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Research Article

Fixed Point Results for Generalized (α, β)-Nonexpansive Type-1 Mapping in Hyperbolic Space

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Abstract: In this article, we define several fundamental characteristics and put forward some basic fixed point results in the context of hyperbolic space for generalized (α, β)-nonexpansive type-1 mappings. Additionally, we present Δ-convergence and strong convergence results within the framework of hyperbolic space for these types of mappings. Lastly, we provide some numerical examples to highlight our main result and compare the iterative procedures that we use in our study with various iterative techniques from the literature. The results in this study enhance, broaden and unite corresponding results in the literature.

Keywords: Hyperbolic space, generalized (α, β)-nonexpansive type-1 mapping, fixed point, strong and Δ-convergence results

MSC: 47H09, 47H10

1. Introduction

The existence of fixed point for nonexpansive mapping was initiated by Browder [1], Gohde [2] and Kirk [3] independently in 1965. Browder proved the fixed point theorem for nonexpansive mapping on a convex, bounded and closed subset of a Hilbert space. Browder and Gohde [1, 2] swiftly generalized a similar conclusion from a Hilbert space to a uniformly convex Banach space. For a further generalization of nonexpansive mappings, see [4–6]. The question of whether fixed points exist in Banach spaces and how to approximate them for firmly nonexpansive mappings was investigated by Kohsaka and Takahashi [7] in 2008.

Since nonexpansive mappings are unquestionably one of the most important issues in the field of metric fixed-point theory, there is a sizable corpus of study on more general types of mappings than nonexpansive ones in the literature.

Let $Q \neq \emptyset$ be a subset of a Banach space E and g is a self-map on Q. Also, the set of fixed points denoted by $Fix(g) \neq \emptyset$ if

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$$\|g\eta - s\| \le \|\eta - s\|,$$

for all $\eta \in Q,\ s \in Fix(g)$, then g is called quasi-nonexpansive mapping, in addition, if the following inequality

$$\|g\eta - g\zeta\| \le \alpha\|\eta - \zeta\| + \beta\left(\|\eta - g\eta\| + \|\zeta - g\zeta\|\right) + \gamma(\|\eta - g\zeta\| + \|\zeta - g\eta\|),$$

for all $\eta,\ \zeta \in Q$, where $\alpha,\ \beta,\ \gamma \ge 0$ and $\alpha + 2\beta + 2\gamma \le 1$, holds g is called generalized nonexpansive mapping [8].

Suzuki [9] defined Condition (C) as a generalization of nonexpansive mapping, that is,

$$\frac{1}{2}\|g\eta - \eta\| \le \|\eta - \zeta\| \Rightarrow \|g\eta - g\zeta\| \le \|\eta - \zeta\|,\ \forall \eta,\ \zeta \in Q. \tag{1}$$

Aoyama et al. [10] presented α-nonexpansive in Banach spaces, they established certain fixed point results for such mappings. A mapping $g:\ Q \to Q$ is called α-nonexpansive, if for all $\eta,\ \zeta \in Q$ and $\alpha < 1$

$$\|g\eta - g\zeta\|^2 \le \alpha\left(\|g\eta - \zeta\|^2 + \|\eta - g\zeta\|^2\right) + (1 - 2\alpha)\|\eta - \zeta\|^2.$$

The generalized α-nonexpansive mapping was introduced by the Pant et al. [11], they also discovered several fixed point results for these mappings. A mapping $g : Q \to Q$ is called generalized α-nonexpansive mapping for all $\eta,\ \zeta \in Q,\ \alpha < 1$ if $\frac{1}{2}\|\eta - g\eta\| \le \|\eta - \zeta\|$ implies

$$\|g\eta - g\zeta\| \le \alpha\left(\|\eta - g\zeta\| + \|\zeta - g\eta\|\right) + (1 - 2\alpha)\|\eta - \zeta\|.$$

Akutsah et al. [12] presented generalized $(\alpha,\ \beta)$-nonexpansive type-1 mapping, that is, Let Q be a nonempty subset of a Banach space E. The mapping $g : Q \to Q$ is called generalized $(\alpha,\ \beta)$-nonexpansive type-1 mapping, if $\alpha,\ \beta,\ \lambda \in [0,\ 1)$ exist, such that $\alpha \le \beta$, and $\alpha + \beta < 1$, for any $\eta,\ \zeta \in Q$ if $\lambda\|g\eta - g\zeta\| \le \|\eta - \zeta\|$, then

$$\|g\eta - g\zeta\| \le \alpha\|g\eta - \zeta\| + \beta\|\eta - g\zeta\| + (1 - (\alpha + \beta))\|\eta - \zeta\|.$$

For the approximation of nonexpansive mappings the following iterations taken from [13] are being used:

In 1953, Mann proposed an iteration for the calculation of fixed points for nonexpansive mappings. The iteration that named as Mann iteration is stated as follows:

$$\eta_{j+1} = \beta_j\eta_j + (1 - \beta_j)g\eta_j,$$

for each $j > 1$ and $\{\beta_j\} \subset (0,1)$.

Ishikawa gave an iteration in 1994 for calculating fixed point for nonexpansive mappings as below:

$$\begin{cases} \eta_1 = \eta \ \in Q \\ \eta_{j+1} = ((1-\alpha_j)\eta_j + \alpha_j g\zeta_j) \\ \zeta_j = ((1-\beta_j)\eta_j + \beta_j g\eta_j), \end{cases}$$

where α_j and β_j are in $(0,\ 1)$.

Noor introduced the three-step method of iteration in 2000 as follows:

$$\begin{cases} \eta_1 = \eta \in Q \\ \eta_{j+1} = ((1-\alpha_j)\eta_j + \alpha_j g\zeta_j) \\ \zeta_j = ((1-\beta_j)\eta_j + \beta_j g z_j) \\ z_j = ((1-\gamma_j)\eta_j + \gamma_j g\eta_j), \end{cases}$$

where $\alpha_j,\ \beta_j$ and γ_j are in $(0,\ 1)$.

Agrwal purposed an iteration in 2007 for calculating fixed point for nonexpansive mappings as below:

$$\begin{cases} \eta_1 = \eta \in Q \\ \eta_{j+1} = ((1-\alpha_j)g\eta_j + \alpha_j g\zeta_j) \\ \zeta_j = ((1-\beta_j)\eta_j + \beta_j g\eta_j), \end{cases}$$

where α_j and β_j are in $(0,\ 1)$.

Abbas et al introduced the three-step method of iteration in 2014 as follows:

$$\begin{cases} \eta_1 = \eta \in Q \\ \eta_{j+1} = ((1-\alpha_j)g z_j + \alpha_j g\zeta_j) \\ \zeta_j = ((1-\beta_j)g\eta_j + \beta_j g z_j) \\ z_j = ((1-\gamma_j)\eta_j + \gamma_j g\eta_j), \end{cases}$$

where $\alpha_j,\ \beta_j$ and γ_j are in $(0,\ 1)$.

