow {\scriptstyle -\partial_r} \\ \searrow {\scriptstyle \partial_r}\end{array} & \uparrow{\scriptstyle -\underline{\omega}}\ \downarrow{\scriptstyle \underline{\omega}} \\
\delta(\underline{x})^{\vee} = \underline{\omega}\,\delta(\underline{x}) & \underset{-\underline{\omega}\partial_r}{\longrightarrow} & \partial_r\,\delta(\underline{x})
\end{array}$$

Note that, as a corollary to the commutative diagram in the above definition, we have obtained the following action for signumdistributions:

$$(\underline{\omega}\,\partial_r)(\underline{\omega}\,\delta(\underline{x})) = -\,\partial_r\,\delta(\underline{x})\,.$$

Moreover the signum-partner of $\underline{\partial}\,\delta(\underline{x})$ turns out to be, up to a minus sign, the radial derivative of $\delta(\underline{x})$:

$$\underline{\omega}\,\underline{\partial}\,\delta(\underline{x}) = -\,\partial_r\,\delta(\underline{x})\,.$$

More generally the following commutative diagram holds:

$$
\begin{array}{ccc}
\underline{\omega}\,\partial_r^{2k+1}\,\delta(\underline{x}) & \xrightarrow{-\underline{\omega}\partial_r} & \partial_r^{2k+2}\,\delta(\underline{x}) \\
{\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \begin{array}{c}\nearrow {\scriptstyle -\partial_r} \\ \searrow {\scriptstyle \partial_r}\end{array} & {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} \\
-\,\partial_r^{2k+1}\,\delta(\underline{x}) & \xrightarrow[-\underline{\omega}\partial_r]{} & \underline{\omega}\,\partial_r^{2k+2}\,\delta(\underline{x})
\end{array}
$$

Taking into account that $\underline{x}\,\delta(\underline{x}) = 0$, the following definition is obvious.

Definition 15 The signumdistribution $r\,\delta(\underline{x})$ is defined to be the zero signumdistribution, according to the commutative diagram

$$
\begin{array}{ccc}
\delta(\underline{x}) & \xrightarrow{-\underline{x}} & 0 \\
{\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \begin{array}{c}\nearrow {\scriptstyle -r} \\ \searrow {\scriptstyle r}\end{array} & {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} \\
\underline{\omega}\,\delta(\underline{x}) & \xrightarrow[-\underline{x}]{} & 0
\end{array}
$$

Note that, at the same time, we have obtained that, such as $\delta(\underline{x})$ itself, its associated signumdistribution $\underline{\omega}\,\delta(\underline{x})$ is annihilated by multiplication by the vector variable $\underline{x}$:

$$\underline{x}\,(\underline{\omega}\,\delta(\underline{x})) = 0\,.$$

Now consider the distribution $\underline{\partial}\,\delta(\underline{x}) = (\underline{\omega}\,\partial_r)\,\delta(\underline{x})$. As $\underline{x}\,\underline{\partial}\,\delta(\underline{x}) = m\,\delta(\underline{x})$, we first define the signumdistribution $r\,\underline{\partial}\,\delta(\underline{x})$ by

$$\langle\, r\,\underline{\partial}\,\delta(\underline{x}),\ \underline{\omega}\,\varphi(\underline{x})\,\rangle = m\,\langle\,\delta(\underline{x}),\ \varphi(\underline{x})\,\rangle$$

In view of (13) we obtain the following identity of signumdistributions:

$$r\,(\underline{\omega}\,\partial_r)\,\delta(\underline{x}) = (-m)\,\underline{\omega}\,\delta(\underline{x}) \tag{15}$$

which is in accordance with the commutative diagram

$$
\begin{array}{ccc}
(\underline{\omega}\,\partial_r)\,\delta(\underline{x}) = \underline{\partial}\,\delta(\underline{x}) & \xrightarrow{-\underline{x}} & -m\,\delta(\underline{x}) \\
{\scriptstyle -\underline{\omega}}\uparrow\ \downarrow{\scriptstyle \underline{\omega}} & \overset{-r}{\nearrow}\ \underset{r}{\searrow} & {\scriptstyle -\underline{\omega}}\uparrow\ \downarrow{\scriptstyle \underline{\omega}} \\
-\partial_r\,\delta(\underline{x}) & \xrightarrow[-\underline{x}]{} & (-m)\,\underline{\omega}\,\delta(\underline{x})
\end{array}
$$

More generally, the distribution

$$\underline{x}\,\underline{\partial}^{2k+1}\,\delta(\underline{x}) = (m+2k)\,\underline{\partial}^{2k}\,\delta(\underline{x})$$

allows for the definition of the signumdistribution $r\,\underline{\partial}^{2k+1}\,\delta(\underline{x})$.

Definition 16 Multiplication by the radial distance r of an odd power of the Dirac operator acting on the delta distribution $\delta(\underline{x})$ results into the signumdistribution $r\,\underline{\partial}^{2k+1}\,\delta(\underline{x})$ given by

$$\langle\, r\,\underline{\partial}^{2k+1}\,\delta(\underline{x}),\ \underline{\omega}\,\varphi(\underline{x})\,\rangle = (m+2k)\,\langle\, \underline{\partial}^{2k}\,\delta(\underline{x}),\ \varphi(\underline{x})\,\rangle\,.$$

Invoking the formulæ obtained in Corollary 3, this action can be rewritten as:

$$\langle\, r\,(\underline{\omega}\,\partial_r^{2k+1})\,\delta(\underline{x}),\ \underline{\omega}\,\varphi(\underline{x})\,\rangle = (m+2k)\,\langle\, \partial_r^{2k}\,\delta(\underline{x}),\ \varphi(\underline{x})\,\rangle$$

which implies the following identity of signumdistributions :

$$r\,\underline{\omega}\,\partial_r^{2k+1}\,\delta(\underline{x}) = -\,(m+2k)\,\underline{\omega}\,\partial_r^{2k}\,\delta(\underline{x})\,.$$

In a similar way we find

$$\partial_r\,\underline{\partial}\,\delta(\underline{x}) = \underline{\omega}\,\partial_r^2\,\delta(\underline{x}) = \frac{1}{2}\,(m+1)\,\underline{\omega}\,\Delta\,\delta(\underline{x})$$

and more generally

$$\partial_r\,\underline{\partial}^{2k+1}\,\delta(\underline{x}) = (-1)^k\,\frac{1}{2}\,\frac{1}{k+1}\,(m+2k+1)\,\underline{\omega}\,\Delta^{k+1}\,\delta(\underline{x})\,.$$

Next we consider distribution $\Delta\,\delta(\underline{x}) = -\,\underline{\partial}^2\,\delta(\underline{x})$. We know that multiplication by the radial distance r results into a signumdistribution, which we now define bearing in mind that $\underline{x}\,\Delta\,\delta(\underline{x}) = -2\,\underline{\partial}\,\delta(\underline{x})$.

Definition 17 The signumdistribution $r\,\Delta\,\delta(\underline{x})$ is defined by

$$\langle\, r\Delta\,\delta(\underline{x}),\ \underline{\omega}\,\varphi(\underline{x})\,\rangle = -2\,\langle\, \underline{\partial}\delta(\underline{x}),\ \varphi(\underline{x})\,\rangle\,.$$

Clearly it holds that

$$r\Delta\,\delta(\underline{x}) = -2\,\partial_r\,\delta(\underline{x})$$

from which it also follows, invoking the results in Proposition 9, that

$$r\left(\frac{1}{r}\,\partial_r\right)\delta(\underline{x}) = \partial_r\,\delta(\underline{x})$$

and that

$$r\,(\partial_r^2)\,\delta(\underline{x}) = -(m+1)\,\partial_r\,\delta(\underline{x})\,.$$

More generally, the formula $\underline{x}\,\underline{\partial}^{2k}\,\delta(\underline{x}) = (2k)\,\underline{\partial}^{2k-1}\,\delta(\underline{x})$ allows for the definition of the signumdistribution $r\,\underline{\partial}^{2k}\,\delta(\underline{x})$.

Definition 18 The multiplication by the radial distance r of the distribution $\underline{\partial}^{2k}\,\delta(\underline{x})$ results into the signumdistribution given by

$$\langle\, r\,\underline{\partial}^{2k}\,\delta(\underline{x}),\ \underline{\omega}\,\varphi(\underline{x})\,\rangle = (2k)\,\langle\, \underline{\partial}^{2k-1}\,\delta(\underline{x}),\ \varphi(\underline{x})\,\rangle\,.$$

Note that this action can be rephrased as:

$$r\,\underline{\partial}^{2k}\,\delta(\underline{x}) = -\,(2k)\,\underline{\omega}\,\underline{\partial}^{2k-1}\,\delta(\underline{x})\,.$$

Again invoking the formulæ obtained in Corollary 3, it follows that

$$\langle\, r\,\partial_r^{2k}\,\delta(\underline{x}),\ \underline{\omega}\,\varphi(\underline{x})\,\rangle = -(m+2k-1)\,\langle\, (\underline{\omega}\,\partial_r^{2k-1})\,\delta(\underline{x}),\ \varphi(\underline{x})\,\rangle$$

or

$$r\,\partial_r^{2k}\,\delta(\underline{x}) = -\,(m+2k-1)\,\partial_r^{2k-1}\,\delta(\underline{x})\,.$$

Also the multiplication of a distribution by the singular vector variable $\underline{\omega}$ results into a signumdistribution. We are now ready to define the signumdistribution $\underline{\omega}\,\underline{\partial}^{2k}\delta(\underline{x})$.

Definition 19 The signumdistribution $\underline{\omega}\,\underline{\partial}^{2k}\delta(\underline{x})$ associated to the distribution $\underline{\partial}^{2k}\delta(\underline{x})$ is given by

$$\langle\, \underline{\omega}\, \underline{\partial}^{2k} \delta(\underline{x}),\ \underline{\omega}\, \varphi(\underline{x})\, \rangle = -\langle\, \underline{\partial}^{2k} \delta(\underline{x}),\ \varphi(\underline{x})\, \rangle\,.$$

Note that this action can be rewritten as:

$$\frac{1}{2^k\, k!}\,(m+1)(m+3)\cdots(m+2k-1)\,\underline{\omega}\,\underline{\partial}^{2k}\,\delta(\underline{x}) = (-1)^k\,\underline{\omega}\,\partial_r^{2k}\,\delta(\underline{x})$$

and, in particular, for $k = 1$:

$$\underline{\omega}\,\underline{\partial}^{2}\,\delta(\underline{x}) = -\frac{2}{m+1}\,\underline{\omega}\,\partial_r^{2}\,\delta(\underline{x})\,.$$

Finally we consider the radial derivative of the distribution $\underline{\partial}^{2k}\delta(\underline{x})$, which is, as expected, a signumdistribution. It is defined as follows.

Definition 20 The signumdistribution $\partial_r\,\underline{\partial}^{2k}\delta(\underline{x})$ is defined by

$$\langle\, \partial_r\,\underline{\partial}^{2k}\delta(\underline{x}),\ \underline{\omega}\,\varphi(\underline{x})\, \rangle = \langle\, (\underline{\omega}\,\partial_r)\,\underline{\partial}^{2k}\delta(\underline{x}),\ \varphi(\underline{x})\, \rangle\,.$$

This leads to the following expression for the signum-partner of $\underline{\partial}^{2k+1}\,\delta(\underline{x})$:

$$\partial_r\,\underline{\partial}^{2k}\,\delta(\underline{x}) = -\,\underline{\omega}\,\underline{\partial}^{2k+1}\,\delta(\underline{x}) = (-1)^k\,\frac{2^k\,k!}{(m+1)(m+3)\cdots(m+2k-1)}\,\partial_r^{2k+1}\,\delta(\underline{x})$$

and, in particular, for $k = 1$,

$$\partial_r\,\underline{\partial}^{2}\,\delta(\underline{x}) = -\,\underline{\omega}\,\underline{\partial}^{3}\,\delta(\underline{x}) = -\frac{2}{m+1}\,\partial_r^{3}\,\delta(\underline{x})\,.$$

The above results can be summarized in the following commutative diagram:

$$\begin{array}{ccc}
\underline{\partial}^{2k}\delta(\underline{x}) & \overset{-\underline{\omega}\partial_r}{\longrightarrow} & -\,\underline{\partial}^{2k+1}\delta(\underline{x}) = (-1)^{k+1}\,\dfrac{2^k\,k!}{(m+1)(m+3)\cdots(m+2k-1)}\,\underline{\omega}\,\partial_r^{2k+1}\,\delta(\underline{x}) \\
{\scriptstyle -\underline{\omega}}\uparrow\ \downarrow{\scriptstyle \underline{\omega}} & \nearrow{\scriptstyle -\partial_r}\ \searrow{\scriptstyle \partial_r} & \uparrow{\scriptstyle -\underline{\omega}}\ \downarrow{\scriptstyle \underline{\omega}} \\
\underline{\omega}\,\underline{\partial}^{2k}\delta(\underline{x}) & \underset{-\underline{\omega}\partial_r}{\longrightarrow} & -\,\underline{\omega}\,\underline{\partial}^{2k+1}\delta(\underline{x}) = (-1)^{k}\,\dfrac{2^k\,k!}{(m+1)(m+3)\cdots(m+2k-1)}\,\partial_r^{2k+1}\,\delta(\underline{x})
\end{array}$$

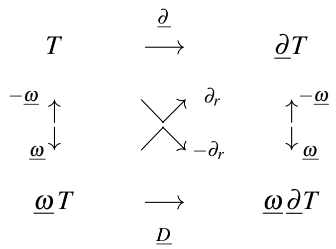
We also find, as could be expected, that

$$\partial_r \, \partial_r^{2k} \, \delta(\underline{x}) = \partial_r^{2k+1} \, \delta(\underline{x}) \, .$$

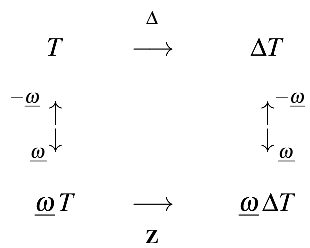
Remark 6 For $k = 0$ we obtain that $r\delta(\underline{x})$ is the zero signumdistribution. In fact, this product is also defined within the framework of standard distributions since the delta distribution is of finite order zero and the function r is continuous in $\mathbb{R}^m$.

