Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

On Existence of Fixed Points for Multivalued Contractive Type Mappings in Quasi-Metric Spaces

Abdul Latif[1]*, Nadiah Zafer Al-shehri[2], Monairah Omar Alansari[3]

[1] Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
[2] Department of Mathematics, King Khalid University, P.O. Box 960, Abha 61421, Saudi Arabia
[3] Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21598, Saudi Arabia
E-mail: alatif@kau.edu.sa

Received: 7 April 2024; **Revised:** 23 May 2024; **Accepted:** 6 June 2024

Abstract: We present fixed point results for multivalued contractive type mappings via Q-function on quasi-metric spaces. Our main results supported with example. Consequently, our results either improve or generalize several known results of metric fixed point theory.

Keywords: quasi-metric space, fixed points, multivalued mappings, Q-function, $\Omega[0, F]$

MSC: 47H09, 54H25

1. Introduction

Applying the Hausdorff-Pompeiu metric, Nadler [1] established a multivalued version of the well-known Banach Contraction Principle (BCP) [2]. Due to this valuable contribution to the metric fixed point theory, several useful generalizations appeared in the literature, see [3–5] and references therein. On the other hand, without using the Hausdorff-Pompeiu metric, the existence part have been studied by several researchers of the area, see [6–8] and others.

In [9], Kada et al. introduced a notion of w-distance on metric spaces and improved several results replacing the involved metric by a generalized distance. While, Suzuki and Takahashi [10] introduced notions of single-valued and multivalued weakly contractive mappings via w-distance and proved fixed point results for such mappings without using the Hausdorff-Pompeiu metric. Consequently, they improved the Banach contraction principle and the Nadler fixed point theorem. Much work has been done in this direction, see; [11–13] and others. On the other hand, Alhomidan et al. [14] introduced a notion of Q-function on quasi metric spaces, which is a generalization of the w-distance and then they improved a number of known fixed point results via Q-function.

In this paper, first we present some useful notions and facts followed by existing related fixed point results for multivalued mappings. In section 2, we establish some new fixed point results for contractive type multivalued mappings via Q-function in the framework of quasi-metric spaces. In support of our main results, an example is also provided. Finally, we conclude that our results either improve or generalize many known fixed point results including the corresponding results due to Latif and Abdou [12, 13], Pathak and Shahzad [15], Ćirić [16] and Klim and Wardowski [17].

We start with some related notions, facts and results.

DOI: https://doi.org/10.37256/cm.5420244704

Let S be a metric space with metric d. Let 2^S be a collection of nonempty subsets of S, $Cl(S)$ a collection of nonempty closed subsets of S, $CB(S)$ a collection of nonempty closed bounded subsets of S and $\mathbb{R}^+ = [0, +\infty)$. An element $s \in S$ is called a fixed point of a multivalued mapping $\Gamma : S \to 2^S$ if $s \in \Gamma(s)$. We denote $Fix(\Gamma) = \{s \in S : s \in \Gamma(s)\}$. A sequence $\{s_n\}$ in S is called an orbit of Γ at $s_0 \in S$ if $s_n \in \Gamma(s_{n-1})$ for all $n \geq 1$. A real-valued function β on S is called lower semicontinuous if for any sequence $\{s_n\} \subset S$ with $s_n \to s \in S$ one has that $\beta(s) \leq \liminf_{n\to\infty} \beta(s_n)$.

In the sequel till Theorem 6, we consider (S, d) a complete metric space.

In [1], Nadler established a multivalued case of the (BCP) as follows.

Theorem 1 [1] Let $\Gamma : S \to CB(S)$ be a multivalued mapping such that for a fixed constant $h \in (0, 1)$ and for any $s_1, s_2 \in S$,

$$H(\Gamma(s_1), \Gamma(s_2)) \leq h\, d(s_1, s_2), \tag{1}$$

where H is the Hausdorff-Pompeiu metric on $CB(S)$. Then, $Fix(\Gamma) \neq \emptyset$.

In [5], Mizoguchi and Takahashi generalized Theorem 1 as follows.

Theorem 2 [5] Let $\Gamma : S \to CB(S)$ be a multivalued mapping such that for any $s_1, s_2 \in S$,

$$H(\Gamma(s_1), \Gamma(s_2)) \leq \chi(d(s_1, s_2))d(s_1, s_2), \tag{2}$$

where χ is a function from $\mathbb{R}^+$ to $[0, 1)$ with $\limsup_{r\to t^+} \chi(r) < 1$, for all $t \in \mathbb{R}^+$. Then, $Fix(\Gamma) \neq \emptyset$.

Excluding the Hausdorff-Pompeiu metric, Feng and Liu [8] proved the following result, which also extends Theorem 1.

Theorem 3 [8] Let $\Gamma : S \to Cl(S)$ with a lower semicontinuous function β on S given by $\beta(s) = d(s, \Gamma(s))$. Then, $Fix(\Gamma)$ is non-empty. If with provided are constants $c, h \in (0, 1)$, $c > h$, and for any $s_1 \in S$, there is $s_2 \in I_c^{s_1}$ with

$$d(s_2, \Gamma(s_2)) \leq hd(s_1, s_2), \tag{3}$$

where $I_c^{s_1} = \{s_2 \in \Gamma(s_1) : cd(s_1, s_2) \leq d(s_1, \Gamma(s_1))\}$.