Later, in 2023, Dashputre [14] proposed a new iteration procedure and used generalized $(\alpha,\ \beta)$-nonexpansive type-1 mapping in the framework of Banach spaces to show some strong and weak convergence results. The iteration scheme defined by Dashputre's is as follows:

For each $\eta_0 \in Q$ and $\{\alpha_j\},\ \{\beta_j\}$ being sequences in $(0,\ 1)$, let $Q \neq \emptyset$ be a subset of a Banach space E and $g: Q \to Q$ be a mapping which is generalized $(\alpha,\ \beta)$-nonexpansive type-1 mapping they formalize the sequence η_j in iterative manner as:

$$\begin{cases} z_j = ((1-\alpha_j)\eta_j + \alpha_j\eta_j) \\ \zeta_j = ((1-\beta_j)z_j + \beta_j g z_j) \\ \eta_{j+1} = g\zeta_j. \end{cases} \tag{2}$$

2. Preliminaries

In 1970, Takahashi [15] present the definition of convexity in a metric space such as, for all $z, \eta, \zeta, \in M$ and $\lambda \in [0, 1]$, a mapping $W : M^2 \times [0, 1] \to M$ is a convex structure in M if

$$d(z, W(\eta, \zeta, \lambda)) \leq \lambda d(z, \eta) + (1-\lambda)d(z, \zeta).$$

A nonempty subset Q of a convex metric space M is convex if $W(\eta, \zeta, \lambda) \in Q$ for all $\eta, \zeta \in Q$ and $\lambda \in [0, 1]$. The concept of hyperbolic space was introduced by Kohlenbach [16] in 2005. Any metric space (M, d) that meets the following axioms for each $u, \eta, \zeta, z \in M$ and $\alpha, \beta \in [0, 1]$ is deemed as hyperbolic space:

$$d(u, W(\eta, \zeta, \alpha)) \leq (1-\alpha)d(u, \eta) + \alpha d(u, \zeta)$$

$$d(W(\eta, \zeta, \alpha), W(\eta, \zeta, \beta)) = |\alpha - \beta| d(\eta, \zeta)$$

$$W(\eta, \zeta, \alpha) = W(\zeta, \eta, (1-\alpha))$$

$$d(W(\eta, z, \alpha), W(\zeta, u, \alpha)) \leq (1-\alpha)d(\eta, \zeta) + \alpha d(z, u)$$

where d is a metric defined on M.

Many mathematicians approximated fixed points in hyperbolic space, with variety of mappings and iterations, for more on this, readers are referred to [17]. The following notions are taken from [18].

If for all $\eta, \zeta, u \in M$, $m > 0$, there exists a $\sigma \in (0, 1]$ such that $(d\left(W\left(\eta, \zeta, \frac{1}{2}\right), u\right) \leq (1-\sigma)m$ where $d(\eta, u) \leq m$, $d(\zeta, u) \leq m$ and $d(\eta, \zeta) \geq \varepsilon m$, $\varepsilon \in (0, 2]$, then a hyperbolic space (M, d, W) is declared as uniformly convex.

A map $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ that yields such a $\sigma = \eta(m, \varepsilon)$ for $u, \eta, \zeta \in M$, $m > 0$ and $\varepsilon \in (0, 2]$ is demonstrate as modulus of uniform convexity of M.

Let $\{\eta_j\}$ represent any bounded sequence, for $\eta \in M$, one can formalize a continuous functional $m(., \{\eta_j\}) : M \to (0, \infty)$ by

$$m(\eta, \eta_j) = \limsup_{j\to\infty} d(\eta, \eta_j). \tag{3}$$

Then

(a) $r_Q(\eta_j) = \inf\{m(\eta, \eta_j) : \eta \in Q\}$ is declared as asymptotic radius of $\{\eta_j\}$ in respect of Q, where Q is subset of M.

(b) The set $A_Q(\eta_j) = \eta \in M$ such that $m(\eta, \eta_j) \leq m(\zeta, \eta_j)$ for any $\zeta \in Q$ is declared as asymptotic center of $\{\eta_j\}$ in respect of Q, where Q is subset of M.

The asymptotic center and radius are simply represented by $A(\eta_j)$ and $r(\eta_j)$ respectively. The set $A(\eta_j)$ may be empty, a singleton or contain infinitely many points. In a complete uniformly convex hyperbolic space with a monotone modulus of uniform convexity, it is known that bounded sequences have a unique asymptotic center with respect to closed convex subsets [18].

In 1976, Lim [19] gave the notion of Δ-convergence, that is, a sequence $\{\eta_j\}$ in M which is declare as Δ-converge to $\eta \in M$ for every subsequence $\{\eta_{j_i}\}$ of $\{\eta_j\}$, provided that η is the unique asymptotic center. The Δ-limit of $\{\eta_j\}$ is denoted by η and result is expressed as $\Delta - \lim_j \eta_j = \eta$.

Payanak and Kirk [20] modified the results given by Lim for $CAT(0)$ spaces, for more Δ-convergence results in $CAT(0)$ spaces, see [21].

The following lemma will be needed to develop convergence of sequences in our main results.

Lemma 1 [18] Assume that $(M,\ d,\ W)$ is a hyperbolic space which is uniformly convex, with monotone modulus of uniform convexity ρ. Moreover $\eta \in M$ and $\{\rho_j\} \in (0,\ 1)$. If $\{\mu_j\}$ and $\{v_j\}$ are sequences in M such that $\limsup_{j\to\infty} d(\mu_j,\ \eta) \le m$, $\limsup_{j\to\infty} d(v_j,\ \eta) \le m$, and $\lim_{j\to\infty} d(W(\mu_j,\ v_j,\ \rho_j),\ \eta) = m$ for some $m \ge 0$, then $\lim_{j\to\infty} d(\mu_j,\ v_j) = 0$.

3. Main result

Inspired and motivated by results given by Dashuptre, we convert the Dashuptre iteration process into hyperbolic space. We will use the generalized $(\alpha,\ \beta)$-nonexpansive mapping in hyperbolic space $(M,\ d,\ W)$ to generate the new iterative scheme. Next, we will investigate and demonstrate the strong and Δ-convergence of our approach.

Definition 1 Let $Q \neq \emptyset$ be a subset of M where M is a hyperbolic space and d is metric defined on M. The mapping $g: Q \to Q$ is called generalized $(\alpha,\ \beta)$-nonexpansive type-1 mapping, if $\alpha,\ \beta,\ \lambda \in [0,\ 1)$ exist, such that $\alpha \le \beta$, and $\alpha + \beta < 1$, for any $\eta,\ \zeta \in Q$, $\lambda d(g\eta,\ g\zeta) \le d(\eta,\ \zeta)$, then

$$d(g\eta,\ g\zeta) \le \alpha d(g\eta,\ \zeta) + \beta d(\eta,\ g\zeta) + (1 - (\alpha + \beta)) d(\eta,\ \zeta). \tag{4}$$

Using Definition 1, we proposed the new iterative scheme for the approximation of fixed points in hyperbolic space M. Given a hyperbolic space M, let $Q \neq \emptyset$ and $g:\ Q \to Q$ being a generalized $(\alpha,\ \beta)$-nonexpansive type-1 mapping. We define an iterative sequence $\{\eta_j\}$ for each $\eta_0 \in Q$ as follows:

$$\begin{aligned} \eta_{j+1} &= g(\zeta_j) \\ \zeta_j &= g(W(g(z_j),\ z_j,\ \beta_j)) \\ z_j &= g(W(g(\eta_j),\ \eta_j,\ \alpha_j)). \end{aligned} \tag{5}$$

Lemma 2 Given that $Q \neq \emptyset$ be a subset of hyperbolic space M and $g:\ Q \to Q$ be generalized $(\alpha,\ \beta)$-nonexpansive type-1 mapping with $Fix(g) \neq \emptyset$. Then g is quasi-nonexpansive mapping.