17. Action of signum-operators on the delta distribution

In Section 14 we encountered the signumpartners $\underline{D}$ and $\mathbf{Z}$ of operators $\underline{\partial}$ and Δ respectively, leading to the commutative diagrams



and



In particular for the delta distribution, more precisely for its signum-partner $\underline{\omega}\,\delta(\underline{x})$, we obtain

$$\underline{D}\,(\underline{\omega}\,\delta(\underline{x})) = \underline{\omega}\,\underline{\partial}\,\delta(\underline{x})$$

and

$$\mathbf{Z}\,(\underline{\omega}\,\delta(\underline{x})) = \underline{\omega}\Delta\delta(\underline{x})\,.$$

Because operators $\underline{D}$ and $\mathbf{Z} = -\underline{D}^2$ are Cartesian operators, also the signum-pairs $[\underline{D},\, \underline{\partial}]$ and $[\mathbf{Z},\, \Delta]$ hold. However, because the delta distribution is radial and homogeneous of degree $(-m)$, the actions of operators $\underline{D}$ and $\mathbf{Z}$ on $\delta(\underline{x})$ will be uniquely determined, leading to the commutative diagrams

$$\begin{array}{ccc} \delta(\underline{x}) & \xrightarrow{\;\underline{D}\;} & \underline{D}\,\delta(\underline{x}) \\ {\scriptstyle -\underline{\omega}}\big\uparrow\big\downarrow{\scriptstyle \underline{\omega}} & & \big\uparrow{\scriptstyle -\underline{\omega}}\big\downarrow{\scriptstyle \underline{\omega}} \\ \underline{\omega}\,\delta(\underline{x}) & \xrightarrow[\;\underline{\partial}\;]{} & \underline{\partial}\,(\underline{\omega}\,\delta(\underline{x})) \end{array}$$

with

$$\underline{D}\,\delta(\underline{x}) = \frac{1}{m}\,\underline{\partial}\,\delta(\underline{x}) \quad \text{and} \quad \underline{\partial}\,(\underline{\omega}\,\delta(\underline{x})) = \frac{1}{m}\,\underline{\omega}\,\underline{\partial}\,\delta(\underline{x}) = -\frac{1}{m}\,\partial_r\,\delta(\underline{x})$$

and

$$\begin{array}{ccc} \delta(\underline{x}) & \xrightarrow{\;\mathbf{Z}\;} & \mathbf{Z}\,\delta(\underline{x}) \\ {\scriptstyle -\underline{\omega}}\big\uparrow\big\downarrow{\scriptstyle \underline{\omega}} & & \big\uparrow{\scriptstyle -\underline{\omega}}\big\downarrow{\scriptstyle \underline{\omega}} \\ \underline{\omega}\,\delta(\underline{x}) & \xrightarrow[\;\Delta\;]{} & \Delta\,(\underline{\omega}\,\delta(\underline{x})) \end{array}$$

with

$$\mathbf{Z}\,\delta(\underline{x}) = \frac{m+1}{2m}\,\Delta\,\delta(\underline{x}) \quad \text{and} \quad \Delta\,(\underline{\omega}\,\delta(\underline{x})) = \frac{m+1}{2m}\,\underline{\omega}\,\Delta\,\delta(\underline{x})\,.$$

In [10] we introduced the vector operator

$$\underline{F} = \underline{D} + \frac{1}{r}\,\partial_{\underline{\omega}} = \underline{\omega}\,\partial_r + (m-1)\,\frac{1}{r}\,\underline{\omega}$$

which is, as a matter of speaking, intermediate between $\underline{\partial}$ and $\underline{D}$. It holds that

$$\underline{\partial} + \underline{D} - \underline{F} = \underline{\omega}\,\partial_r\,.$$

The operator $\underline{F}$ is signum-invariant:

$$-\,\underline{\omega}\,\underline{F}\,\underline{\omega} = \underline{F}$$

and Cartesian:

$$\underline{F} = -\left[\frac{1}{\underline{x}}\left(\mathbb{E} + (m-1)\right)\right].$$

Moreover it holds that
(i) if the distribution T^{rad} is radial, then $\underline{F}\, T^{rad} = \underline{D}\, T^{rad}$;
(ii) if the distribution T^{srad} is signum-radial, then $\underline{F}\, T^{srad} = \underline{\partial}\, T^{srad}$.
In particular we have the commutative diagram

$$\begin{array}{ccc} \delta(\underline{x}) & \stackrel{\underline{F}}{\longrightarrow} & \underline{D}\,\delta(\underline{x}) \\ {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & & \uparrow\downarrow{\scriptstyle -\underline{\omega},\ \underline{\omega}} \\ \underline{\omega}\,\delta(\underline{x}) & \underset{\underline{F}}{\longrightarrow} & \underline{\partial}\left(\underline{\omega}\,\delta(\underline{x})\right) \end{array}$$

with

$$\underline{F}\,\delta(\underline{x}) = \underline{D}\,\delta(\underline{x}) = \frac{1}{m}\,\underline{\partial}\delta(\underline{x})$$

and

$$\underline{F}\left(\underline{\omega}\,\delta(\underline{x})\right) = \underline{\partial}\left(\underline{\omega}\,\delta(\underline{x})\right) = \frac{1}{m}\,\underline{\omega}\,\underline{\partial}\,\delta(\underline{x}) = -\frac{1}{m}\,\partial_r\,\delta(\underline{x}).$$

The square of the $\underline{F}$-operator is a purely radial operator:

$$-\underline{F}^2 = \partial_r^2 + 2(m-1)\,\frac{1}{r}\,\partial_r + (m-1)(m-2)\,\frac{1}{r^2}$$

and it holds that:

$$-\underline{F}^2\,\delta(\underline{x}) = -\frac{1}{m}\,\underline{\partial}^2\,\delta(\underline{x}) = \frac{1}{m}\,\Delta\,\delta(\underline{x})$$

and

$$-\underline{F}^2\left(\underline{\omega}\,\delta(\underline{x})\right) = -\frac{1}{m}\,\underline{\omega}\,\underline{\partial}^2\,\delta(\underline{x}) = \frac{1}{m}\,\underline{\omega}\,\Delta\,\delta(\underline{x}).$$

Recalling that

$$\partial_r^2\,\delta(\underline{x}) = \frac{1}{2}\,(m+1)\,\Delta\,\delta(\underline{x})$$

$$\frac{1}{r}\,\partial_r\,\delta(\underline{x}) = -\,\frac{1}{2}\,\Delta\,\delta(\underline{x})$$

$$\frac{1}{r^2}\,\Delta^*\,\delta(\underline{x}) = 0$$

$$\frac{1}{r^2}\,\mathbf{Z}^*\,\delta(\underline{x}) = -\,\frac{m-1}{2m}\,\Delta\,\delta(\underline{x})$$

$$\frac{1}{r^2}\,\delta(\underline{x}) = \frac{1}{2}\,\frac{1}{m}\,\Delta\,\delta(\underline{x})$$

it is straightforwardly verified that indeed

$$\mathbf{Z}\,\delta(\underline{x}) = \frac{m+1}{2m}\,\Delta\,\delta(\underline{x})$$

and

$$-\,\underline{F}^2\,\delta(\underline{x}) = \frac{1}{m}\,\Delta\,\delta(\underline{x})\,.$$

18. Division by natural powers of the radial distance *r*

Because the division of the delta distribution and its Dirac derivatives by natural powers of the vector variable $\underline{x}$ is uniquely determined, see Section 7, this will also be the case for the corresponding cross operation of division by the radial distance r. Based on the result of Section 7:

$$\frac{1}{\underline{x}}\,\delta(\underline{x}) = \frac{1}{m}\,\underline{\partial}\,\delta(\underline{x})$$

we can define the signumdistribution $\frac{1}{r}\delta(\underline{x})$.

Definition 21 The signumdistribution $\frac{1}{r}\delta(\underline{x})$ is defined by

$$\langle\,\frac{1}{r}\,\delta(\underline{x}),\;\underline{\omega}\,\varphi(\underline{x})\,\rangle = \langle\,-\,\frac{1}{\underline{x}}\,\delta(\underline{x}),\;\varphi(\underline{x})\,\rangle = \langle\,-\,\frac{1}{m}\,\underline{\partial}\,\delta(\underline{x}),\;\varphi(\underline{x})\,\rangle,\quad \forall\,\underline{\omega}\,\varphi(\underline{x}) \in \Omega(\mathbb{R}^m)\,.$$

In view of

$$\langle\, \partial_r\, \delta(\underline{x}),\ \underline{\omega}\, \varphi(\underline{x})\, \rangle = \langle\, \underline{\partial}\, \delta(\underline{x}),\ \varphi(\underline{x})\, \rangle$$

we obtain the signumdistributional identity

$$\frac{1}{\underline{x}}\, (\underline{\omega}\, \delta(\underline{x})) = \frac{1}{r}\, \delta(\underline{x}) = -\frac{1}{m}\, \partial_r\, \delta(\underline{x})\,.$$

More generally, we have the following definition.

Definition 22 The signumdistribution $\dfrac{1}{r^{2k+1}}\, \delta(\underline{x})$ is defined by

$$\langle\, \frac{1}{r^{2k+1}}\, \delta(\underline{x}),\ \underline{\omega}\, \varphi(\underline{x})\, \rangle = \langle\, (-1)^{k+1}\, \frac{1}{\underline{x}^{2k+1}}\, \delta(\underline{x}),\ \varphi(\underline{x})\, \rangle$$

$$= (-1)^{k+1}\, \langle\, \frac{1}{2^k k!(m+2k)(m+2k-2)\cdots(m)}\, \underline{\partial}^{2k+1} \delta(\underline{x}),\ \varphi(\underline{x})\, \rangle, \quad \forall\, \underline{\omega}\, \varphi(\underline{x}) \in \Omega(\mathbb{R}^m)\,.$$

Through the commutative diagram

$$\begin{array}{ccc}
\delta(\underline{x}) & \xrightarrow{(-1)^k \frac{1}{\underline{x}^{2k+1}}} & (-1)^k\, \frac{1}{\underline{x}^{2k+1}}\, \delta(\underline{x}) = \dfrac{(-1)^k}{2^k k!(m+2k)(m+2k-2)\cdots(m)}\, \underline{\partial}^{2k+1} \delta(\underline{x}) \\
{\scriptstyle -\underline{\omega}} \uparrow \downarrow {\scriptstyle \underline{\omega}} & & \uparrow {\scriptstyle -\underline{\omega}} \ \downarrow {\scriptstyle \underline{\omega}} \\
\underline{\omega}\, \delta(\underline{x}) & \xrightarrow[(-1)^k \frac{1}{\underline{x}^{2k+1}}]{} & (-1)^k\, \frac{1}{\underline{x}^{2k+1}}\, (\underline{\omega}\, \delta(\underline{x})) = \dfrac{(-1)^k}{2^k k!(m+2k)(m+2k-2)\cdots(m)}\, \underline{\omega}\, \underline{\partial}^{2k+1} \delta(\underline{x})
\end{array}$$

it becomes apparent that

$$(-1)^k\, \frac{1}{\underline{x}^{2k+1}}\, (\underline{\omega}\, \delta(\underline{x})) = \frac{1}{r^{2k+1}}\, \delta(\underline{x}) = -\frac{(m-1)!}{(m+2k)!}\, \partial_r^{2k+1}\, \delta(\underline{x})\,.$$

Note that division by an even natural power of r is a Cartesian operation. So $\dfrac{1}{r^{2k}}\, \delta(\underline{x})$ has to be a distribution; its definition runs as follows.