Later, Theorem 3 generalized in [17] as under.

Theorem 4 [17] Let $\Gamma : S \to Cl(S)$ with a lower semicontinuous function β on S given by $\beta(s) = d(s, \Gamma(s))$. Then, $Fix(\Gamma)$ is non-empty, provided there is some $c \in (0, 1)$ such that for any $s_1 \in S$, there is $s_2 \in \Gamma(s_1)$ satisfying

$$\begin{aligned} &cd(s_1, s_2) \leq d(s_1, \Gamma(s_1)), \\ &d(s_2, \Gamma(s_2)) \leq \chi(d(s_1, s_2))d(s_1, s_2), \end{aligned} \tag{4}$$

where χ is a function from $\mathbb{R}^+$ to $[0, c)$ with $\limsup_{r\to t^+} \chi(r) < c$, for all $t \in \mathbb{R}^+$.

Kada et al. [9] introduced a concept of w-distance on metric spaces as follows.

Let (S, d) be a metric space. A function $w : S \times S \to \mathbb{R}^+$ is called w-distance on S if it satisfies, for any $s_1, s_2, s_3 \in S$, the following.

(i) $w(s_1, s_3) \leq w(s_1, s_2) + w(s_2, s_3)$;

(ii) for any $s \in S$, a function $w(s, \cdot) : S \to \mathbb{R}^+$ is lowersemicontinuous;

(iii) for any $\varepsilon > 0$, there is $\delta > 0$ with $w(s_3, s_1) \leq \delta$ and $w(s_3, s_2) \leq$ imply $d(s_1, s_2) \leq \varepsilon$.

Using w-distance, they proved several results via w-distance. It is obvious that for any $s_1, s_2 \in S$, $w(s_1, s_2) \neq w(s_2, s_1)$, and not either of the implications $w(s_1, s_2) = 0$ if and only if $s_1 = s_2$ necessarily hold. Clearly, the metric d is a w-distance on S. Examples and properties of the w-distance (see [9, 10]). Without using the Hausdorff-Pompeiu metric, Suzuki and Takahashi [10] established some metric fixed point results for contractive type mappings with respect to w-distance. They generalized Theorem 1 as follows.

Theorem 5 [10] Let $\Gamma : S \to Cl(S)$ be a multivalued mapping. If there exists a w-distance w on S and a constant $\lambda \in (0, 1)$ such that for any $s_1, s_2 \in S$ and $u \in \Gamma(s_1)$, there is $v \in \Gamma(s_2)$ satisfying

$$w(u, v) \leq \lambda w(s_1, s_2). \tag{5}$$

Then, there exists $s_0 \in S$ such that $s_0 \in \Gamma(s_0)$ and $w(s_0, s_0) = 0$.

While, Latif and Abdou [13] generalized Theorem 4 as follows.

Theorem 6 [13] Let $\Gamma : S \to Cl(S)$ be a multivalued mapping. If there exists a w-distance w on S such that a real-valued function β on S with $\beta(s) = w(s, \Gamma(s))$ is lower semicontinuous. Then, $Fix(\Gamma) \neq \emptyset$ provided there exists $c \in (0, 1)$ such that for any $s_1 \in S$ there is $s_2 \in \Gamma(s_1)$ satisfying

$$cw(s_1, s_2) \leq w(s_1, \Gamma(s_1)),$$

$$w(s_2, \Gamma(s_2)) \leq \chi(w(s_1, s_2))w(s_1, s_2), \tag{6}$$

where χ is a function from $\mathbb{R}^+$ to $[0, c)$ with $\limsup_{r \to t^+} \chi(r) < c$, for all $t \in \mathbb{R}^+$.

Now, let us recall the well-known generalization of the standard metric, known as quasi-metric, see [18] and others.

A quasi-metric on a nonempty set S is a function $\Lambda : S \times S \to \mathbb{R}^+$ if it satisfies, for any $s_1, s_2, s_3 \in S$, the following.

(i) $\Lambda(s_1, s_2) = 0$ if and only if $s_1 = s_2$,

(ii) $\Lambda(s_1, s_2) \leqslant \Lambda(s_1, s_3) + \Lambda(s_3, s_2)$.

The pair (S, Λ) is called a quasi-metric space.

Note that every metric space is a quasi-metric space. For more examples of quasi-metric spaces and related fixed point results, see; [19–26]. The concepts of Cauchy sequences, convergent sequences, and completeness in the frame work of quasi-metric spaces can be defined in the same manner as in the setting of metric spaces, see; [14]. That is; here we follow the technique of the paper [14] rather the usual technique of left/right Cauchy (convergent) sequences.

Al-Homidan et al. [14] introduced a concept of Q-function on quasi-metric spaces as follows.

A Q-function on a quasi-metric space (S, Λ) is a function $q : S \times S \to \mathbb{R}^+$ if it satisfies, for any $s_1, s_2, s_3 \in S$, the following:

(i) $q(s_1, s_2) \leqslant q(s_1, s_3) + q(s_3, s_2)$,

(ii) If $\{s_n\}$ is a sequence in S such that $s_n \to s \in S$ and $q(s_1, s_n) \leqslant M$ for some $M = M(s_1) > 0$, then $q(s_1, s) \leqslant M$;

(iii) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $q(s_3, s_1) \leqslant \delta$ and $q(s_3, s_2) \leqslant \delta$ imply $\Lambda(s_1, s_2) \leqslant \varepsilon$.

