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Review

Review on Jacobi-Galerkin Spectral Method for Linear PDEs in Applied Mathematics

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Abstract: This study explores the spectral Galerkin approach to solving the space-time Schrödinger, wave, Airy, and beam equations. In order to facilitate the creation of a semi-analytical approximation solution, it uses polynomial bases that are formed from a linear combination of Jacobi polynomials (JPs) in both spatial and temporal dimensions. By using these polynomials to expand the exact solution, the paper hopes to satisfy the homogeneous starting and Dirichlet boundary requirements. Notably, the Jacobi Galerkin (JG) method exhibits exponential convergence rates if the solution is sufficiently smooth. This result emphasizes the JG approach's potential as an effective numerical solution method, which has promise for a variety of applications in other domains where these equations occur, such as quantum mechanics, acoustics, optics, and structural mechanics.

***Keywords*:** Galerkin method, Jacobi Polynomials, Schrödinger equation, Airy equation, wave equation, beam model

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1. Introduction

Spectral techniques for differential problems have advanced quickly over the past fifty years. The remarkable accuracy of these makes them fascinating. As a result, they have been successfully used to numerically simulate various issues in science and engineering, as seen in [1–3]. Sturm-Liouville problems are crucial for spectral approaches because the spectral approximation of a differential equation's solution is typically considered a finite expansion of the eigenfunctions of a suitable Sturm-Liouville problem.

Jacobi polynomials $J_n^{(\zeta,\eta)}(x) (n = 0, 1, 2, \ldots)$ are crucial for mathematical analysis and its applications. It is demonstrated that only the JPs can arise as eigenfunctions of a single Sturm-Liouville problem ([3], Chapter 3). All polynomial solutions to singular Sturm-Liouville problems on $[-1, 1]$ can be found in this class of polynomials. Particular instances of the JPs include Chebyshev, Legendre, and ultraspherical polynomials. Jacobi spectral methods and their special cases are of important use in numerical analysis and scientific computing. They are extensively used to handle many models, such as ordinary differential equations, fractional differential equations, delay differential equations, partial differential equations, high-order PDEs, time-dependent PDEs in physics, and many real-world applications, like the Lane-Emden equation in astronomy and the Burger equation in fluid dynamics. See, for instance, [4–9].

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The choice of setting the time domain to $[-1, 1]$ in numerical methods, particularly when using Jacobi polynomials for time discretization, may seem atypical compared to the more common practice of using $[0, 1]$. However, there are reasons behind this choice that are rooted in the mathematical properties and advantages offered by Jacobi polynomials. Jacobi polynomials are a set of mathematical functions that are orthogonal to each other. They are defined on the interval between -1 and 1 (see, [10, 11]). They have desirable properties such as orthogonality, which means that different polynomials in the family are orthogonal to each other with respect to a certain weight function. These properties make Jacobi polynomials well-suited for approximating functions with respect to certain inner products. When using Jacobi polynomials for time discretization, the choice of setting the time domain to $[-1, 1]$ aligns with the domain of orthogonality for these polynomials. This allows us to leverage their orthogonality properties effectively in numerical approximations or spectral methods (see, [12]). By restricting the time domain to $[-1, 1]$, we can utilize the advantageous properties of Jacobi polynomials to accurately and efficiently represent functions in terms of polynomial expansions. Additionally, working with a symmetric interval such as $[-1, 1]$ simplifies the mathematics involved in the numerical methods. It enables symmetries and simplifications in algorithmic implementation and reduces the computational complexity compared to working with an asymmetric interval. While setting the time domain to $[-1, 1]$ may appear atypical compared to the traditional $[0, 1]$ interval, it offers practical advantages in terms of leveraging the properties of Jacobi polynomials, simplifying mathematical formulations, and facilitating efficient numerical computations.

The common spectral technique is only applicable to nonsingular problems on rectangular bounded domains and is based on the Legendre or Chebyshev approximation. However, a variety of problems can be solved using the broader Jacobi technique; see [13–16]. The p-version of the finite element approach, the boundary element method, and a spectral method for axisymmetrical domains, on the other hand, were all examined using some Jacobi approximation results, and various rational spectral methods in [17–23], besides, in the study of quadratures involving the values of derivatives of functions at endpoints. It is crucial to understand that, in the context of the Galerkin technique, the basis functions (BFs) used to determine how well spectral methods approximate problems compared to finite difference and finite element methods [24].

Here, we apply the JG approximations to directly solve linear PDEs with initial and boundary conditions. We offer suitable bases that fulfill these conditions and can be used efficiently as a starting step for handling linear PDEs. These bases resulted in discrete systems with effectively invertible, exceptionally structured matrices.

The motivation to find numerical spectral solutions for linear partial differential equations, such as the Schrödinger, wave, Airy, and beam equations, stems from their extensive applicability in science and engineering. These equations describe diverse aspects of the physical world, from quantum mechanics to wave phenomena, optics, and structural behavior. Numerical solutions are crucial because these equations often lack analytical solutions for complex scenarios, enabling researchers and engineers to model and optimize systems, make predictions in various fields, and facilitate scientific exploration. Additionally, the teaching of numerical methods for solving linear PDEs plays a pivotal role in educating future scientists and engineers, emphasizing their practical importance in addressing real-world challenges when analytical solutions are unavailable. For more details about the applications of linear PDEs, the interested reader is referred to [25, 26]. Solving linear partial differential equations (PDEs) presents several challenges, including complexities in handling complex geometries and boundaries, ensuring numerical stability, minimizing discretization errors, achieving convergence, defining appropriate boundary conditions, managing high-dimensional and time-dependent problems, addressing nonlinearity within linear PDEs, and optimizing parallel computing resources. Additionally, challenges arise in applications such as optimization and sensitivity analysis, adaptive methods, heterogeneous media, memory and storage management, and leveraging high-performance computing. Successfully tackling these challenges requires expertise in numerical methods, computational science, and domain-specific knowledge to obtain accurate and efficient solutions for a wide range of real-world problems.

This paper is organized as follows: Section 2 is dedicated to the preliminary properties of JPs. Section 3 is the central part, providing in-depth, new spectral Galerkin algorithms for solving the Schrödinger equation, Airy equation, wave equation, and beam model. Section 4 is a brief overview of how to easily handle nonhomogeneous conditions. Section 5 is dedicated to finding an upper estimate for the unknown expansion coefficients and the rate of convergence for truncation errors. Section 6 presents some numerical test problems along with comparisons, and finally, Section 7 contains concluding remarks.

2. Some properties of JPs

The JPs with the real parameters $(\zeta > -1,\ \eta > -1)$ (see, Luke [27] and Szegö [28]), are a sequence of polynomials $J_n^{(\zeta,\,\eta)}(z)(n = 0,\ 1,\ 2,\ ...)$, constitute an orthogonal system concerning the weight function $w^{(\zeta,\,\eta)}(z) = (1-z)^{\zeta}(z+1)^{\eta}$, that is,

$$\int_{-1}^{1} w^{(\zeta,\,\eta)}(z) J_m^{(\zeta,\,\eta)}(z) J_n^{(\zeta,\,\eta)}(z)\,dz = \gamma_n^{(\zeta,\,\eta)}\delta_{mn},$$

where δ_{mn} is the Kronecker function and

$$\gamma_n^{(\zeta,\,\eta)} = \frac{2^{\eta+\zeta+1}\,\Gamma(n+\zeta+1)\,\Gamma(n+\eta+1)}{(2n+\eta+\zeta+1)\,n!\,\Gamma(n+\eta+\zeta+1)}\ . \tag{1}$$

Standardizing the JPs makes sense for our current needs so that

$$J_n^{(\zeta,\,\eta)}(1) = \frac{(\zeta+1)_n}{n!}, \quad J_n^{(\zeta,\,\eta)}(-1) = \frac{(-1)^n\,(\eta+1)_n}{n!}, \tag{2}$$

where $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$. These polynomials may be generated using the standard three-term difference equation of JPs with initials $J_0^{(\zeta,\,\eta)}(z) = 1$ and $J_1^{(\zeta,\,\eta)}(z) = \frac{1}{2}[\zeta-\eta+(1+\lambda)z]$, or obtained from Rodrigue's formula

$$J_n^{(\zeta,\,\eta)}(z) = \frac{(-1)^n}{2^n\ n!}(1-z)^{-\zeta}(z+1)^{-\eta}D^n\left[(1-z)^{\zeta+n}(z+1)^{\eta+n}\right],$$

where $\lambda = \eta+\zeta+1$, and denote D as the differential operator $\frac{d}{dz}$. When $\eta = \zeta$, one retrieves the ultraspherical polynomials, also known as symmetric Jacobi polynomials. Similarly, when $\eta = \zeta = \pm\frac{1}{2}$ or $\eta = \zeta = 0$, one obtains the Legendre polynomials, the Chebyshev polynomials of the first- and second-kinds, respectively. Additionally, for nonsymmetric Jacobi polynomials, the special cases $\zeta = -\eta = \pm\frac{1}{2}$ correspond to the Chebyshev polynomials of the third- and fourth-kinds.

The special values

$$D^q J_n^{(\zeta,\,\eta)}(1) = \prod_{i=0}^{q-1}\frac{\Gamma(n+\zeta+1)(n+\lambda+i)}{2^q(n-q)!\,\Gamma(q+\zeta+1)}, \quad D^q J_n^{(\zeta,\,\eta)}(-1) = (-1)^{n+q} D^q J_n^{(\eta,\,\zeta)}(1),$$

will be of important use later. The Jacobi-Gauss quadrature is commonly used to evaluate the previous integrals. For any $\varphi \in P_{2N+1}(\Lambda)$ and $\Lambda = (-1,\ 1)$ denotes a bounded domain, we have

$$\int_{-1}^{1} \varphi(z) w^{(\zeta,\,\eta)}(z) dz = \sum_{i=0}^{N} \varpi_i^{(\zeta,\,\eta)} \varphi(z_i^{(\zeta,\,\eta)}),$$

$P_N(\Lambda)$ represents the set of polynomials with a degree up to N. The symbols $\varpi_i^{(\zeta,\,\eta)}$ and $z_i^{(\zeta,\,\eta)}$ (where $0 \le i \le N$) represent the Christoffel numbers and the nodes in Λ, respectively, following standard conventions. In the Jacobi-Gauss scheme, the values of $z_i^{(\zeta,\,\eta)}$ (where $0 \le i \le N$) correspond to the zeros of $J_{N+1}^{(\zeta,\,\eta)}(z)$ and the weights

$$\varpi_i^{(\zeta,\,\eta)} = \frac{G_N^{(\zeta,\,\eta)}}{(1-(z_i^{(\zeta,\,\eta)})^2)\left[\partial_z J_{N+1}^{(\zeta,\,\eta)}(z_i^{(\zeta,\,\eta)})\right]^2}, \quad 0 \le i \le N,$$

where

$$G_N^{(\zeta,\,\eta)} = \frac{2^{\zeta+\eta+1}\Gamma(N+\zeta+2)\Gamma(N+\eta+2)}{(N+1)!\Gamma(2N+\zeta+\eta+2)}.$$

The following lemma is essential in our sequel work.

Lemma 1 The qth derivative of $J_n^{(\zeta,\,\eta)}(z)$ can be written as

$$D^q J_k^{(\zeta,\,\eta)}(z) = \sum_{i=0}^{k-q} C_q(k,\,i,\,\zeta,\,\eta) J_i^{(\zeta,\,\eta)}(z), \tag{3}$$

where

$$C_q(k,\,i,\,\zeta,\,\eta) = \frac{(k+\lambda)_q (k+\lambda+q)_i (i+\zeta+q+1)_{k-i-q}\,\Gamma(i+\lambda)}{2^q (k-i-q)!\,\Gamma(2i+\lambda)}$$

$$\times\, {}_3F_2\left(\begin{matrix} -k+i+q, & k+i+\lambda+q, & i+\zeta+1 \\ i+\zeta+q+1, & 2i+\lambda+1 & \end{matrix} ;1\right),$$

where in this context, let $\lambda = \zeta+\eta+1$, and $(\cdot)_i$ denotes the Pochhammer symbol. Refer to [27] for the definition of generalized hypergeometric functions, including the special ${}_3F_2$.

3. Linear PDEs

In this section, using the Galerkin technique and Jacobi expansions in both time and space, we construct a time-space discretization for the following linear partial differential equations. The space-time spectral Galerkin method to hyperbolic and advection-reaction-diffusion equations are studied in [14] and [29], respectively. Here are some common linear PDEs used in various applications: Schrodinger, wave, Airy, and beam equations. We consider the simplest case where both spatial and temporal domains are within the range of -1 to 1. There is no loss of generality, as this can always be achieved by simply changing the variables. Because this can always be accomplished by simply changing the variables, there is no loss of generality.

3.1 *Schrödinger equation*

The linear Schrödinger equation is

$$\frac{\partial u}{\partial t} - I\frac{\partial^2 u}{\partial z^2} = g(z,\, t),\;\; (z,\, t) \in \Omega\colon = \Lambda^2, \tag{4}$$

with boundary conditions

$$u(\pm 1,\, t) = 0,\;\; t \in (-1,\, 1), \tag{5}$$

together with initial condition

$$u(z,\, -1) = 0,\;\; z \in (-1,\, 1), \tag{6}$$

where $I = \sqrt{-1}$. We insert a proposed solution into equations (4)-(6) and demand that the overall residual resulting from projecting onto the space formed by the test functions equals zero. Here, $P_N(\Lambda)$ and $P_M(\Lambda)$ represent the collection of polynomials with degrees up to N in space and M in time, respectively. As we consider $u(z,\, -1) \equiv 0$ as well as $u(\pm 1,\, t) \equiv 0$, we choose suitable basis for the time ansatz from

$$P_M^t(\Lambda) = \{y \in P_M(\Lambda)\,|y(-1) \equiv 0\},$$

as well as for space

$$P_N^s(\Lambda) = \{y \in P_M(\Lambda)\,|y(\pm 1) \equiv 0\},$$

For the sake of convenience, define

$$S_L\colon = P_N(\Lambda) \otimes P_M(\Lambda),$$

$$W_L\colon = P_N^s(\Lambda) \otimes P_M^t(\Lambda),$$

where the multiindex $L = (N,\, M)$. In order to represent the integrals used in the JG spectral formulation of the model equations, we introduce the notation below

$$\langle\langle\langle\cdot\rangle\rangle\rangle \equiv \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \cdot\, w^{(\zeta,\,\eta)}(z)\, w^{(\zeta,\,\eta)}(y)\, w^{(\zeta,\,\eta)}(t)\, dz\, dy\, dt,$$

$$\langle\langle\cdot\rangle\rangle \equiv \int_{-1}^{1}\int_{-1}^{1} \cdot\, w^{(\zeta,\,\eta)}(z)\, w^{(\zeta,\,\eta)}(t)\, dz\, dt,$$

$$\langle\cdot\rangle_z \equiv \int_{-1}^{1} \cdot\, w^{(\zeta,\,\eta)}(z)\, dz,$$

$$\langle\cdot\rangle_t \equiv \int_{-1}^{1} \cdot\, w^{(\zeta,\,\eta)}(t)\, dt.$$

The discrete solution is written as follows in terms of a matrix $\mathbf{U}_{ij}^{(1)}$ with unknown coefficients:

$$u(z,\,t) \backsimeq \hat{u}_1(z,\,t) = \sum_{i=0}^{N-2}\sum_{j=0}^{M-1} \phi_i^{(\zeta,\,\eta)}(z)\mathbf{U}_{ij}^{(1)}\,\psi_j^{(\zeta,\,\eta)}(t). \tag{7}$$

where $\phi_i^{(\zeta,\,\eta)}(z) \in P_N^s(\Lambda)$ and $\psi_j^{(\zeta,\,\eta)}(t) \in P_M^t(I)$.