Definition 23 The distribution $\dfrac{1}{r^{2k}}\, \delta(\underline{x})$ is defined by

$$\frac{1}{r^{2k}}\,\delta(\underline{x}) = (-1)^k\,\frac{1}{\underline{x}^{2k}}\,\delta(\underline{x})$$

$$= (-1)^k\,\frac{1}{2^k\,k!\,m(m+2)\cdots(m+2k-2)}\,\underline{\partial}^{2k}\,\delta(\underline{x})$$

$$= \frac{(m-1)!}{(m+2k-1)!}\,\partial_r^{2k}\,\delta(\underline{x})\,.$$

On the contrary, $\frac{1}{r^{2k}}\,(\underline{\omega}\,\delta(\underline{x}))$ has to be a signumdistribution; its definition runs as follows.
Definition 24

$$\frac{1}{r^{2k}}\,(\underline{\omega}\,\delta(\underline{x})) = (-1)^k\,\frac{1}{\underline{x}^{2k}}\,(\underline{\omega}\,\delta(\underline{x}))$$

$$= (-1)^k\,\frac{1}{2^k\,k!\,m(m+2)\cdots(m+2k-2)}\,\underline{\omega}\,\underline{\partial}^{2k}\,\delta(\underline{x})$$

$$= \frac{(m-1)!}{(m+2k-1)!}\,\underline{\omega}\,\partial_r^{2k}\,\delta(\underline{x})\,.$$

Similarly, based upon the following formulæ for division by $\underline{x}$:

$$\frac{1}{\underline{x}}\,\underline{\partial}^{2k+1}\delta(\underline{x}) = \frac{1}{2k+2}\,\underline{\partial}^{2k+2}\delta(\underline{x})$$

$$\frac{1}{\underline{x}}\,\underline{\partial}^{2k}\delta(\underline{x}) = \frac{1}{m+2k}\,\underline{\partial}^{2k+1}\delta(\underline{x})$$

we may establish the commutative diagrams

$$\begin{array}{ccccc}
\underline{\partial}^{2k}\,\delta(\underline{x}) & \overset{\frac{1}{\underline{x}}}{\longrightarrow} & \frac{1}{m+2k}\,\underline{\partial}^{2k+1}\,\delta(\underline{x}) = \frac{1}{m+2k}\,\frac{(-1)^k\,2^k\,k!}{(m+1)(m+3)\cdots(m+2k-1)}\,\underline{\omega}\,\partial_r^{2k+1}\delta(\underline{x}) \\
{\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \begin{array}{c}\nearrow {\scriptstyle -\frac{1}{r}} \\ \searrow {\scriptstyle \frac{1}{r}}\end{array} & \uparrow{\scriptstyle -\underline{\omega}}\ \downarrow{\scriptstyle \underline{\omega}} \\
\underline{\omega}\,\underline{\partial}^{2k}\,\delta(\underline{x}) & \underset{\frac{1}{\underline{x}}}{\longrightarrow} & \frac{1}{m+2k}\,\underline{\omega}\,\underline{\partial}^{2k+1}\,\delta(\underline{x}) = \frac{1}{m+2k}\,\frac{(-1)^{k+1}\,2^k\,k!}{(m+1)(m+3)\cdots(m+2k-1)}\,\partial_r^{2k+1}\delta(\underline{x})
\end{array}$$

and

$$\begin{array}{ccc}
\underline{\partial}^{2k+1}\,\delta(\underline{x}) & \overset{\frac{1}{\underline{x}}}{\longrightarrow} & \dfrac{1}{2k+2}\,\underline{\partial}^{2k+2}\,\delta(\underline{x}) = \dfrac{(-1)^{k+1}\,2^k\,k!}{(m+1)(m+3)\cdots(m+2k+1)}\,\partial_r^{2k+2}\delta(\underline{x}) \\
{\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \overset{-\frac{1}{r}\,\nearrow}{\underset{\searrow\,\frac{1}{r}}{\times}} & \uparrow\downarrow{\scriptstyle -\underline{\omega}}\ {\scriptstyle \underline{\omega}} \\
\underline{\omega}\,\underline{\partial}^{2k+1}\,\delta(\underline{x}) & \underset{\frac{1}{\underline{x}}}{\longrightarrow} & \dfrac{1}{2k+2}\,\underline{\omega}\,\underline{\partial}^{2k+2}\,\delta(\underline{x}) = \dfrac{(-1)^{k+1}\,2^k\,k!}{(m+1)(m+3)\cdots(m+2k+1)}\,\underline{\omega}\,\partial_r^{2k+2}\delta(\underline{x})
\end{array}$$

showing that

$$\frac{1}{r}\,\underline{\partial}^{2k}\,\delta(\underline{x}) = \frac{1}{m+2k}\,\underline{\omega}\,\underline{\partial}^{2k+1}\,\delta(\underline{x}) \quad \text{and} \quad \frac{1}{r}\,\partial_r^{2k}\,\delta(\underline{x}) = -\frac{1}{m+2k}\,\partial_r^{2k+1}\,\delta(\underline{x})$$

and

$$\frac{1}{r}\,\underline{\partial}^{2k+1}\,\delta(\underline{x}) = \frac{1}{2k+2}\,\underline{\omega}\,\underline{\partial}^{2k+2}\,\delta(\underline{x}) \quad \text{and} \quad \frac{1}{r}\,(\underline{\omega}\,\partial_r^{2k+1})\,\delta(\underline{x}) = -\frac{1}{m+2k+1}\,\underline{\omega}\,\partial_r^{2k+2}\,\delta(\underline{x})\,.$$

More generally, based on the formulæ obtained in Section 5, one has the following formulæ:
(i)

$$\frac{1}{r^{2k+1}}\,\underline{\partial}^{2\ell}\,\delta(\underline{x}) = (-1)^k\,\frac{1}{2^k\,(\ell+1)(\ell+2)\cdots(\ell+k)(m+2\ell)(m+2\ell+2)\cdots(m+2\ell+2k)}\,\underline{\omega}\,\underline{\partial}^{2\ell+2k+1}\,\delta(\underline{x})$$

$$\frac{1}{r^{2k+1}}\,\partial_r^{2\ell}\,\delta(\underline{x}) = -\,\frac{(m+2\ell-1)!}{(m+2\ell+2k)!}\,\partial_r^{2\ell+2k+1}\,\delta(\underline{x})$$

(ii)

$$\frac{1}{r^{2k}}\,\underline{\partial}^{2\ell}\,\delta(\underline{x}) = (-1)^k\,\frac{1}{2^k\,(\ell+1)(\ell+2)\cdots(\ell+k)(m+2\ell)(m+2\ell+2)\cdots(m+2\ell+2k-2)}\,\underline{\partial}^{2\ell+2k}\,\delta(\underline{x})$$

$$\frac{1}{r^{2k}}\,\partial_r^{2\ell}\,\delta(\underline{x}) = \frac{(m+2\ell-1)!}{(m+2\ell+2k-1)!}\,\partial_r^{2\ell+2k}\,\delta(\underline{x})$$

(iii)

$$\frac{1}{r^{2k+1}}\,\underline{\partial}^{2\ell+1}\,\delta(\underline{x}) = (-1)^k\,\frac{1}{2^{k+1}\,(\ell+1)(\ell+2)\cdots(\ell+k+1)(m+2\ell+2)(m+2\ell+4)\cdots(m+2\ell+2k)}\,\underline{\omega}\,\underline{\partial}^{2\ell+2k+2}\,\delta(\underline{x})$$

$$\frac{1}{r^{2k+1}}\,\partial_r^{2\ell+1}\,\delta(\underline{x}) = -\,\frac{(m+2\ell)!}{(m+2\ell+2k+1)!}\,\partial_r^{2\ell+2k+2}\,\delta(\underline{x})$$

(iv)

$$\frac{1}{r^{2k}}\,\underline{\partial}^{2\ell+1}\,\delta(\underline{x}) = (-1)^k\,\frac{1}{2^k\,(\ell+1)(\ell+2)\cdots(\ell+k)(m+2\ell+2)(m+2\ell+4)\cdots(m+2\ell+2k)}\,\underline{\partial}^{2\ell+2k+1}\,\delta(\underline{x})$$

$$\frac{1}{r^{2k}}\,\partial_r^{2\ell+1}\,\delta(\underline{x}) = \frac{(m+2\ell)!}{(m+2\ell+2k)!}\,\partial_r^{2\ell+2k+1}\,\delta(\underline{x})$$

19. Two fundamental sequences of (signum)distributions

There is a fundamental sequence of derivatives of the delta distribution, which are alternatively scalar and vector valued, and which is generated by the action of the operator $(\underline{\omega}\,\partial_r)$:

$$\delta \to (\underline{\omega}\,\partial_r)\,\delta \to \partial_r^2\,\delta \to \ldots \to (\underline{\omega}\,\partial_r^{2k-1})\,\delta \to \partial_r^{2k}\,\delta \to (\underline{\omega}\,\partial_r^{2k+1})\,\delta \to \ldots.$$

For each of the distributions in this sequence we defined, in the examples of the foregoing section, through the action of $\underline{\omega}$, a specific associated signumdistribution, yielding in this way a parallel sequence of signumdistributions:

$$\underline{\omega}\,\delta \to \partial_r\,\delta \to (\underline{\omega}\,\partial_r^2)\,\delta \to \ldots \to \partial_r^{2k-1}\,\delta \to (\underline{\omega}\,\partial_r^{2k})\,\delta \to \partial_r^{2k+1}\,\delta \to \ldots.$$

Let us recall these definitions. The initial definition is the following.
Definition 25

$$\langle\, \underline{\omega}\,\partial_r^{2k}\,\delta(\underline{x}),\ \underline{\omega}\,\varphi(\underline{x})\,\rangle = \langle\, -\partial_r^{2k}\,\delta(\underline{x}),\ \varphi(\underline{x})\,\rangle \tag{16}$$

$$\langle\, \underline{\omega}\,(\underline{\omega}\,\partial_r^{2k-1}\,\delta(\underline{x})),\ \underline{\omega}\ \varphi(\underline{x})\,\rangle = \langle\, -(\underline{\omega}\,\partial_r^{2k-1})\,\delta(\underline{x}),\ \varphi(\underline{x})\,\rangle\,. \tag{17}$$

Whereupon we introduce the signumdistribution $\partial_r^{2k-1}\,\delta(\underline{x})$ by
Definition 26

$$\partial_r^{2k-1}\,\delta(\underline{x}) = -\,\underline{\omega}\,(\underline{\omega}\,\partial_r^{2k-1}\,\delta(\underline{x}))$$

such that Definition 17 may be rephrased as

$$\langle\, \partial_r^{2k-1}\, \delta(\underline{x}),\ \underline{\omega}\, \varphi(\underline{x})\, \rangle = \langle\, (\underline{\omega}\, \partial_r^{2k-1})\, \delta(\underline{x}),\ \varphi(\underline{x})\, \rangle\,. \tag{18}$$

There are two other actions on each of the distributions of the first sequence, yielding a signumdistribution of the second sequence, viz. the actions by r and by ∂_r. Indeed, by an appropriate combination of the above definitions, we obtain the following calculus rules.

Property 1 One has

$$r\,(\underline{\omega}\, \partial_r^{2k+1})\, \delta(\underline{x}) = -(m+2k)\, \underline{\omega}\, \partial_r^{2k}\, \delta(\underline{x})$$

$$r\, \partial_r^{2k}\, \delta(\underline{x}) = -(m+2k-1)\, \partial_r^{2k-1}\, \delta(\underline{x})$$

$$\partial_r\,(\underline{\omega}\, \partial_r^{2k+1})\, \delta(\underline{x}) = \underline{\omega}\, \partial_r^{2k+2}\, \delta(\underline{x})$$

$$\partial_r\, \partial_r^{2k}\, \delta(\underline{x}) = \partial_r^{2k+1}\, \delta(\underline{x})\,.$$

One may wonder if there are actions transforming the signumdistributions from the second sequence back into distributions from the first sequence and the answer is positive. Indeed, the same actions apply on the signumdistributions from the second sequence. The basic action is again through the operator $\underline{\omega}$, which yields the following definitions.

