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Research Article

Optical Solitons for the Dispersive Concatenation Model with Polarization Mode Dispersion by Sardar's Sub-Equation Approach

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Abstract: This paper recovers the spectrum of optical solitons for the dispersive concatenation model with the application of the Sardar sub-equation approach. The single soliton solutions that emerge from this scheme are bright, dark, and of singular type. The parameter constraints for the existence of such solitons are also included.

***Keywords*:** solitons, birefringence, Sardar, concatenation

MSC: 78A60

1. Introduction

It was during 2004 that the concept of the concatenation model emerged [1, 2]. This model is formed by conjoining the familiar nonlinear Schrödinger's equation (NLSE), Lakshmanan-Porsezian-Daniel (LPD) equation, and Sasa-Satsuma equation [3, 4]. This model itself later served as an independent equation to address the propagation dynamics of optical solitons across trans-oceanic and trans-continental distances [5, 6]. Thereafter, another form of the concatenation model was proposed [7, 8]. This is similarly formed from the three pre-existing models: the Schrödinger-Hirota equation, LPD model, and the fifth-order NLSE [9, 10]. Having higher-order dispersion terms, this model is therefore referred to as the dispersive concatenation model [11, 12].

There are several results that have emerged after extensive studies with the two models [13, 14]. They include the retrieval of the full spectrum of soliton solutions to the models using a plethora of integration schemes, locating the conservation laws, addressing the Internet bottleneck effect with spatio-temporal dispersion, studying the model for gap solitons, the perturbed version of the concatenation model using semi-inverse variation. The numerical simulations of

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the model have also been studied with Kerr and power laws by the aid of the Laplace-Adomian decomposition method (LADM). Several additional features have been covered, such as studying the model with multiplicative white noise and so on. Later, the concatenation model was also studied with differential group delay, and its soliton solutions have been recovered.

Similarly, the dispersive concatenation model was also covered analytically and numerically using a variety of integration algorithms [7]. The LADM scheme has been lately implemented to address the soliton solutions to the dispersive concatenation model. Very recently, the dispersive concatenation model was studied with polarization-mode dispersion, and some preliminary results have been reported. The current paper revisits the model to recover the soliton solutions with a completely different approach, namely the Sardar sub-equation algorithm. This would provide a fresher perspective on the model from a totally different angle. Additionally, the paper not only identifies the solitons but also delineates the essential parameter constraints for their existence. These constraints provide important insights into the conditions necessary for soliton phenomena, enhancing our understanding of the model's behavior. The results are recovered and displayed after a quick introduction to the model.

1.1 *Governing model*

The dimensionless form of the scalar version of the dispersive concatenation model with Kerr law nonlinearity is structured as [12–14]:

$$
\begin{aligned}
& iq_t + aq_{xx} + b|q|^2 q - i\delta_1 \left[\sigma_1 q_{xxx} + \sigma_2 |q|^2 q_x\right] \\
& + \delta_2 \left[\sigma_3 q_{xxxx} + \sigma_4 |q|^2 q_{xx} + \sigma_5 |q|^4 q + \sigma_6 |q_x|^2 q + \sigma_7 q_x^2 q^* + \sigma_8 q_{xx}^* q^2\right] \\
& - i\delta_3 \left[\sigma_9 q_{xxxxx} + \sigma_{10} |q|^2 q_{xxx} + \sigma_{11} |q|^4 q_x + \sigma_{12} q q_x q_{xx}^* + \sigma_{13} q^* q_x q_{xx} + \sigma_{14} q q_x^* q_{xx} + \sigma_{15} q_x^2 q_x^*\right] = 0.
\end{aligned} \tag{1}
$$