In the setting of metric space, if we replace (q_2) with the condition of lower semi-continuity of the map $q(s, .) : S \to \mathbb{R}^+$, then the Q-function reduces to w-distance. It has been observed that every w-distance is a Q-function, but the converse may not be true, see; [14]. It is also worth to mention that the concepts of a Q-function and a quasi-metric are not comparable, see; [14, Example 3.1 and Example 3.2]. Each discrete metric on quasi-metric space (S, Λ) is a Q-function. For other examples of Q-functions, see; [24]. In [14], Al-Homidan et al. studied a number of important fixed point results via Q-function.

Using the technique as in [27], the following result is obvious.

Lemma 1 Let N be a closed subset of a quasi-metric space $(S,\ \Lambda)$ and q be a Q-function on S. Suppose that there exists $s_1 \in S$ such that $q(s_1,\ s_1)=0$. Then, $q(s_1,\ N)=0$ if and only if $s_1 \in N$, where $q(s_1,\ N)=\inf\{q(s_1,\ s_2): s_2 \in N\}$.

The following result is an analog of the Lemma [9, Lemma 2.6], stated and used in [14, 28].

Lemma 2 [14, 28] Let $(S,\ \Lambda)$ be a quasi-metric space and q be a Q-function on S. Let $\{s_n\}$ and $\{s_n^{'}\}$ be sequences in S, let $\{\alpha_n\}$ and $\{\gamma_n\}$ be sequences in $\mathbb{R}^+$ converging to zero. Then, for any $s_1,\ s_2,\ s_3 \in S$, the following hold.

(i) if $q(s_n,\ s_2) \le \alpha_n$ and $q(s_n,\ s_3) \le \beta_n$ for any $n \in \mathbb{N}$, then $s_2 = s_3$. In particular, if $q(s_1,\ s_2)=0$ and $q(s_1,\ s_3)=0$, then $s_2 = s_3$;

(ii) if $q(s_n,\ s_n^{'}) \le \alpha_n$ and $q(s_n,\ s_3) \le \gamma_n$ for any $n \in \mathbb{N}$, then $\Lambda(s_n^{'},\ s_3) \to 0$;

(iii) if $q(s_n,\ s_m) \le \alpha_n$ for any $n,\ m \in \mathbb{N}$ with $m > n$, then $\{s_n\}$ is a Cauchy sequence;

(iv) if $q(s_2,\ s_n) \le \alpha_n$ for any $n \in \mathbb{N}$, then $\{s_n\}$ is a Cauchy sequence.

In [14], Al-Homidan et al. generalized Theorem 1 with respect to Q-function.

Theorem 7 [14] Let $\Gamma: S \to Cl(S)$, where $(S,\ \Lambda)$ is a complete quasi-metric space. If there exists a Q-function q on S and a constant $\lambda \in (0,\ 1)$, such that for any $s_1,\ s_2 \in S$ and $u \in \Gamma(s_1)$, there is $v \in \Gamma(s_2)$ satisfying

$$q(u,\ v) \le \lambda\ q(s_1,\ s_2). \tag{7}$$

Then, there exists $s_0 \in S$ such that $s_0 \in \Gamma(s_0)$ and $q(s_0,\ s_0)=0$.

To see further results in this area, we refer [19, 20, 24] and references therein.

Let $F \in (0,\ +\infty]$. Let $\xi : [0,\ F) \to \mathbb{R}$ satisfy that

(i) $\xi(0)=0$ and $\xi(t)>0$ for all $t \in (0,\ F)$;

(ii) ξ is non-decreasing on $[0,\ F)$;

(iii) ξ is sub-additive; that is, $\xi(t_1+t_2) \le \xi(t_1)+\xi(t_2)$, for all $t_1,\ t_2 \in (0,\ F)$.

We consider $\Omega[0,\ F)=\{\xi : \xi$ satisfies (i)-(iii) above$\}$.

Note that for any Q-function q, $\xi \circ q$ is also a Q-function [14, Remark 2.2].

(i) It follows from property (ii) of ξ, for each $t_1,\ t_2 \in (0,\ F)$;

$$\xi(t_1) < \xi(t_2) \text{ imply } t_1 < t_2. \tag{8}$$

(ii) If $\xi \in \Omega[0,\ F)$ and ξ is continuous at 0, then ξ is continuous at each point of $[0,\ F)$, see [12, 29].

(iii) If $\xi \in \Omega[0,\ F)$ and $\{\alpha_n\}$ in $[0,\ F)$ is any sequence with $\lim_{n\to\infty} \xi(\alpha_n)=0$, then $\lim_{n\to\infty} \alpha_n = 0$.

For a quasi-metric space $(S,\ D)$, we denote the diameter of S by

$$\delta(S)=\sup\{D(s_1,\ s_2): s_1,\ s_2 \in S\}.$$

From now on we denote $F=\delta(S)$ if $\delta(S)=+\infty$, while $F>\delta(S)$ if $\delta(S)<+\infty$.