The Galerkin problem is then presented by finding $\hat{u}_1 \in W_L$ in such a way

$$\langle\langle \hat{v}_1 \partial_t \hat{u}_1\rangle\rangle - I\left\langle\left\langle \hat{v}_1 \partial_x^2 \hat{u}_1\right\rangle\right\rangle = \langle\langle g\hat{v}_1\rangle\rangle\,,\ \ \forall \hat{v}_1 \in W_L, \tag{8}$$

The linear system associated with (8) relies on the selection of basis functions $\phi_i^{(\zeta,\,\eta)}(z)$ and $\psi_j^{(\zeta,\,\eta)}(t)$ over W_L. To ensure that the resulting system remains sparse and easily invertible, it is essential to meticulously choose a suitable basis for both spatial and temporal domains. Hence, we explore basis functions in the form of a compact array of Jacobi polynomials. Therefore, we choose the basis functions of expansion $\phi_i^{(\zeta,\,\eta)}(z)$ and $\psi_j^{(\zeta,\,\eta)}(t)$ to be of the form:

$$\phi_i^{(\zeta,\,\eta)}(z) = J_i^{(\zeta,\,\eta)}(z) + \varepsilon_i J_{i+1}^{(\zeta,\,\eta)}(z) + \varsigma_i J_{i+2}^{(\zeta,\,\eta)}(z),$$

$$\psi_j^{(\zeta,\,\eta)}(t) = J_j^{(\zeta,\,\eta)}(t) + \rho_j J_{j+1}^{(\zeta,\,\eta)}(t),$$

where the parameters $\{\varepsilon_i,\ \varsigma_i\}$ and $\{\rho_j\}$ are selected to meet the Dirichlet border and homogenous beginning conditions such that $\phi_i^{(\zeta,\,\eta)}(z) \in P_N^s(\Lambda)$ for $i = 0,\,1,\,\ldots,\,N-2$ and $\psi_j^{(\zeta,\,\eta)}(t) \in P_M^t(I)$ for $j = 0,\,1,\,\ldots,\,N-1$. While the choice of these modal BFs may seem random or unexplained, there is actually a specific reason for their selection, it can be demonstrated that they provide an appropriate basis that enables straightforward evaluation of the involved derivatives when combined with (3).

Lemma 2 For all $i \geq 0$, there exists a unique set of $\{\varepsilon_i,\ \varsigma_i,\ \rho_i\}$ such that

$$\phi_i^{(\zeta,\eta)}(z) = J_i^{(\zeta,\eta)}(z) + \varepsilon_i J_{i+1}^{(\zeta,\eta)}(z) + \varsigma_i J_{i+2}^{(\zeta,\eta)}(z),$$

$$\psi_i^{(\zeta,\eta)}(t) = J_i^{(\zeta,\eta)}(t) + \rho_i J_{i+1}^{(\zeta,\eta)}(t),$$

verify the boundary-initial conditions in (5) and (6).

Proof. From the boundary-initial conditions; $\phi_i^{(\zeta,\eta)}(-1) = \phi_i^{(\zeta,\eta)}(1) = 0$ and the two relations (2), we have the following system

$$-\varepsilon_i \frac{(i+\eta+1)}{(i+1)} + \varsigma_i \frac{(i+\eta+1)(i+\eta+2)}{(i+1)(i+2)} = -1,$$

$$\varepsilon_i \frac{(i+\zeta+1)}{(i+1)} + \varsigma_i \frac{(i+\zeta+1)(i+\zeta+2)}{(i+1)(i+2)} = -1.$$

Therefore, ε_i and ς_i can be determined in a unique manner to provide the solution

$$\varepsilon_i = -\frac{(i+1)(\eta-\zeta)(2i+\zeta+\eta+3)}{(i+\zeta+1)(i+\eta+1)(2i+\zeta+\eta+4)},$$

$$\varsigma_i = -\frac{(i+1)(i+2)(2i+\zeta+\eta+2)}{(i+\zeta+1)(i+\eta+1)(2i+\zeta+\eta+4)}.$$

Also, from the initial condition $\psi_j^{(\zeta,\eta)}(0) = 0$ and the relation (2), we have that

$$\rho_j \frac{(j+\eta+1)}{(j+1)} = 1.$$

Hence ρ_j can be uniquely determined to give

$$\rho_j = \frac{(j+1)}{(j+\eta+1)}.$$

□

It is clear that $\phi_i^{(\zeta,\eta)}(z) \in P_{N-2}^s(\Lambda)$ and $\psi_j^{(\zeta,\eta)}(t) \in P_{M-1}^t(\Lambda)$ are two linearly independent basis function sets. Therefore, by dimension argument, we have

$$P_M^t(\Lambda) = \text{span}\left\{\psi_j^{(\zeta,\eta)}(t):\ j = 0,\ 1,\ 2,\ \dots,\ M-1\right\}, \qquad P_N^s(\Lambda) = \text{span}\left\{\phi_j^{(\zeta,\eta)}(z):\ j = 0,\ 1,\ 2,\ \dots,\ N-2\right\}.$$

It is clear that the Galerkin formulation of (8) is equivalent to following discrete discretization

$$\left\langle\left\langle \phi_i^{(\zeta,\,\eta)}(z)\partial_t \hat{u}_1(z,\,t)\psi_j^{(\zeta,\,\eta)}(t)\right\rangle\right\rangle - I\left\langle\left\langle \phi_i^{(\zeta,\,\eta)}(z)\partial_x^2 \hat{u}_1(z,\,t)\psi_j^{(\zeta,\,\eta)}(t)\right\rangle\right\rangle = \left\langle\left\langle \phi_i^{(\zeta,\,\eta)}(z)g(z,\,t)\psi_j^{(\zeta,\,\eta)}(t)\right\rangle\right\rangle,$$

for $0 \le i \le N-2$ and $0 \le j \le M-1$, we establish the following conventions for clarity: the indices i and r are assumed to range from 0 to $N-2$, while the indices j and s range from 0 to $M-1$. Additionally, we adopt the convention of summing over repeated indices.

The discretization method explained earlier can be expressed in the form of a matrix:

$$\left\langle \phi_i^{(\zeta,\,\eta)}\phi_r^{(\zeta,\,\eta)}\right\rangle_z \mathbf{U}_{ij}^{(1)}\left\langle \frac{d\psi_j^{(\zeta,\,\eta)}}{dt}\psi_s^{(\zeta,\,\eta)}\right\rangle_t - I\left\langle \phi_i^{(\zeta,\,\eta)}\frac{d^2\phi_r^{(\zeta,\,\eta)}}{dz^2}\right\rangle_z \mathbf{U}_{ij}^{(1)}\left\langle \psi_{\mathscr{T},\,j}^{(\zeta,\,\eta)}\psi_{\mathscr{T},\,s}^{(\zeta,\,\eta)}\right\rangle_t$$

$$= \left\langle\left\langle \phi_i^{(\zeta,\,\eta)}(z)g(z,\,t)\psi_j^{(\zeta,\,\eta)}(t)\right\rangle\right\rangle.$$

Let us denote

$$\mathbf{U}^{(1)} = \left(\mathbf{U}_{ij}^{(1)}\right),\ \mathbf{G} = (\mathbf{G}_{ij}),\ \mathbf{A} = (\mathbf{A}_{ir}),\ \mathbf{B} = (\mathbf{B}_{ir}),\ \mathbf{D} = (\mathbf{D}_{ir}),\ \mathbf{E} = (\mathbf{E}_{js}),$$

where $\mathbf{U}$ is the matrix of unknown coefficients, $\mathbf{G}$ is the reaction matrix whose entries are $\mathbf{G}_{ij} = \langle\langle\, \phi_i^{(\zeta,\,\eta)}(z)g(z,\,t)\psi_j^{(\zeta,\,\eta)}$ $(t)\rangle\rangle$, and the nonzero elements of the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{D}$ and $\mathbf{E}$ are given altogether in Theorem 1. The previous integral can be evaluated with the Jacobi-Gauss quadrature. The JG discretization of the Schrödinger equation (4) is equivalent to the following matrix equation (ME)

$$\mathbf{A}\mathbf{U}^{(1)}\mathbf{D} - I\mathbf{B}\mathbf{U}^{(1)}\mathbf{E} = \mathbf{G}. \tag{9}$$

To get the solution of (9), we rewrite it in a more suitable form using the Kronecker product (KP) (denoted by $\otimes$). Let $\mathbf{F} \in \mathbb{R}^{n,\,m}$ and $\mathbf{G} \in \mathbb{R}^{q,\,p}$, then the KP of $\mathbf{F}$ and $\mathbf{G}$ is defined as the matrix

$$\mathbf{F} \otimes \mathbf{G} = \begin{pmatrix} f_{11}\mathbf{G} & f_{12}\mathbf{G} & \cdots & f_{1m}\mathbf{G} \\ f_{21}\mathbf{G} & f_{22}\mathbf{G} & \cdots & f_{2m}\mathbf{G} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}\mathbf{G} & f_{n2}\mathbf{G} & \cdots & f_{nm}\mathbf{G} \end{pmatrix} \in \mathbb{R}^{nq,\,mp}.$$

Let $f_i \in \mathbb{R}^n$ denote the columns of $\mathbf{F} \in \mathbb{R}^{n,\,m}$, such that $\mathbf{F} = [f_1,\,\ldots,\,f_m]$. The vector formed by stacking the columns of $\mathbf{F}$ on top of one another is called $\mathrm{vec}(\mathbf{F})$, which is an nm-vector, i.e.

$$\mathrm{vec}(\mathbf{F}) = \begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix} \in \mathbb{R}^{nm}.$$

The following helpful attribute of the KP and the vec operator will be used in the following discussions. We have the matrix product **FGH** for any three matrices **F**, **G** and **H** for which

$$\mathrm{vec}(\mathbf{FGH}) = \left(\mathbf{H}^T \otimes \mathbf{F}\right) \mathrm{vec}(\mathbf{G}),$$

we can express the set of discrete equations (9) in the following matrix form (MF) where T denotes the transpose.

$$\left(\mathbf{D}^T \otimes \mathbf{A} - I\mathbf{E}^T \otimes \mathbf{B}\right) \mathrm{vec}(\mathbf{U}^{(1)}) = \mathrm{vec}(\mathbf{G}).$$

The solution to this linear system can be obtained using an appropriate iterative method to compute the numerical solution (7). In our implementation, we solved these systems using the Mathematica function FindRoot with zero initial approximation.

Theorem 1 Let

$$\mathbf{A}_{ir} = \left\langle \phi_i^{(\zeta,\,\eta)} \phi_r^{(\zeta,\,\eta)} \right\rangle_z, \quad \mathbf{B}_{ir} = \left\langle \phi_i^{(\zeta,\,\eta)} \frac{d^2 \phi_r^{(\zeta,\,\eta)}}{dz^2} \right\rangle_z,$$

$$\mathbf{D}_{js} = \left\langle \frac{d\psi_j^{(\zeta,\,\eta)}}{dt} \psi_s^{(\zeta,\,\eta)} \right\rangle_t, \quad \mathbf{E}_{js} = \left\langle \psi_j^{(\zeta,\,\eta)} \psi_s^{(\zeta,\,\eta)} \right\rangle_t.$$

Then the nonzero elements $\mathbf{A}_{ir}$, $\mathbf{B}_{ir}$, $\mathbf{D}_{js}$ and $\mathbf{E}_{js}$ are given by

$$\mathbf{A}_{ii} = \gamma_i^{(\zeta,\,\eta)} + \varepsilon_i^2 \gamma_{i+1}^{(\zeta,\,\eta)} + \varsigma_i^2 \gamma_{i+2}^{(\zeta,\,\eta)},$$

$$\mathbf{A}_{i+1,\,i} = \mathbf{A}_{i,\,i+1} = \varepsilon_i \gamma_{i+1}^{(\zeta,\,\eta)} + \varsigma_i \varepsilon_{i+1} \gamma_{i+2}^{(\zeta,\,\eta)},$$

$$\mathbf{A}_{i+2,\,i} = \mathbf{A}_{i,\,i+2} = \varsigma_i \gamma_{i+2}^{(\zeta,\,\eta)},$$

$$\mathbf{B}_{ii} = \varsigma_i \mathscr{A}_2(i+2,\, i,\, \zeta,\, \eta) \gamma_i^{(\zeta,\,\eta)},$$

$$\mathbf{B}_{ir} = O_2(r,\, i,\, \zeta,\, \eta) \gamma_i^{(\zeta,\,\eta)} + O_2(r,\, i+1,\, \zeta,\, \eta) \varepsilon_i \gamma_{i+1}^{(\zeta,\,\eta)}$$

$$+ O_2(r,\, i+2,\, \zeta,\, \eta) \varsigma_i \gamma_{i+2}^{(\zeta,\,\eta)}, \quad r = i+n, \quad n \geq 1,$$

$$\mathbf{D}_{jj} = \rho_j \mathscr{A}_1(j+1,\, j,\, \zeta,\, \eta) \gamma_j^{(\zeta,\,\eta)},$$

$$\mathbf{D}_{js} = \chi_1(s,\, j,\, \zeta,\, \eta) \gamma_j^{(\zeta,\,\eta)} + \chi_1(s,\, j+1,\, \zeta,\, \eta) \rho_j \gamma_{j+1}^{(\zeta,\,\eta)}, \quad s = j+n, \quad n \geq 1,$$

$$\mathbf{E}_{jj} = \gamma_j^{(\zeta,\,\eta)} + \rho_j^2 \gamma_{j+1}^{(\zeta,\,\eta)},$$

$$\mathbf{E}_{j+1,\,j} = \mathbf{E}_{j,\,j+1} = \rho_j \gamma_{j+1}^{(\zeta,\,\eta)},$$

where

$$O_\sigma(r,\,i,\,\zeta,\,\eta) = \mathscr{A}_\sigma(r,\,i,\,\zeta,\,\eta) + \varepsilon_r \mathscr{A}_\sigma(r+1,\,i,\,\zeta,\,\eta) + \varsigma_r \mathscr{A}_\sigma(r+2,\,i,\,\zeta,\,\eta),$$

$$\chi_\sigma(s,\,j,\,\zeta,\,\eta) = \mathscr{A}_\sigma(s,\,j,\,\zeta,\,\eta) + \rho_s \mathscr{A}_\sigma(s+1,\,j,\,\zeta,\,\eta),$$

and $\gamma_i^{(\zeta,\,\eta)}$ is given by (1).