Here, $q(x, t)$ represents the complex-valued function that signifies the wave amplitude, with the independent variables x and t denoting the spatial and temporal variables, respectively. The coefficient of a corresponds to the chromatic dispersion (CD), while the coefficient of b accounts for the self-phase modulation (SPM) arising from the Kerr law of nonlinearity. Additionally, $i = \sqrt{-1}$ denotes the imaginary unit. The first five terms originate from the Schrödinger-Hirota equation (SHE), whereas the coefficients of δ_j for $j = 1, 2$ stem from the Lakshmanan-Porsezian-Daniel (LPD) and Sasa-Satsuma equation (SSE), respectively. It is this equation that will be split into two components, and the corresponding governing mode will be derived for birefringent fibers. For pulse splitting purposes, setting $q(x, t) = u(x, t) + v(x, t)$ and neglecting the four-wave mixing (4WM) terms result in coupled-mode equations along the two components of a birefringent fiber as follows [10, 11]:

$$
\begin{aligned}
& iu_t + a^{(1)} u_{xx} + \left(b_1^{(1)} |u|^2 + b_2^{(1)} |v|^2\right) u \\
& - i\delta_1^{(1)} \left[\sigma_1^{(1)} u_{xxx} + \left(\sigma_{21}^{(1)} |u|^2 + \sigma_{22}^{(1)} |v|^2\right) u_x\right] \\
& + \delta_2^{(1)} \left[\sigma_3^{(1)} u_{xxxx} + \left(\sigma_{41}^{(1)} |u|^2 + \sigma_{42}^{(1)} |v|^2\right) u_{xx} + \left(\sigma_{51}^{(1)} |u|^4 + \sigma_{52}^{(1)} |u|^2 |v|^2 + \sigma_{53}^{(1)} |v|^4\right) u
\end{aligned}
$$

$$+\left(\sigma_{61}^{(1)}\left|u_x\right|^2+\sigma_{62}^{(1)}\left|v_x\right|^2\right)u+\left(\sigma_{71}^{(1)}u_x^2+\sigma_{72}^{(1)}v_x^2\right)u^*+\left(\sigma_{81}^{(1)}u_{xx}^*+\sigma_{82}^{(1)}v_{xx}^*\right)u^2\Big]$$

$$-i\delta_3^{(1)}\Big[\sigma_9^{(1)}u_{xxxxx}+\left(\sigma_{101}^{(1)}|u|^2+\sigma_{102}^{(1)}|v|^2\right)u_{xxx}+\left(\sigma_{111}^{(1)}|u|^4+\sigma_{112}^{(1)}|u|^2|v|^2+\sigma_{113}^{(1)}|v|^4\right)u_x$$

$$+\left(\sigma_{121}^{(1)}uu_x+\sigma_{122}^{(1)}vv_x\right)u_{xx}^*+\left(\sigma_{131}^{(1)}u^*u_x+\sigma_{132}^{(1)}v^*v_x\right)u_{xx}$$

$$+\left(\sigma_{141}^{(1)}uu_x^*+\sigma_{142}^{(1)}vv_x^*\right)u_{xx}+\left(\sigma_{151}^{(1)}\left|u_x\right|^2+\sigma_{152}^{(1)}\left|v_x\right|^2\right)u_x\Big]=0, \tag{2}$$

and

$$iv_t+a^{(2)}v_{xx}+\left(b_1^{(2)}|v|^2+b_2^{(2)}|u|^2\right)v$$

$$-i\delta_1^{(2)}\left[\sigma_1^{(2)}v_{xxx}+\left(\sigma_{21}^{(2)}|v|^2+\sigma_{22}^{(2)}|u|^2\right)v_x\right]$$

$$+\delta_2^{(2)}\Big[\sigma_3^{(2)}v_{xxxx}+\left(\sigma_{41}^{(2)}|v|^2+\sigma_{42}^{(2)}|u|^2\right)v_{xx}+\left(\sigma_{51}^{(2)}|v|^4+\sigma_{52}^{(2)}|v|^2|u|^2+\sigma_{53}^{(2)}|u|^4\right)v$$

$$+\left(\sigma_{61}^{(2)}\left|v_x\right|^2+\sigma_{62}^{(2)}\left|u_x\right|^2\right)v+\left(\sigma_{71}^{(2)}v_x^2+\sigma_{72}^{(2)}u_x^2\right)v^*+\left(\sigma_{81}^{(2)}v_{xx}^*+\sigma_{82}^{(2)}u_{xx}^*\right)v^2\Big]$$

$$-i\delta_3^{(2)}\Big[\sigma_9^{(2)}v_{xxxxx}+\left(\sigma_{101}^{(2)}|v|^2+\sigma_{102}^{(2)}|u|^2\right)v_{xxx}+\left(\sigma_{111}^{(2)}|v|^4+\sigma_{112}^{(2)}|v|^2|u|^2+\sigma_{113}^{(2)}|u|^4\right)v_x$$