Definition 27

$$\langle\, \underline{\omega}\,(\underline{\omega}\, \partial_r^{2k})\, \delta(\underline{x}),\ \varphi(\underline{x})\, \rangle = \langle\, \underline{\omega}\, \partial_r^{2k}\, \delta(\underline{x}),\ \underline{\omega}\, \varphi(\underline{x})\, \rangle \tag{19}$$

$$\langle\, \underline{\omega}\,(\partial_r^{2k-1}\, \delta(\underline{x})),\ \varphi(\underline{x})\, \rangle \langle\, \partial_r^{2k-1}\, \delta(\underline{x}),\ \underline{\omega}\, \varphi(\underline{x})\, \rangle\,. \tag{20}$$

Comparing Definitions 19 and 16, it is clear that the distribution $\underline{\omega}\,(\underline{\omega}\, \partial_r^{2k})\, \delta(\underline{x})$ is nothing else but the distribution $-\, \partial_r^{2k}\, \delta(\underline{x})$, whereas comparing Definitions 20 and 18 shows that the distribution $\underline{\omega}\,(\partial_r^{2k-1}\, \delta(\underline{x}))$ is indeed the distribution $(\underline{\omega}\, \partial_r^{2k-1})\, \delta(\underline{x})$.

For the actions of operators r and ∂_r on signumdistributions, which are defined in a similar way as the actions of r and ∂_r on distributions, we obtain the following computation rules.

Property 2 One has the following formulæ:

$$r\,\partial_r^{2k+1}\,\delta(\underline{x}) = -(m+2k)\,\partial_r^{2k}\,\delta(\underline{x})$$

$$r\,(\underline{\omega}\,\partial_r^{2k})\,\delta(\underline{x}) = -(m+2k-1)\,(\underline{\omega}\,\partial_r^{2k-1})\,\delta(\underline{x})$$

$$r\left(\frac{1}{r}\,\partial_r^{2k}\delta(\underline{x})\right) = \partial_r^{2k}\delta(\underline{x})$$

$$r\left(\frac{1}{r}\,\underline{\omega}\,\partial_r^{2k+1}\delta(\underline{x})\right) = \underline{\omega}\,\partial_r^{2k+1}\delta(\underline{x})$$

$$\partial_r\,(\partial_r^{2k+1})\,\delta(\underline{x}) = \partial_r^{2k+2}\,\delta(\underline{x})$$

$$\partial_r\,(\underline{\omega}\,\partial_r^{2k})\,\delta(\underline{x}) = (\underline{\omega}\,\partial_r^{2k+1})\,\delta(\underline{x})\,.$$

From those fomulæthe following completely symmetric picture can be deduced:

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & (\underline{\omega}\,\partial_r^{2k-1})\,\delta & \longrightarrow & \partial_r^{2k}\,\delta & \longrightarrow & (\underline{\omega}\,\partial_r^{2k+1})\,\delta & \longrightarrow \ldots \\
 & {\scriptstyle\underline{\omega}}\uparrow\downarrow{\scriptstyle\underline{\omega}} & {\scriptstyle r}\nwarrow\nearrow{\scriptstyle\partial_r}\ \ {\scriptstyle r}\swarrow\searrow{\scriptstyle\partial_r} & {\scriptstyle\underline{\omega}}\uparrow\downarrow{\scriptstyle\underline{\omega}} & {\scriptstyle r}\nwarrow\nearrow{\scriptstyle\partial_r}\ \ {\scriptstyle r}\swarrow\searrow{\scriptstyle\partial_r} & {\scriptstyle\underline{\omega}}\uparrow\downarrow{\scriptstyle\underline{\omega}} & \\
\ldots \longrightarrow & \partial_r^{2k-1}\,\delta & \longrightarrow & (\underline{\omega}\,\partial_r^{2k})\,\delta & \longrightarrow & \partial_r^{2k+1}\,\delta & \longrightarrow \ldots
\end{array}$$

20. Composition of spherical operators

When composing two operators out of the set of operators: r, ∂_r and $\underline{\omega}$, six operators originate: r^2, $r\,\partial_r$, $r\,\underline{\omega}$, ∂_r^2, $\underline{\omega}\,\partial_r$ and $\underline{\omega}^2$, which are traditional operators whose actions on distributions are well-defined. This means that the consecutive action by any two of the operators r, ∂_r and $\underline{\omega}$ should lead to a known result, which is a serious test for all calculus rules established above. We now prove that this is indeed the case.

(i) By the calculus rules we have

$$r^2(\underline{\omega}\,\partial_r^{2k+1}\,\delta) = -(m+2k)r(\underline{\omega}\,\partial_r^{2k}\,\delta) = (m+2k-1)(m+2k)(\underline{\omega}\,\partial_r^{2k-1})\,\delta$$

and

$$r^2(\partial_r^{2k}\,\delta) = -(m+2k-1)r(\partial_r^{2k-1}\,\delta) = (m+2k-2)(m+2k-1)\partial_r^{2k-2}\,\delta\,.$$

On the other hand, invoking the identities

$$\underline{x}^2 \underline{\partial}^{2k+1}\, \delta(\underline{x}) = (m+2k)(2k)\, \underline{\partial}^{2k-1}\, \delta(\underline{x}) \tag{21}$$

$$\underline{x}^2 \underline{\partial}^{2k}\, \delta(\underline{x}) = (m+2k-2)(2k)\, \underline{\partial}^{2k-2}\, \delta(\underline{x}) \tag{22}$$

and the formulæ of Corollary 3, we have

$$r^2(\underline{\omega}\, \partial_r^{2k+1}\, \delta) = -\frac{(-1)^k}{2^k k!}\, (m+1)(m+3)\cdots(m+2k-1)\, \underline{x}^2 \underline{\partial}^{2k+1}\, \delta$$

$$= (m+2k-1)(m+2k)(\underline{\omega}\, \partial_r^{2k-1})\, \delta$$

and

$$r^2(\partial_r^{2k}\, \delta) = -\frac{(-1)^k}{2^k k!}\, (m+1)(m+3)\cdots(m+2k-1)\, \underline{x}^2 \underline{\partial}^{2k}\, \delta$$

$$= (m+2k-2)(m+2k-1)(\underline{\omega}\, \partial_r^{2k-2})\, \delta\, .$$

(ii) The Euler operator measures the degree of homogeneity and thus

$$r\, \partial_r\, (\underline{\omega}\, \partial_r^{2k+1}\, \delta) = -(m+2k+1)\, \underline{\omega}\, \partial_r^{2k+1}\, \delta$$

$$r\, \partial_r\, (\partial_r^{2k}\, \delta) = -(m+2k) \partial_r^{2k}\, \delta$$

whereas the calculus rules lead to

$$r\, \partial_r\, (\underline{\omega}\, \partial_r^{2k+1}\, \delta) = r(\underline{\omega}\, \partial_r^{2k+2}\, \delta) = -(m+2k+1)\, \underline{\omega}\, \partial_r^{2k+1}\, \delta$$

$$r\, \partial_r\, (\partial_r^{2k}\, \delta) = r(\partial_r^{2k+1}\, \delta) = -(m+2k) \partial_r^{2k}\, \delta\, .$$

(iii) By the calculus rules we obtain

$$r\, \underline{\omega}\, (\underline{\omega}\, \partial_r^{2k+1}\, \delta) = -\, r\, \partial_r^{2k+1}\, \delta = (m+2k)\, \partial_r^{2k}\, \delta$$

$$r\, \underline{\omega}\, (\partial_r^{2k}\, \delta) = -(m+2k-1)\, \underline{\omega}\, \partial_r^{2k-1}\, \delta$$

whereas invoking identities (21) and (22) respectively, leads to

$$r\,\underline{\omega}\,(\underline{\omega}\,\partial_r^{2k+1}\,\delta) = \frac{(-1)^k}{2^k k!}\,(m+1)(m+3)\cdots(m+2k-1)\,\underline{x}\,\underline{\partial}^{2k+1}\,\delta$$

$$= (m+2k)\,\partial_r^{2k}\,\delta$$

and

$$r\,\underline{\omega}\,(\partial_r^{2k}\,\delta) = \frac{(-1)^k}{2^k k!}\,(m+1)(m+3)\cdots(m+2k-1)\,\underline{x}\,\underline{\partial}^{2k}\,\delta$$

$$= -(m+2k-1)\,\underline{\omega}\,\partial_r^{2k-1}\,\delta\,.$$

(iv) The calculus rules lead to

$$\partial_r^2\,(\underline{\omega}\,\partial_r^{2k+1}\,\delta) = \underline{\omega}\,\partial_r^{2k+3}\,\delta$$

$$\partial_r^2\,(\partial_r^{2k}\,\delta) = \partial_r^{2k+2}\,\delta\,.$$

On the other hand we can make use of the identities

$$(\underline{\omega}\,\partial_r)\,\underline{\partial}^{2k}\,\delta(\underline{x}) = \underline{\partial}^{2k+1}\,\delta(\underline{x})$$

$$(\underline{\omega}\,\partial_r)\,\underline{\partial}^{2k+1}\,\delta(\underline{x}) = \frac{m+2k+1}{2(k+1)}\,\underline{\partial}^{2k+2}\,\delta(\underline{x})$$

to obtain

$$\partial_r^2\,(\underline{\omega}\,\partial_r^{2k+1}\,\delta) = \frac{(-1)^k}{2^k k!}\,(m+1)(m+3)\cdots(m+2k-1)\,(-1)\,\frac{m+2k+1}{2(k+1)}\,\underline{\partial}^{2k+3}\,\delta$$

$$= \frac{(-1)^{k+1}}{2^{k+1}(k+1)!}\,(m+1)(m+3)\cdots(m+2k-1)(m+2k+1)\,\underline{\partial}^{2k+3}\,\delta = \underline{\omega}\,\partial_r^{2k+3}\,\delta$$

and

$$\partial_r^2 (\partial_r^{2k} \delta) = \frac{(-1)^k}{2^k k!} (m+1)(m+3)\cdots(m+2k-1)(-1) \frac{m+2k+1}{2(k+1)} \underline{\partial}^{2k+2} \delta$$

$$= \frac{(-1)^{k+1}}{2^{k+1}(k+1)!} (m+1)(m+3)\cdots(m+2k-1)(m+2k+1) \underline{\partial}^{2k+2} \delta = \partial_r^{2k+2} \delta .$$

(v) On the one hand we have by the calculus rules

$$\underline{\omega}\, \partial_r (\underline{\omega}\, \partial_r^{2k+1} \delta) = \underline{\omega} (\underline{\omega}\, \partial_r^{2k+2} \delta) = -\, \partial_r^{2k+2} \delta$$

$$\underline{\omega}\, \partial_r (\partial_r^{2k} \delta) = \underline{\omega} (\partial_r^{2k+1} \delta) = \underline{\omega}\, \partial_r^{2k+1} \delta$$

and on the other

$$\underline{\omega}\, \partial_r (\underline{\omega}\, \partial_r^{2k+1} \delta) = \underline{\omega}\, \partial_r (-1)^k (\underline{\omega}\, \partial_r)^{2k+1} \delta = (-1)^k (\underline{\omega}\, \partial_r)^{2k+2} \delta = -\, \partial_r^{2k+2} \delta$$

$$\underline{\omega}\, \partial_r (\partial_r^{2k} \delta) = \underline{\omega}\, \partial_r (-1)^k (\underline{\omega}\, \partial_r)^{2k} \delta = (-1)^k (\underline{\omega}\, \partial_r)^{2k+1} \delta = \underline{\omega}\, \partial_r^{2k+1} \delta .$$

(vi) The action by $\underline{\omega}^2 = -1$ is trivial.

21. Two families of specific distributions

In a series of papers, see [11, 13] and the references therein, several specific families of distributions in Euclidean space $\mathbb{R}^m$ were thoroughly studied. Of particular importance are distribution families T_λ and U_λ, λ being a complex parameter, appearing in harmonic analysis. They are defined as follows, using spherical coordinates $\underline{x} = r\,\underline{\omega}$, $r = |\underline{x}|$, $\underline{\omega} \in S^{m-1}$, S^{m-1} being the unit sphere in $\mathbb{R}^m$.

Definition 28 For all $\lambda \in \mathbb{C}$ and for all test functions $\varphi(\underline{x}) \in \mathscr{D}(\mathbb{R}^m)$ the distributions T_λ and U_λ are defined by

$$\langle\, T_\lambda ,\ \varphi(\underline{x})\, \rangle := a_m \langle\, \mathrm{Fp}\, r_+^{\lambda+m-1},\ \Sigma^0[\varphi](r)\, \rangle_r$$

and

$$\langle\, U_\lambda ,\ \varphi(\underline{x})\, \rangle := a_m \langle\, \mathrm{Fp}\, r_+^{\lambda+m-1},\ \Sigma^1[\varphi](r)\, \rangle_r$$

where, recall, the spherical means Σ^0 and Σ^1 are given by

$$\Sigma^0[\varphi](r) = \frac{1}{a_m} \int_{S^{m-1}} \varphi(r\underline{\omega})\, dS(\underline{\omega})$$

and

$$\Sigma^1[\varphi](r) = \frac{1}{a_m}\int_{S^{m-1}} \underline{\omega}\, \varphi(r\underline{\omega})\, dS(\underline{\omega})$$

with $a_m = \dfrac{2\pi^{m/2}}{\Gamma(m/2)}$ the area of the unit sphere S^{m-1}, and where $\mathrm{Fp}\, r_+^{\mu}$ stands for the *Finite Part* distribution on the one-dimensional r-axis.