Proof. The basis functions $\phi_i^{(\zeta,\,\eta)}(z)$ and $\psi_j^{(\zeta,\,\eta)}(t)$ are selected such that $\phi_i^{(\zeta,\,\eta)}(z) \in P_N^s(\Lambda)$ for $i = 0, 1, \ldots, N-2$ and $\psi_j^{(\zeta,\,\eta)}(t) \in P_M^t(I)$ for $j = 0, 1, \ldots, N-1$. Additionally, it is evident that $\phi_i^{(\zeta,\,\eta)}(z)$ and $\psi_j^{(\zeta,\,\eta)}(t)$ are linearly independent, and the dimensions of $P_N^s(\Lambda)$ and $P_M^t(I)$ are both $(N-1)$ and (N) respectively. The nonzero elements $\mathbf{A}_{ir}$ for $0 \leqslant i,\, r \leqslant N-2$ can be computed directly using the properties of Jacobi polynomials. Specifically, the diagonal elements of the matrix $\mathbf{A}$ have the form:

$$\mathbf{A}_{ii} = \gamma_i^{(\zeta,\,\eta)} + \varepsilon_i^2 \gamma_{i+1}^{(\zeta,\,\eta)} + \varsigma_i^2 \gamma_{i+2}^{(\zeta,\,\eta)},$$

Furthermore, all other formulas can be derived through direct computations utilizing the properties of Jacobi polynomials. □

Corollary 1 (Legendre Case) If $\eta = \zeta = 0$, then the nonzero elements $\mathbf{A}_{ir}$, $\mathbf{B}_{ir}$, $\mathbf{D}_{js}$ and $\mathbf{E}_{js}$ are expressed as follows:

$$\mathbf{A}_{ii} = \frac{4(2i+3)}{(2i+1)(2i+5)}, \qquad \mathbf{A}_{i,\,i+2} = \mathbf{A}_{i+2,\,i} = -\frac{2}{2i+5},$$

$$\mathbf{B}_{ii} = -2(2i+3),$$

$$\mathbf{D}_{jj} = 2, \qquad \mathbf{D}_{js} = 4,\ \ s = j+n,\ \ n \geq 1,$$

$$\mathbf{E}_{jj} = \frac{8(j+1)}{(2j+1)(2j+3)}, \qquad \mathbf{E}_{j,\,j+1} = \mathbf{E}_{j+1,\,j} = \frac{2}{(2j+3)}.$$

Corollary 2 (ChebyshevU Case) If $\eta = \zeta = \frac{1}{2}$, then the nonzero elements $\mathbf{A}_{ir}$, $\mathbf{B}_{ir}$, $\mathbf{D}_{js}$ and $\mathbf{E}_{js}$ are expressed as follows:

$$\mathbf{A}_{ii} = \frac{4\left(i^2+4i+5\right)\Gamma\left(i+\frac{3}{2}\right)^2}{(i+3)^2\Gamma(i+2)^2}, \qquad \mathbf{A}_{i,\, i+2} = \mathbf{A}_{i+2,\, i} = \frac{2\Gamma\left(i+\frac{3}{2}\right)\Gamma\left(i+\frac{7}{2}\right)}{(i+3)\Gamma(i+1)\Gamma(i+4)},$$

$$\mathbf{B}_{ii} = -\frac{8(i+2)\Gamma\left(i+\frac{3}{2}\right)^2}{(i+3)\Gamma(i+1)^2},$$

$$\mathbf{B}_{ir} = \frac{2^{1-2i}\sqrt{\pi}\Gamma(i+\frac{2n+3}{2})\Gamma(2i+5)}{(r+3)(2i+3)\Gamma(i+1)\Gamma(i+2)\Gamma(r+2)}, \ r = i+n, \ n \text{ even number},$$

$$\mathbf{D}_{jj} = \frac{4(j+1)\Gamma\left(j+\frac{3}{2}\right)^2}{\Gamma(j+1)\Gamma(j+3)}, \qquad \mathbf{D}_{js} = \frac{8\Gamma(j+\frac{2n+5}{2})\Gamma(j+\frac{3}{2})}{\Gamma(j+1)\Gamma(s+3)}, s = j+n, \ n \geq 1,$$

$$\mathbf{E}_{jj} = \frac{4\left(2j^2+6j+5\right)\Gamma\left(j+\frac{3}{2}\right)^2}{(2j+3)\Gamma(j+3)^2}, \quad \mathbf{E}_{j,\, j+1} = \mathbf{E}_{j+1,\, j} = \frac{4(j+1)\Gamma\left(j+\frac{5}{2}\right)^2}{(2j+3)\Gamma(j+3)^2}.$$

Corollary 3 (ChebyshevT Case) If $\eta = \zeta = -\frac{1}{2}$, then the nonzero elements $\mathbf{A}_{ir}$, $\mathbf{B}_{ir}$, $\mathbf{D}_{js}$ and $\mathbf{E}_{js}$ are expressed as follows:

$$\mathbf{A}_{ii} = \frac{\Gamma\left(i+\frac{1}{2}\right)^2}{\Gamma(i+1)^2}, \qquad \mathbf{A}_{i,\, i+2} = \mathbf{A}_{i+2,\, i} = -\frac{\Gamma\left(i+\frac{1}{2}\right)\Gamma\left(i+\frac{5}{2}\right)}{2\Gamma(i+1)\Gamma(i+3)},$$

$$\mathbf{B}_{ii} = -\frac{2(i+1)(i+2)\Gamma\left(i+\frac{1}{2}\right)^2}{\Gamma(i+1)^2},$$

$$\mathbf{B}_{ir} = -\frac{4(i+1)\Gamma(i+\frac{1}{2})\Gamma(i+\frac{2n+1}{2})}{\Gamma(i+1)\Gamma(r+1)}, \ r = i+n, \ n \text{ even number},$$

$$\mathbf{D}_{jj} = \frac{(j+1)\Gamma\left(j+\frac{1}{2}\right)^2}{\Gamma(j+1)^2}, \qquad \mathbf{D}_{js} = \frac{2\Gamma(j+\frac{2n+3}{2})\Gamma(j+\frac{1}{2})}{\Gamma(j+1)\Gamma(s+1)}, s = j+n, \ n \geq 1,$$

$$\mathbf{E}_{jj} = \frac{\Gamma\left(j+\frac{1}{2}\right)^2}{\Gamma(j+1)^2}, \qquad \mathbf{E}_{j,\, j+1} = \mathbf{E}_{j+1,\, j} = \frac{\Gamma\left(j+\frac{3}{2}\right)^2}{(2j+1)\Gamma(j+1)\Gamma(j+2)}.$$

3.2 *Wave equation*

The numerical solution of the linear wave equation in the following form is examined in this section:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial z^2} = g(z,\, t), \text{ on}(-1,\, 1)^2, \tag{10}$$

with boundary conditions

$$u(\pm 1, t) = 0, \ t \in (-1, 1),$$

and initial conditions

$$u(z, -1) = u_t(z, -1) = 0, \ z \in (-1, 1).$$

The goal of our approach is to provide an extended solution by combining the BFs of JPs, in the form

$$u(z, t) \backsimeq \hat{u}_2(z, t) = \sum_{i=0}^{N-2} \sum_{j=0}^{M-2} \phi_i^{(\zeta, \eta)}(z) \mathbf{U}_{ij}^{(2)} \widehat{\psi}_j^{(\zeta, \eta)}(t).$$

Now, we choose the BFs $\phi_i^{(\zeta, \eta)}(z)$ and $\widehat{\psi}_j^{(\zeta, \eta)}(t)$ to be of the form

$$\phi_i^{(\zeta, \eta)}(z) = J_i^{(\zeta, \eta)}(z) + \varepsilon_i J_{i+1}^{(\zeta, \eta)}(z) + \varsigma_i J_{i+2}^{(\zeta, \eta)}(z), \quad i = 0, 1, \ldots, N-2,$$

$$\widehat{\psi}_j^{(\zeta, \eta)}(t) = J_j^{(\zeta, \eta)}(t) + \widehat{\rho}_j J_{j+1}^{(\zeta, \eta)}(t) + \widehat{\varrho}_j J_{j+2}^{(\zeta, \eta)}(t), \quad j = 0, 1, \ldots, M-2.$$

It is not difficult to show that the BFs $\phi_i^{(\zeta, \eta)}(z) \in P_{N+2}^s(\Lambda)$ and $\widehat{\psi}_j^{(\zeta, \eta)}(t) \in P_{M+2}^t(\Lambda)$ are given by

$$\begin{aligned}\phi_i^{(\zeta, \eta)}(z) =& J_i^{(\zeta, \eta)}(z) - \frac{(i+1)(\eta-\zeta)(\eta+\zeta+2i+3)}{(\zeta+i+1)(\eta+i+1)(\eta+\zeta+2i+4)} J_{i+1}^{(\zeta, \eta)}(z) \\ & - \frac{(i+1)(i+2)(\eta+\zeta+2i+2)}{(\zeta+i+1)(\eta+i+1)(\eta+\zeta+2i+4)} J_{i+2}^{(\zeta, \eta)}(z),\end{aligned}$$

$$\begin{aligned}\widehat{\psi}_j^{(\zeta, \eta)}(t) =& J_j^{(\zeta, \eta)}(t) + \frac{2(j+1)(\eta+\zeta+2j+3)}{(\eta+j+1)(\eta+\zeta+2j+4)} J_{j+1}^{(\zeta, \eta)}(t) \\ & + \frac{(j+1)(j+2)(\eta+\zeta+2j+2)}{(\eta+j+1)(\eta+j+2)(\eta+\zeta+2j+4)} J_{j+2}^{(\zeta, \eta)}(t).\end{aligned}$$

In the wave equation, the Jacobi-Galerkin (JG) technique (10) is then equivalent to

$$
\begin{aligned}
&\left\langle \phi_i^{(\zeta,\,\eta)} \phi_l^{(\zeta,\,\eta)} \right\rangle_z \mathbf{U}_{ij}^{(2)} \left\langle \frac{d^2 \widehat{\psi}_j^{(\zeta,\,\eta)}}{dt^2} \widehat{\psi}_m^{(\zeta,\,\eta)} \right\rangle_t - \left\langle \phi_i^{(\zeta,\,\eta)} \frac{d^2 \phi_l^{(\zeta,\,\eta)}}{dz^2} \right\rangle_z \mathbf{U}_{ij}^{(2)} \left\langle \widehat{\psi}_j^{(\zeta,\,\eta)} \widehat{\psi}_m^{(\zeta,\,\eta)} \right\rangle_t \\
&= \left\langle \left\langle \phi_i^{(\zeta,\,\eta)}(z) g(z,\,t) \widehat{\psi}_j^{(\zeta,\,\eta)}(t) \right\rangle \right\rangle.
\end{aligned} \tag{11}
$$

The following ME corresponds to the wave equation's JG discretization (11):

$$
\mathbf{A}\mathbf{U}^{(2)}\mathbf{R} - \mathbf{B}\mathbf{U}^{(2)}\mathbf{S} = \mathbf{G}. \tag{12}
$$

The collection of discrete equations (12) can be expressed in the MF shown below:

$$
\left(\mathbf{R}^T \otimes \mathbf{A} - \mathbf{S}^T \otimes \mathbf{B}\right) \mathrm{vec}(\mathbf{U}^{(2)}) = \mathrm{vec}(\mathbf{G}).
$$

Theorem 2 Let

$$
\mathbf{R}_{jm} = \left\langle \frac{d^2 \widehat{\psi}_j^{(\zeta,\,\eta)}}{dt^2} \widehat{\psi}_m^{(\zeta,\,\eta)} \right\rangle_t, \quad \mathbf{S}_{jm} = \left\langle \widehat{\psi}_j^{(\zeta,\,\eta)} \widehat{\psi}_m^{(\zeta,\,\eta)} \right\rangle_t.
$$

The nonzero elements $\mathbf{R}_{jm}$ and $\mathbf{S}_{jm}$ can be obtained using the following equations:

$$
\begin{aligned}
&\mathbf{R}_{jj} = \widehat{\varrho}_j \mathscr{A}_2(j+2,\, j,\, \zeta,\, \eta) \gamma_j^{(\zeta,\,\eta)}, \\
&\mathbf{R}_{jm} = \lambda_2(m,\, j,\, \zeta,\, \eta) \gamma_j^{(\zeta,\,\eta)} + \lambda_2(m,\, j+1,\, \zeta,\, \eta) \widehat{\rho}_j \gamma_{j+1}^{(\zeta,\,\eta)} \\
&\qquad + \lambda_2(m,\, j+2,\, \zeta,\, \eta) \widehat{\varrho}_j \gamma_{j+2}^{(\zeta,\,\eta)}, \quad m = j+n, \quad n \geq 1, \\
&\mathbf{S}_{jj} = \gamma_j^{(\zeta,\,\eta)} + \widehat{\rho}_j^2 \gamma_{j+1}^{(\zeta,\,\eta)} + \widehat{\varrho}_j^2 \gamma_{j+2}^{(\zeta,\,\eta)}, \\
&\mathbf{S}_{j+1,\,j} = \mathbf{S}_{j,\,j+1} = \widehat{\rho}_j \gamma_{j+1}^{(\zeta,\,\eta)} + \widehat{\varrho}_j \widehat{\rho}_{j+1} \gamma_{j+2}^{(\zeta,\,\eta)}, \\
&\mathbf{S}_{j+2,\,j} = \mathbf{S}_{j,\,j+2} = \widehat{\varrho}_j \gamma_{j+2}^{(\zeta,\,\eta)},
\end{aligned}
$$

where

$$
\lambda_\sigma(m,\, j,\, \zeta,\, \eta) = \mathscr{A}_\sigma(m,\, j,\, \zeta,\, \eta) + \widehat{\rho}_m \mathscr{A}_\sigma(m+1,\, j,\, \zeta,\, \eta) + \widehat{\varrho}_m \mathscr{A}_\sigma(m+2,\, j,\, \zeta,\, \eta).
$$

Proof. The nonzero entries of $\mathbf{R}$ and $\mathbf{S}$ can be readily obtained by using the properties of Jacobi polynomials, which are presented in Section 2. $\square$

The following corollaries report specific orthogonal functions that are produced by studying the class of JPs as direct special cases:

Corollary 4 (Legendre Case) If $\eta = \zeta = 0$, then the nonzero elements $\mathbf{R}_{jm}$ and $\mathbf{S}_{jm}$ are expressed as follows:

$$\mathbf{R}_{jj} = \frac{2(j+1)(2j+3)}{(j+2)},$$

$$\mathbf{R}_{jm} = \frac{2n(2j+3)(2j+2n+3)(2j+n+3)}{(j+2)(j+n+2)}, \ m = j+n, \ n \geq 1,$$

$$\mathbf{S}_{jj} = \frac{12(j+1)(2j+3)}{(j+2)(2j+1)(2j+5)}, \quad \mathbf{S}_{j,\,j+1} = \mathbf{S}_{j+1,\,j} = \frac{4}{j+3}, \quad \mathbf{S}_{j,\,j+2} = \mathbf{S}_{j+2,\,j} = \frac{2(j+1)}{(j+2)(2j+5)}.$$