$$+\left(\sigma_{121}^{(2)}vv_x+\sigma_{122}^{(2)}uu_x\right)v_{xx}^*+\left(\sigma_{131}^{(2)}v^*v_x+\sigma_{132}^{(2)}u^*u_x\right)v_{xx}$$

$$+\left(\sigma_{141}^{(2)}vv_x^*+\sigma_{142}^{(2)}uu_x^*\right)v_{xx}+\left(\sigma_{151}^{(2)}\left|v_x\right|^2+\sigma_{152}^{(2)}\left|u_x\right|^2\right)v_x\Big]=0. \tag{3}$$

In equations (2) and (3), the components $u(x, t)$ and $v(x, t)$ represent the wave amplitudes of the split pulses arising from birefringence. The first terms with coefficients i denote the linear temporal evolution along the two components of the pulses. The second terms in (2) and (3), denoted by $a^{(j)}$, are the coefficients of chromatic dispersion along the two components of birefringent fibers. Then, $b_1^{(j)}$ and $b_2^{(j)}$ for $j=1, 2$, are the coefficients of self-phase modulation (SPM) and cross-phase modulation (XPM) effects, respectively. Subsequently, $\sigma_{21}^{(j)}$ and $\sigma_{22}^{(j)}$ represent SPM and XPM for intermodal dispersions, while $\sigma_3^{(j)}$ indicates fourth-order dispersions along the two components of a birefringent fiber. Additionally, $\sigma_{41}^{(j)}$ and $\sigma_{42}^{(j)}$ denote SPM and XPM for second-order dispersions, respectively. Moreover, $\sigma_{51}^{(j)}$ stands for SPM with quintic form of nonlinearity, whereas $\sigma_{52}^{(j)}$ and $\sigma_{53}^{(j)}$ account for XPM with quintic nonlinearity. Further, $\sigma_{61}^{(j)}$, $\sigma_{62}^{(j)}$, $\sigma_{71}^{(j)}$, $\sigma_{72}^{(j)}$, $\sigma_{81}^{(j)}$, and $\sigma_{82}^{(j)}$ arise from the radiative effect of the solitons, originating from four-wave mixing (4WM) effect and other sources of small-amplitude dispersive effects. Subsequently, $\sigma_9^{(j)}$ accounts for fifth-order dispersions along the two components of birefringence. Additionally, $\sigma_{101}^{(j)}$ and $\sigma_{102}^{(j)}$ represent SPM and XPM components stemming from third-order dispersions along the two components. Similarly, $\sigma_{111}^{(j)}$ denotes SPM coefficients due to inter-modal dispersion along the two components, while $\sigma_{112}^{(j)}$ and $\sigma_{113}^{(j)}$ are the coefficients of XPM due to inter-modal dispersion. Finally, $\sigma_{121}^{(j)}$, $\sigma_{122}^{(j)}$, $\sigma_{131}^{(j)}$, $\sigma_{132}^{(j)}$, $\sigma_{141}^{(j)}$, $\sigma_{142}^{(j)}$, $\sigma_{151}^{(j)}$, and $\sigma_{152}^{(j)}$ are additional terms from soliton radiation along the two components, emerging from multi-wave mixing and other sources. It is worth mentioning that the effect of 4WM arising from XPM is ignored in the derivation of the model equations (2) and (3) from (1), aiming to maintain the coupled system's simplicity without additional nonlinear effects. However, apart from neglecting the 4WM effects, the remaining terms included are equally important as they directly arise from the dispersive concatenation model given by (1).

To tackle the interconnected nature of systems (2) and (3), we adopt the assumption of a particular solution structure:

$$u(x, t) = U_1(\zeta)\, e^{i\phi(x, t)}, \tag{4}$$

and

$$v(x, t) = U_2(\zeta)\, e^{i\phi(x, t)}. \tag{5}$$

This structure serves as a foundational framework for addressing the complexities inherent in both systems simultaneously. By delineating a clear solution structure, we aim to provide a systematic approach that facilitates the analysis and resolution of the coupled dynamics present in systems (2) and (3).