Zhang [30] obtained some results for single-valued mappings involving some contractive type condition with the function ξ. For multivalued mappings, using the function ξ, Theorem 4 generalized in [15] and while Latif and Abdou [12] extended Theorem 6. Here, We establish some general fixed point results for contractive type multivalued mappings including $\xi \circ q$, where $\xi \in \Omega[0,\ F)$ with the Q-function q. In fact, new results either improve or extend several results of the metric fixed point theory.

2. Results

In this section, we consider (S, Λ) is a quasi-metric space with Q-function q. Before establishing our main results, we present the key result.

Lemma 3 Let $\Gamma : S \to 2^S$ be a multivalued mapping. Suppose that

(i) there exists $b \in (0, 1)$ and $\chi : \mathbb{R}^+ \to [0, b)$ satisfying for all $t \in \mathbb{R}^+$

$$b > \limsup_{r \to t^+} \chi(r), \tag{9}$$

(ii) there is $\xi \in \Omega[0, F)$ such that for $s_1 \in S$, we have $s_2 \in \Gamma(s_1)$ satisfying

$$b\xi(q(s_1, s_2)) \leq \xi(q(s_1, \Gamma(s_1))),$$

$$\xi(q(s_2, \Gamma(s_2))) \leq \chi(q(s_1, s_2))\xi(q(s_1, s_2)). \tag{10}$$

Then, there is an orbit $\{s_n\}$ of Γ in S which turns out as a Cauchy sequence such that the sequences of reals $\{q(s_n, s_{n+1})\}$ and $\{q(s_n, \Gamma(s_n))\}$ are convergent.

Proof. For any $s_0 \in S$ there is $s_1 \in \Gamma(s_0)$ with

$$b\xi(q(s_0, s_1)) \leq \xi(q(s_0, \Gamma(s_0))), \tag{11}$$

$$\xi(q(s_1, \Gamma(s_1))) \leq \chi(q(s_0, s_1))\xi(q(s_0, s_1)), \; \chi(q(s_0, s_1)) < b. \tag{12}$$

From (11) and (12), we get

$$\xi(q(s_0, \Gamma(s_0))) - \xi(q(s_1, \Gamma(s_1))) \geq b\xi(q(s_0, s_1)) - \chi(q(s_0, s_1))\xi(q(s_0, s_1))$$

$$= [b - \chi(q(s_0, s_1))]\xi(q(s_0, s_1)) > 0.$$

Finally, we have an orbit $\{s_n\}$ of Γ in S at s_0 such that

$$b\xi(q(s_n, s_{n+1})) \leq \xi(q(s_n, \Gamma(s_n))),$$

$$\xi(q(x_{n+1}, \Gamma(s_{n+1}))) \leq \chi(q(s_n, s_{n+1}))\xi(q(s_n, s_{n+1})), \; \chi(q(s_n, s_{n+1})) < b. \tag{13}$$

From (13), we get

$$\xi(q(s_n, \Gamma(s_n))) - \xi(q(s_{n+1}, \Gamma(s_{n+1}))) \geq [b - \chi(q(s_n, s_{n+1}))]\xi(q(s_n, s_{n+1})). \tag{14}$$

For any n,

$$\xi(q(s_{n+1},\ \Gamma(s_{n+1}))) \le \xi(q(s_n,\ \Gamma(s_n))),$$

$$\xi(q(s_n,\ s_{n+1})) \le \xi(q(s_{n-1},\ s_n)). \tag{15}$$

Thus, the sequence of nonnegative real numbers $\{\xi(q(s_n,\ \Gamma(s_n)))\}$ and $\{\xi(q(s_n,\ s_{n+1}))\}$ are decreasing and hence we deduce that the sequences $\{q(s_n,\ \Gamma(s_n))\}$ and $\{q(s_n,\ s_{n+1})\}$ are convergent. Now, note that there is $\alpha \in [0,\ b)$ with

$$\limsup_{n\to\infty} \chi(q(s_n,\ s_{n+1})) = \alpha. \tag{16}$$

Now, for $b_0 \in (\alpha,\ b)$, there is $n_0 \in \mathbb{N}$ such that for each for each $n > n_0$,

$$\chi(q(s_n,\ s_{n+1})) < b_0, \tag{17}$$

and then we get

$$\chi(q(s_n,\ s_{n+1})) \times \ldots \times \chi(q(s_{n_0+1},\ s_{n_0+2})) < b_0^{n-n_0}. \tag{18}$$

Also, it follows from (14) that for any $n > n_0$,

$$\xi(q(s_n,\ \Gamma(s_n))) - \xi(q(s_{n+1},\ \Gamma(s_{n+1}))) \ge \gamma\xi(q(s_n,\ s_{n+1})), \tag{19}$$

where $\gamma = b - b_0$. Note that for any $n > n_0$, we have

$$
\begin{aligned}
\xi(q((s_{n+1},\, \Gamma(s_{n+1}))) &\le \chi(q(s_n,\, s_{n+1}))\,\xi(q(s_n,\, s_{n+1})) \\
&\le \frac{1}{b}\chi(q(s_n,\, s_{n+1}))\,\xi(q(s_n,\, \Gamma(s_n))) \\
&\le \frac{1}{b^2}\chi(q(s_n,\, s_{n+1}))\,\chi(q(s_{n-1},\, s_n))\,\xi(q(s_{n-1},\, \Gamma(s_{n-1}))) \\
&\ \vdots \\
&\le \frac{1}{b^n}\left[\chi(q(s_n,\, s_{n+1}))\times\cdots\times\chi(q(s_1,\, s_2))\right]\xi(q(s_1,\, \Gamma(s_1))) \\
&= \frac{\chi(q(s_n,\, s_{n+1}))\times\cdots\times\chi\left(q(s_{n_0+1},\, s_{n_0+2})\right)}{b^{n-n_0}} \\
&\quad \times \frac{\chi\left(q(s_{n_0},\, s_{n_0+1})\right)\times\cdots\times\chi(q(s_1,\, s_2))\,\xi(q(s_1,\, \Gamma(s_1)))}{b^{n_0}}.
\end{aligned} \tag{20}
$$