Proof. Since with $Fix(g) \neq \emptyset$, let $\zeta \in Fix(g)$ and $\eta \in Q$. Since g is generalized $(\alpha,\ \beta)$-nonexpansive type-1 mappings so equation (4) implies

$$(1-\alpha) d(g\eta,\ g\zeta) \le (1-\alpha) d(\eta,\ \zeta),$$

since $\zeta \in Fix(g)$, it follows

$$d(g\eta,\ g\zeta) \leq d(\eta,\ \zeta).$$

□

Lemma 3 Let $Q \neq \emptyset$ be the subset of hyperbolic space M and $g:\ Q \to Q$ be a generalized $(\alpha,\ \beta)$-nonexpansive type-1 mapping. Then for each and every $\eta,\ \zeta \in Q$

$$d(\eta,\ g\zeta) \leq \left(\frac{1+\alpha}{1-\beta}\right) d(\eta,\ g\eta) + d(\eta,\ \zeta).$$

Proof. Consider

$$d(\eta,\ g\zeta) \leq d(\eta,\ g\eta) + d(g\eta,\ g\zeta),$$

thus, equation (4) implies,

$$(1-\beta)d(\eta,\ g\zeta) \leq (1+\alpha)d(\eta,\ g\eta) + (1-\beta)d(\eta,\ \zeta).$$

which completes the proof. □

Lemma 4 For $Q \neq \emptyset$, be a closed and convex subset of M where M is a hyperbolic space. Let $g:\ Q \to Q$ be a generalized $(\alpha,\ \beta)$-nonexpansive type-1 mapping. For all bounded sequence $\{\eta_j\}$ in Q such that $\lim d(\eta_j,\ g\eta_j) = 0$, then g contain a fixed point that is $g\eta = \eta$.

Proof. Given that$\{\eta_j\}$ is bounded sequence in Q such that $\lim d(\eta_j,\ g\eta_j) = 0$. Let $\eta \in A_Q(\{\eta_j\})$ and by (3) we have

$$m(g\eta,\ \eta_j) = \limsup_{j\to\infty} d(\eta_j,\ g\eta),$$

Lemma 3 implies

$$m(g\eta,\ \eta_j) \ \leq\ \left(\frac{1+\alpha}{1-\beta}\right) \limsup_{j\to\infty} d(\eta_j,\ g\eta_j) + \limsup_{j\to\infty} d(\eta_j,\ \eta)$$

$$m(g\eta,\ \eta_j) \leq \limsup_{j\to\infty} d(\eta_j,\ \eta).$$

This implies that $g\eta \in A_Q(\{\eta_j\})$. By uniqueness of asymptotic center it is shown that $g\eta = \eta$. □

Theorem 1 Assume M is hyperbolic space and $Q \neq \emptyset$, be a closed and convex subset of M. Let $g:\ Q \to Q$ is generalized $(\alpha,\ \beta)$-nonexpansive type-1 mapping with $Fix(g) \neq \emptyset$. If $\{\eta_j\}$ be a sequence formalize by (5), then

(a) $\lim_{j\to\infty} d(\eta_j,\ s)$ exists for all $s \in Fix(g)$.

(b) $\lim_{j\to\infty} d(\eta_j,\ g\eta_j) = 0$.

Proof. (a) Let $s \in Fix(g)$ and for the sequence formalize in equation (5), consider

$$d(z_j,\, s) = d\,(g(W(g(\eta_j,\, \eta_j,\, \alpha_j),\, s)\,.$$

Since g is generalized $(\alpha,\ \beta)$-nonexpansive type-1 mapping, from Lemma 2 it follows that

$$d(z_j,\, s) \leq d\,(W(g(\eta_j,\, \eta_j,\, \alpha_j),\, s)\,,$$

by the convexity of metric space and Lemma 2, we get

$$\begin{aligned} d(z_j,\, s) \ &\leq \ \alpha_j d(g\eta_j,\, s) + (1-\alpha_j) d(\eta_j,\, s) \\ &\leq d(\eta_j,\, s). \end{aligned} \tag{6}$$

Consider $\{\zeta_j\}$ from equation (5) and by using equation (6), we have the form

$$\begin{aligned} d(\zeta_j,\, s) =& d\,(g(W(gz_j,\, z_j,\, \beta_j)),\, s) \\ & (W(z_j,\, z_j,\, \beta_j),\, s) \\ & \leq d(z_j,\, s). \end{aligned} \tag{7}$$

Consider the sequence $\{\eta_{j+1}\}$ from equation (5)-(7) and Lemma 2, we obtained the form

$$\begin{aligned} d(\eta_{j+1},\, s) &= d\,(g\zeta_j,\, s) \\ &\leq d\,(g(g(W(gz_j,\, z_j,\, \beta_j)),\, s) \\ d(\eta_{j+1},\, s) &\leq d(z_j,\, s), \end{aligned} \tag{8}$$

this shows that $d(\eta_j,\, s)$ is bounded and nonincreasing for all $s \in Fix(g)$. Thus $\{\eta_j\}$ is bounded and $\lim_{j\to\infty} d(\eta_j,\, s)$ exists for all $s \in Fix(g)$. Hence the result holds.

(b) Now assume that $\lim_{j\to\infty} d(\eta_j,\, s) = m$. Thus, two cases arises:

(i) if $m = 0$, then $\lim_{j\to\infty} d(\eta_j,\, s) = 0$ and

$$\begin{aligned} \lim_{j\to\infty} d(\eta_j,\, g\eta_j) &\leq \lim_{j\to\infty} d(\eta_j,\, s) + \lim_{j\to\infty} d(g\eta_j,\, s) \\ &\leq \lim_{j\to\infty} d(\eta_j,\, s) + \lim_{j\to\infty} d(\eta_j,\, s) = 0, \end{aligned}$$

(ii) if $m \neq 0$, we have $d(z_j,\ s) \leq d(\eta_j,\ s)$. Then by applying $\limsup_{j\to\infty}$ on both sides, we have

$$\limsup_{j\to\infty} d(z_j,\ s) \leq \limsup_{j\to\infty} d(\eta_j,\ s) \leq m$$

$$\liminf_{j\to\infty} d(\eta_{j+1},\ s) \leq \liminf_{j\to\infty} d(z_j,\ s),$$

since $\lim_{j\to\infty} d(\eta_j,\ s) = r$ and the sequence $\{\eta_j\}$ is nonincreasaing. Thus $\lim_{j\to\infty} d(\eta_{j+1},\ s) = m$ and we obtain

$$m = \liminf_{j\to\infty} d(\eta_{j+1},\ s) \leq \liminf_{j\to\infty} d(z_j,\ s).$$

Thus, by Lemma 1, result holds. □

Lemma 5 Assume that $Q \neq \emptyset$ be a convex subset of hyperbolic space M. If g be generalized $(\alpha,\ \beta)$-nonexpansive type-1 mapping on Q and $Fix(g)$ is nonempty, then $Fix(g)$ is closed.

Proof. Assume that $\{\eta_j\} \subseteq Fix(g)$ and $\{\eta_j\}$ converges to some $\zeta \in Fix(g)$. According to Lemma 3

$$d(\eta,\ g\zeta) \leq ((1+\alpha)/(1-\beta))\, d(\eta,\ g\eta) + d(\eta,\ \zeta),$$

taking $\limsup$ and using Theorem 1, we get

$$\limsup_{j\to\infty} d(\eta_j,\ g\zeta) \leq \left(\frac{1+\alpha}{1-\beta}\right) \limsup_{j\to\infty} d(\eta_j,\ g\eta_j) + \limsup_{j\to\infty} d(\eta_j,\ \zeta)$$

$$\limsup_{j\to\infty} d(\eta_j,\ g\zeta) \leq \limsup_{j\to\infty} d(\eta_j,\ \zeta).$$

Since $Q \subseteq M$ has unique limit point of Q. Therefore, $g\zeta = \zeta$ and $Fix(g)$ is closed. □

The following definitions are needed for further use.