Corollary 5 (ChebyshevU Case) If $\eta = \zeta = \frac{1}{2}$, then the nonzero elements $\mathbf{R}_{jm}$ and $\mathbf{S}_{jm}$ are expressed as follows:

$$\mathbf{R}_{jj} = \frac{8(j+2)(2j+3)\Gamma\left(j+\frac{3}{2}\right)^2}{(j+3)(2j+5)\Gamma(j+1)^2},$$

$$\mathbf{R}_{jm} = \frac{32(2nj+(n^2+4n-1))(j+2)(j+n+2)\Gamma(j+\frac{3}{2})\Gamma(j+\frac{(2n+5)}{2})}{(2j+5)\Gamma(j+1)\Gamma(j+n+4)}, \ m = j+n, \ n \geq 1,$$

$$\mathbf{S}_{jj} = \frac{12\left(4j^4+32j^3+95j^2+124j+63\right)\Gamma\left(j+\frac{3}{2}\right)^2}{(j+3)^2(2j+5)^2\Gamma(j+2)^2},$$

$$\mathbf{S}_{j,\,j+1} = \mathbf{S}_{j+1,\,j} = \frac{8\left(4j^2+20j+27\right)\Gamma\left(j+\frac{3}{2}\right)\Gamma\left(j+\frac{5}{2}\right)}{(2j+5)(2j+7)\Gamma(j+1)\Gamma(j+4)},$$

$$\mathbf{S}_{j,\,j+2} = \mathbf{S}_{j+2,\,j} = \frac{2(j+1)(j+2)\Gamma\left(j+\frac{5}{2}\right)^2}{\Gamma(j+4)^2}.$$

Corollary 6 (ChebyshevT Case) If $\eta = \zeta = -\frac{1}{2}$, then the nonzero elements $\mathbf{R}_{jm}$ and $\mathbf{S}_{jm}$ are expressed as follows:

$$\mathbf{R}_{jj} = \frac{2(j+1)(j+2)(2j+1)\Gamma\left(j+\frac{1}{2}\right)^2}{(2j+3)\Gamma(j+1)^2},$$

$$\mathbf{R}_{jm} = \frac{8(2nj+(n^2+2n+1))\Gamma(j+\frac{1}{2})\Gamma(j+\frac{(2n+3)}{2})}{(2j+3)\Gamma(j+1)\Gamma(j+n+1)}, \ m = j+n, \ n \geq 1,$$

$$\mathbf{S}_{jj} = \frac{\left(12j^2+24j+13\right)\Gamma\left(j+\frac{1}{2}\right)^2}{(2j+3)^2\Gamma(j+1)^2},$$

$$\mathbf{S}_{j,\,j+1} = \mathbf{S}_{j+1,\,j} = \frac{2\left(4j^2+12j+7\right)\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(j+\frac{3}{2}\right)}{(2j+3)(2j+5)\Gamma(j+1)\Gamma(j+2)},$$

$$\mathbf{S}_{j,\,j+2} = \mathbf{S}_{j+2,\,j} = \frac{\Gamma\left(j+\frac{3}{2}\right)^2}{2\Gamma(j+1)\Gamma(j+3)}.$$

3.3 *Airy equation*

Let's discuss the Airy equation

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial z^3} = g(z,\,t),\ on\ (-1,\,1)^2, \tag{13}$$

with boundary conditions

$$u(\pm 1,\,t) = u_z(1,\,t) = 0,\ \ t \in (-1,\,1),$$

and initial conditions

$$u(z,\,-1) = 0,\ \ z \in (-1,\,1).$$

The goal of our approach is to obtain an extended solution utilizing a combination of JPs' BFs, in the form

$$u(z,\,t) \simeq \hat{u}_3(z,\,t) = \sum_{i=0}^{N-3}\sum_{j=0}^{M-1} \widehat{\phi}_i^{(\zeta,\,\eta)}(z)\mathbf{U}_{ij}^{(3)}\psi_j^{(\zeta,\,\eta)}(t).$$

Now, we choose the BFs $\widehat{\phi}_i^{(\zeta,\,\eta)}(z)$ and $\psi_j^{(\zeta,\,\eta)}(t)$ to be of the form

$$\widehat{\phi}_i^{(\zeta,\,\eta)}(z) = J_i^{(\zeta,\,\eta)}(z) + \widehat{\varepsilon}_i J_{i+1}^{(\zeta,\,\eta)}(z) + \widehat{\varsigma}_i J_{i+2}^{(\zeta,\,\eta)}(z) + \widehat{\delta}_i J_{i+3}^{(\zeta,\,\eta)}(z), \qquad i = 0,\,1,\,\ldots,\,N-3,$$

$$\psi_j^{(\zeta,\,\eta)}(t) = J_j^{(\zeta,\,\eta)}(t) + \rho_j J_{j+1}^{(\zeta,\,\eta)}(t), \qquad j = 0,\,1,\,\ldots,\,M-1.$$

It is not difficult to show that the BFs $\widehat{\phi}_i^{(\zeta,\,\eta)}(z) \in P_{N+3}^s(\Lambda)$ and $\psi_j^{(\zeta,\,\eta)}(t) \in P_{M+1}^t(\Lambda)$ are given by

$$
\begin{aligned}
\phi_i^{(\zeta,\,\eta)}(z) =& J_i^{(\zeta,\,\eta)}(z) - \frac{(i+1)(\eta+\zeta+2i+3)(-\zeta+2\eta+i+1)}{(\zeta+i+1)(\eta+i+1)(\eta+\zeta+2i+5)} J_{i+1}^{(\zeta,\,\eta)}(z) \\
&- \frac{(i+1)(i+2)(2\zeta-\eta+i+3)(\eta+\zeta+2i+2)}{(\zeta+i+1)(\zeta+i+2)(\eta+i+1)(\eta+\zeta+2i+6)} J_{i+2}^{(\zeta,\,\eta)}(z) \\
&+ \frac{(i+1)(i+2)(i+3)(\eta+\zeta+2i+2)(\eta+\zeta+2i+3)}{(\zeta+i+1)(\zeta+i+2)(\eta+i+1)(\eta+\zeta+2i+5)(\eta+\zeta+2i+6)} J_{i+3}^{(\zeta,\,\eta)}(z),
\end{aligned}
$$

$$
\psi_j^{(\zeta,\,\eta)}(t) = J_j^{(\zeta,\,\eta)}(t) + \frac{(j+1)}{(\eta+j+1)} J_{j+1}^{(\zeta,\,\eta)}(t).
$$

Consequently, the Airy equation's JG scheme (13) is equivalent to

$$
\begin{aligned}
&\left\langle \widehat{\phi}_i^{(\zeta,\,\eta)} \widehat{\phi}_l^{(\zeta,\,\eta)} \right\rangle_z \mathbf{U}_{ij}^{(3)} \left\langle \frac{d\psi_j^{(\zeta,\,\eta)}}{dt} \psi_m^{(\zeta,\,\eta)} \right\rangle_t + \left\langle \widehat{\phi}_i^{(\zeta,\,\eta)} \frac{d^3\widehat{\phi}_l^{(\zeta,\,\eta)}}{dz^3} \right\rangle_z \mathbf{U}_{ij}^{(3)} \left\langle \psi_j^{(\zeta,\,\eta)} \psi_m^{(\zeta,\,\eta)} \right\rangle_t \\
&= \left\langle \left\langle \widehat{\phi}_i^{(\zeta,\,\eta)}(z) g(z,\,t) \psi_j^{(\zeta,\,\eta)}(t) \right\rangle \right\rangle.
\end{aligned} \tag{14}
$$

The following ME corresponds to the Jacobi-Galerkin discretization of the Airy equation (14)

$$
\mathbf{P}\mathbf{U}^{(3)}\mathbf{D} + \mathbf{Q}\mathbf{U}^{(3)}\mathbf{E} = \mathbf{G}. \tag{15}
$$

The set of discrete equations (15) can be expressed in the MF shown below:

$$
\left(\mathbf{D}^T \otimes \mathbf{P} + \mathbf{E}^T \otimes \mathbf{Q}\right) \mathrm{vec}(\mathbf{U}^{(3)}) = \mathrm{vec}(\mathbf{G}).
$$

Theorem 3 Let

$$
\mathbf{P}_{im} = \left\langle \widehat{\phi}_i^{(\zeta,\,\eta)} \widehat{\phi}_m^{(\zeta,\,\eta)} \right\rangle_z, \quad \mathbf{Q}_{im} = \left\langle \frac{d^3\widehat{\phi}_i^{(\zeta,\,\eta)}}{dz^3} \widehat{\phi}_m^{(\zeta,\,\eta)} \right\rangle_z.
$$

Then the nonzero elements $\mathbf{P}_{im}$ and $\mathbf{Q}_{im}$ are given by

$$\mathbf{P}_{ii} = \gamma_i^{(\zeta,\,\eta)} + \widehat{\varepsilon}_i^2 \gamma_{i+1}^{(\zeta,\,\eta)} + \widehat{\varsigma}_i^2 \gamma_{i+2}^{(\zeta,\,\eta)} + \widehat{\delta}_i^2 \gamma_{i+3}^{(\zeta,\,\eta)},$$

$$\mathbf{P}_{i+1,\,i} = \mathbf{P}_{i,\,i+1} = \widehat{\varepsilon}_i \gamma_{i+1}^{(\zeta,\,\eta)} + \widehat{\varsigma}_i \varepsilon_{i+1} \gamma_{i+2}^{(\zeta,\,\eta)} + \widehat{\delta}_i \widehat{\varsigma}_{i+1} \gamma_{i+3}^{(\zeta,\,\eta)},$$

$$\mathbf{P}_{i+2,\,i} = \mathbf{P}_{i,\,i+2} = \widehat{\varsigma}_i \gamma_{i+2}^{(\zeta,\,\eta)} + \widehat{\delta}_i \widehat{\varepsilon}_{i+2} \gamma_{i+3}^{(\zeta,\,\eta)},$$

$$\mathbf{P}_{i+3,\,i} = \mathbf{P}_{i,\,i+3} = \widehat{\delta}_i \gamma_{i+3}^{(\zeta,\,\eta)},$$

$$\mathbf{Q}_{ii} = \widehat{\delta}_i \mathscr{A}_3(i+3,\, i,\, \zeta,\, \eta) \gamma_i^{(\zeta,\,\eta)},$$

$$\begin{aligned}\mathbf{Q}_{ir} = {} & \varpi_3(r,\, i,\, \zeta,\, \eta) \gamma_i^{(\zeta,\,\eta)} + \varpi_3(r,\, i+1,\, \zeta,\, \eta) \widehat{\varepsilon}_i \gamma_{i+1}^{(\zeta,\,\eta)} \\ & + \varpi_3(r,\, i+2,\, \zeta,\, \eta) \widehat{\varsigma}_i \gamma_{i+2}^{(\zeta,\,\eta)} + \varpi_3(r,\, i+3,\, \zeta,\, \eta) \widehat{\delta}_i \gamma_{i+3}^{(\zeta,\,\eta)}, \quad r = i+n, \quad n \geq 1,\end{aligned}$$

where

$$\varpi_\sigma(r,\, i,\, \zeta,\, \eta) = \mathscr{A}_\sigma(r,\, i,\, \zeta,\, \eta) + \widehat{\varepsilon}_r \mathscr{A}_\sigma(r+1,\, i,\, \zeta,\, \eta) + \widehat{\varsigma}_r \mathscr{A}_\sigma(r+2,\, i,\, \zeta,\, \eta) + \widehat{\delta}_r \mathscr{A}_\sigma(r+3,\, i,\, \zeta,\, \eta),$$

Proof. The JPs' properties in Section 2 facilitate determining nonzero entries of **P** and **Q**. □

The following corollaries report specific orthogonal functions that are produced by studying the class of JPs as direct special cases:

Corollary 7 (Legendre Case) If $\eta = \zeta = 0$, then the nonzero elements $\mathbf{P}_{ir}$ and $\mathbf{Q}_{ir}$ are expressed as follows:

$$\mathbf{P}_{ii} = \frac{16(i+2)(2i+3)}{(2i+1)(2i+5)(2i+7)},$$

$$\mathbf{P}_{i,\,i+1} = \mathbf{P}_{i+1,\,i} = -\frac{2}{2i+7},$$

$$\mathbf{P}_{i,\,i+2} = \mathbf{P}_{i+2,\,i} = -\frac{8(i+3)}{(2i+5)(2i+9)},$$

$$\mathbf{P}_{i,\,i+3} = \mathbf{P}_{i+3,\,i} = \frac{2(2i+3)}{(2i+5)(2i+7)},$$

$$\mathbf{Q}_{ii} = 2(2i+3)^2,$$

$$\mathbf{Q}_{ir} = (-1)^n 4(2i+3)(2i+2n+3), \;\; r = i+n, \;\; n \geq 1.$$

Corollary 8 (ChebyshevU Case)) If $\eta = \zeta = \frac{1}{2}$, then the nonzero elements $\mathbf{P}_{ir}$ and $\mathbf{Q}_{ir}$ are expressed as follows:

$$\mathbf{P}_{ii} = \frac{8\left(i^4+10i^3+38i^2+65i+45\right)\Gamma\left(i+\frac{3}{2}\right)^2}{(i+3)^2(i+4)^2\Gamma(i+2)^2},$$

$$\mathbf{P}_{i,\,i+1} = \mathbf{P}_{i+1,\,i} = -\frac{2(i(i+6)+12)\Gamma\left(i+\frac{3}{2}\right)\Gamma\left(i+\frac{5}{2}\right)}{(i+4)\Gamma(i+1)\Gamma(i+5)},$$

$$\mathbf{P}_{i,\,i+2} = \mathbf{P}_{i+2,\,i} = -\frac{(2i+3)(2i+5)\left(i^2+7i+13\right)\Gamma\left(i+\frac{3}{2}\right)^2}{(i+3)\Gamma(i+1)\Gamma(i+6)},$$

$$\mathbf{P}_{i,\,i+3} = \mathbf{P}_{i+3,\,i} = \frac{(i+1)(i+2)^2(2i+5)(2i+7)\Gamma\left(i+\frac{5}{2}\right)^2}{(2i+3)\Gamma(i+5)^2},$$

$$\mathbf{Q}_{ii} = \frac{16(i+2)^2\Gamma\left(i+\frac{3}{2}\right)^2}{(i+4)\Gamma(i+1)^2},$$

$$\mathbf{Q}_{ir} = -\frac{16(i+2)(i+n+2)(2i^3+(4n+15)i^2+(12n^2+28n+17)i+(8n^3+18n^2+22n+6))}{\Gamma(i+1)\,\Gamma(i+n+5)}$$

$$\times\,\Gamma\left(i+\frac{3}{2}\right)\Gamma\left(i+\frac{(2n+3)}{2}\right),\; r=i+n,\; n=1,\,3,\,5,\,\cdots,$$

$$\mathbf{Q}_{ir} = \frac{16(i+2)(i+n+2)(2i^3+(4n+15)i^2+(12n^2+40n+37)i+(8n^3+30n^2+46n+30))}{\Gamma(i+1)\,\Gamma(i+n+5)}$$

$$\times\,\Gamma\left(i+\frac{3}{2}\right)\Gamma\left(i+\frac{(2n+3)}{2}\right),\; r=i+n,\; n=2,\,4,\,6,\,\cdots.$$