An alternative, and handy, notation could be $T_\lambda = \mathrm{Fp}\, r^\lambda$ and $U_\lambda = \underline{\omega}\,\mathrm{Fp}\, r^\lambda$. For the sake of completeness we give the definition of the one-dimensional Finite Part distribution.

Definition 29 The distribution $\mathrm{Fp} x_+^{\mu}$ is defined for the complex parameter μ such that $-n-1 < \Re\mu < -n$ and all $n \in \mathbb{N}$, by

$$\begin{aligned} \langle\, \mathrm{Fp} x_+^{\mu}\,,\ \phi(x)\,\rangle &= \int_0^{+\infty} x^{\mu} \left(\phi(x) - \phi(0) - \frac{\phi'(0)}{1!}\, x - \cdots - \frac{\phi^{(n-1)}(0)}{(n-1)!}\, x^{n-1} \right) dx \\ &= \lim_{\varepsilon\to 0+} \left(\int_{\varepsilon}^{+\infty} x^{\mu}\, \phi(x)\, dx + \phi(0)\, \frac{\varepsilon^{\mu+1}}{\mu+1} + \cdots + \frac{\phi^{(n-1)}(0)}{(n-1)!}\, \frac{\varepsilon^{\mu+n}}{\mu+n} \right). \end{aligned} \tag{23}$$

Let us comment on the Definition 28; for more details we refer to the above mentioned series of papers.

A priori, distributions T_λ and U_λ are not defined at the simple poles of the one-dimensional distribution $\mathrm{Fp}\ r_+^{\lambda+m-1}$ on the r-axis, viz., $\lambda = -m-n+1$, $n \in \mathbb{N}$. To cope with these singularities, the distributions $\mathrm{Fp}\ r_+^{-n}$, $n \in \mathbb{N}$ are interpreted as the so-called *monomial pseudofunctions*, see [11].

Distributions T_λ are standard scalar distributions well known in harmonic analysis. They are radial and homogeneous of degree λ. As meromorphic functions of $\lambda \in \mathbb{C}$ they show genuine simple poles at $\lambda = -m,\ -m-2,\ -m-4,\ \ldots$. This is due to the fact that the singular points $\lambda = -m-2\ell-1$, $\ell = 0,\ 1,\ 2,\ \ldots$ are removable, since spherical mean $\Sigma^{(0)}[\phi]$ has its odd order derivatives vanishing at $r = 0$. So we can define

$$\langle T_{-m-2\ell-1},\ \phi\rangle = \lim_{\mu\to -2\ell-2} a_m \langle \mathrm{Fp}\ r_+^{\mu},\ \Sigma^{(0)}[\phi]\rangle$$

but, remarkably, this limit is precisely $a_m\langle \mathrm{Fp}\ r_+^{-2\ell-2},\ \Sigma^{(0)}[\phi]\rangle$, with $\mathrm{Fp}\ r_+^{-2\ell-2}$ the monomial pseudofunction. The most important distribution is this family is $T_{-m+2} = \dfrac{1}{r^{m-2}}$, which is, up to a constant, the fundamental solution of Laplace operator Δ. Also note the special cases $T_0 = 1$, $T_{2\ell} = r^{2\ell} = (-1)^\ell \underline{x}^{2\ell}$, and $T_{2\ell+1} = r^{2\ell+1}$, $\ell = 0,\ 1,\ 2,\ \ldots$.

Distributions U_λ are typical Clifford analysis constructs. They are homogeneous of degree λ. As vector-valued meromorphic functions of $\lambda \in \mathbb{C}$ they show genuine simple poles at $\lambda = -m-1,\ -m-3,\ -m-5,\ \ldots$. This is due to the fact that the singular points $\lambda = -m-2\ell$, $\ell = 0,\ 1,\ 2,\ \ldots$ are removable, since spherical mean $\Sigma^{(1)}[\phi]$ has its even order derivatives vanishing at $r = 0$. So we can define

$$\langle U_{-m-2\ell},\ \phi\rangle = \lim_{\mu\to -2\ell-1} a_m \langle \mathrm{Fp}\ r_+^{\mu},\ \Sigma^{(1)}[\phi]\rangle$$

but this limit is precisely $a_m \langle \mathrm{Fp}\, r_+^{-2\ell-1},\, \Sigma^{(1)}[\phi] \rangle$, with $\mathrm{Fp}\, r_+^{-2\ell-1}$ the monomial pseudofunction. The most important distribution in this family is $U_{-m+1} = \dfrac{\underline{\omega}}{r^{m-1}} = \dfrac{\underline{x}}{r^m}$, which is, up to a constant, the fundamental solution of Dirac operator $\underline{\partial}$. Also note the special cases $U_0 = \underline{\omega}$, $U_{2\ell} = \underline{\omega}\, r^{2\ell}$ and $U_{2\ell+1} = \underline{\omega}\, r^{2\ell+1} = (-1)^{\ell}\, \underline{x}^{2\ell+1},\ \ell = 0,\ 1,\ 2,\ \ldots$.

It is important to note that, although distributions T_λ and U_λ are also defined in their respective singularities, these exceptional values do *not* turn these distributions into entire functions of the parameter $\lambda \in \mathbb{C}$.

When restricted to the half-plane $\mathbb{R}e\, \lambda > -m$, distributions T_λ and U_λ are regular, i.e. they are locally integrable functions. From [4] we know that a locally integrable function can be seen as a signumdistribution as well. This inspires the definition of the following two families of signumdistributions:

$$\langle\, {}^sT_\lambda,\, \underline{\omega}\, \varphi(\underline{x})\, \rangle := a_m \langle\, \mathrm{Fp}\, r_+^{\lambda+m-1},\, \Sigma^1[\varphi](r)\, \rangle_r\, ;$$

$$\langle\, {}^sU_\lambda,\, \underline{\omega}\, \varphi(\underline{x})\, \rangle := -a_m \langle\, \mathrm{Fp}\, r_+^{\lambda+m-1},\, \Sigma^0[\varphi](r)\, \rangle_r\, .$$

It is clear that

$$T_\lambda^{\vee} = {}^sU_\lambda, \quad {}^sU_\lambda^{\wedge} = T_\lambda$$

and

$$U_\lambda^{\vee} = -{}^sT_\lambda, \quad {}^sT_\lambda^{\wedge} = -U_\lambda\, .$$

In this way ${}^sT_\lambda$ inherits the simple poles of U_λ, viz., $\lambda = -m-1,\ -m-3,\ -m-5,\ \ldots$, whereas ${}^sU_\lambda$ inherits the simple poles of T_λ, viz., $\lambda = -m,\ -m-2,\ -m-4,\ \ldots$.

Because distributions T_λ are radial, by Definition 12, signumdistributions ${}^sU_\lambda$ are signum-radial. Because signumdistributions ${}^sT_\lambda$ are radial, by the same definition, distributions U_λ are signum-radial.

Now we will systematically compute the actions on T_λ, U_λ, ${}^sT_\lambda$ and ${}^sU_\lambda$ of all operators introduced in the preceding sections, paying attention to the uniqueness of the expressions obtained.

Multiplication operator $\underline{x}$ is a Cartesian operator whose actions are uniquely determined, quite naturally. It holds for all $\lambda \in \mathbb{C}$ that

$$\underline{x}\, T_\lambda = U_{\lambda+1}, \quad \underline{x}\, U_\lambda = -\, T_{\lambda+1}\, . \tag{24}$$

Based on the commutative diagram

$$\begin{array}{ccc} T_\lambda & \overset{-\underline{x}}{\longrightarrow} & -\, U_{\lambda+1} \\ {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \begin{smallmatrix} \nearrow\, -r \\ \searrow\, r \end{smallmatrix} & \uparrow\downarrow\, {\scriptstyle -\underline{\omega},\ \underline{\omega}} \\ {}^sU_\lambda & \underset{-\underline{x}}{\longrightarrow} & {}^sT_{\lambda+1} \end{array}$$

we find the additional formulæ

$$r\,T_\lambda = {}^sT_{\lambda+1} \quad \text{and} \quad r\,{}^sU_\lambda = U_{\lambda+1}, \quad \lambda \in \mathbb{C}$$

and

$$\underline{x}\,{}^sU_\lambda = -\,{}^sT_{\lambda+1}, \quad \lambda \in \mathbb{C}.$$

In a similar way, based on the commutative diagram

$$\begin{array}{ccc}
-\,U_\lambda & \xrightarrow{\ \underline{x}\ } & T_{\lambda+1} \\
{\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \begin{array}{c}\nearrow^{\,r} \\ \searrow_{\,-r}\end{array} & \uparrow\downarrow{\scriptstyle -\underline{\omega},\ \underline{\omega}} \\
{}^sT_\lambda & \xrightarrow[\ \underline{x}\ ]{} & {}^sU_{\lambda+1}
\end{array}$$

we obtain the additional formulæ

$$r\,{}^sT_\lambda = T_{\lambda+1} \quad \text{and} \quad r\,U_\lambda = {}^sU_{\lambda+1}, \quad \lambda \in \mathbb{C}$$

and also

$$\underline{x}\,{}^sT_\lambda = {}^sU_{\lambda+1}, \quad \lambda \in \mathbb{C}.$$

Iterated action of multiplication operator $\underline{x}$ results into

$$r^2\,T_\lambda = T_{\lambda+2}, \quad r^2\,U_\lambda = U_{\lambda+2}, \quad \lambda \in \mathbb{C}$$

and

$$r^2\,{}^sT_\lambda = {}^sT_{\lambda+2}, \quad r^2\,{}^sU_\lambda = {}^sU_{\lambda+2}, \quad \lambda \in \mathbb{C}.$$

As is the case for multiplication operator $\underline{x}$, also Dirac operator $\underline{\partial}$ intertwines the T_λ and U_λ distribution families. It clearly is a Cartesian operator whose action is uniquely determined. It holds that

$$\underline{\partial}\,T_\lambda = \lambda\,U_{\lambda-1}, \quad \lambda \neq -m,\ -m-2,\ -m-4,\ \ldots \tag{25}$$

and

$$\underline{\partial}\, U_\lambda = -(\lambda+m-1)\, T_{\lambda-1}, \quad \lambda \neq -m+1,\ -m-1,\ -m-3,\ \ldots \tag{26}$$

whereas for $\ell = 0,\ 1,\ 2,\ \ldots$

$$\underline{\partial}\, T_{-m-2\ell} = -\,(m+2\ell)\, U_{-m-2\ell-1} + (-1)^{\ell+1}\, \frac{1}{C(m,\ \ell)}\, a_m\, \underline{\partial}^{2\ell+1}\, \delta(\underline{x}) \tag{27}$$

and

$$\underline{\partial}\, U_{-m-2\ell+1} = (2\ell)\, T_{-m-2\ell} + (-1)^{\ell-1}\, \frac{m+2\ell}{C(m,\ \ell)}\, a_m\, \underline{\partial}^{2\ell}\, \delta(\underline{x}) \tag{28}$$

with

$$C(m,\ \ell) = 2^{2\ell+1}\, \ell!\, \frac{\Gamma\left(\frac{m}{2}+\ell+1\right)}{\Gamma\left(\frac{m}{2}\right)} = 2^{\ell}\, \ell!\, m(m+2)(m+4)\cdots(m+2\ell)\,.$$

In particular, for $\ell = 0$, it holds that

$$\underline{\partial}\, T_{-m} = (-m)\, U_{-m-1} - \frac{1}{m}\, a_m\, \underline{\partial}\, \delta(\underline{x})\,; \tag{29}$$

$$\underline{\partial}\, U_{-m+1} = -\, a_m\, \delta(\underline{x})\,. \tag{30}$$

Formula (30) expresses the well known fact that $-\dfrac{1}{m}\, U_{-m+1}$ is indeed the fundamental solution of Dirac operator $\underline{\partial}$.