Definition 2 [22] Let Q be a nonempty subset of a Banach space, a mapping $g:\ Q \to Q$ satisfies Condition (I) if there exists a nondecreasing function $f : [0,\ \infty) \to [0,\ \infty)$ such that $f(0) = 0$ and $f(t) > 0$ for all $t \in (0,\ \infty)$ and $\|\eta - g(\eta)\| \geq f(d(\eta,\ Fix(g)))$ for all $x \in Q$ where $d(\eta,\ Fix(g))$ denotes distance from η to $Fix(g)$.

Definition 3 If $\{g\eta_j\}$ has a convergent subsequence in Q for a bounded sequence $\{\eta_j\} \in Q$, then g is said to be completely continuous.

Theorem 2 In a hyperbolic space M, let $Q \neq \emptyset$, be a closed and convex subset of M. Let $g:\ Q \to Q$ be a mapping that fulfills properties of generalized $(\alpha,\ \beta)$-nonexpansive type-1 mapping, where $Fix(g) \neq \emptyset$. Given two sequences $\{\alpha_j\}$, $\{\beta_j\}$ in $(0,\ 1)$ and $\alpha_j + \beta_j < 1$, let $\{\alpha_j\}$ and $\{\beta_j\}$ satisfy

(a) $0 < \liminf_{j\to\infty} \alpha_j \leq \limsup_{j\to\infty} \alpha_j + \beta_j < 1$

(b) $\liminf_{j\to\infty} d(\eta_j,\ Fix(g)) = 0$

(c) g satisfies Condition (I).

If g is completely continuous, then sequence $\{\eta_j\}$ which is formalize in equation (5), strongly convergent to some fixed point of g.

Proof. As g is completely continuous, there is a subsequence $\{\eta_{j_i}\}$ of $\{\eta_j\}$ such that $\{g\eta_j\}$ converges and by Theorem 1, $\{\eta_j\} \in Q$ is bounded. Thus, we have $\lim_{j\to\infty} d(\eta_j,\ g\eta_j) = 0$.

Let $\lim_{j\to\infty} \eta_{j_i} = q$. By continuity of g and $\lim_{j\to\infty} d(\eta_{j_i},\ g\eta_{j_i}) = 0$, we obtain

$$\lim_{j\to\infty} \eta_{j_i} = \lim_{j\to\infty} g\eta_{j_i}.$$

Hence $q \in Fix(g)$ and $\lim_{j\to\infty} d(\eta_j,\ s)$ exists for all $s \in Fix(g)$, so $\{\eta_j\}$ converges. □

Theorem 3 Let $Q \neq \emptyset$, be a closed and convex subset of hyperbolic space M which is uniformly convex with monotone mudulus of uniform convexity ρ and $g:\ Q \to Q$ be a mapping that have properties of generalized $(\alpha,\ \beta)$-nonexpansive type-1 mapping with $Fix(g)$ is nonempty if and only if $\liminf_{j\to\infty} d(\eta_j,\ Fix(g)) = 0$ the sequence $\{\eta_j\}$ which is specified by equation (5) strongly convergent to some fixed point of g where we formulize $d(\eta_j,\ Fix(g))$ as $\inf_{\eta\in Fix(g)} d(\eta_j,\ \eta)$.

Proof. Suppose $\{\eta_j\}$ converges to some fixed point of g, say s. Then $\lim_{j\to\infty} d(\eta_j,\ s) = 0$ and since $0 \leq d(\eta_j,\ Fix(g)) \leq d(\eta_j,\ s)$, we have

$$0 \leq \lim_{j\to\infty} d(\eta_j,\ Fix(g)) \leq \lim_{j\to\infty} d(\eta_j,\ s)$$

$$0 \leq \lim_{j\to\infty} d(\eta_j,\ Fix(g)) \leq 0,$$

which implies that $\lim_{j\to\infty} d(\eta_j,\ Fix(g)) = 0$.

Conversely, assume that $\liminf_{j\to\infty} d(\eta_j,\ Fix(g)) = 0$. Then there is a subsequence $\{\mu_j\}$ of $\{\eta_j\}$ such that $\lim_{j\to\infty} d(\mu_j,\ Fix(g)) = 0$. Now we will prove that $\{\eta_j\}$ converges to $s \in Fix(g)$.

Consider any arbitrary subsequence $\{\mu_{j_i}\}$ of $\{\mu_j\}$ and $\{\nu_i\}$ sequence belongs to $Fix(g)$ in such a way that $d(\mu_{j_i},\ \nu_i) \leq \frac{1}{2i}$ for all $i \geq 1$. By Theorem 1, $\{\eta_j\}$ is bounded and nonincreasing, so the subsequence $\{\mu_{j_i}\}$ of $\{\eta_j\}$ is also nonincreasing and bounded, we have

$$d(\mu_{j_{i+1}},\ \nu_i) \leq d(\mu_{j_i},\ \nu_i) \leq \frac{1}{2^i}$$

$$d(\nu_{i+1},\ \nu_i) \leq d(\nu_{i+1},\ \mu_{i+1}) + d(\mu_{i+1},\ \nu_i)$$

$$\leq \frac{1}{2^{i-1}},$$

this proves that $\{\nu_i\}$ is a Cauchy sequence belongs to $Fix(g)$. Since $Fix(g)$ has proved closed by Lemma 5, $\{\nu_i\}$ is converges to s. Since $\lim_{j\to\infty} d(\eta_{j_i},\ s)$ is the sequence $\{\eta_j\}$ converges to s. Thus $\lim_{j\to\infty} d(\mu_{j_i},\ s) = 0$. Hence the sequence $\{\eta_j\}$ converges to s. □

Theorem 4 Given that $Q \neq \emptyset$ be a closed and convex subset of M being a hyperbolic space. Assume that $g:\ Q \to Q$ is a generalized $(\alpha,\ \beta)$-nonexpansive mapping such that $\frac{\lambda}{2} \in \left[0,\ \frac{1}{2}\right]$ with $Fix(g) \neq \emptyset$ and $\{\eta_n\}$ be a sequence formalize by (5), then Δ-converges to the unique fixed point of g.

Proof. Since, every bounded sequence have unique asymptotic center in respect of $Q \subset M$ and $\{\eta_j\}$ is bounded, thus any subsequences $\{\mu_j\}$ and $\{v_j\}$ of $\{\eta_j\}$ also has unique asymptotic center. Also, by Theorem 1, $\lim_{j\to\infty} d(\eta_j, g\eta_j) = 0$, we have

$$\lim_{j\to\infty} d(\mu_j, g\mu_j) = 0 \text{ and } \lim_{j\to\infty} d(v_j, gv_j) = 0,$$

thus by using Lemma 4, $\mu, v \in Fix(g)$. Now, assume that $\mu \neq v$ and

$$\limsup_{j\to\infty} d(\eta_j, \mu) = \limsup_{j\to\infty} d(\mu_j, \mu),$$

by asymptotic center, we obtain

$$\begin{aligned}\limsup_{j\to\infty} d(\mu_j, \mu) &\leq \limsup_{j\to\infty} d(\mu_j, v)\\ &= \limsup_{j\to\infty} d(\eta_j, v)\\ &= \limsup_{j\to\infty} d(v_j, v)\\ &\leq \limsup_{j\to\infty} d(\eta_j, \mu).\end{aligned}$$

which is contradiction to our supposition, therefore $\mu = v$. Then sequence $\{\eta_j\}$ Δ-converges to unique fixed point of g.□

4. Numerical example

Now, we will provide examples of generalized (α, β)-nonexpansive type-1 mapping and compare the convergence behavior of the iteration method employed in this study with the iterations proposed by Mann, Ishikawa, Noor, Agrawal, and Abbas.