Corollary 9 (ChebyshevT Case)) If $\eta = \zeta = -\frac{1}{2}$, then the nonzero elements $\mathbf{P}_{ir}$ and $\mathbf{Q}_{ir}$ are expressed as follows:

$$\mathbf{P}_{ii} = \frac{\left(2i^2+6i+5\right)\Gamma\left(i+\frac{1}{2}\right)^2}{(i+2)^2\Gamma(i+1)^2},$$

$$\mathbf{P}_{i,\,i+1} = \mathbf{P}_{i+1,\,i} = -\frac{\left(i^2+4i+2\right)\Gamma\left(i+\frac{1}{2}\right)\Gamma\left(i+\frac{3}{2}\right)}{2\Gamma(i+1)\Gamma(i+4)},$$

$$\mathbf{P}_{i,\,i+2} = \mathbf{P}_{i+2,\,i} = -\frac{(i+1)\left(2i^2+10i+11\right)\Gamma\left(i+\frac{1}{2}\right)\Gamma\left(i+\frac{5}{2}\right)}{2(i+4)\Gamma(i+3)^2},$$

$$\mathbf{P}_{i,\, i+3} = \mathbf{P}_{i+3,\, i} = \frac{(i+1)^2 \Gamma\left(i+\frac{1}{2}\right) \Gamma\left(i+\frac{7}{2}\right)}{2\Gamma(i+3)\Gamma(i+4)},$$

$$\mathbf{Q}_{ii} = \frac{4(i+1)^2(i+3)\Gamma\left(i+\frac{1}{2}\right)^2}{\Gamma(i+1)^2},$$

$$\mathbf{Q}_{ir} = -\frac{4(i+1)(2i^3+(8n+9)i^2+(36n^2+12n-1)i+(24n^3+18n^2))}{(i+n+2)\Gamma(i+1)\Gamma(i+n+1)}$$

$$\times \Gamma\left(i+\frac{1}{2}\right) \Gamma\left(i+\frac{(2n+1)}{2}\right), \; r = i+n, \; n = 1, 3, 5, \cdots,$$

$$\mathbf{Q}_{ir} = \frac{4(i+1)(2i^3+(8n+9)i^2+(36n^2+48n+13)i+(24n^3+54n^2+36n+6))}{(i+n+2)\Gamma(i+1)\Gamma(i+n+1)}$$

$$\times \Gamma\left(i+\frac{1}{2}\right) \Gamma\left(i+\frac{(2n+1)}{2}\right), \; r = i+n, \; n = 2, 4, 6, \cdots.$$

3.4 *Beam equation*

Lastly, take into account the fourth-order beam equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = g(x,\, t), \, on \, (-1,\, 1)^2, \tag{16}$$

with boundary conditions

$$u(\pm 1,\, t) = u_x(\pm 1,\, t) = 0, \; t \in (-1,\, 1),$$

and initial conditions

$$u(x,\, -1) = u_t(x,\, -1) = 0, \; x \in (-1,\, 1).$$

The goal of our approach is to obtain an extended solution utilizing a combination of JPs' BFs, in the form

$$u(x,\, t) \backsimeq \hat{u}_4(x,\, t) = \sum_{i=0}^{N-4} \sum_{j=0}^{M-2} \widetilde{\phi}_i^{(\zeta,\, \eta)}(x) \mathbf{U}_{ij}^{(4)} \widehat{\psi}_j^{(\zeta,\, \eta)}(t).$$

Currently, we select the BFs $\widetilde{\phi}_i^{(\zeta,\, \eta)}(x)$ and $\widehat{\psi}_j^{(\zeta,\, \eta)}(t)$ to have the quality

$$\widetilde{\phi}_i^{(\zeta,\eta)}(x) = J_i^{(\zeta,\eta)}(x) + \widetilde{\varepsilon}_i J_{i+1}^{(\zeta,\eta)}(x) + \widetilde{\varsigma}_i J_{i+2}^{(\zeta,\eta)}(x) + \widetilde{\delta}_i J_{i+3}^{(\zeta,\eta)}(x) + \widetilde{\tau}_i J_{i+4}^{(\zeta,\eta)}(x), \quad i = 0, 1, \ldots, N-4,$$

$$\widehat{\psi}_j^{(\zeta,\eta)}(t) = J_j^{(\zeta,\eta)}(t) + \widehat{\rho}_j J_{j+1}^{(\zeta,\eta)}(t) + \widehat{\varrho}_j J_{j+2}^{(\zeta,\eta)}(t) \qquad j = 0, 1, \ldots, M-2.$$

It is simple to demonstrate how the basis works $\widetilde{\phi}_i^{(\zeta,\eta)}(x) \in P_{N+4}^s(\Lambda)$ and $\widehat{\psi}_j^{(\zeta,\eta)}(t) \in P_{M+2}^t(\Lambda)$ are given by

$$\begin{aligned}
\widetilde{\phi}_i^{(\zeta,\eta)}(x) =& J_i^{(\zeta,\eta)}(x) + \frac{2(i+1)(\zeta-\eta)(\eta+\zeta+2i+3)}{(\zeta+i+1)(\eta+i+1)(\eta+\zeta+2i+6)} J_{i+1}^{(\zeta,\eta)}(x) \\
& - \frac{(i+1)(i+2)(\eta+\zeta+2i+2)(\eta+\zeta+2i+5)}{(\zeta+i+1)(\zeta+i+2)(\eta+i+1)(\eta+i+2)(\eta+\zeta+2i+6)(\eta+\zeta+2i+7)} \\
& \times \left(-\zeta^2 + 4\zeta\eta + 5\zeta - \eta^2 + 5\eta + 2i^2 + 2\zeta i + 2\eta i + 10i + 12\right) J_{i+2}^{(\zeta,\eta)}(x) \\
& - \frac{2(i+1)(i+2)(i+3)(\zeta-\eta)(\eta+\zeta+2i+2)(\eta+\zeta+2i+3)}{(\zeta+i+1)(\zeta+i+2)(\eta+i+1)(\eta+i+2)(\eta+\zeta+2i+6)(\eta+\zeta+2i+8)} J_{i+3}^{(\zeta,\eta)}(x) \\
& + \frac{(i+1)(i+2)(i+3)(i+4)(\eta+\zeta+2i+2)(\eta+\zeta+2i+3)(\eta+\zeta+2i+4)}{(\zeta+i+1)(\zeta+i+2)(\eta+i+1)(\eta+i+2)(\eta+\zeta+2i+6)(\zeta+\eta+2i+7)(\eta+\zeta+2i+8)} \\
& \times J_{i+4}^{(\zeta,\eta)}(x),
\end{aligned}$$

$$\widehat{\psi}_j^{(\zeta,\eta)}(t) = J_j^{(\zeta,\eta)}(t) + \frac{2(j+1)(\eta+\zeta+2j+3)}{(\eta+j+1)(\eta+\zeta+2j+4)} J_{j+1}^{(\zeta,\eta)}(t) + \frac{(j+1)(j+2)(\eta+\zeta+2j+2)}{(\eta+j+1)(\eta+j+2)(\eta+\zeta+2j+4)} J_{j+2}^{(\zeta,\eta)}(t).$$

The JG scheme (16) used in the beam equation can be considered to be equal to

$$\begin{aligned}
& \left\langle \widetilde{\phi}_i^{(\zeta,\eta)} \widetilde{\phi}_l^{(\zeta,\eta)} \right\rangle_x \mathbf{U}_{ij}^{(4)} \left\langle \frac{d^2 \widehat{\psi}_j^{(\zeta,\eta)}}{dt^2} \widehat{\psi}_m^{(\zeta,\eta)} \right\rangle_t + \left\langle \widetilde{\phi}_i^{(\zeta,\eta)} \frac{d^4 \widetilde{\phi}_l^{(\zeta,\eta)}}{dx^4} \right\rangle_x \mathbf{U}_{ij}^{(4)} \left\langle \widehat{\psi}_j^{(\zeta,\eta)} \widehat{\psi}_m^{(\zeta,\eta)} \right\rangle_t \\
&= \left\langle \left\langle \widetilde{\phi}_i^{(\zeta,\eta)}(x) g(x,t) \widehat{\psi}_j^{(\zeta,\eta)}(t) \right\rangle \right\rangle.
\end{aligned} \tag{17}$$

The Jacobi-Galerkin discretization of the beam equation (17) corresponds to the following ME

$$\mathbf{K}\mathbf{U}^{(4)}\mathbf{R} + \mathbf{L}\mathbf{U}^{(4)}\mathbf{S} = \mathbf{G}. \tag{18}$$

The set of discrete equations (18) can be expressed in the MF shown below:

$$\left(\mathbf{R}^T \otimes \mathbf{K} + \mathbf{S}^T \otimes \mathbf{L}\right) \operatorname{vec}(\mathbf{U}^{(4)}) = \operatorname{vec}(\mathbf{G}).$$

Theorem 4 Let

$$\mathbf{K}_{im} = \left\langle \widetilde{\phi}_i^{(\zeta,\,\eta)} \widetilde{\phi}_m^{(\zeta,\,\eta)} \right\rangle_x, \quad \mathbf{L}_{im} = \left\langle \frac{d^4 \widetilde{\phi}_i^{(\zeta,\,\eta)}}{dx^4} \widetilde{\phi}_m^{(\zeta,\,\eta)} \right\rangle_x.$$

Then the nonzero elements $\mathbf{K}_{im}$ and $\mathbf{L}_{im}$ are given by

$$\mathbf{K}_{ii} = \gamma_i^{(\zeta,\,\eta)} + \widetilde{\varepsilon}_i^2 \gamma_{i+1}^{(\zeta,\,\eta)} + \widetilde{\varsigma}_i^2 \gamma_{i+2}^{(\zeta,\,\eta)} + \widetilde{\delta}_i^2 \gamma_{i+3}^{(\zeta,\,\eta)} + \widetilde{\tau}_i^2 \gamma_{i+4}^{(\zeta,\,\eta)},$$

$$\mathbf{K}_{i+1,\,i} = \mathbf{K}_{i,\,i+1} = \widetilde{\varepsilon}_i \gamma_{i+1}^{(\zeta,\,\eta)} + \widetilde{\varsigma}_i \varepsilon_{i+1} \gamma_{i+2}^{(\zeta,\,\eta)} + \widetilde{\delta}_i \widetilde{\varsigma}_{i+1} \gamma_{i+3}^{(\zeta,\,\eta)} + \widetilde{\tau}_i \widetilde{\delta}_{i+1} \gamma_{i+4}^{(\zeta,\,\eta)},$$

$$\mathbf{K}_{i+2,\,i} = \mathbf{K}_{i,\,i+2} = \widetilde{\varsigma}_i \gamma_{i+2}^{(\zeta,\,\eta)} + \widetilde{\delta}_i \widetilde{\varepsilon}_{i+2} \gamma_{i+3}^{(\zeta,\,\eta)} + \widetilde{\tau}_i \widetilde{\varsigma}_{i+2} \gamma_{i+4}^{(\zeta,\,\eta)},$$

$$\mathbf{K}_{i+3,\,i} = \mathbf{K}_{i,\,i+3} = \widetilde{\delta}_i \gamma_{i+3}^{(\zeta,\,\eta)} + \widetilde{\tau}_i \widetilde{\varepsilon}_{i+3} \gamma_{i+4}^{(\zeta,\,\eta)},$$

$$\mathbf{K}_{i+4,\,i} = \mathbf{K}_{i,\,i+4} = \widetilde{\tau}_i \gamma_{i+4}^{(\zeta,\,\eta)},$$

$$\mathbf{L}_{ii} = \widetilde{\tau}_i \mathscr{A}_4(i+4,\, i,\, \zeta,\, \eta) \gamma_i^{(\zeta,\,\eta)},$$

$$\mathbf{L}_{ir} = \varsigma_4(r,\, i,\, \zeta,\, \eta) \gamma_i^{(\zeta,\,\eta)} + \varsigma_4(r,\, i+1,\, \zeta,\, \eta) \widetilde{\varepsilon}_i \gamma_{i+1}^{(\zeta,\,\eta)} + \varsigma_4(r,\, i+2,\, \zeta,\, \eta) \widetilde{\varsigma}_i \gamma_{i+2}^{(\zeta,\,\eta)}$$

$$+ \varsigma_4(r,\, i+3,\, \zeta,\, \eta) \widetilde{\delta}_i \gamma_{i+3}^{(\zeta,\,\eta)} + \varsigma_4(r,\, i+4,\, \zeta,\, \eta) \widetilde{\tau}_i \gamma_{i+4}^{(\zeta,\,\eta)}, \quad r = i+n, \quad n \geq 1,$$

where

$$\varsigma_\sigma(r,\, i,\, \zeta,\, \eta) = \mathscr{A}_\sigma(r,\, i,\, \zeta,\, \eta) + \widetilde{\varepsilon}_r \mathscr{A}_\sigma(r+1,\, i,\, \zeta,\, \eta) + \widetilde{\varsigma}_r \mathscr{A}_\sigma(r+2,\, i,\, \zeta,\, \eta) + \widetilde{\delta}_r \mathscr{A}_\sigma(r+3,\, i,\, \zeta,\, \eta)$$

$$+ \widetilde{\tau}_r \mathscr{A}_\sigma(r+4,\, i,\, \zeta,\, \eta).$$

Proof. The properties of JPs presented in Section 2 can be used to easily obtain the nonzero entries of **K** and **L**. □

The following corollaries report specific orthogonal functions that are produced by studying the class of JPs as direct special cases:

Corollary 10 (Legendre Case) If $\eta = \zeta = 0$, then the nonzero elements $\mathbf{K}_{ir}$ and $\mathbf{L}_{ir}$ are expressed as follows:

$$\mathbf{K}_{ii} = \frac{12(2i+3)(2i+5)}{(2i+1)(2i+7)(2i+9)},$$

$$\mathbf{K}_{i,\,i+1} = \mathbf{K}_{i+1,\,i} = 0,$$

$$\mathbf{K}_{i,\,i+2} = \mathbf{K}_{i+2,\,i} = -\frac{8}{2i+11},$$

$$\mathbf{K}_{i,\,i+3} = \mathbf{K}_{i+3,\,i} = 0,$$

$$\mathbf{K}_{i,\,i+4} = \mathbf{K}_{i+4,\,i} = \frac{2(2i+3)}{(2i+7)(2i+9)},$$

$$\mathbf{L}_{ii} = 2(2i+3)^2(2i+5),$$

$$\mathbf{L}_{ir} = 0,\ \ r = i+n,\ \ n \geq 1.$$

Corollary 11 (ChebyshevU Case)) If $\eta = \zeta = \frac{1}{2}$, then the nonzero elements $\mathbf{K}_{ir}$ and $\mathbf{L}_{ir}$ are expressed as follows:

$$\mathbf{K}_{ii} = \frac{12\left(i^4+12i^3+53i^2+102i+84\right)\Gamma\left(i+\frac{3}{2}\right)^2}{(i+4)^2(i+5)^2\Gamma(i+2)^2},$$

$$\mathbf{K}_{i,\,i+1} = \mathbf{K}_{i+1,\,i} = 0,$$

$$\mathbf{K}_{i,\,i+2} = \mathbf{K}_{i+2,\,i} = \frac{8\left(i^2+8i+18\right)\Gamma\left(i+\frac{3}{2}\right)\Gamma\left(i+\frac{7}{2}\right)}{\Gamma(i+1)\Gamma(i+7)},$$

$$\mathbf{K}_{i,\,i+3} = \mathbf{K}_{i+3,\,i} = 0,$$

$$\mathbf{K}_{i,\,i+4} = \mathbf{K}_{i+4,\,i} = \frac{2(i+1)(i+2)^2(i+3)\Gamma\left(i+\frac{3}{2}\right)\Gamma\left(i+\frac{11}{2}\right)}{\Gamma(i+6)^2},$$

$$\mathbf{L}_{ii} = \frac{32(i+2)^2(i+3)\Gamma\left(i+\frac{3}{2}\right)^2}{(i+5)\Gamma(i+1)^2},$$

$$\mathbf{L}_{ir} = 0,\ \ r = i+n,\ \ n = 1,\,3,\,5,\,\cdots,$$

$$\mathbf{L}_{ir} = -\frac{128(i+2)(i+3)(i^2+(n+6)i+(n^2+3n+8))\Gamma\left(i+\frac{3}{2}\right)\Gamma\left(i+\frac{(4n+3)}{2}\right)}{(i+2n+4)(i+2n+5)\Gamma(i+1)\Gamma(i+2n+2)}, r = i+2n, n = 1, 2, 3, \cdots.$$