In this context, ζ, which we have designated as our primary wave variable, is formally delineated as

$$\zeta = k(x - \nu t). \tag{6}$$

Here, the variables $U_j(\zeta)$ (where $j = 1, 2$) serve as explicit representations of the amplitude component inherent in the soliton solution, while ν serves as a clear indicator of the soliton's velocity. Moreover, the phase $\phi(x, t)$ is meticulously characterized as

$$\phi(x, t) = -\kappa x + \omega t + \theta_0. \tag{7}$$

In this particular context, κ embodies the frequency attributed to the solitons, while ω corresponds to the wave number, and θ_0 signifies the phase constant. Upon substituting equations (4) and (5) with equations (2) and (3) respectively, and subsequently decomposing them into their constituent real and imaginary components, the resulting real parts yield

$$-\kappa^3\sigma_{122}^{(1)}{U_2}^3\delta_3^{(1)} - 2k^2\kappa\sigma_{122}^{(1)}U_2{U_{2'}}^2\delta_3^{(1)} - 2k^2\kappa\sigma_{121}^{(1)}U_1U_{1'}U_{2'}\delta_3^{(1)}$$

$$+2k^2\kappa\left(\sigma_{132}^{(1)}+\sigma_{142}^{(1)}\right)U_2{U_1}'{U_2}'\delta_3^{(1)} + k^2\kappa\sigma_{122}^{(1)}{U_2}^2{U_2}''\delta_3^{(1)} + \left(\kappa\delta_3^{(1)}\sigma_{111}^{(1)} - \delta_2^{(1)}\sigma_{51}^{(1)}\right){U_1}^5 + \left(\kappa\delta_3^{(1)}\sigma_{113}^{(1)} - \delta_2^{(1)}\sigma_{53}^{(1)}\right)U_1{U_2}^4$$

$$+\left(\kappa^2\left[\delta_2^{(1)}\left(\sigma_{41}^{(1)} - \sigma_{61}^{(1)} + \sigma_{71}^{(1)} + \sigma_{81}^{(1)}\right) - \kappa\delta_3^{(1)}\left(\sigma_{101}^{(1)} + \sigma_{131}^{(1)} - \sigma_{141}^{(1)} - \sigma_{151}^{(1)}\right)\right] - b_1^{(1)}\right){U_1}^3$$

$$+\left[\kappa^2\left\{\delta_2^{(1)}\left(\sigma_{42}^{(1)}-\sigma_{62}^{(1)}+\sigma_{72}^{(1)}\right)-\kappa\delta_3^{(1)}\left(\sigma_{102}^{(1)}+\sigma_{132}^{(1)}-\sigma_{142}^{(1)}-\sigma_{152}^{(1)}\right)\right\}-b_2^{(1)}\right]U_1{U_2}^2$$

$$-k^2\left[\delta_2^{(1)}\left(\sigma_{61}^{(1)}+\sigma_{71}^{(1)}\right)-\kappa\delta_3^{(1)}\left(2\sigma_{131}^{(1)}+2\sigma_{141}^{(1)}+\sigma_{151}^{(1)}\right)\right]U_1{U_{1'}}^2$$

$$-k^4\left(\delta_2^{(1)}\sigma_3^{(1)}-5\kappa\delta_3^{(1)}\sigma_9^{(1)}\right){U_1}^{(4)}-k^2\left(\delta_2^{(1)}\left(\sigma_{62}^{(1)}+\sigma_{72}^{(1)}\right)-\kappa\delta_3^{(1)}\sigma_{152}^{(1)}\right)U_1{U_2}'^2$$

$$-k^2\left(\delta_2^{(1)}\sigma_{42}^{(1)}-\kappa\delta_3^{(1)}\left[3\sigma_{102}^{(1)}+\sigma_{132}^{(1)}-\sigma_{142}^{(1)}\right]\right){U_2}^2{U_1}''$$

$$+\left(\kappa^2a^{(1)}-\kappa^3\delta_1^{(1)}\sigma_1^{(1)}-\kappa^4\delta_2^{(1)}\sigma_3^{(1)}+\kappa^5\delta_3^{(1)}\sigma_9^{(1)}+\omega\right)U_1$$