Thus,

$$
\xi(q(s_{n+1},\, \Gamma(s_{n+1}))) \le \left(\frac{b_0}{b}\right)^{n-n_0} k, \tag{21}
$$

where $k = \chi(q(s_{n_0},\, s_{n_{0+1}}))\times....\times\chi(q(s_1,\, s_2))\xi(q(s_1,\, \Gamma(s_1)))/b^{n_0}$. Now, since $b_0 < b$, we have $\lim_{n\to\infty}(b_0/b)^{n-n_0} = 0$, and we get the decreasing sequence $\{\xi(q(s_n,\, \Gamma(s_n)))\}$ converging to 0. Thus, we have

$$
q(s_n,\, \Gamma(s_n)) \to 0. \tag{22}
$$

For any $n > n_0$,

$$
\xi(q(s_n,\, s_{n+1})) < c^n \xi(q(s_0,\, s_1)), \tag{23}
$$

where $c = b_0/b < 1$. Thus, for $m > n > n_0,\ n,\ m \in \mathbb{N}$,

$$
\xi(q(s_n,\, s_m)) \le \sum_{j=1}^{m-1} \xi(q(s_j,\, s_{j+1})) < \frac{c^n}{1-c}\xi(q(s_0,\, s_1)). \tag{24}
$$

Clearly, $\lim_{n,m\to\infty} \xi(q(s_n,\, s_m)) = 0$, and thus we get that

$$\lim_{n,m\to\infty} q(s_n,\ s_m) = 0, \tag{25}$$

that is, $\{s_n\}$ is Cauchy sequence in S. □

Using Lemma 3, we obtain the following fixed point result.

Theorem 8 Assume that the assumptions of Lemma 3 hold. If S is complete, then there exists an orbit of Γ which converges in S. Further, if there is a lower semicontinuous function β on S with $\beta(s) = q(s,\ \Gamma(s))$, then, there exists $u_0 \in S$ such that $\beta(u_0) = 0$. Also, if the mapping Γ is closed valued and $q(u_0,\ u_0) = 0$ then $u_0 \in \Gamma(u_0)$.

Proof. Following the Lemma 3, we get a Cauchy sequence $\{s_n\}$ of S. Thus, there is some $u_0 \in S$ with $\lim_{n\to\infty} s_n = u_o$. The lower semi-continuity of β and (22), yield

$$0 \le \beta(u_0) \le \liminf_{n\to\infty} \beta(s_n) = \liminf_{n\to\infty} q(s_n,\ \Gamma(s_n)) = 0, \tag{26}$$

and thus, $\beta(u_0) = q(u_0,\ \Gamma(u_0)) = 0$. Since $q(u_0,\ u_0) = 0$, and $\Gamma(u_0)$ is closed, thus it follows by Lemma 1 that $u_0 \in \Gamma(u_0)$.

Consequently, we obtain the following result as a special case.

Corollary 1 Let $\Gamma : S \to Cl(S)$ be such that for $0 < b,\ h < 1$ with $h < b$, $s_1 \in S$ we get $s_2 \in \Gamma(s_1)$ with

$$b\xi(q(s_1,\ s_2)) \le \xi(q(s_1,\ \Gamma(s_1))),$$

$$\xi(q(s_2,\ \Gamma(s_2))) \le h\xi(q(s_1,\ s_2)). \tag{27}$$

If S is complete, suppose that a real-valued function β on S defined by $\beta(s) = q(s,\ \Gamma(s))$ is lower semicontinuous. Then, there exists $u_0 \in S$ such that $\beta(u_0) = 0$. Further, if $q(u_0,\ u_0) = 0$, then $u_0 \in Fix(\Gamma)$.

We also obtain the following intriguing result by substituting a different natural condition for the function β's lower semicontinuity in Theorem 8.

Theorem 9 Assume that the assumptions of Theorem 3 without the assumption of the real-valued function β hold. Further, consider that

$$\inf\{\xi(q(s,\ u)) + \xi(q(s,\ \Gamma(s))) : s \in S\} > 0, \tag{28}$$

for all $u \in S$ with $u \notin \Gamma(u)$ and ξ is a continuous function at 0. Then, $Fix(\Gamma) \neq \emptyset$.