Corollary 12 (ChebyshevT Case)) If $\eta = \zeta = -\frac{1}{2}$, then the nonzero elements $\mathbf{K}_{ir}$ and $\mathbf{L}_{ir}$ are expressed as follows:

$$\mathbf{K}_{ii} = \frac{\left(3i^2+12i+13\right)\Gamma\left(i+\frac{1}{2}\right)^2}{(i+3)^2\Gamma(i+1)^2},$$

$$\mathbf{K}_{i,\,i+1} = \mathbf{K}_{i+1,\,i} = 0,$$

$$\mathbf{K}_{i,\,i+2} = \mathbf{K}_{i+2,\,i} = -\frac{2(i+4)(2i+3)\left(i^2+6i+7\right)\Gamma\left(i+\frac{3}{2}\right)^2}{(2i+1)\Gamma(i+1)\Gamma(i+6)},$$

$$\mathbf{K}_{i,\,i+3} = \mathbf{K}_{i+3,\,i} = 0,$$

$$\mathbf{K}_{i,\,i+4} = \mathbf{K}_{i+4,\,i} = \frac{(i+1)^2(i+2)\Gamma\left(i+\frac{1}{2}\right)\Gamma\left(i+\frac{9}{2}\right)}{2\Gamma(i+4)\Gamma(i+5)},$$

$$\mathbf{L}_{ii} = \frac{8(i+1)^2(i+2)(i+4)\Gamma\left(i+\frac{1}{2}\right)^2}{\Gamma(i+1)^2},$$

$$\mathbf{L}_{ir} = 0,\ r = i+n,\ n = 1, 3, 5, \cdots,$$

$$\mathbf{L}_{ir} = \frac{32(i+2)(i+2)(i^2+(3n+4)i+3(n+1)^2)\Gamma\left(i+\frac{1}{2}\right)\Gamma\left(i+\frac{(4n+1)}{2}\right)}{(i+2n+3)\Gamma(i+1)\Gamma(i+2n+1)},\ r = i+2n,\ n = 1, 2, 3, \cdots.$$

4. Nonhomogeneous boundary conditions

We outline a method for effectively converting problems with nonhomogeneous boundary-initial circumstances into problems with homogeneous boundary-initial conditions (see [30–32]). Let us consider for instance the Schrödinger equation (4) and Wave equation (10). Nonhomogeneous boundary-initial conditions for Airy and Beam equations can be treated similarly.

If the solution u of equation (4) is subjected to non-homogeneous boundary-initial conditions:

$$u(\pm 1, t) = a_{\pm}(t),\ t \in (-1, 1),$$

$$u(z, -1) = b_{-}(z),\ z \in (-1, 1).$$

Presently, assume the accompanying transformation

$$u(z,t)=\tilde{u}(z,t)+\xi_0(t)+z\xi_1(t)+(z+1)(z-1)\xi(z)=\tilde{u}(z,t)+u_e(z,t),$$

where

$$\xi_0(t)=\frac{a_+(t)+a_-(t)}{2},\ \ \xi_1(t)=\frac{a_+(t)-a_-(t)}{2},\ \ \xi(z)=\frac{b_-(z)-z\xi_1(-1)-\xi_0(-1)}{(z+1)(z-1)},$$

and $\tilde{u}$ is an unidentified auxiliary function that fulfills the modified problem

$$\partial_t\tilde{u}(z,t)+I\partial_z^2\tilde{u}(z,t)=g^*(z,t),\ \ (z,t)\in\Omega,$$

depending on the homogenous boundary-initial conditions

$$\tilde{u}(\pm 1,t)=0,\ \ t\in(-1,1),$$

$$\tilde{u}(z,-1)=0,\ \ z\in(-1,1),$$

where

$$g^*(z,t)=g(z,t)-\partial_t u_e(z,t)+I\partial_z^2 u_e(z,t),$$

while $u_e(z,t)$ is a function of any type that satisfies the initial nonhomogeneous boundary requirements.

Hence the solution u of (10) is subject to the nonhomogeneous boundary-initial conditions:

$$u(\pm 1,t)=c_\pm(t),\ \ t\in(-1,1),$$

$$u(z,-1)=d_-(z),\ \ u_t(z,-1)=e_-(z),\ \ z\in(-1,1).$$

We move forward as follows:

Setting

$$u(z,t)=\tilde{u}(z,t)+\alpha_0(t)+z\alpha_1(t)+(z+1)(z-1)(\beta_0(z)+t\beta_1(z))=\tilde{u}(z,t)+u_e(z,t),$$

where

$$\alpha_0(t) = \frac{c_+(t)+c_-(t)}{2}, \;\; \alpha_1(t) = \frac{c_+(t)-c_-(t)}{2},$$

$$\beta_0(z) = \frac{d_-(z)-\alpha_0(-1)-z\alpha_1(-1)+e_-(z)-\partial_t\alpha_0(-1)-z\partial_t\alpha_1(-1)}{(z+1)(z-1)},$$

$$\beta_1(z) = \frac{e_-(z)-\partial_t\alpha_0(-1)-z\partial_t\alpha_1(-1)}{(z+1)(z-1)},$$

that fulfills the modified problem

$$\frac{\partial^2\tilde{u}}{\partial t^2} - \frac{\partial^2\tilde{u}}{\partial z^2} = g_*(z, t), \;\text{ on } (-1, 1)^2,$$

depending on the homogenous boundary-initial conditions

$$\tilde{u}(\pm 1, t) = 0, \;\; t \in (-1, 1),$$

$$\tilde{u}(z, -1) = \tilde{u}_t(z, -1) = 0, \;\; z \in (-1, 1).$$

where

$$g_*(z, t) = g(z, t) - \frac{\partial^2 u_e}{\partial t^2} + \frac{\partial^2 u_e}{\partial z^2}, \;\text{ on } (-1, 1)^2.$$

5. Convergence analysis and truncation error estimate

In this section, we use the findings of Hafez and Youssri [33] to determine the convergence rate of the unknown solution coefficients and estimate the truncation error in the Jacobi Spectral expansion method for solving the reaction-subdiffusion equation. We apply these results to all the linear PDEs discussed in Section 3 to establish their convergence rate and truncation error.

Lemma 3 The following formulas for connections are valid:

$$\phi_i^{(\zeta,\,\eta)}(x) = \nu_{1i}(1-x^2)J_i^{(\zeta+1,\,\eta+1)}(x),$$

$$\widehat{\phi}_i^{(\zeta,\,\eta)}(x) = \nu_{2i}(1-x)^2(x+1)J_i^{(\zeta+2,\,\eta+1)}(x),$$

$$\widetilde{\phi}_i^{(\zeta,\,\eta)}(x) = \nu_{3i}(1-x^2)^2J_i^{(\zeta+2,\,\eta+2)}(x),$$

$$\psi_j^{(\zeta,\,\eta)}(t) = \nu_{4j}(1+t)J_j^{(\zeta,\,\eta+1)}(t),$$

$$\widehat{\psi}_j^{(\zeta,\,\eta)}(t) = v_{5j}(1+t)^2 J_j^{(\zeta,\,\eta+2)}(t),$$

where

$$v_{1i} = \frac{(2i+\eta+\zeta+2)(2i+\eta+\zeta+3)}{4(i+\zeta+1)(i+\eta+1)},$$

$$v_{2i} = \frac{(2i+\eta+\zeta+2)(2i+\eta+\zeta+3)(2i+\eta+\zeta+4)}{8(i+\zeta+1)(i+\zeta+2)(i+\eta+1)},$$

$$v_{3i} = \frac{(2i+\eta+\zeta+2)(2i+\eta+\zeta+3)(2i+\zeta+\eta+4)(2i+\eta+\zeta+5)}{16(i+\zeta+1)(i+\zeta+2)(i+\eta+1)(i+\eta+2)},$$

$$v_{4j} = \frac{(2j+\eta+\zeta+2)}{2(j+\eta+1)},$$

$$v_{5j} = \frac{(2j+\eta+\zeta+2)(2j+\eta+\zeta+3)}{4(j+\eta+1)(j+\eta+2)}.$$

Proof. The proof follows directly from the definition of generalized JPs presented in [15]. □

Lemma 4 The following orthogonality relations are accurate:

$$A)\int_{-1}^{1} \phi_i^{(\zeta,\,\eta)}(x)\phi_j^{(\zeta,\,\eta)}(x)w^{(\zeta-1,\,\eta-1)}(x)\,dx = v_{1i}^2\gamma_i^{(\zeta+1,\,\eta+1)}\delta_{ij},$$

$$B)\int_{-1}^{1} \widehat{\phi}_i^{(\zeta,\,\eta)}(x)\widehat{\phi}_j^{(\zeta,\,\eta)}(x)w^{(\zeta-2,\,\eta-1)}(x)\,dx = v_{2i}^2\gamma_i^{(\zeta+2,\,\eta+1)}\delta_{ij},$$

$$C)\int_{-1}^{1} \widetilde{\phi}_i^{(\zeta,\,\eta)}(x)\widetilde{\phi}_j^{(\zeta,\,\eta)}(x)w^{(\zeta-2,\,\eta-2)}(x)\,dx = v_{3i}^2\gamma_i^{(\zeta+2,\,\eta+2)}\delta_{ij},$$

$$D)\int_{-1}^{1} \psi_i^{(\zeta,\,\eta)}(t)\psi_j^{(\zeta,\,\eta)}(t)w^{(\zeta,\,\eta-1)}(t)\,dt = v_{4i}^2\gamma_i^{(\zeta,\,\eta+1)}\delta_{ij},$$

$$E)\int_{-1}^{1} \widehat{\psi}_i^{(\zeta,\,\eta)}(t)\widehat{\psi}_j^{(\zeta,\,\eta)}(t)w^{(\zeta,\,\eta-1)}(t)\,dt = v_{5i}^2\gamma_i^{(\zeta,\,\eta+2)}\delta_{ij}.$$

Proof. If we simplify the product of the basis functions and use the orthogonality relation of the Jacobi polynomials, we can easily derive all of these orthogonality relations. □

Lemma 5 [28] The following inequalities are valid:

$$A) \mid J_i^{(\zeta,\,\eta)}(x) \mid \lesssim i^q, \quad q = max(\zeta,\, \eta,\, -\frac{1}{2}),$$

$$B) \mid \phi_i^{(\zeta,\,\eta)}(x) \mid \lesssim i^{q_1}, \quad q_1 = max(\zeta+1,\, \eta+1,\, -\frac{1}{2}),$$

$$C) \mid \widehat{\phi}_i^{(\zeta,\,\eta)}(x) \mid \lesssim i^{q_2}, \quad q_2 = max(\zeta+2,\, \eta+2,\, -\frac{1}{2}),$$

$$D) \mid \widetilde{\phi}_i^{(\zeta,\,\eta)}(x) \mid \lesssim i^{q_3}, \quad q_3 = max(\zeta+3,\, \eta+3,\, -\frac{1}{2}),$$

$$E) \mid \psi_j^{(\zeta,\,\eta)}(t) \mid \lesssim j^{q_4}, \quad q_4 = max(\zeta,\, \eta+1,\, -\frac{1}{2}),$$

$$F) \mid \widehat{\psi}_j^{(\zeta,\,\eta)}(t) \mid \lesssim j^{q_5}, \quad q_5 = max(\zeta,\, \eta+2,\, -\frac{1}{2}).$$

In [33], Hafez and Youssri proved the following convergence result. If $u(x,\, t) = x(\mathscr{L} - x)t\, f(x)g(t)$ and $|f'''(x)| \leq A,\, |g'''(t)| \leq B$ is approximated by $\tilde{u}(x,\, t)$, then the expansion coefficients c_{ij} satisfy the following estimate

$$\left|c_{ij}\right| = \mathscr{O}\left(i^{-\frac{5}{2}} j^{-\frac{5}{2}}\right) \forall i,\, j > 3.$$

Let us use the notations $U_{ij}^{(r)}$ to represent the expansion coefficients of the Schrödinger equation, Airy equation, wave equation, and beam model, where $r = 1,\, 2,\, 3,\, 4$, respectively. The exact solutions can be expressed as separable in the form $u^{(r)}(x,\, t) = f_r(x)\, g_r(t)$, where $r = 1, 2, 3, 4$, respectively, and f_r and g_r possess $r+2$ bounded continuous derivatives. Finally, based on the result of Theorem 3 in [33] (mentioned above), we can obtain the following estimates.

Theorem 5 The estimates provided below are deemed to be accurate and valid

$$A) \mid \mathbf{U}_{ij}^{(1)} \mid \lesssim i^{-\frac{5}{2}} j^{-\frac{5}{2}} \quad \mid f_1^{(3)} \mid < M_{11} \quad \mid g_1^{(3)} \mid < M_{12},$$

$$B) \mid \mathbf{U}_{ij}^{(2)} \mid \lesssim i^{-\frac{7}{2}} j^{-\frac{7}{2}} \quad \mid f_2^{(4)} \mid < M_{21} \quad \mid g_2^{(4)} \mid < M_{22},$$

$$C) \mid \mathbf{U}_{ij}^{(3)} \mid \lesssim i^{-\frac{9}{2}} j^{-\frac{5}{2}} \quad \mid f_3^{(5)} \mid < M_{31} \quad \mid g_3^{(3)} \mid < M_{32},$$

$$D) \mid \mathbf{U}_{ij}^{(4)} \mid \lesssim i^{-\frac{11}{2}} j^{-\frac{7}{2}} \quad \mid f_4^{(6)} \mid < M_{41} \quad \mid g_4^{(4)} \mid < M_{42}.$$

Based on Theorem 4 in [33] and Lemma 5, we can estimate as follows.