Example 1 Let $M = \mathbb{R}^2$ be a hyperbolic space and $Q = [0, 1]$ is a closed and convex subset of M. We defined a metric on M as $d(\eta, \zeta) = |\eta_1 - \zeta_1| + |\eta_1\zeta_1 - \eta_2\zeta_2|$, for every $(\eta_1, \eta_2), (\zeta_1, \zeta_2) \in Q$ with $(\eta_1 = \frac{\zeta_1}{2})$ and $(\eta_2 = \frac{\zeta_2}{2})$. Now define a map $g: Q \to Q$ such that

$$g(\eta_1, \eta_2) = \left(\frac{2\eta_1}{3} + \alpha\beta, \frac{2\eta_2}{3}\right), \forall (\eta_1, \eta_2) \in Q. \tag{9}$$

Then g is generalized (α, β)-nonexpansive type-1 mapping for every $\alpha, \beta, \lambda \in \left[0, \frac{1}{6}\right]$, $\alpha \geq \beta$.

Solution First step demonstrate that $\lambda d(g\eta, g\zeta) \leq d(\eta, \zeta)$. Second step shows that

$$d(g\eta,\ g\zeta) \le \alpha d(g\eta,\ \zeta) + \beta d(\eta,\ g\zeta) + (1 - (\alpha + \beta))d(\eta,\ \zeta).$$

Step 1 Let any arbitrary $\eta = (\eta_1,\ \eta_2),\ \zeta = (\zeta_1,\ \zeta_2) \in Q$,

$$d(g\eta,\ g\zeta) = \left|\frac{2}{3}(\eta_1 - \zeta_1)\right| + \left|\frac{4}{9}(\eta_1\zeta_1 - \eta_2\zeta_2) - \left(\frac{2}{3}\alpha\beta(\eta_1 + \zeta_1 + \alpha\beta)\right)\right|,$$

we have

$$\left|\frac{4}{9}(\eta_1\zeta_1 - \eta_2\zeta_2) - \left(\frac{2}{3}\alpha\beta(\eta_1 + \zeta_1 + \alpha\beta)\right)\right| \le |(\eta_1\zeta_1 - \eta_2\zeta_2)|,$$

consequently, it results

$$\lambda d(g\eta,\ g\zeta) \le d(\eta,\ \zeta).$$

Step 2 Assuming

$$\alpha d(g\eta,\ \zeta) + \beta d(\eta,\ g\zeta) + (1 - (\alpha + \beta))d(\eta,\ \zeta), \tag{10}$$

determining values of $\alpha d(g\eta,\ \zeta)$ and $\beta d(g\zeta,\ \eta)$

$$\alpha d(g\eta,\ \zeta) = \alpha\left(\left|\frac{2\eta_1}{3} + \alpha\beta - \zeta_1\right| + \left|\left(\frac{2\eta_1}{3} + \alpha\beta\right)\zeta_1 - \frac{2\eta_2\zeta_2}{3}\right|\right)$$

$$\beta d(\eta,\ \zeta) = \beta\left(\left|\frac{2\zeta_1}{3} + \alpha\beta - \eta_1\right| + \left|\left(\frac{2\zeta_1}{3} + \alpha\beta\right)\eta_1 - \frac{2\zeta_2\eta_2}{3}\right|\right),$$

applying $\alpha \ge \beta,\ \eta_1 = \dfrac{\zeta_1}{2}$ and $\eta_1 = \dfrac{\zeta_1}{2}$, we get that

$$\alpha d(g\eta,\ \zeta) \ge \beta d(\eta,\ \zeta),$$

using $\alpha \ge \beta$ and $\beta d(\eta,\ g\zeta) \le \alpha d(g\eta,\ \zeta)$ in (10) implies that

$$\alpha d(g\eta,\ \zeta) - \alpha d(\eta,\ \zeta) + \beta d(\eta,\ g\zeta) - \beta d(\eta,\ \zeta) + d(\eta,\ \zeta) \ge d(\eta,\ \zeta),$$

this implies that

$$\alpha d(g\eta,\ \zeta)-\alpha d(\eta,\ \zeta)+\beta d(\eta,\ g\zeta)-\beta d(\eta,\ \zeta)+d(\eta,\ \zeta)\geq d(g\eta,\ g\zeta),$$

hence proved that

$$d(g\eta,\ g\zeta)\leq \alpha d(g\eta,\ \zeta)-\alpha d(\eta,\ \zeta)+\beta d(\eta,\ g\zeta)-\beta d(\eta,\ \zeta)+d(\eta,\ \zeta)d(\eta,\ \zeta),$$

thus the mapping g is generalized $(\alpha,\ \beta)$-nonexpansive type-1 mapping.

The following Tables 1 and 2, and Please replace the word figures with Figure 1, 2 and 3 shows convergence behavior of iterative procedures with initial value $\eta_1=(0,\ 1)$ and $\alpha=0.5,\ \beta=0.3,\ \gamma=0.3$ on x-axis and y-axis, respectively.

Table 1. Comparison of iterations on x-axis

Mann	Ishikawa	Agarwal	Noor	Abbas	New
0.075000000000000	0.090000000000000	0.165000000000000	0.093000000000000	0.222000000000000	0.350000000000000
0.137500000000000	0.162000000000000	0.269500000000000	0.166780000000000	0.334480000000000	0.350000000000000
0.189583333333333	0.219600000000000	0.335683333333333	0.225312133333333	0.391469866666667	0.445061728395062
0.232986111111111	0.265680000000000	0.377599444444444	0.271747625777778	0.420344732444444	0.448902606310014
0.269155092592593	0.302544000000000	0.404146314814815	0.308586449783704	0.434974664438518	0.449756134735559
0.299295910493827	0.332035200000000	0.420959332716049	0.337811916828405	0.442387163315516	0.449945807719013
0.324413258744856	0.355628160000000	0.431607577386831	0.337811916828405	0.446142829413195	0.449987957270892
0.345344382287380	0.374502528000000	0.438351465678326	0.379391313520313	0.446142829413195	0.449997323837976
0.362786985239483	0.389602022400000	0.442622594929607	0.393983775392782	0.449009821452916	0.449999405297328
0.377322487699569	0.401681617920000	0.445327643455418	0.405560461811607	0.449498309536144	0.449999867843851
0.389435406416308	0.411345294336000	0.447040840855098	0.414744633037208	0.449745810164980	0.449999970631967
0.399529505346923	0.419076235468800	0.448125865874895	0.422030742209518	0.449871210483590	0.449999993473770
0.407941254455769	0.425260988375040	0.448813048387434	0.427811055486218	0.449934746645019	0.449999998549727
0.414951045379808	0.430208790700032	0.449248263978708	0.432396770685733	0.449966938300143	0.449999999677717
0.420792537816506	0.434167032560026	0.449248263978708	0.436034771410681	0.449983248738739	0.449999999928381
0.425660448180422	0.437333626048020	0.449698470329237	0.438920918652474	0.449991512694294	0.449999999984085
0.429717040150352	0.439866900838416	0.449809031208517	0.441210595464296	0.449995699765109	0.449999999996463
0.433097533458626	0.441893520670733	0.449879053098727	0.443027072401675	0.449997821214322	0.449999999999214
0.435914611215522	0.443514816536586	0.449923400295861	0.444468144105329	0.449998896081923	0.449999999999825
0.438262176012935	0.444811853229269	0.449951486854045	0.445611394323561	0.449999440681508	0.449999999999961
0.440218480010779	0.445849482583415	0.449969275007562	0.446518372830025	0.449999716611964	0.449999999999991
0.441848733342316	0.446679586066732	0.449980540838122	0.447237909111820	0.449999856416728	0.449999999999998
0.443207277785263	0.447343668853386	0.449987675864144	0.447808741228710	0.449999927251142	0.450000000000000
0.444339398154386	0.447874935082709	0.449992194713958	0.448261601374777	0.449999963140579	0.450000000000000