Through the signum-pair of operators $(\underline{\partial},\ \underline{D})$, the corresponding formulæfor signumdistributions ${}^sT_\lambda$ and ${}^sU_\lambda$ are readily obtained:

$$\underline{D}\, {}^sU_\lambda = -\,\lambda\, {}^sT_{\lambda-1}, \quad \lambda \neq -m,\ -m-2,\ -m-4,\ \ldots$$

and

$$\underline{D}\, {}^sT_\lambda = (\lambda+m-1)\, {}^sU_{\lambda-1}, \quad \lambda \neq -m+1,\ -m-1,\ -m-3,\ \ldots$$

whereas for $\ell = 0,\ 1,\ 2,\ \ldots$

$$\underline{D}^s U_{-m-2\ell} = (m+2\ell)^s T_{-m-2\ell-1} + (-1)^{\ell+1} \frac{1}{C(m,\ \ell)} a_m \underline{\omega}\, \underline{\partial}^{2\ell+1} \delta(\underline{x})$$

and

$$\underline{D}^s T_{-m-2\ell+1} = -(2\ell)^s U_{-m-2\ell} + (-1)^{\ell} \frac{m+2\ell}{C(m,\ \ell)} a_m \underline{\omega}\, \underline{\partial}^{2\ell} \delta(\underline{x})$$

and in particular, for $\ell = 0$,

$$\underline{D}^s U_{-m} = m^s T_{-m-1} - \frac{1}{m} a_m \underline{\omega}\, \underline{\partial} \delta(\underline{x})$$

$$\underline{D}^s T_{-m+1} = a_m \underline{\omega}\, \delta(\underline{x}) .$$

Iterated action of Dirac operator $\underline{\partial}$ results into formulæ for the action of the Laplace operator on distributions. It holds that

$$\Delta T_\lambda = \lambda\,(\lambda + m - 2)\, T_{\lambda-2}, \quad \lambda \neq -m+2,\ -m,\ -m-2,\ \ldots \tag{31}$$

$$\Delta U_\lambda = (\lambda - 1)(\lambda + m - 1)\, U_{\lambda-2}, \quad \lambda \neq -m+1,\ -m-1,\ \ldots \tag{32}$$

and

$$\Delta T_{-m-2\ell} = (m+2\ell)(2\ell+2)\, T_{-m-2\ell-2} + (-1)^{\ell} a_m \frac{(m+4\ell+2)(m+2\ell+2)}{C(m,\ \ell+1)} \underline{\partial}^{2\ell+2} \delta(\underline{x}) \tag{33}$$

$$\Delta U_{-m-2\ell+1} = (m+2\ell)(2\ell)\, U_{-m-2\ell-1} + (-1)^{\ell} a_m \frac{m+4\ell}{C(m,\ \ell)} \underline{\partial}^{2\ell+1} \delta(\underline{x}) \tag{34}$$

and, in particular, for $\ell = 0$

$$\Delta T_{-m} = 2m\, T_{-m-2} + \frac{m+2}{2m} a_m \underline{\partial}^2 \delta(\underline{x}) \tag{35}$$

$$\Delta U_{-m+1} = a_m \underline{\partial}\, \delta(\underline{x}) \tag{36}$$

and also

$$\Delta T_{-m+2} = -(m-2)\, a_m\, \delta(\underline{x}) \tag{37}$$

this last formula expressing the fact that $-\dfrac{1}{m-2}\dfrac{1}{a_m} T_{-m+2}$ is, as expected, the fundamental solution of the Laplace operator.

Continuing the iterated action by Dirac operator $\underline{\partial}$ leads to the fundamental solutions of the natural powers of $\underline{\partial}$. We find

$$\underline{\partial}^{2\ell} E_{2\ell} = \delta(\underline{x})$$

with

$$E_{2\ell} = \frac{1}{2^{\ell-1}(\ell-1)!\,(m-2)(m-4)\cdots(m-2\ell)}\frac{1}{a_m} T_{-m+2\ell} = \frac{1}{2^{2\ell}}\frac{1}{(\ell-1)!}\frac{\Gamma\left(\frac{m}{2}-\ell\right)}{\pi^{m/2}} T_{-m+2\ell} \tag{38}$$

and

$$\underline{\partial}^{2\ell+1} E_{2\ell+1} = \delta(\underline{x})$$

with

$$E_{2\ell+1} = -\frac{1}{2^{\ell}(\ell)!\,(m-2)(m-4)\cdots(m-2\ell)}\frac{1}{a_m} U_{-m+2\ell+1} = -\frac{1}{2^{2\ell+1}}\frac{1}{\ell!}\frac{\Gamma\left(\frac{m}{2}-\ell\right)}{\pi^{m/2}} U_{-m+2\ell+1}\,. \tag{39}$$

If dimension m is odd, then the above formulæ are valid for all natural values of ℓ. However if dimension m is even, then these expressions are only valid for $\ell < m/2$; this already becomes clear from the fundamental solution E_m of the operator $\underline{\partial}^m$ which is logarithmic in nature:

$$\underline{\partial}^m E_m = \underline{\partial}^m \left(-\frac{1}{2^{m-1}\pi^{m/2}\Gamma(m/2)}\ln r\right) = \delta(\underline{x}), \quad m \text{ even}\,.$$

More generally, it holds for all $k \in \mathbb{N}$ and still for m even, that, see [13],

$$E_{m+2k-1} = (p_{2k-1}\ln r + q_{2k-1})\,\frac{\pi^{\frac{m}{2}+k}}{\Gamma\left(\frac{m}{2}+k\right)} U_{2k-1} \tag{40}$$

$$E_{m+2k} = (p_{2k}\ln r + q_{2k})\,\frac{\pi^{\frac{m}{2}+k}}{\Gamma\left(\frac{m}{2}+k\right)} T_{2k} \tag{41}$$

the constants $(p_{2k-1},\ q_{2k-1})$ and $(p_{2k},\ q_{2k})$ satisfying the recurrence relations

$$\begin{cases} p_{2k} = \dfrac{1}{2k}\, p_{2k-1} \\[2ex] q_{2k} = \dfrac{1}{2k}\left(q_{2k-1} - \dfrac{1}{2k}\, p_{2k-1}\right) \end{cases}$$

and

$$\begin{cases} p_{2k+1} = -\dfrac{1}{2\pi}\, p_{2k} \\[2ex] q_{2k+1} = -\dfrac{1}{2\pi}\left(q_{2k} - \dfrac{1}{m+2k}\, p_{2k}\right) \end{cases}$$

with initial values

$$p_0 = -\frac{1}{2^{m-1}\,\pi^m}, \quad q_0 = 0\,.$$

Through the signum-pair of operators $(\Delta,\ \mathbf{Z})$ or, equivalently, by iterated action of operator $\underline{D}$, we obtain the corresponding formulæ for the operator $\mathbf{Z}$ acting between signum-distributions:

$$\mathbf{Z}^s U_\lambda = \lambda\,(\lambda+m-2)\,^s U_{\lambda-2}, \quad \lambda \neq -m+2,\ -m,\ -m-2,\ \ldots \tag{42}$$

$$\mathbf{Z}^s T_\lambda = (\lambda-1)(\lambda+m-1)\,^s T_{\lambda-2}, \quad \lambda \neq -m+1,\ -m-1,\ \ldots \tag{43}$$

and

$$\mathbf{Z}^s U_{-m-2\ell} = (m+2\ell)(2\ell+2)\,^s U_{-m-2\ell-2} + (-1)^\ell\, a_m\, \frac{(m+4\ell+2)(m+2\ell+2)}{C(m,\ \ell+1)}\, \underline{\omega}\,\underline{\partial}^{2\ell+2}\,\delta(\underline{x}) \tag{44}$$

$$\mathbf{Z}^s T_{-m-2\ell+1} = (m+2\ell)(2\ell)\,^s T_{-m-2\ell-1} + (-1)^{\ell+1}\, a_m\, \frac{m+4\ell}{C(m,\ \ell)}\, \underline{\omega}\,\underline{\partial}^{2\ell+1}\,\delta(\underline{x}) \tag{45}$$

and, in particular, for $\ell = 0$

$$\mathbf{Z}^s U_{-m} = 2m\,^s U_{-m-2} + a_m\, \frac{m+2}{2m}\, \underline{\omega}\,\underline{\partial}^2\,\delta(\underline{x}) \tag{46}$$

$$\mathbf{Z}^s T_{-m+1} = a_m\, \partial_r\, \delta(\underline{x}) \tag{47}$$

and also

$$\mathbf{Z}^s U_{-m+2} = -(m-2)\, a_m\, \underline{\omega}\, \delta(\underline{x})\,. \tag{48}$$

22. Negative integer powers of the Dirac operator

Firstly let us concentrate on operator $\underline{\partial}^{-1}$. Consider test function $\varphi(\underline{x}) \in \mathscr{D}(\mathbb{R}^m)$; it is a well known result in Clifford analysis that the so-called T-operator, given by

$$T(\varphi)(\underline{x}) = \int_{\mathbb{R}^m} E_1(\underline{x}-\underline{y})\, \varphi(\underline{y})\, d\underline{y}$$

with

$$E_1(\underline{x}) = -\frac{1}{a_m}\frac{\underline{x}}{r^m} = -\frac{1}{a_m} U_{-m+1}$$

the fundamental solution to the Dirac operator, see (30), is an inverse operator to Dirac operator $\underline{\partial}$. We could as well have written:

$$\underline{\partial}^{-1}\, \varphi(\underline{x}) = E_1 \star \varphi\,(\underline{x})$$

from which it easily follows that indeed

$$\underline{\partial}\left(\underline{\partial}^{-1}\, \varphi(\underline{x})\right) = \underline{\partial}\, E_1 \star \varphi\,(\underline{x}) = \delta \star \varphi\,(\underline{x}) = \varphi\,(\underline{x})\,.$$

Quite naturally $\underline{\partial}^{-1}\, \delta(\underline{x})$ is not uniquely determined, since Ker $\underline{\partial} \neq \{0\}$, but, instead, consists of all entire monogenic functions in $\mathbb{R}^m$, which compels us to make a choice. Whence the following definition of $\underline{\partial}^{-1}\, \delta(\underline{x})$.

Definition 30 For all test functions $\varphi(\underline{x}) \in \mathscr{D}(\mathbb{R}^m)$ one defines

$$\langle\, \underline{\partial}^{-1}\, \delta(\underline{x}),\ \varphi(\underline{x})\,\rangle = -\langle\, \delta(\underline{x}),\ E_1 \star \varphi\,(\underline{x})\,\rangle\,.$$

It follows at once that

$$
\begin{aligned}
\langle\, \underline{\partial}^{-1}\, \delta(\underline{x}),\ \varphi(\underline{x})\,\rangle &= -\langle\, \delta(\underline{x}),\ E_1 \star \varphi\,(\underline{x})\,\rangle \\
&= -\int_{\mathbb{R}^m} E_1(-\underline{y})\,\varphi(\underline{y})\,d\underline{y} \\
&= \int_{\mathbb{R}^m} E_1(\underline{y})\,\varphi(\underline{y})\,d\underline{y} \\
&= \langle E_1(\underline{x}),\ \varphi(\underline{x})\,\rangle
\end{aligned}
$$

in other words: $\underline{\partial}^{-1}\,\delta(\underline{x})$ is defined to be the regular distribution $E_1(\underline{x})$. This inspires the following definition.

Definition 31 For each $n \in \mathbb{N}$, distribution $\underline{\partial}^{-n}\delta(\underline{x})$ is the regular distribution E_n, E_n being the fundamental solution of the operator $\underline{\partial}^n$.

In Section 21 we have established the explicit formulæ (38), (39), (40) and (41) for those fundamental solutions E_n. Note that, for each $n \in \mathbb{N}$, it indeed holds that:

$$\underline{\partial}^n\left(\underline{\partial}^{-n}\delta(\underline{x})\right) = \underline{\partial}^n E_n(\underline{x}) = \delta(\underline{x})\,.$$

Now recall the signum-partner $\underline{D}$ to the Dirac operator:

$$\underline{D} = \underline{\omega}\,\underline{\partial}\,(-\underline{\omega}) = \underline{\omega}\,\partial_r - \frac{1}{r}\,\partial_{\underline{\omega}} + (m-1)\,\frac{1}{r}\,\underline{\omega}\,.$$

It was shown in [6] that

$$\underline{D}\,{}^sT_{-m+1} = a_m\,\underline{\omega}\,\delta(\underline{x})$$

whence the following definition.

Definition 32 Signumdistribution $\underline{D}^{-1}(\underline{\omega}\,\delta(\underline{x}))$ is the regular signumdistribution $\underline{\omega}\,E_1(\underline{x}) = \dfrac{1}{a_m}\,{}^sT_{-m+1}$. More generally, for each $n \in \mathbb{N}$, signumdistribution $\underline{D}^{-n}(\delta(\underline{x}))$ is the regular signumdistribution $\underline{\omega}\,E_n(\underline{x})$.

Definitions 31 and 32 correspond to each other through the following commutative diagram:

$$
\begin{array}{ccc}
\delta(\underline{x}) & \xrightarrow{\ \underline{\partial}^{-n}\ } & E_n(\underline{x}) \\
{\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \nearrow^{-r}\ \searrow_{r} & {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} \\
\underline{\omega}\,\delta(\underline{x}) & \xrightarrow[\ \underline{D}^{-n}\ ]{} & \underline{\omega}\,E_n(\underline{x})
\end{array}
$$

with

$$\underline{D}^{-n} = \underline{\omega}\, \underline{\partial}^{-n} (-\underline{\omega})$$

Let us open a parenthesis here and wonder if it is possible to define negative entire powers of the Dirac operator acting on a more general distribution. Because, for each test function $\varphi(\underline{x}) \in \mathscr{D}(\mathbb{R}^m)$, it holds that $\underline{\partial}^{-1} \varphi = E_1 \star \varphi \notin \mathscr{D}(\mathbb{R}^m)$, it is clear at once that this will not be the case for the most general distribution. We propose the following definition.