$$+\kappa^2\left(\delta_2^{(1)}\sigma_{82}^{(1)}-\kappa\delta_3^{(1)}\sigma_{121}^{(1)}\right){U_1}^2U_2-k^2\left(\delta_2^{(1)}\left(\sigma_{41}^{(1)}+\sigma_{81}^{(1)}\right)-\kappa\delta_3^{(1)}\left[3\sigma_{101}^{(1)}+\sigma_{131}^{(1)}-\sigma_{141}^{(1)}\right]\right){U_1}^2{U_1}''$$

$$+k^2\left(\kappa\delta_3^{(1)}\sigma_{121}^{(1)}-\delta_2^{(1)}\sigma_{82}^{(1)}\right){U_1}^2{U_2}''-k^2\left(a^{(1)}-3\kappa\delta_1^{(1)}\sigma_1^{(1)}-6\kappa^2\delta_2^{(1)}\sigma_3^{(1)}+10\kappa^3\delta_3^{(1)}\sigma_9^{(1)}\right){U_1}''$$

$$+\left(\kappa\delta_3^{(1)}\sigma_{112}^{(1)}-\delta_2^{(1)}\sigma_{52}^{(1)}\right){U_1}^3{U_2}^2=0, \tag{8}$$

and

$$\kappa^3\sigma_{122}^{(2)}{U_1}^3\delta_3^{(2)}+2k^2\kappa\sigma_{122}^{(2)}U_1{U_1}'^2\delta_3^{(2)}-2k^2\kappa\left(\sigma_{132}^{(2)}+\sigma_{142}^{(2)}\right)U_1{U_1}'{U_2}'\delta_3^{(2)}+k^4\left(\delta_2^{(2)}\sigma_3^{(2)}-5\kappa\delta_3^{(2)}\sigma_9^{(2)}\right){U_2}^{(4)}$$

$$+2k^2\kappa\sigma_{121}^{(2)}U_2U_{1'}{U_2}'\delta_3^{(2)}-k^2\kappa\sigma_{122}^{(2)}{U_1}^2U_{1''}\delta_3^{(2)}+\left(\delta_2^{(2)}\sigma_{51}^{(2)}-\kappa\delta_3^{(2)}\sigma_{111}^{(2)}\right){U_2}^5+\left(\delta_2^{(2)}\sigma_{52}^{(2)}-\kappa\delta_3^{(2)}\sigma_{112}^{(2)}\right){U_1}^2{U_2}^3$$

$$+\left(b_1^{(2)}+\kappa^2\left\{\kappa\delta_3^{(2)}\left(\sigma_{101}^{(2)}+\sigma_{131}^{(2)}-\sigma_{141}^{(2)}-\sigma_{151}^{(2)}\right)-\delta_2^{(2)}\left[\sigma_{41}^{(2)}-\sigma_{61}^{(2)}+\sigma_{71}^{(2)}+\sigma_{81}^{(2)}\right]\right\}\right){U_2}^3$$

$$+\kappa^2\left(\kappa\delta_3^{(2)}\sigma_{121}^{(2)}-\delta_2^{(2)}\sigma_{82}^{(2)}\right)U_1{U_2}^2+k^2\left(\delta_2^{(2)}\left(\sigma_{62}^{(2)}+\sigma_{72}^{2}\right)-\kappa\delta_3^{(2)}\sigma_{152}^{(2)}\right)U_2{U_1}'^2$$

$$+k^2\left(\delta_2^{(2)}\left(\sigma_{61}^{(2)}+\sigma_{71}^{(2)}\right)-\kappa\delta_3^{(2)}\left[2\sigma_{131}^{(2)}+2\sigma_{141}^{(2)}+\sigma_{151}^{(2)}\right]\right)U_2{U_2}'^2+\left(\delta_2^{(2)}\sigma_{53}^{(2)}-\kappa\delta_3^{(2)}\sigma_{113}^{(2)}\right){U_1}^4U_2$$

$$+\left(b_2^{(2)}+\kappa^2\left\{\kappa\delta_3^{(2)}\left(\sigma_{102}^{(2)}+\sigma_{132}^{(2)}-\sigma_{142}^{(2)}-\sigma_{152}^{(2)}\right)-\delta_2^{(2)}\left[\sigma_{42}^{(2)}-\sigma_{62}^{(2)}+\sigma_{72}^{2}\right]\right\}\right){U_1}^2U_2$$