Theorem 6 If $\zeta < \eta < \frac{1}{2}$ then

$$A) \mid \mathbf{U}_{ij}^{(1)} \phi_i^{(\zeta,\,\eta)}(x) \psi_j^{(\zeta,\,\eta)}(t) \mid \lesssim (ij)^{\eta-\frac{3}{2}},$$

$$B) \mid \mathbf{U}_{ij}^{(2)} \phi_i^{(\zeta,\,\eta)}(x) \widehat{\psi}_j^{(\zeta,\,\eta)}(t) \mid \lesssim (ij)^{\eta-\frac{3}{2}},$$

$$C) \mid \mathbf{U}_{ij}^{(3)} \widehat{\phi}_i^{(\zeta,\,\eta)}(x) \psi_j^{(\zeta,\,\eta)}(t) \mid \lesssim (ij)^{\eta-\frac{3}{2}},$$

$$D) \mid \mathbf{U}_{ij}^{(4)} \widetilde{\phi}_i^{(\zeta,\,\eta)}(x) \widehat{\psi}_j^{(\zeta,\,\eta)}(t) \mid \lesssim (ij)^{\eta-\frac{3}{2}}.$$

Finally combining the results of Theorems 5 and 6, we have the following truncation error estimate.

Theorem 7 For 3.1, 3.2, 3.3 and 3.4 $\|u_{exact} - u_{app}\|_2 \lesssim (NM)^{-\frac{3}{2}}$, where N, M are the numbers of retained modes in the truncated approximate solution.

6. Numerical results

Example 1 We consider the following Schrödinger equation [34]:

$$\frac{\partial u}{\partial t} - I\frac{\partial^2 u}{\partial x^2} = g(x,\, t), \quad \text{on}(-1,\, 1)^2,$$

with the boundary conditions:

$$u(\pm 1,\, t) = 0, \quad t \in (-1,\, 1),$$

and the initial condition:

$$u(x,\, -1) = -e^{x-1}\sin(\pi x), \quad x \in (-1,\, 1).$$

The smooth analytic solution of this problem is $u = e^{x+t}\sin(\frac{\pi t}{2})\sin \pi x$. The known source function is given by $g = \frac{1}{2}e^{t+x}\left(\pi\cos\left(\frac{\pi t}{2}\right)\sin(\pi x) - 2I\sin\left(\frac{\pi t}{2}\right)\left(-2\pi\cos(\pi x) + \left(\pi^2 + (-1+I)\right)\sin(\pi x)\right)\right)$.

Table 1 shows the maximum absolute errors for Example 1 with with various choices of N, M, ζ and η. Figure 1 shows the space-time of absolute errors for Example 1 with $\zeta = \frac{1}{2}$, $\eta = -\frac{1}{2}$ and $N = M = 18$. Figure 2 shows the convergence rates for Example 1 with $\zeta = \frac{1}{2}$, $\eta = -\frac{1}{2}$.

Example 2 We consider the following wave equation [34]:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = g(x,\, t), \quad \text{on}(-1,\, 1)^2,$$

with the boundary conditions:

$$u(\pm 1, t) = 0, \quad t \in (-1, 1),$$

and the initial condition:

$$u(x, -1) = -e^{x-1}\sin(\pi x), \quad x \in (-1, 1),$$

and

$$u_t(x, -1) = -e^{x-1}\sin(\pi x), \quad x \in (-1, 1).$$

Table 1. MAE with various choices of N, M, ζ and η for Example 1

$N = M$	ζ	η	Real part	Imaginary part	ζ	η	Real part	Imaginary part
6			6.96×10^{-3}	9.32×10^{-4}			1.53×10^{-2}	8.96×10^{-3}
12	0	0	3.27×10^{-8}	5.61×10^{-9}	$\frac{1}{2}$	$\frac{1}{2}$	8.62×10^{-8}	1.11×10^{-7}
18			1.73×10^{-15}	3.88×10^{-15}			2.26×10^{-14}	8.01×10^{-15}
6			7.23×10^{-3}	4.90×10^{-3}			3.65×10^{-3}	1.39×10^{-3}
12	$-\frac{1}{2}$	$-\frac{1}{2}$	7.23×10^{-8}	1.20×10^{-8}	$\frac{1}{2}$	0	7.98×10^{-8}	3.09×10^{-7}
18			7.53×10^{-14}	5.10×10^{-14}			5.29×10^{-13}	3.17×10^{-13}
6			2.39×10^{-2}	1.08×10^{-2}			2.81×10^{-2}	2.20×10^{-2}
12	0	$\frac{1}{2}$	3.26×10^{-8}	1.82×10^{-7}	1	1	3.30×10^{-7}	3.49×10^{-7}
18			5.68×10^{-13}	3.15×10^{-13}			7.47×10^{-14}	3.76×10^{-14}

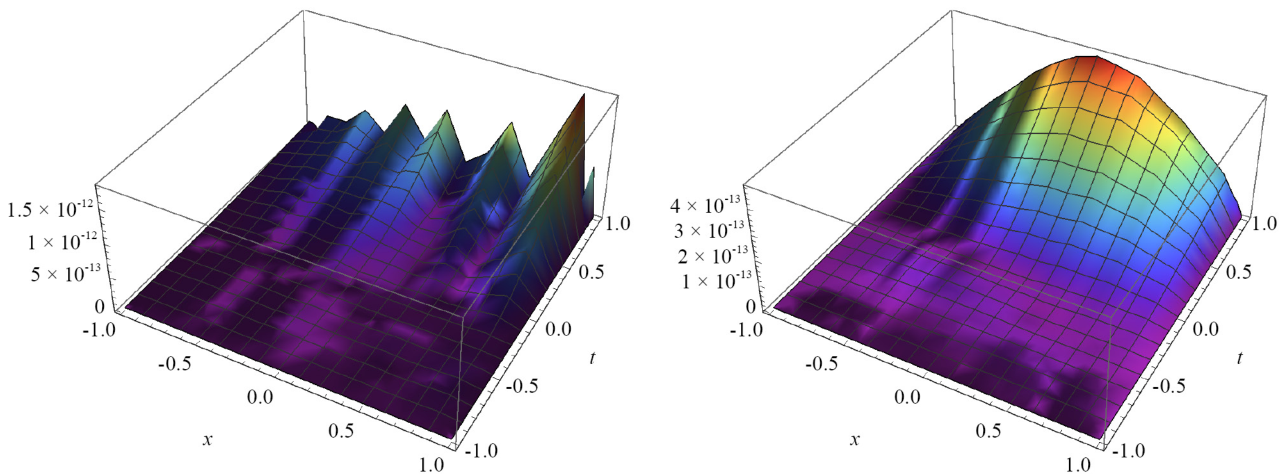

Figure 1. The graphs of absolute errors obtained for Example 1 with $\zeta = \frac{1}{2}$, $\eta = -\frac{1}{2}$ and $N = M = 18$ (left panel for real part and right panel for imaginary part)

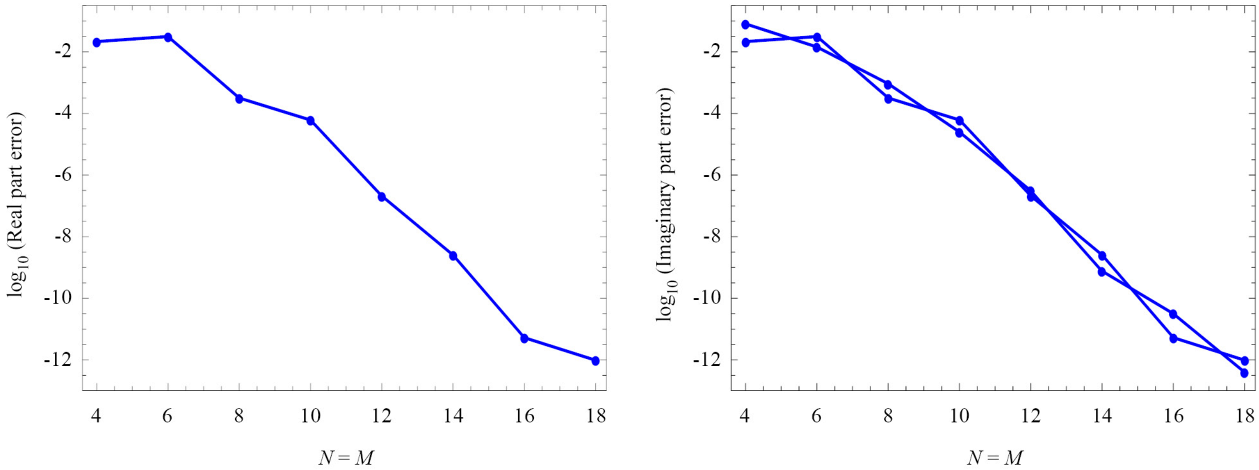

Figure 2. Exponential convergence in $N = M$ for Example 1 with $\zeta = -\frac{1}{2},\ \eta = \frac{1}{2}$ (left panel for real part and right panel for imaginary part)

The exact solution of the above problem is $u(x, t) = e^{x+t}\sin(\frac{\pi t}{2})\sin\pi x$. The source function is given by $g(x, t) = \frac{1}{4}\pi e^{t+x}\left(4\cos\left(\frac{\pi t}{2}\right)\sin(\pi x) + \sin\left(\frac{\pi t}{2}\right)(3\pi\sin(\pi x) - 8\cos(\pi x))\right)$.

Table 2 shows the maximum absolute errors and CPU time in seconds for Example 2 with with various choices of N, M, ζ and η. For various choices of N and M, the graphs of the absolute error functions with $\zeta = -\eta = 1/2$ are displayed in Figure 3.

Table 2. Maximum absolute errors and CPU time with various choices of N, M, ζ and η for Example 2

$N = M$	ζ	η	JG	CPU time	$N = M$	JG	CPU time
	1	1	4.1921×10^{-1}			2.3195×10^{-4}	
	$\frac{1}{2}$	$\frac{1}{2}$	3.5439×10^{-1}			1.7511×10^{-4}	
4	0	0	4.2676×10^{-1}	0.015625	8	1.1932×10^{-4}	0.03125
	$-\frac{1}{2}$	$\frac{1}{2}$	2.9024×10^{-1}			9.9212×10^{-4}	
	$-\frac{1}{2}$	$-\frac{1}{2}$	9.0489×10^{-1}			4.8919×10^{-5}	
	1	1	2.1459×10^{-7}			2.5435×10^{-11}	
	$\frac{1}{2}$	$\frac{1}{2}$	5.9406×10^{-8}			6.6404×10^{-12}	
12	0	0	3.7341×10^{-8}	0.078125	16	1.5477×10^{-12}	0.109375
	$-\frac{1}{2}$	$\frac{1}{2}$	4.4371×10^{-7}			1.6799×10^{-11}	
	$-\frac{1}{2}$	$-\frac{1}{2}$	7.2687×10^{-8}			3.3394×10^{-12}	

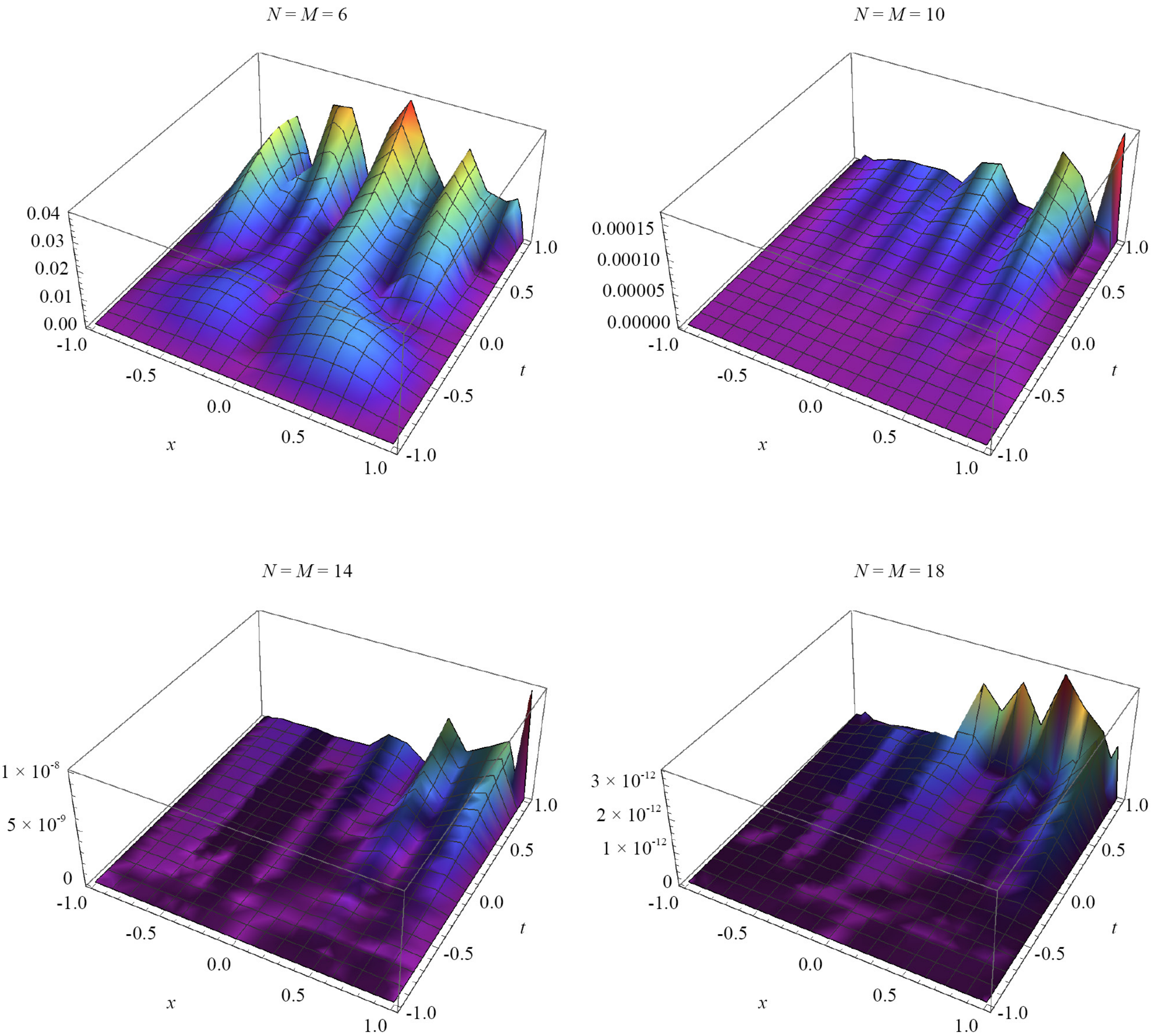

Figure 3. The graphs of the absolute error functions for Example 2 at various choices of $N = M$ with $\zeta = -\eta = 1/2$

Example 3 We consider the following Airy equation [34]:

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = g(x, t), \quad \text{on}(-1, 1)^2,$$

with the boundary conditions:

$$u(\pm 1, t) = 0, \; u_x(1, t) = \pi\left(-e^{t+1}\right)\sin\left(\frac{\pi t}{2}\right), \quad t \in (-1, 1),$$

and the initial condition:

$$u(x, -1) = -e^{x-1}\sin(\pi x), \quad x \in (-1, 1),$$

with the same exact solution as before. The source function is given by

$$g(x, t) = \frac{1}{2}e^{t+x}\left(\pi\cos\left(\frac{\pi t}{2}\right)\sin(\pi x) - 2\sin\left(\frac{\pi t}{2}\right)\left(\left(3\pi^2 - 2\right)\sin(\pi x) + \pi\left(\pi^2 - 3\right)\cos(\pi x)\right)\right).$$