Definition 33 For Dirac operator $\underline{\partial}$ one defines its inverse $\underline{\partial}^{-1}$ to be the convolution operator given by

$$\underline{\partial}^{-1} T = E_1 \star T$$

for all distributions T for which this convolution is meaningful.

Note that it holds indeed that

$$\underline{\partial}\, (\underline{\partial}^{-1} T) = \underline{\partial}\, E_1 \star T = \delta(\underline{x}) \star T = T.$$

Note also that Definition 33 applies to all tempered distributions $T \in \mathscr{S}'(\mathbb{R}^m)$.

More generally, we have the following definition.

Definition 34 For Dirac operator $\underline{\partial}$ one defines the inverse operators $\underline{\partial}^{-n}$, $n \in \mathbb{N}$ to be the convolution operators given by

$$\underline{\partial}^{-n} T = E_n \star T$$

for all distributions T for which this convolution is meaningful .

The following commutative diagram holds:

$$\begin{array}{ccc} T & \xrightarrow{\underline{\partial}^{-n}} & E_n \star T \\ {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \nearrow^{-r} \searrow_{r} & \uparrow\downarrow{\scriptstyle -\underline{\omega},\ \underline{\omega}} \\ \underline{\omega} T & \xrightarrow[\underline{D}^{-n}]{} & \underline{\omega} E_n \star T \end{array}$$

with

$$\underline{\partial}^n E_n = \delta(\underline{x})$$

and

$$\underline{D}^n(\underline{\omega} E_n) = \underline{\omega}\, \delta(\underline{x}).$$

Note that, when $\underline{\partial}^{-n} T$ is defined, we indeed have

$$\underline{\partial}^n (\underline{\partial}^{-n} T) = \underline{\partial}^n (E_n \star T) = (\underline{\partial}^n E_n) \star T = \delta(\underline{x}) \star T = T.$$

Note also that

$$\underline{\partial}^{-n} (\underline{\partial}^n \delta(\underline{x})) = E_n \star (\underline{\partial}^n \delta(\underline{x})) = (E_n \underline{\partial}^n) \star \delta(\underline{x}) = \delta(\underline{x})$$

which means that, if we drop the local integrability condition, we may qualify $E_{-n} = \underline{\partial}^n \delta(\underline{x})$ as the *fundamental solution* of the operator $\underline{\partial}^{-n}$, where the notation E_{-n} is introduced for symmetry reasons. In this way we obtain the following three sequences:

• the operators

$$\ldots,\ \underline{\partial}^{-n},\ \ldots,\ \underline{\partial}^{-1},\ \mathbf{1},\ \underline{\partial},\ \ldots,\ \underline{\partial}^{\mathbf{n}},\ \ldots$$

• their action on the delta distribution

$$\ldots,\ E_n,\ \ldots,\ E_1,\ E_0,\ E_{-1},\ \ldots,\ E_{-n},\ \ldots$$

• and their fundamental solutions

$$\ldots,\ E_{-n},\ \ldots,\ E_{-1},\ E_0,\ E_1,\ \ldots,\ E_n,\ \ldots$$

and so, in words: *the action of the operator $\underline{\partial}^k$, $k \in \mathbb{Z}$, on the delta distribution $\delta(\underline{x})$, equals the fundamental solution of the inverse operator $\underline{\partial}^{-k}$.*

23. Regularization of the distributions T_λ and U_λ

In Section 21 we saw that, when considered as functions of the complex parameter λ, distributions T_λ show simple poles at $\lambda = -m,\ -m-2,\ -m-4,\ \ldots$, whereas distributions U_λ show simple poles at $\lambda = -m-1,\ -m-3,\ -m-5,\ \ldots$. In these singular points the distributions T_λ and U_λ were defined through monomial pseudofunctions.

There is, however, a second option to cope with these singularities: removing the singularities by dividing distributions T_λ and U_λ by an appropriate Gamma-function, which gives rise to the so-called normalized distributions T_λ^* and U_λ^*. Their definition runs as follows.

The normalized distributions T_λ^* are defined by

$$\begin{cases} T^*_\lambda = \pi^{\frac{\lambda+m}{2}} \dfrac{T_\lambda}{\Gamma\left(\dfrac{\lambda+m}{2}\right)}, & \lambda \neq -m-2l \\ T^*_{-m-2l} = \dfrac{\pi^{\frac{m}{2}-l}}{2^{2l}\Gamma\left(\dfrac{m}{2}+l\right)} (-\Delta)^l \delta(\underline{x}), & l \in \mathbb{N}_0 \end{cases}$$

and we call *Riesz potential of the first kind* $\mathscr{P}^\gamma_T$, $\gamma \in \mathbb{C}$, the scalar valued convolution operator given by

$$\mathscr{P}^\gamma_T[f] = T^*_{\gamma-m} * f\,, \quad f \in \mathscr{S}\,.$$

For $\gamma \neq -2l$, $l \in \mathbb{N}_0$, we have more explicitly:

$$\mathscr{P}^\gamma_T[f] = \frac{\pi^{\frac{\gamma}{2}}}{\Gamma\left(\dfrac{\gamma}{2}\right)}\, Fp \int_{\mathbb{R}^m} |\underline{x}-\underline{y}|^{\gamma-m} f(\underline{y})\, d\underline{y} \tag{49}$$

whereas for $\gamma = -2l$, $l \in \mathbb{N}_0$ we have

$$\mathscr{P}^{-2l}_T[f] = \frac{\pi^{\frac{m}{2}-l}}{2^{2l}\,\Gamma\left(\dfrac{m}{2}+l\right)}\,(-\Delta)^l f = \frac{\pi^{\frac{m}{2}-l}}{2^{2l}\,\Gamma\left(\dfrac{m}{2}+l\right)}\,\underline{\partial}^{2l} f\,.$$

So

$$\mathscr{P}^\gamma_T[f] = \frac{2^\gamma \pi^{\frac{\gamma+m}{2}}}{\Gamma\left(\dfrac{m-\gamma}{2}\right)}\, I^\gamma[f]\,, \quad \gamma \neq m+2k,\ k \in \mathbb{N}_0,$$

where $I^\gamma[f]$ is the traditional Riesz potential.

Note that $\mathscr{P}^\gamma_T[f]$ is an entire function of γ, whereas $I^\gamma[f]$ shows simple poles at $\gamma = m+2k,\ k \in \mathbb{N}_0$.

The normalized distributions U^*_λ are defined by

$$\begin{cases} U^*_\lambda = \pi^{\frac{\lambda+m+1}{2}} \dfrac{U_\lambda}{\Gamma\left(\dfrac{\lambda+m+1}{2}\right)}, & \lambda \neq -m-2l-1 \\ U^*_{-m-2l-1} = -\dfrac{\pi^{\frac{m}{2}-l}}{2^{2l+1}\,\Gamma\left(\dfrac{m}{2}+l+1\right)}\, \underline{\partial}^{2l+1}\delta(\underline{x}), & l \in \mathbb{N}_0 \end{cases}$$

and we call *Riesz potential of the second kind* $\mathscr{P}^\gamma_U$, $\gamma \in \mathbb{C}$, the Clifford-vector valued convolution operator

$$\mathscr{P}_U^{\gamma}[f] = U^*_{\gamma-m} * f, \quad f \in \mathscr{S}.$$

For $\gamma \neq -2l-1$, $l = 0,\ 1,\ 2,\ \ldots$, we have more explicitly:

$$\mathscr{P}_U^{\gamma}[f] = \pi^{\frac{\gamma+1}{2}} \frac{U_{\gamma-m}}{\Gamma\left(\frac{\gamma+1}{2}\right)} * f$$

$$= \frac{\pi^{\frac{\gamma+1}{2}}}{\Gamma\left(\frac{\gamma+1}{2}\right)} Fp \int_{\mathbb{R}^m} |\underline{x}-\underline{y}|^{\gamma-m} (\underline{\omega} - \underline{\xi}) f(\underline{y})\, d\underline{y}$$

whereas for $\gamma = -2l-1$, $l = 0,\ 1,\ 2,\ \ldots$, we have

$$\mathscr{P}_U^{-2l-1}[f] = U^*_{-m-2l-1} * f = -\frac{\pi^{\frac{m}{2}-l}}{2^{2l+1}\Gamma\left(\frac{m}{2}+l+1\right)} \underline{\partial}^{2l+1} f.$$

We also put

$$\mathscr{P}_U^{\gamma}[f] = -\frac{2^{\gamma}\pi^{\frac{\gamma+m+1}{2}}}{\Gamma\left(\frac{m-\gamma+1}{2}\right)} J^{\gamma}[f], \quad \gamma \neq m+2k+1,\ k \in \mathbb{N}_0.$$

Note that $\mathscr{P}_U^{\gamma}[f]$ is an entire function of γ, whereas $J^{\gamma}[f]$ shows simple poles at $\gamma = m+2k+1,\ k \in \mathbb{N}_0$.

Normalized distributions T^*_{λ} and U^*_{λ} are holomorphic mappings from $\lambda \in \mathbb{C}$ to the space $\mathscr{S}'(\mathbb{R}^m)$ of tempered distributions. They are intertwined by the actions of multiplication operator $\underline{x}$ and of the Dirac operator, according to the following formulæ: for all $\lambda \in \mathbb{C}$ one has

(i) $\underline{x}\, T^*_{\lambda} = \dfrac{\lambda+m}{2\pi} U^*_{\lambda+1}$; $\underline{x}\, U^*_{\lambda} = U^*_{\lambda}\, \underline{x} = -T^*_{\lambda+1}$;

(ii) $\underline{\partial}\, T^*_{\lambda} = \lambda\, U^*_{\lambda-1}$; $\underline{\partial}\, U^*_{\lambda} = U^*_{\lambda}\, \underline{\partial} = -2\pi\, T^*_{\lambda-1}$;

(iii) $\Delta T^*_{\lambda} = 2\pi\lambda T^*_{\lambda-2}$; $\Delta U^*_{\lambda} = 2\pi(\lambda-1) U^*_{\lambda-2}$;

(iv) $\mathscr{F}\left[T^*_{\lambda}\right] = T^*_{-\lambda-m}$; $\mathscr{F}\left[U^*_{\lambda}\right] = -i\, U^*_{-\lambda-m}$.

For property (iv) recall from Section 2 the following definition of the Fourier transformation:

$$\mathscr{F}[f(\underline{x})](\underline{y}) = \int_{\mathbb{R}^m} f(\underline{x}) \exp\left(-2\pi i \langle \underline{x},\ \underline{y} \rangle\right) d\underline{x}.$$

24. Complex powers of the Dirac operator

In [8], complex powers $\underline{\partial}^{\mu}$, $\mu \in \mathbb{C}$, of the Dirac operator were defined as convolution operators acting on tempered distributions in the following way:

$$\underline{\partial}^{\mu} f = \frac{1+\exp(i\pi\mu)}{2} \, 2^{\mu} \, \frac{\Gamma\left(\frac{m+\mu}{2}\right)}{\pi^{\frac{m-\mu}{2}}} \, T^{*}_{-m-\mu} \star f - \frac{1-\exp(i\pi\mu)}{2} \, 2^{\mu} \, \frac{\Gamma\left(\frac{m+\mu+1}{2}\right)}{\pi^{\frac{m-\mu+1}{2}}} \, U^{*}_{-m-\mu} \star f, \quad f \in \mathscr{S}'(\mathbb{R}^m)$$

where T^*_λ and U^*_λ are normalized versions of distributions T_λ and U_λ respectively, introduced in Section 23.

In particular for delta distribution $\delta(\underline{x})$ it thus holds that

$$\underline{\partial}^{\mu} \, \delta(\underline{x}) = \frac{1+\exp(i\pi\mu)}{2} \, 2^{\mu} \, \frac{\Gamma\left(\frac{m+\mu}{2}\right)}{\pi^{\frac{m-\mu}{2}}} \, T^{*}_{-m-\mu} - \frac{1-\exp(i\pi\mu)}{2} \, 2^{\mu} \, \frac{\Gamma\left(\frac{m+\mu+1}{2}\right)}{\pi^{\frac{m-\mu+1}{2}}} \, U^{*}_{-m-\mu} \, . \tag{50}$$

Clearly expression (50) is valid for most complex values of the parameter μ, *but not for all*. Firstly, let us verify that for natural values of μ we indeed recover the natural powers of $\underline{\partial}$ acting on $\delta(\underline{x})$.