Table 2. Comparison of iterations on y-axis

Mann	Ishikawa	Agarwal	Noor	Abbas	New
0.833333333333333	0.800000000000000	0.633333333333333	0.793333333333333	0.506666666666667	0.222222222222222
0.694444444444444	0.640000000000000	0.401111111111111	0.629377777777778	0.256711111111111	0.049382716049383
0.578703703703704	0.512000000000000	0.254037037037037	0.499306370370370	0.130066962962963	0.010973936899863
0.482253086419753	0.409600000000000	0.160890123456790	0.396116387160494	0.065900594567901	0.002438652644414
0.401877572016461	0.327680000000000	0.101897078189300	0.314252333813992	0.033389634581070	0.000541922809870
0.334897976680384	0.262144000000000	0.064534816186557	0.249306851492433	0.016917414854409	0.000120427291082
0.000039305567154	0.209715200000000	0.040872050251486	0.197783435517331	0.008571490192900	0.000026761620240
0.279081647233653	0.167772160000000	0.025885631825941	0.156908192177082	0.004342888364403	0.000005947026720
0.232568039361378	0.134217728000000	0.016394233489763	0.124480499127152	0.002200396771297	0.000001321561493
0.193806699467815	0.107374182400000	0.010383014543516	0.098754529307540	0.001114867697457	0.000000293680332
0.161505582889846	0.085899345920000	0.006575909210894	0.078345259917315	0.000564866300045	0.000000065262296
0.134587985741538	0.068719476736000	0.004164742500233	0.062153906201070	0.000286198925356	0.000000014502732
0.112156654784615	0.054975581388800	0.002637670250147	0.049308765586182	0.000145007455514	0.000000003222829
0.093463878987179	0.043980465111040	0.001670524491760	0.039118287365038	0.000073470444127	0.000000000716184
0.077886565822649	0.035184372088832	0.001057998844781	0.031033841309597	0.000037225025024	0.000000000159152
0.064905471518874	0.028147497671066	0.000670065935028	0.024620180772280	0.000018860679346	0.000000000035367
0.054087892932395	0.022517998136852	0.000424375092185	0.019532010079342	0.000009556077535	0.000000000007859
0.045073244110329	0.018014398509482	0.000268770891717	0.015495394662945	0.000004841745951	0.000000000001747
0.037561036758608	0.014411518807586	0.000170221564754	0.012293013099270	0.000002453151282	0.000000000000388
0.031300863965507	0.011529215046068	0.000107806991011	0.009752457058754	0.000001242929983	0.000000000000086
0.026084053304589	0.009223372036855	0.000068277760974	0.007736949266611	0.000000629751191	0.000000000000019
0.021736711087157	0.007378697629484	0.000043242581950	0.006137979751512	0.000000319073937	0.000000000000004
0.018113925905964	0.005902958103587	0.000027386968568	0.004869463936199	0.000000161664128	0.000000000000001
0.015094938254970	0.004722366482870	0.000017345080093	0.003863108056051	0.000000081909825	0.000000000000000
0.012579115212475	0.003777893186296	0.000010985217392	0.003064732391134	0.000000041500978	0.000000000000000
0.010482596010396	0.003022314549037	0.000006957304349	0.002431354363633	0.000000021027162	0.000000000000000
0.008735496675330	0.002417851639229	0.000004406292754	0.001928874461816	0.000000010653762	0.000000000000000
0.007279580562775	0.001934281311383	0.000002790652078	0.001530240406374	0.000000005397906	0.000000000000000
0.006066317135646	0.001547425049107	0.000001767412982	0.001213990722390	0.000000002734939	0.000000000000000
0.005055264279705	0.001237940039285	0.000001119361556	0.000963099306429	0.000000001385702	0.000000000000000
0.004212720233087	0.000792281625143	0.000000708928985	0.000764058783101	0.000000000702089	0.000000000000000
0.003510600194240	0.000990352031428	0.000000448988357	0.000606153301260	0.000000000355725	0.000000000000000
0.002925500161866	0.000633825300114	0.000000284359293	0.000480881618999	0.000000000180234	0.000000000000000
0.002437916801555	0.000507060240091	0.000000180094219	0.000381499417740	0.000000000091319	0.000000000000000

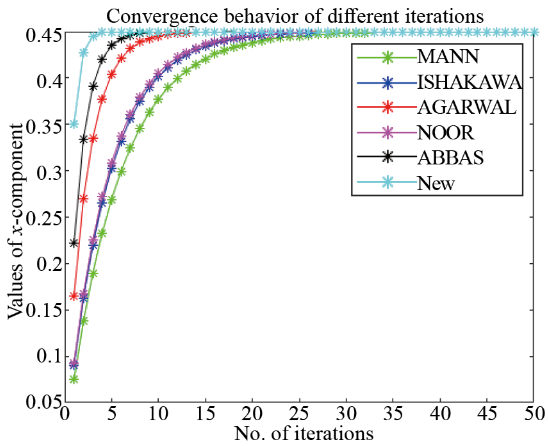

Figure 1. Convergence behavior of iterations

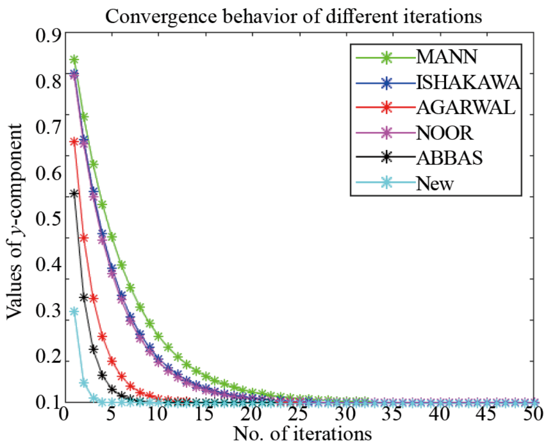

Figure 2. Convergence behavior of iterations

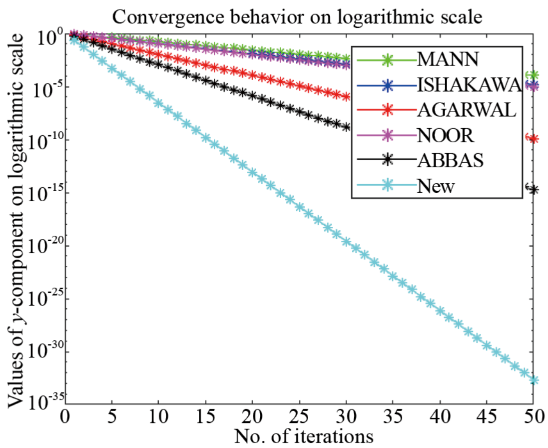

Figure 3. Convergence behavior of iterations on logarithmic scale

Example 2 Assume that $Q = [0,\ 1]$ is a closed and convex subset of $M = \mathbb{R}$ with usual metric. Now formalize a map $g:\ Q \to Q$ such that

$$g\eta = \frac{e^{\frac{\eta}{2}} - 1}{2}, \tag{11}$$

then g is generalized $(\alpha,\ \beta)$-nonexpansive type-1 mapping for every $\eta,\ \zeta \in Q,\ \eta \leq \zeta$ and $\alpha,\ \beta,\ \lambda \in \left[0,\ \frac{1}{6}\right],\ \alpha \leq \beta$.