$$+\left(-\kappa^2a^{(2)}+\kappa^3\delta_1^{(2)}\sigma_1^{(2)}+\kappa^4\delta_2^{(2)}\sigma_3^{(2)}-\kappa^5\delta_3^{(2)}\sigma_9^{(2)}-\omega\right)U_2+k^2\left(\delta_2^{(2)}\sigma_{82}^{(2)}-\kappa\delta_3^{(2)}\sigma_{121}^{(2)}\right){U_2}^2{U_1}''$$

$$+k^2\left(\delta_2^{(2)}\sigma_{42}^{(2)}-\kappa\delta_3^{(2)}\left[3\sigma_{102}^{(2)}+\sigma_{132}^{(2)}-\sigma_{142}^{(2)}\right]\right)U_1{}^2U_{2''}+k^2\left(a^{(2)}-3\kappa\delta_1^{(2)}\sigma_1^{(2)}-6\kappa^2\delta_2^{(2)}\sigma_3^{(2)}+10\kappa^3\delta_3^{(2)}\sigma_9^{(2)}\right)U_2{}''$$

$$+k^2\left(\delta_2^{(2)}\left(\sigma_{41}^{(2)}+\sigma_{81}^{(2)}\right)-\kappa\delta_3^{(2)}\left[3\sigma_{101}^{(2)}+\sigma_{131}^{(2)}-\sigma_{141}^{(2)}\right]\right)U_2{}^2U_2{}''=0, \tag{9}$$

and the imaginary parts yield

$$kU_1{}'\left(2\kappa a^{(1)}+5\kappa^4\delta_3^{(1)}\sigma_9^{(1)}-4\kappa^3\delta_2^{(1)}\sigma_3^{(1)}-3\kappa^2\delta_1^{(1)}\sigma_1^{(1)}+\nu\right)$$

$$+k^5\delta_3^{(1)}\sigma_9^{(1)}U_1{}^{(5)}+\delta_1^{(1)}\sigma_{21}^{(1)}U_1{}^3+k^3U_1{}^{(3)}\left(4\kappa\delta_2^{(1)}\sigma_3^{(1)}+\delta_1^{(1)}\sigma_1^{(1)}+\delta_3^{(1)}\left[\sigma_{102}^{(1)}U_2{}^2-10\kappa^2\sigma_9^{(1)}\right]\right)$$

$$+k^3\delta_3^{(1)}\left(\sigma_{131}^{(1)}+\sigma_{141}^{(1)}\right)U_1U_{1'}U_{1''}+k^3\delta_3^{(1)}\left(\sigma_{132}^{(1)}+\sigma_{142}^{(1)}\right)U_2U_2{}'U_1{}''+k^3\delta_3^{(1)}\sigma_{121}^{(1)}U_1U_1{}'U_2{}''$$

$$+k^3\delta_3^{(1)}\sigma_{122}^{(1)}U_2U_2{}'U_2{}''+k^3\delta_3^{(1)}\sigma_{151}^{(1)}U_1{}'^3+k^3\delta_3^{(1)}\sigma_{101}^{(1)}U_1{}^2U_1{}^{(3)}-\kappa^2k\delta_3^{(1)}\sigma_{121}^{(1)}U_1U_2U_{1'}$$

$$+\kappa^2k\delta_3^{(1)}\sigma_{122}^{(1)}U_2{}^2U_{2'}+k\delta_3^{(1)}\sigma_{111}^{(1)}U_1{}^4U_1{}'+k\delta_3^{(1)}\sigma_{113}^{(1)}U_2{}^4U_1{}'$$

$$+kU_1{}^2U_1{}'\left(2\kappa\delta_2^{(1)}\left(\sigma_{41}^{(1)}+\sigma_{71}^{(1)}-\sigma_{81}^{(1)}\right)+\delta_3^{(1)}\left[-3\kappa^2\sigma_{101}^{(1)}-3\kappa^2\sigma_{131}^{(1)}+\kappa^2\sigma_{141}^{(1)}+\kappa^2\sigma_{151}^{(1)}+\sigma_{112}^{(1)}U_2{}^2\right]\right)$$