For $\mu = 2\ell+1, \ \ell = 0, \ 1, \ 2, \ \ldots$ the right-hand side of expression (50) takes the form

$$-\, 2^{2\ell+1} \, \frac{\Gamma\left(\frac{m+2\ell+2}{2}\right)}{\pi^{\frac{m-2\ell}{2}}} \, U^{*}_{-m-2\ell-1}$$

which indeed reduces to $\underline{\partial}^{2\ell+1} \, \delta(\underline{x}) = E_{-2\ell-1}$.

For $\mu = 2\ell, \ \ell = 0, \ 1, \ 2, \ \ldots$ the right-hand side of expression (50) takes the form

$$2^{2\ell} \, \frac{\Gamma\left(\frac{m+2\ell}{2}\right)}{\pi^{\frac{m-2\ell}{2}}} \, T^{*}_{-m-2\ell}$$

which reduces to $(-\Delta)^{\ell} \, \delta(\underline{x}) = \underline{\partial}^{2\ell} \, \delta(\underline{x}) = E_{-2\ell}$.

Now we focus on the negative entire values of the parameter μ.

For $\mu = -2\ell - 1$ expression (50) takes the form

$$\underline{\partial}^{-2\ell-1} \, \delta(\underline{x}) = -\, 2^{-2\ell-1} \, \frac{\Gamma\left(\frac{m-2\ell}{2}\right)}{\pi^{\frac{m+2\ell+1}{2}}} \, U^{*}_{-m+2\ell+1}$$

$$= -\, \frac{1}{2^{\ell} \, \ell! \, (m-2) \cdots (m-2\ell)} \, \frac{1}{a_m} \, U_{-m+2\ell+1}$$

in which we recognize fundamental solution $E_{2\ell+1}$ of operator $\underline{\partial}^{2\ell+1}$.

For $\mu = -2\ell - 2$ expression (50) takes the form

$$\underline{\partial}^{-2\ell-2}\,\delta(\underline{x}) = 2^{-2\ell-2}\,\frac{\Gamma\left(\dfrac{m-2\ell-2}{2}\right)}{\pi^{\frac{m+2\ell+2}{2}}}\,T^{*}_{-m+2\ell+2}$$

$$= \frac{1}{2^{\ell}\,\ell!\,(m-2)\cdots(m-2\ell-2)}\,\frac{1}{a_m}\,T_{-m+2\ell+2}$$

in which we recognize fundamental solution $E_{2\ell+2}$ of operator $\underline{\partial}^{2\ell+2}$.

At least these are correct statements for all natural values of ℓ as long as dimension m is odd; if dimension m is even, then the above statements are correct as long as $\ell < m/2$.

Because the right-hand side of expression (50) is well-defined for all complex, non-integer values of parameter μ, it follows that the complex powers $\underline{\partial}^{\mu}$ of the Dirac operator acting on delta distribution $\delta(\underline{x})$ are well-defined by (50) either

(i) for all $\mu \in \mathbb{C}$ when dimension m is *odd*, or

(ii) for all $\mu \in \mathbb{C}\backslash\{-m,\ -m-1,\ -m-2,\ \ldots\}$ when dimension m is *even*.

We complete the definition of $\underline{\partial}^{\mu}\,\delta(\underline{x})$ by putting

$$\underline{\partial}^{-m-k}\,\delta(\underline{x}) = E_{m+k}, \quad m \text{ even}, \quad k = 0,\ 1,\ 2,\ \ldots$$

where E_{m+k} is the fundamental solution to operator $\underline{\partial}^{m+k}$ given by (40) and (41).

Now we prove that $\underline{\partial}^{-\mu}\,\delta(\underline{x})$ is the fundamental solution of operator $\underline{\partial}^{\mu}$. In fact this was already proven in Section 22 for all integer values of μ. So we may restrict ourselves now to the case where $\mu \in \mathbb{C}\backslash\mathbb{Z}$. We find, making use of the convolution formulæ established in [13],

$$\begin{aligned}
\underline{\partial}^{\mu}\,(\underline{\partial}^{-\mu}\,\delta(\underline{x})) = {} & \frac{1+\exp(i\pi\mu)}{2}\,2^{\mu}\,\frac{\Gamma\left(\dfrac{m+\mu}{2}\right)}{\pi^{\frac{m-\mu}{2}}}\,T^{*}_{-m-\mu} \star \frac{1+\exp(-i\pi\mu)}{2}\,2^{-\mu}\,\frac{\Gamma\left(\dfrac{m-\mu}{2}\right)}{\pi^{\frac{m+\mu}{2}}}\,T^{*}_{-m+\mu} \\
& + \frac{1-\exp(i\pi\mu)}{2}\,2^{\mu}\,\frac{\Gamma\left(\dfrac{m+\mu+1}{2}\right)}{\pi^{\frac{m-\mu+1}{2}}}\,U^{*}_{-m-\mu} \star \frac{1-\exp(-i\pi\mu)}{2}\,2^{-\mu}\,\frac{\Gamma\left(\dfrac{m-\mu+1}{2}\right)}{\pi^{\frac{m+\mu+1}{2}}}\,U^{*}_{-m+\mu} \\
& - \frac{1+\exp(i\pi\mu)}{2}\,2^{\mu}\,\frac{\Gamma\left(\dfrac{m+\mu}{2}\right)}{\pi^{\frac{m-\mu}{2}}}\,T^{*}_{-m-\mu} \star \frac{1-\exp(-i\pi\mu)}{2}\,2^{-\mu}\,\frac{\Gamma\left(\dfrac{m-\mu+1}{2}\right)}{\pi^{\frac{m+\mu+1}{2}}}\,U^{*}_{-m+\mu} \\
& - \frac{1-\exp(i\pi\mu)}{2}\,2^{\mu}\,\frac{\Gamma\left(\dfrac{m+\mu+1}{2}\right)}{\pi^{\frac{m-\mu+1}{2}}}\,U^{*}_{-m-\mu} \star \frac{1+\exp(-i\pi\mu)}{2}\,2^{-\mu}\,\frac{\Gamma\left(\dfrac{m-\mu}{2}\right)}{\pi^{\frac{m+\mu}{2}}}\,T^{*}_{-m+\mu}
\end{aligned}$$

or

$$\underline{\partial}^{\mu}\,(\underline{\partial}^{-\mu}\,\delta(\underline{x})) = \frac{1+\exp(i\pi\mu)}{2}\,2^{\mu}\,\frac{\Gamma\left(\frac{m+\mu}{2}\right)}{\pi^{\frac{m-\mu}{2}}}\,\frac{1+\exp(-i\pi\mu)}{2}\,2^{-\mu}\,\frac{\Gamma\left(\frac{m-\mu}{2}\right)}{\pi^{\frac{m+\mu}{2}}}\,\pi^{m/2}\,\frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m+\mu}{2}\right)\Gamma\left(\frac{m-\mu}{2}\right)}\,T^{*}_{-m}$$

$$+\frac{1-\exp(i\pi\mu)}{2}\,2^{\mu}\,\frac{\Gamma\left(\frac{m+\mu+1}{2}\right)}{\pi^{\frac{m-\mu+1}{2}}}\,\frac{1-\exp(-i\pi\mu)}{2}$$

$$2^{-\mu}\,\frac{\Gamma\left(\frac{m-\mu+1}{2}\right)}{\pi^{\frac{m+\mu+1}{2}}}\,\frac{2\pi}{m}\,\pi^{m/2}\,\frac{\Gamma\left(\frac{m}{2}+1\right)}{\Gamma\left(\frac{m+\mu+1}{2}\right)\Gamma\left(\frac{m-\mu+1}{2}\right)}\,T^{*}_{-m}$$

$$-\frac{1+\exp(i\pi\mu)}{2}\,2^{\mu}\,\frac{\Gamma\left(\frac{m+\mu}{2}\right)}{\pi^{\frac{m-\mu}{2}}}\,\frac{1-\exp(-i\pi\mu)}{2}$$

$$2^{-\mu}\,\frac{\Gamma\left(\frac{m-\mu+1}{2}\right)}{\pi^{\frac{m+\mu+1}{2}}}\,\pi^{m/2}\,\frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m-\mu+1}{2}\right)\Gamma\left(\frac{m+\mu}{2}\right)}\,U^{*}_{-m}$$

$$-\frac{1-\exp(i\pi\mu)}{2}\,2^{\mu}\,\frac{\Gamma\left(\frac{m+\mu+1}{2}\right)}{\pi^{\frac{m-\mu+1}{2}}}\,\frac{1+\exp(-i\pi\mu)}{2}$$

$$2^{-\mu}\,\frac{\Gamma\left(\frac{m-\mu}{2}\right)}{\pi^{\frac{m+\mu}{2}}}\,\pi^{m/2}\,\frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m+\mu+1}{2}\right)\Gamma\left(\frac{m-\mu}{2}\right)}\,U^{*}_{-m}$$

or still

$$\underline{\partial}^{\mu}\,(\underline{\partial}^{-\mu}\,\delta(\underline{x})) = \frac{\Gamma\left(\frac{m}{2}\right)}{\pi^{m/2}}\,T^{*}_{-m}$$

or finally

$$\underline{\partial}^{\mu}\,(\underline{\partial}^{-\mu}\,\delta(\underline{x}))\;=\;\delta(\underline{x})\,.$$

Conflict of interest

The author declares no competing financial interest.

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Appendix: Spherical means

Let $\phi(\underline{x})$ be a scalar test function in $\mathbb{R}^m$. Introduce spherical coordinates $\underline{x} = r\,\underline{\omega}$, $r = |\underline{x}|$, $\underline{\omega} \in S^{m-1}$ and denote by a_m the area of the unit sphere S^{m-1}.

Definition 35 Spherical means $\Sigma^{(0)}[\phi]$ and $\Sigma^{(1)}[\phi]$ are defined by

$$\Sigma^{(0)}[\phi](r) = \frac{1}{a_m}\int_{S^{m-1}} \phi(r\,\underline{\omega})\,dS_{\underline{\omega}}$$

and

$$\Sigma^{(1)}[\phi](r) = \frac{1}{a_m}\int_{S^{m-1}} \underline{\omega}\,\phi(r\,\underline{\omega})\,dS_{\underline{\omega}}\,.$$

Spherical mean $\Sigma^{(0)}[\phi]$ is a classical concept. It is a scalar function of radial distance r for which

$$\Sigma^{(0)}[\phi](0) = \phi(0)\,.$$

It can be defined for $r < 0$ through an even extension.

Spherical mean $\Sigma^{(1)}[\phi]$ is a Clifford vector valued function for which

$$\Sigma^{(1)}[\phi](0) = 0\,.$$

It can be defined for $r < 0$ through an odd extension.

Further properties of the spherical means are listed in the following sequence of lemmata, see also [11].

Lemma 5 One has

$$\Sigma^{(0)}[\underline{\omega}\,\phi] = \Sigma^{(1)}[\phi]$$

and

$$\Sigma^{(1)}[\underline{\omega}\,\phi] = -\,\Sigma^{(0)}[\phi]\,.$$

Lemma 6 One has

$$\Sigma^{(0)}[\partial_{\underline{\omega}}\,\phi] = (m-1)\,\Sigma^{(1)}[\phi]$$

and

$$\Sigma^{(1)}[\partial_{\underline{\omega}}\,\phi] = 0\,.$$

Lemma 7 One has

$$\Sigma^{(0)}[\underline{\partial}\,\phi] = \left(\partial_r + (m-1)\,\frac{1}{r}\right)\Sigma^{(1)}[\phi]$$

and

$$\Sigma^{(1)}[\underline{\partial}\,\phi] = -\,\partial_r\,\Sigma^{(0)}[\phi]\,.$$

Lemma 8 One has

$$\{\partial_r^{2\ell}\,\Sigma^0[\varphi](r)\}|_{r=0} = (-1)^{\ell}\,\frac{(2\ell)!(m+2\ell)}{C(m,\ \ell)}\,\{\underline{\partial}^{2\ell}\,\varphi(\underline{x})\}|_{\underline{x}=0}$$

$$\{\partial_r^{2\ell+1}\,\Sigma^0[\varphi](r)\}|_{r=0} = 0$$

with

$$C(m,\ \ell) = 2^{\ell}\,\ell!\,m(m+2)\cdots(m+2\ell)\,.$$

Lemma 9 One has

$$\{\partial_r^{2\ell}\,\Sigma^1[\varphi](r)\}|_{r=0} = 0$$

$$\{\partial_r^{2\ell+1}\,\Sigma^1[\varphi](r)\}|_{r=0} = (-1)^{\ell}\,\frac{(2\ell+1)!}{C(m,\ \ell)}\,\{\underline{\partial}^{2\ell+1}\,\varphi(\underline{x})\}|_{\underline{x}=0}\,.$$