Solution First, demonstrate that g is generalized $(\alpha,\ \beta)$-nonexpansive type-1 mapping. subsequently the convergence the behavior of the iteration used in this study is being analyzed and compared with the convergence behavior of other iterations.

It must be shown that if $\lambda d(g\eta,\ g\zeta) \leq d(\eta,\ \zeta)$ then

$$d(g\eta,\ g\zeta) \leq \alpha d(g\eta,\ \zeta) + \beta d(\eta,\ g\zeta) + (1 - (\alpha + \beta))d(\eta,\ \zeta),$$

First it is necessary to establish that $\lambda d(g\eta,\ g\zeta) \leq d(\eta,\ \zeta)$

$$d(g\eta,\ g\zeta) = \left| \frac{e^{\frac{\eta}{2}} - e^{\frac{\zeta}{2}}}{2} \right|,$$

as the function $g(\eta)$ is continuous on $[0,\ 1]$ and differentiable on $(0,\ 1)$ then by Mean value theorem there exists $c \in [0,\ 1]$ such that

$$g(\zeta) - g(\eta) = g'(c)(\zeta - \eta)$$

$$\frac{e^{\frac{\zeta}{2}} - e^{\frac{\eta}{2}}}{2} = \frac{e^{\frac{c}{2}}}{4}(\zeta - \eta),$$

as $\frac{e^{\frac{1}{2}}}{4} < \frac{1}{2}$ so,

$$\frac{e^{\frac{\zeta}{2}} - e^{\frac{\eta}{2}}}{2} \leq \frac{1}{2}(\zeta - \eta),$$

taking absolute value

$$\left| \frac{e^{\frac{\zeta}{2}} - e^{\frac{\eta}{2}}}{2} \right| \leq \frac{1}{2} |(\zeta - \eta)|$$

$$d(g\eta,\ g\zeta) \leq d(\eta,\ \zeta),$$

this implies that

$$\lambda d(g\eta,\ g\zeta) \leq d(\eta,\ \zeta).$$

Now it is necessary to prove that

$$d(g\eta,\ g\zeta) \leq \alpha d(g\eta,\ \zeta) + \beta d(\eta,\ g\zeta) + (1-(\alpha+\beta))d(\eta,\ \zeta),$$

consider

$$\alpha d(g\eta,\ \zeta) + \beta d(\eta,\ g\zeta) + (1-(\alpha+\beta))d(\eta,\ \zeta), \tag{12}$$

using $\alpha \leq \beta,\ \zeta \geq \eta$, we get that

$$\alpha\left(\left|\frac{e^{\frac{\eta}{2}}-1-2\zeta}{2}\right|\right) \leq \beta\left(\left|\frac{e^{\frac{\zeta}{2}}-1-2\eta}{2}\right|\right),$$

utilizing $\alpha \leq \beta$ and $\beta d(\eta,\ g\zeta) \geq \alpha d(g\eta,\ \zeta)$ in equation (12) implies that

$$\alpha d(g\eta,\ \zeta) - \alpha d(\eta,\ \zeta) + \beta d(\eta,\ g\zeta) - \beta d(\eta,\ \zeta) + d(\eta,\ \zeta) \geq d(\eta,\ \zeta),$$

hence it is shown that

$$\alpha d(g\eta,\ \zeta) - \alpha d(\eta,\ \zeta) + \beta d(\eta,\ g\zeta) - \beta d(\eta,\ \zeta) + d(\eta,\ \zeta) \geq d(g\eta,\ g\zeta),$$

thus g is generalized $(\alpha,\ \beta)$-nonexpansive type-1 mapping.

The following Please replace the word tables with Table 3 and Table 4 and Please replace the word figures with Figure 4 and Figure 5 show the convergence behavior of iterative procedures with initial value $\eta = 0.1,\ \zeta = 0.99$ and $\alpha = 0.12,\ \beta = 0.12$ and $\gamma = 0.2$.

Table 3. Comparison of iterations by numerical values

Mann	Ishikawa	Agarwal	Noor	Abbas	New
0.033071993369211	0.090795454803432	0.025354737208883	0.090780783623989	0.019802212099950	0.001334010239514
0.010810262187702	0.082432591527308	0.006309888848529	0.082406045850667	0.003849372992689	0.000017267805236
0.003519920545003	0.074835435169912	0.001562884134784	0.074799407067613	0.000745549626480	0.000000223430378
0.001144671514698	0.067934690970084	0.000386651349045	0.067891220816200	0.000144295957703	0.000000002890980
0.000372091959189	0.061667189355495	0.000095628063811	0.061618012101584	0.000027923634022	0.000000000037407
0.000120937675168	0.055975370971915	0.000023649382003	0.055921956607780	0.000005403536993	0.000000000000484
0.000039305567154	0.050806810092987	0.000005848526530	0.050750399014035	0.000001045639738	0.000000000000006
0.000012774396228	0.046113774510544	0.000001446342712	0.046055408422506	0.000000202341794	0.000000000000000
0.000004151687953	0.041852819887808	0.000000357680681	0.041793368801019	0.000000039155162	0.000000000000000
0.000001349299554	0.037984416503060	0.000000088454440	0.037924602299959	0.000000007576916	0.000000000000000
0.000000438522458	0.034472606304578	0.000000021874784	0.034413023304229	0.000000001466209	0.000000000000000
0.000000142519810	0.031284688226765	0.000000005409634	0.031225821117824	0.000000000283726	0.000000000000000
0.000000046318939	0.028390929772750	0.000000001337803	0.028333169240589	0.000000000054904	0.000000000000000
0.000000015053655	0.025764302942955	0.000000000330839	0.025707959276543	0.000000000010624	0.000000000000000
0.000000004892438	0.023380242675961	0.000000000081816	0.023325557605098	0.000000000002056	0.000000000000000
0.000000001590042	0.021216426063069	0.000000000020233	0.021163583045800	0.000000000000398	0.000000000000000
0.000000000516764	0.019252570697400	0.000000000005004	0.019201703850539	0.000000000000077	0.000000000000000
0.000000000167948	0.017470250619584	0.000000000001237	0.017421452461680	0.000000000000015	0.000000000000000
0.000000000054583	0.015852728422906	0.000000000000306	0.015806056578251	0.000000000000001	0.000000000000000
0.000000000017739	0.014384802179622	0.000000000000076	0.014340285173763	0.000000000000003	0.000000000000000
0.000000000005765	0.013052665945950	0.000000000000019	0.013010308207202	0.000000000000001	0.000000000000000
0.000000000001874	0.011843782695169	0.000000000000005	0.011803568862613	0.000000000000001	0.000000000000000
0.000000000000609	0.010746768615734	0.000000000000001	0.010708667241915	0.000000000000001	0.000000000000000
0.000000000000198	0.009751287794151	0.000000000000000	0.009715254519852	0.000000000000001	0.000000000000000
0.000000000000064	0.008847956380234	0.000000000000000	0.008813936649227	0.000000000000001	0.000000000000000
0.000000000000021	0.008028255405347	0.000000000000000	0.007996186778678	0.000000000000001	0.000000000000000
0.000000000000007	0.007284451492351	0.000000000000000	0.007254265614373	0.000000000000001	0.000000000000000
0.000000000000002	0.006609524759315	0.000000000000000	0.006581149021233	0.000000000000001	0.000000000000000
0.000000000000001	0.005997103277791	0.000000000000000	0.005970462218827	0.000000000000001	0.000000000000000