$$+\kappa kU_2{}^2U_1{}'\left(2\delta_2^{(1)}\sigma_{42}^{(1)}-\kappa\delta_3^{(1)}\left[3\sigma_{102}^{(1)}+2\sigma_{132}^{(1)}-2\sigma_{142}^{(1)}-\sigma_{152}^{(1)}\right]\right)$$

$$+\delta_1^{(1)}\sigma_{22}^{(1)}U_1U_2{}^2+k^3\delta_3^{(1)}\sigma_{152}^{(1)}U_1{}'U_2{}'^2+2\kappa kU_1{}^2U_2{}'\left(\kappa\delta_3^{(1)}\sigma_{121}^{(1)}-\delta_2^{(1)}\sigma_{82}^{(1)}\right)$$

$$-\kappa kU_1U_2U_2{}'\left(\kappa\delta_3^{(1)}\left(\sigma_{132}^{(1)}+\sigma_{142}^{(1)}\right)-2\delta_2^{(1)}\sigma_{72}^{(1)}\right)=0, \tag{10}$$

and

$$-kU_2{}'\left(2\kappa a^{(2)}+5\kappa^4\delta_3^{(2)}\sigma_9^{(2)}-4\kappa^3\delta_2^{(2)}\sigma_3^{(2)}-3\kappa^2\delta_1^{(2)}\sigma_1^{(2)}+\nu\right)$$

$$-k\delta_3^{(2)}\sigma_{113}^{(2)}U_1{}^4U_2{}'-k\delta_3^{(2)}\sigma_{111}^{(2)}U_2{}^4U_2{}'-k^5\delta_3^{(2)}\sigma_9^{(2)}U_2{}^{(5)}$$

$$-k^3U_2{}^{(3)}\left(4\kappa\delta_2^{(2)}\sigma_3^{(2)}+\delta_1^{(2)}\sigma_1^{(2)}+\delta_3^{(2)}\left[\sigma_{102}^{(2)}U_1{}^2-10\kappa^2\sigma_9^{(2)}\right]\right)$$

$$-\delta_1^{(2)}\sigma_{22}^{(2)}U_1{}^2U_2-k^3\delta_3^{(2)}\sigma_{122}^{(2)}U_1U_1{}'U_1{}''-k^3\delta_3^{(2)}\sigma_{121}^{(2)}U_2U_2{}'U_1{}''-k^3\delta_3^{(2)}\left(\sigma_{132}^{(2)}+\sigma_{142}^{(2)}\right)U_1U_1{}'U_2{}''$$

$$-k^3\delta_3^{(2)}\left(\sigma_{131}^{(2)}+\sigma_{141}^{(2)}\right)U_2U_2{}'U_2{}''-k^3\delta_3^{(2)}\sigma_{151}^{(2)}U_2{}'^3-k^3\delta_3^{(2)}\sigma_{152}^{(2)}U_1{}'^2U_2{}'$$

$$+U_2{}^2\left(-k^3\delta_3^{(2)}\sigma_{101}^{(2)}U_2{}^{(3)}-k\delta_3^{(2)}\sigma_{112}^{(2)}U_1{}^2U_{2'}\right)-\kappa^2k\delta_3^{(2)}\sigma_{122}^{(2)}U_1{}^2U_1{}'+\kappa^2k\delta_3^{(2)}\sigma_{121}^{(2)}U_1U_2U_2{}'$$

$$+2\kappa kU_2{}^2U_1{}'\left(\delta_2^{(2)}\sigma_{82}^{(2)}-\kappa\delta_3^{(2)}\sigma_{121}^{(2)}\right)+\kappa kU_1U_2U_1{}'\left(\kappa\delta_3^{(2)}\left(\sigma_{132}^{(2)}+\sigma_{142}^{(2)}\right)-2\sigma_{72}^2\delta_2^{(2)}\right)$$

$$+\kappa kU_2{}^2U_2{}'\left(\kappa\delta_3^{(2)}\left(3\sigma_{101}^{(2)}+3\sigma_{131}^{(2)}-\sigma_{141}^{(2)}-\sigma_{151}^{(2)}\right)-2\delta_2^{(2)}\left[\sigma_{41}^{(2)}+\sigma_{71}^{(2)}-\sigma_{81}^{(2)}\right]\right)$$