Proof. As in Theorem 3, we get a sequence $\{s_n\}$ which becomes a Cauchy sequence with $s_n \in \Gamma(s_{n-1})$. Since S is complete, there exists $u_0 \in S$ such that the sequence $\{s_n\}$ converges to u_0. Since q is a Q-function, we have for any $n > n_0$

$$\xi(q(s_n,\ u_0)) \le \frac{c^n}{1-c} \xi(q(s_0,\ s_1)), \tag{29}$$

where $c = b_0/b < 1$. Since $q(s_n,\ \Gamma(s_n)) \le q(s_n,\ s_{n+1})$, for any n, and the function ξ is non-decreasing, we have

$$\xi(q(s_n,\ \Gamma(s_n))) \le \xi(q(s_n,\ s_{n+1})), \tag{30}$$

and thus by using (23), we get

$$\xi(q(s_n,\, \Gamma(s_n))) \le c^n \xi(q(s_0,\, s_1)), \tag{31}$$

Assume that $u_0 \notin \Gamma(u_0)$. Then, we have

$$\begin{aligned} 0 &< \inf\{\xi(q(s,\, u_0)) + \xi(q(s,\, \Gamma(s))) : s \in S\} \\ &\le \inf\{\xi(q(s_n,\, u_0)) + \xi(q(s_n,\, \Gamma(s_n))) : n > n_0\} \\ &\le \inf\left\{\frac{c^n}{1-c}\xi(q(s_0,\, s_1)) + c^n \xi(q(s_0,\, s_1)) : n > n_0\right\} \\ &= \left\{\frac{2-c}{1-c}\xi(q(s_0,\, s_1))\right\} \inf\{c^n : n > n_0\} = 0, \end{aligned} \tag{32}$$

which is impossible, and hence $u_0 \in Fix(\Gamma)$. □

Theorem 10 Let $\Gamma : S \to Cl(S)$ be a multivalued mapping. Assume that the following conditions hold:

(i) there exists a function $\chi : \mathbb{R}^+ \to [0,\, 1)$ such that for any $t \in \mathbb{R}^+$

$$\limsup_{r \to t^+} \chi(r) < 1; \tag{33}$$

(ii) there exists a function $\xi \in \Omega[0,\, F)$ such that for any $s_1 \in S$, there exists $s_2 \in \Gamma(s_1)$ satisfying

$$\begin{aligned} &\xi(q(s_1,\, s_2)) = \xi(q(s_1,\, \Gamma(s_1))), \\ &\xi(q(s_2,\, \Gamma(s_2))) \le \chi(q(s_1,\, s_2))\xi(q(s_1,\, s_2)); \end{aligned} \tag{34}$$

(iii) a real-valued function β on S defined by $\beta(s) = q(s,\, \Gamma(s))$ is lower semicontinuous.

If S is complete, then there exists $u_0 \in S$ such that $\beta(u_0) = 0$. Further, if $q(u_0,\, u_0) = 0$ then $u_0 \in \Gamma(u_0)$.

Proof. Let $s_0 \in S$ be any initial point. We use the same approach as in the proof of Theorem 3 to show that there is a Cauchy sequence $\{s_n\}$ with $s_n \in \Gamma(s_{n-1})$ and

$$\begin{aligned} &\xi(q(s_n,\, s_{n+1})) = \xi(q(s_n,\, \Gamma(s_n))), \\ &\xi(q(s_{n+1},\, \Gamma(s_{n+1}))) \le \chi(q(s_n,\, s_{n+1}))\xi(q(s_n,\, s_{n+1})),\ \chi(q(s_n,\, s_{n+1})) < 1. \end{aligned} \tag{35}$$

Consequently, there exists $u_0 \in S$ such that $\lim\limits_{n \to \infty} s_n = u_0$. Since β is lower semicontinuous, we have

$$0 \leq \beta(u_0) \leq \liminf_{n\to\infty} \beta(s_n) = 0, \tag{36}$$

thus, $\beta(u_0) = q(u_0,\ \Gamma(u_0)) = 0$. Further, by closedness of $\Gamma(u_0)$ and since $q(u_0,\ u_0) = 0$, It follows from Lemma 1 that $u_0 \in \Gamma(u_0)$. □

The following fixed point result can be obtained by applying the same methodology as in the Theorem 9 proof. Suppose that the assumptions of Theorem 10 without (iii) hold. Assume that

$$\inf\{\xi(q(s,\ u)) + \xi(q(s,\ \Gamma(s))) : s \in S\} > 0, \tag{37}$$

for any $u \in S$ with $u \notin \Gamma(u)$ and the function ξ is continuous at 0. Then, $Fix(\Gamma) \neq \emptyset$.

(i) Theorem 8 extends and generalizes [28, Theorem 5]. Indeed, if we consider $\xi(t) = t$ for each $t \in (0,\ F)$ in Theorem 2.1, then we can get Theorem 5 due to Latif and Al-Mezel [28].

(ii) Theorem 8 contains [15, Theorem 4] of Pathak and Shahzad as a special case.

(iii) Corollary 1 extends and generalizes [28, Theorem 3] due to Latif and Al-Mezel; it also generalizes [31, Theorem 3.3].

(iv) Theorem 10 extends and generalizes fixed point results [16, Theorem 7] and [12, Theorem 7], and improves fixed point result [15, Theorem 6].