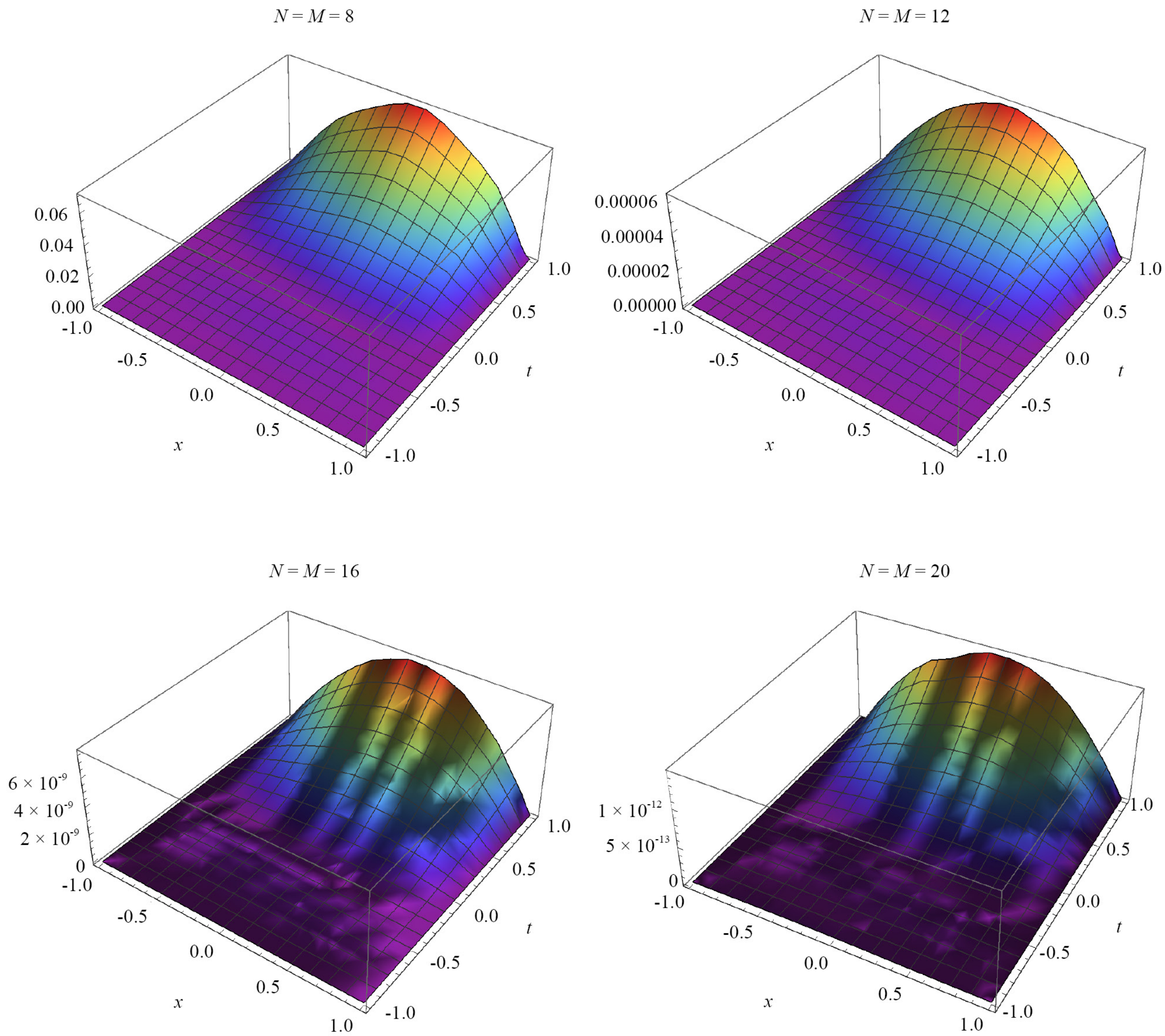

Figure 4. The graphs of the absolute error functions for Example 3 at various choices of $N = M$ with $\eta = \zeta = 1/2$

Table 3 lists the maximum absolute errors, using the Jacobi Galerkin (JG) method with various choices of N, M, ζ and η. For various choices of N and M, the graphs of the absolute error functions with $\eta = \zeta = 1/2$ are displayed in Figure 4.

Table 3. Maximum absolute errors with various choices of N, M, ζ and η for Example 3

$N = M$	ζ	η	JG	$N = M$	JG
8	1	1	9.8166×10^{-2}	12	9.8454×10^{-5}
	$\frac{1}{2}$	$\frac{1}{2}$	7.0324×10^{-2}		5.8614×10^{-5}
	0	0	4.4119×10^{-2}		2.9304×10^{-5}
	$-\frac{1}{2}$	$\frac{1}{2}$	8.2491×10^{-3}		5.2676×10^{-6}
	$-\frac{1}{2}$	$-\frac{1}{2}$	2.1501×10^{-2}		1.0440×10^{-5}
16	1	1	1.4523×10^{-8}	20	2.8848×10^{-12}
	$\frac{1}{2}$	$\frac{1}{2}$	7.7420×10^{-9}		1.4583×10^{-12}
	0	0	3.4268×10^{-9}		5.6665×10^{-13}
	$-\frac{1}{2}$	$\frac{1}{2}$	4.0132×10^{-10}		8.7041×10^{-14}
	$-\frac{1}{2}$	$-\frac{1}{2}$	1.0816×10^{-9}		1.3677×10^{-13}

Example 4 We consider the following beam equation [34]:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = g(x, t), \quad \text{on}(-1, 1)^2,$$

with the boundary conditions:

$$u(\pm 1, t) = 0 = u_x(\pm 1, t), \quad t \in (-1, 1),$$

and the initial conditions:

$$u(x, -1) = -e^{x-1} \sin^2(\pi x), \; u_t(x, -1) = -e^{x-1} \sin^2(\pi x), \quad x \in (-1, 1).$$

The exact solution of the above problem is $u(x, t) = e^{t+x} \sin\left(\frac{\pi t}{2}\right) \sin^2(\pi x)$. The source function is given by $g(x, t) = \frac{1}{8} e^{t+x} \left(8\pi \cos\left(\frac{\pi t}{2}\right) \sin^2(\pi x) - \sin\left(\frac{\pi t}{2}\right) \left(32\pi \left(4\pi^2 - 1\right) \sin(2\pi x) + \left(8 - 97\pi^2 + 64\pi^4\right) \cos(2\pi x) + \pi^2 - 8\right)\right)$. Table 4 lists the maximum absolute errors, using the Jacobi Galerkin (JG) method with various choices of N, M, ζ and η. For various choices of N and M, the graphs of the absolute error functions with $\eta = \zeta = 0$ are displayed in Figure 5.

Table 4. Maximum absolute errors with various choices of N, M, ζ and η for Example 4

$N=M$	ζ	η	JG	$N=M$	JG
	1	1	4.5627×10^{-3}		2.3549×10^{-5}
	$\frac{1}{2}$	$\frac{1}{2}$	1.2765×10^{-3}		5.8096×10^{-6}
12	0	0	2.3181×10^{-4}	16	1.0249×10^{-6}
	$-\frac{1}{2}$	$\frac{1}{2}$	1.6071×10^{-2}		2.8658×10^{-5}
	$-\frac{1}{2}$	$-\frac{1}{2}$	5.8618×10^{-4}		2.3902×10^{-6}
	1	1	2.2599×10^{-8}		6.0718×10^{-12}
	$\frac{1}{2}$	$\frac{1}{2}$	5.1018×10^{-9}		6.8922×10^{-13}
20	0	0	7.4227×10^{-10}	24	6.1146×10^{-14}
	$-\frac{1}{2}$	$\frac{1}{2}$	1.0349×10^{-9}		1.4156×10^{-11}
	$-\frac{1}{2}$	$-\frac{1}{2}$	1.6901×10^{-9}		1.8600×10^{-13}

Example 5 We consider the following wave equation [35]:

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=0, \quad (x,t)\in[0,\pi]\times[0,1],$$

with the boundary conditions:

$$u(0,t)=u(\pi,t)=0, \quad t\in[0,1],$$

and the initial condition:

$$u(x,0)=\sin(x), \quad x\in[0,\pi],$$

and

$$u_t(x,0)=0, \quad x\in[0,\pi].$$

The exact solution of the above problem is $u(x,t)=\sin(x)\cos t$.

Table 5 exhibits a comparison between the error obtained by using JG method and the Taylor matrix (TM) [35] and Bernolli collocation (BC) [36] . The numerical results show that JG method is more accurate than the existing methods.

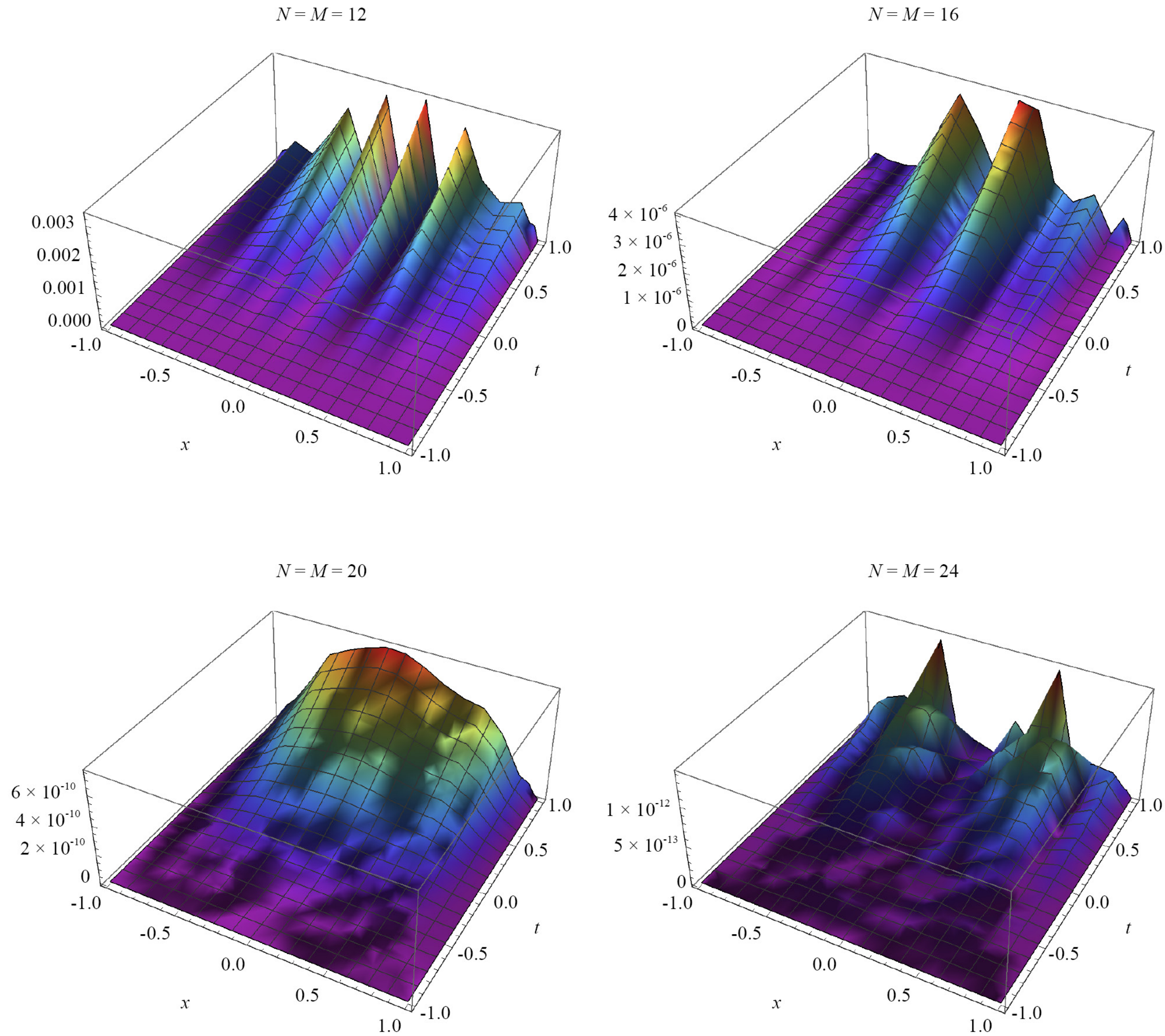

Figure 5. The graphs of the absolute error functions for Example 4 at different choices of $N = M$ with $\eta = \zeta = 0$

Table 5. Maximum absolute errors with various choices of ζ, η and $x = t = 0.6, 0.7, 0.8, 0.9, 1$ for Example 5

$(\frac{2}{\pi}x - 1, 2t - 1)$	ζ	η	JG method $N = M = 12$	JG method $N = M = 16$	TM method $(N = 12)$ [35]	BC method $(N = 12)$ [36]
$(-0.618028, 0.2)$	$\frac{1}{2}$	$\frac{1}{2}$	8.50×10^{-13}	7.49×10^{-16}	4.63×10^{-11}	2.51×10^{-11}
	0	0	3.30×10^{-13}	4.16×10^{-17}		
	$-\frac{1}{2}$	$-\frac{1}{2}$	7.99×10^{-13}	8.18×10^{-16}		
$(-0.554366, 0.4)$	$\frac{1}{2}$	$\frac{1}{2}$	2.38×10^{-12}	2.22×10^{-16}	3.27×10^{-10}	2.29×10^{-11}
	0	0	7.14×10^{-13}	5.55×10^{-17}		
	$-\frac{1}{2}$	$-\frac{1}{2}$	1.27×10^{-12}	6.93×10^{-16}		
$(-0.490704, 0.6)$	$\frac{1}{2}$	$\frac{1}{2}$	1.85×10^{-12}	3.05×10^{-16}	1.75×10^{-9}	2.52×10^{-11}
	0	0	2.04×10^{-14}	1.11×10^{-16}		
	$-\frac{1}{2}$	$-\frac{1}{2}$	3.22×10^{-13}	1.97×10^{-15}		
$(-0.427042, 0.8)$	$\frac{1}{2}$	$\frac{1}{2}$	3.71×10^{-13}	1.97×10^{-15}	7.61×10^{-9}	2.22×10^{-11}
	0	0	2.40×10^{-13}	8.32×10^{-16}		
	$-\frac{1}{2}$	$-\frac{1}{2}$	1.15×10^{-13}	1.11×10^{-15}		
$(1, 1)$	$\frac{1}{2}$	$\frac{1}{2}$	0.00×10^{-00}	0.00×10^{-00}	2.76×10^{-9}	1.81×10^{-11}
	0	0	0.00×10^{-00}	0.00×10^{-00}		
	$-\frac{1}{2}$	$-\frac{1}{2}$	0.00×10^{-00}	0.00×10^{-00}		

7. Concluding remarks

In this study, we focused on solving linear PDEs with their initial and boundary conditions using the JG method. We introduce appropriate bases that meet these criteria and serve as a foundational step in tackling linear PDEs. These bases yield discrete systems characterized by highly structured, practically invertible matrices. The outcomes are encouraging. Our future objective is to expand the existing algorithms to address the fractional versions of these PDEs.

Conflict of interest

The authors declare no competing financial interest.

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