Table 4. CPU time comparison for Figure 4 and Figure 5

Iterations	Figure 1	Figure 2
Mann	0.29576 sec	1.1877 sec
Ishikawa	0.2051 sec	1.1656 sec
Agarwal	0.29942 sec	1.1772 sec
Noor	0.19578 sec	1.1824 sec
Abbas	0.19838 sec	1.1625 sec
New	0.20245sec	1.1598sec

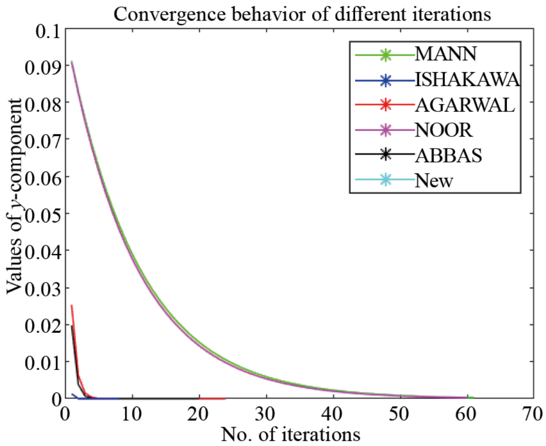

Figure 4. Convergence behavior of iterations

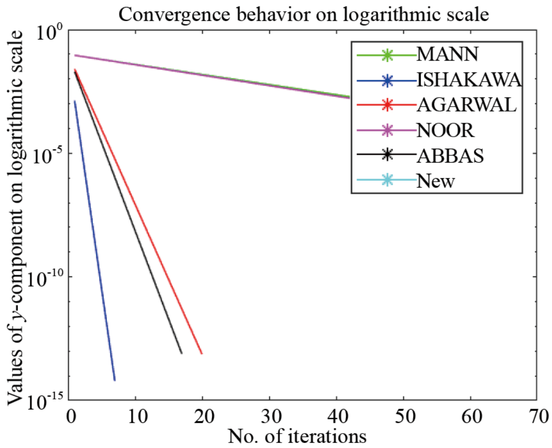

Figure 5. Convergence behavior of iterations in logarithmic scale

Now we will prove the existence of the solution of a nonlinear quadratic integral equation by applying our convergence result.

Application Let $C(Q)$ be the set of all continuous functions defined on $Q = [0, 1]$ and $d : C(Q)\times C(Q) \to \mathbb{R}$ defined by

$$d(\eta, \zeta) = \sup_{t\in Q} |\eta(t) - \zeta(t)|, \ \forall\, \eta, \zeta \in C(Q)$$

$(C(Q), d, H)$ is a hyperbolic space with modulus of uniform convexity where H is a convex structure. Assume S is the set of functions $T : [0, +\infty) \to [0, +\infty)$ which satisfies the following condition:

T is nondecreasing and $T(t) \le t$ for all $t \in [0, +\infty)$.

Consider the nonlinear quadratic equation as follows:

$$\eta(t) = x(t) + \alpha \int_0^1 (h(t,\ s) f(s,\ \eta(s)) ds,\ t \in Q,\ \alpha \geq 0, \tag{13}$$

where $x:\ Q \to \mathbb{R}$, $h: Q \times \mathbb{R} \to \mathbb{R}$ and $f: Q \times Q \to \mathbb{R}$.

Now let a mapping $g:\ C(Q) \to C(Q)$ defined as follows:

$$g(\eta)(t) = x(t) + \alpha \int_0^1 (h(t,\ s) f(s,\ \eta(s))) ds,\ t \in Q,\ \alpha \geq 0.$$

Suppose that the following condition holds:

1. $x: Q \to \mathbb{R}$ is continuous

2. $f: Q \times Q \to \mathbb{R}$ is continuous, $f(t,\ \zeta) \geq 0$ and there exists $P \geq 0$ and $T \in S$ such that for all $t \in Q$ and $a,\ b \in \mathbb{R}$, $|f(t,\ a) - f(t,\ b)| \leq PT(|a-b|)$;

3. $h:\ Q \times \mathbb{R} \to \mathbb{R}$ is continuous at $t \in Q$ for every $s \in Q$ and measurable at $s \in Q$ for all $t \in Q$ such that $h(t,\ s) \geq 0$ and $\int_0^1 (h(t,\ s) \leq K$;

4. $\lim_{j\to\infty} d(\eta_j,\ g\eta_j) = 0$.

Theorem 5 Under conditions (1)-(4), the integral equation (13) has a solution in $C(Q)$.

Proof. Using conditions 2 we obtain

$$|g(\eta)(t) - g(\zeta)(t)| \leq \alpha \int_0^1 (h(t,\ s) PT(|(\eta)(t) - (\zeta)(t|))) ds.$$

Since T is nondecreasing, we have

$$g(|(\eta)(t) - (\zeta)(t)|) \leq g\left(\sup_{t\in Q} |\eta(t) - \zeta(t)|\right) = T(d(\eta,\ \zeta))$$

From equation 3 we obtain that

$$|g(\eta)(t) - g(\zeta)(t)| \leq \alpha PT(d(\eta,\ \zeta))$$

Therefore,

$$\begin{aligned} d(g\eta,\ g\zeta) &= \sup_{t\in Q} |\eta(t) - \zeta(t)| \\ &\leq \alpha KPT(d(\eta,\ \zeta)) \\ &\leq (d(\eta,\ \zeta)) \end{aligned}$$

and for $\lambda \in (0,\ 1]$

$$d(g\eta,\ g\zeta)(\eta,\ \zeta) \tag{14}$$

$$\lambda d(g\eta,\ g\zeta) \leq d(\eta,\ \zeta)$$

Now, consider equation (14) and using triangular inequality, we have

$$d(g\eta,\ g\zeta) \leq \alpha d(g\eta,\ \zeta) + \beta d(\eta,\ g\zeta) + (1-(\alpha+\beta))d(\eta,\ \zeta).$$

Hence the mapping g on the set of all continuous functions is generalized $(\alpha,\ \beta)$-nonexpansive type-1 mapping. By Lemma 4, we have that $Fix(g)$ is nonempty. Now we have all the conditions of Theorem 4 so in the view of Theorem 4 we get sequence formalized by equation (5) Δ-converges to the unique fixed point of g in $C(Q)$. This implies that the integral equation (13) has a solution in $C(Q)$.

5. Conclusion

In this article, we generalized the iteration scheme in hyperbolic space and proved some strong Δ-convergence results using generalized $(\alpha,\ \beta)$-nonexpansive type-1 mapping. Our results generalized the results in linear space. At the end, we presented some examples of our results and gave a comparison with some other iterative schemes using computational tools.

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Conflicts of interest

The authors declare no conflicts of interest.

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