In support of our main results Theorems 8 and Theorem 10, we present the following example. Let $S = [0,\ 1]$. Define $\Lambda : S \times S \to \mathbb{R}^+$ as follows:

$$\Lambda(s_1,\ s_2) = \begin{cases} 0; & \text{if } s_1 = s_2, \\ s_2; & \text{otherwise.} \end{cases} \tag{38}$$

Clearly, $(S,\ \Lambda)$ is a quasi-metric space. Define a Q-function $q : S \times S \to \mathbb{R}^+$ by

$$q(s_1,\ s_2) = s_2, \text{ for all} s_1,\ s_2 \in S. \tag{39}$$

Let $\Gamma : S \to Cl(S)$ be defined as

$$\Gamma(s) = \begin{cases} \left\{\frac{1}{2}s^2\right\}; & s \in \left[0,\ \frac{1}{2}\right) \cup \left(\frac{1}{2},\ 1\right], \\ \left\{0,\ \frac{1}{4}\right\}; & s = \frac{1}{2}. \end{cases} \tag{40}$$

Note that $\delta(S) = 1$. Let $F \in [1,\ +\infty)$, $b = 9/10$. Define a function $\xi : [0,\ F) \to \mathbb{R}$ by $\xi(t) = t^{1/2}$. Clearly, $\xi \in \Omega[0,\ F)$. Define $\chi : \mathbb{R}^+ \to [0,\ b)$ by

$$\chi(t)=\begin{cases}(\frac{3}{4})^{1/4}t^{1/2}; & t\in\left[0,\ \frac{1}{2}\right),\\ \frac{3}{8}; & t\in\left[\frac{1}{2},\ +\infty\right).\end{cases} \tag{41}$$

Note that

$$\beta(s)=q(s,\ \Gamma(s))=\begin{cases}\left\{\frac{1}{2}s^2\right\}; & s\in\left[0,\ \frac{1}{2}\right)\cup\left(\frac{1}{2},\ 1\right],\\ 0; & s=\frac{1}{2},\end{cases} \tag{42}$$

and β is lower semicontinuous. Moreover, for any $s_1\in[0,\ 1/2)\cup(1/2,\ 1]$, we have $\Gamma(s_1)=\{(1/2)s_1^2\}$. Take $s_2=(1/2)s_1^2$, then we have

$$\begin{aligned}b\xi(q(s_1,\ s_2)) &= \frac{9}{10}\xi\left(q\left(s_1,\ \frac{1}{2}s_1^2\right)\right)\\ &= \frac{9}{10}\left(\frac{1}{2}s_1^2\right)^{1/2}\le\left(\frac{1}{2}s_1^2\right)^{1/2}\\ &= \xi(q(s_1,\ \Gamma(s_1))),\ \xi(q(s_2,\ \Gamma(s_2)))\\ &= \xi\left(q\left(\frac{1}{2}s_1^2,\ \frac{1}{2}\left(\frac{1}{2}s_1^2\right)^2\right)\right)\le\left(\frac{3}{4}\right)^{1/4}\left(\frac{1}{2}s_1^2\right)^{1/2}\left(\frac{1}{2}s_1^2\right)^{1/2}\\ &= \chi(q(s_1,\ s_2))\xi(q(s_1,\ s_2)).\end{aligned} \tag{43}$$

Thus, for all $s_1\in[0,\ 1]$, $s_1\neq 1/2$, T satisfies all the conditions of Theorem 8. Now, let $s_1=1/2$, then we have $\Gamma(s_1)=\{0,\ 1/4\}$. Clearly, that for $s_1=1/2$, there is $s_2=0\in\Gamma(s_1)$ such that $\xi(q(s_1,\ \Gamma(s_1)))=0$. Now

$$b\xi\left(q\left(s_1,\ s_2\right)\right)=\frac{9}{10}\xi\left(q\left(\frac{1}{2},\ 0\right)\right)=0=\xi\left(q\left(s_1,\ \Gamma(s_1)\right)\right),$$

$$\xi(q(s_2,\ \Gamma(s_2)))=\xi(q(0,\ 0))=\chi(q(s_1,\ s_2))\xi(q(s_1,\ s_2)). \tag{44}$$

Thus, for $s_1=1/2$ all the conditions of Theorem 8 are satisfied and hence $Fix(\Gamma)\neq\emptyset$. Note that $Fix(\Gamma)=\{0\}$. Note that in the above example, the Q-function q is not a w-distance because $(S,\ D)$ is not a metric space.

3. Conclusion and recommendation

Among others, Feng and Liu [8], Klim and Wardowski [17], and Ciric [16] studied the existence of fixed points for multivalued contractive mappings without using the Hausdorff–Pompeiu metric, and consequently, they generalized some classically known fixed point results, including Theorem 1. In this paper, we establish some general fixed point results for multivalued generalized contractive mappings on quasi-metric spaces for the Q-function. Our results generalize and improve a number of known fixed point results, including the corresponding fixed point results which are stated in Section 2. To illustrate our main fixed point theorems, we have also provided an example. This research leads us in the future towards finding results for fixed points in more general spaces for generalized distances.

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgement

The authors thank the referees for their valuable comments and suggestions.

References

[1] Nadler SB. Multi-valued contraction mappings. *Pacific Journal of Mathematics.* 1969; 30(2): 475-488.
[2] Banach S. On operations in abstract sets and their applications in equations whole. *Fundamenta Mathematicae*. 1922; 3(1): 133-181.
[3] Daffer PZ, Kaneko H. Fixed points of generalized contractive multi-valued mappings. *Journal of Mathematical Analysis and Applications*. 1995; 192(2): 655-666.
[4] Du WS, Hung YL. A generalization of Mizoguchi-Takahashi's fixed point theorem and its applications to fixed point theory. *International Journal of Mathematical Analysis.* 2017; 11(4): 151-161.
[5] Mizoguchi N, Takahashi W. Fixed point theorems for multivalued mappings on complete metric spaces. *Journal of Mathematical Analysis and Applications.* 1989; 141(1): 177-188.
[6] Ćirić LB. Common fixed point theorems for multi-valued mappings. *Demonstratio Mathematica.* 2006; 39(2): 419-428.
[7] Ćirić LB. multi-valued nonlinear contraction mappings. *Nonlinear Analysis: Theory, Methods and Applications.* 2009; 71: 2716-2723.
[8] Feng Y, Liu S. Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings. *Journal of Mathematical Analysis and Applications.* 2006; 317(1): 103-112.
[9] Kada O, Suzuki T, Takahashi W. Nonconvex minimization theorems and fixed point theorems in complete metric spaces. *Mathematica Japonicae*. 1996; 44: 381-391.
[10] Suzuki T, Takahashi W. Fixed point theorems and characterizations of metric completeness. *Journal of the Juliusz Schauder Center.* 1996; 8: 371-382.
[11] Kaneko S, Takahashi W, Wen CF, Yao JC. Existence theorems for single-valued and multivalued mappings with w-distances in metric spaces. *Fixed Point Theory and Applications.* 2016; 2016: 1-15.
[12] Latif A, Abdou A. Some new weakly contractive type multimaps and fixed point results in metric spaces. *Fixed Point Theory and Applications.* 2010; 2009: 1-12.
[13] Latif A, Abdou A. Fixed points of generalized contractive maps. *Fixed Point Theory and Applications.* 2009; 2009: 1-9.
[14] Al-Homidan S, Ansari QH, Yao JC. Some generalizations of Ekeland-type variational principle with applications to equilibrium problems and fixed point theory. *Nonlinear Analysis: Theory, Methods and Applications.* 2008; 69(1): 126-139.

[15] Pathak H, Shahzad N. Fixed point results for set-valued contractions by altering distances in complete metric spaces. *Nonlinear Analysis: Theory, Methods and Applications.* 2009; 70(7): 2634-2641.

[16] Ćirić LB. Fixed point theorems for multi-valued contractions in complete metric spaces. *Journal of Mathematical Analysis and Applications.* 2008; 348(1): 499-507.

[17] Klim D, Wardowski D. Fixed point theorems for set-valued contractions in complete metric spaces. *Journal of Mathematical Analysis and Applications.* 2007; 334(1): 132-139.

[18] Wilson WA. On quasi-metric spaces. *American Journal of Mathematics.* 1931; 53(3): 675-684.

[19] Alegre C, Marín J, Romaguera S. A fixed point theorem for generalized contractions involving w-distances on complete quasi-metric spaces. *Fixed Point Theory and Applications.* 2014; 2014: 1-8.

[20] Alegre C, Marín J. Modified *w*-distances on quasi-metric spaces and a fixed point theorem on complete quasi-metric spaces. *Topology and Its Applications.* 2016; 203: 32-41.

[21] Alhazimeh A, Hatamleh R. $H(\omega, \theta)$-Contracion and some new fixed point results in modified ω-Distance mappings via complete quasi metric spaces and application. *Nonlinear Functional Analysis and Applications.* 2023; 28(2): 395-405.

[22] Barootkoob S, Lakzian H. Fixed point results via L-contractions on quasi W-distances. *Journal of Mathematical Extension.* 2021; 15(3): 1-22.

[23] Karapinar E, Pitea A, Shatanawi W. Function weighted quasi-metric spaces and fixed point results. *IEEE Access.* 2019; 7: 89026-89032.

[24] Marin J, Romaguera S, Tirado P. Q-function on quasi-metric spaces and fixed points for multi-valued maps. *Fixed Point Theory and Applications.* 2011; 2011: 1-10.

[25] Romaguera S. Basic contractions of suzuki-type on quasi-metric spaces and fixed point results. *Mathematics*. 2022; 10(21): 3931.

[26] Secelean NA, Mathew S, Wardowski D. New fixed point results in quasi-metric spaces and applications in fractals theory. *Advances in Difference Equations.* 2019; 2019: 1-23.

[27] Lin LJ, Du WS. Some equivalent formulations of the generalized Ekeland's variational principle and their applications. *Nonlinear Analysis: Theory, Methods and Applications.* 2007; 67(1): 187-199.

[28] Latif A, Al-Mezel SA. Fixed point results in quasi-metric spaces. *Fixed Point Theory and Applications.* 2011; 2011: 1-8.

[29] Boyd DW, Wong JS. On nonlinear contractions. *Proceedings of the American Mathematical Society.* 1969; 20(2): 458-464.

[30] Zhang X. Common fixed point theorems for some new generalized contractive type mappings. *Journal of Mathematical Analysis and Applications.* 2007; 333(2): 780-786.

[31] Latif A, Albar WA. Fixed point results in complete metric spaces. *Mathematical Demonstration.* 2008; 41(1): 145-150.