$$-\delta_1^{(2)}\sigma_{21}^{(2)}U_2{}^3+\kappa kU_1{}^2U_2{}'\left(\kappa\delta_3^{(2)}\left(3\sigma_{102}^{(2)}+2\sigma_{132}^{(2)}-2\sigma_{142}^{(2)}-\sigma_{152}^{(2)}\right)-2\delta_2^{(2)}\sigma_{42}^{(2)}\right)=0. \tag{11}$$

This coupled system of equations can be straightforwardly decoupled under the implicit assumption that $U_2=\lambda\, U_1$. As a result, equations (8) and (9) can be reformulated as:

$$-k^2U_1{}''\left(a^{(1)}+10\kappa^3\delta_3^{(1)}\sigma_9^{(1)}-6\kappa^2\delta_2^{(1)}\sigma_3^{(1)}-3\kappa\delta_1^{(1)}\sigma_1^{(1)}\right)$$

$$+U_1\left(\kappa^2a^{(1)}+\kappa^5\delta_3^{(1)}\sigma_9^{(1)}-\kappa^4\delta_2^{(1)}\sigma_3^{(1)}-\kappa^3\delta_1^{(1)}\sigma_1^{(1)}+\omega\right)$$

$$+U_1{}^5\left(\kappa\delta_3^{(1)}\left(\lambda^4\sigma_{113}^{(1)}+\lambda^2\sigma_{112}^{(1)}+\sigma_{111}^{(1)}\right)-\delta_2^{(1)}\left(\lambda^4\sigma_{53}^{(1)}+\lambda^2\sigma_{52}^{(1)}+\sigma_{51}^{(1)}\right)\right)$$

$$+U_1{}^3(-\lambda^2b_2^{(1)}-b_1^{(1)}+\kappa^2[\delta_2^{(1)}\left(\lambda^2\sigma_{42}^{(1)}-\lambda^2\sigma_{62}^{(1)}+\lambda^2\sigma_{72}^{(1)}+\lambda\sigma_{82}^{(1)}+\sigma_{41}^{(1)}-\sigma_{61}^{(1)}+\sigma_{71}^{(1)}+\sigma_{81}^{(1)}\right)$$

$$-\kappa\delta_3^{(1)}\left\{\lambda^3\sigma_{122}^{(1)}+\lambda^2\sigma_{102}^{(1)}+\lambda^2\sigma_{132}^{(1)}-\lambda^2\sigma_{142}^{(1)}-\lambda^2\sigma_{152}^{(1)}+\lambda\sigma_{121}^{(1)}+\sigma_{101}^{(1)}+\sigma_{131}^{(1)}-\sigma_{141}^{(1)}-\sigma_{151}^{(1)}\right\}])$$

$$-k^4U_1{}^{(4)}\left(\delta_2^{(1)}\sigma_3^{(1)}-5\kappa\delta_3^{(1)}\sigma_9^{(1)}\right)+k^2U_1U_{1'}{}^2(\kappa\delta_3^{(1)}(-2\lambda^3\sigma_{122}^{(1)}+2\lambda^2\sigma_{132}^{(1)}$$

$$+2\lambda^2\sigma_{142}^{(1)}+\lambda^2\sigma_{152}^{(1)}-2\lambda\sigma_{121}^{(1)}+2\sigma_{131}^{(1)}+2\sigma_{141}^{(1)}+\sigma_{151}^{(1)})-\delta_2^{(1)}\left[\lambda^2\sigma_{62}^{(1)}+\lambda^2\sigma_{72}^{(1)}+\sigma_{61}^{(1)}+\sigma_{71}^{(1)}\right])$$

$$+k^2U_1{}^2U_1{}''(\kappa\delta_3^{(1)}\left(\lambda^3\sigma_{122}^{(1)}+3\lambda^2\sigma_{102}^{(1)}+\lambda^2\sigma_{132}^{(1)}-\lambda^2\sigma_{142}^{(1)}+\lambda\sigma_{121}^{(1)}+3\sigma_{101}^{(1)}+\sigma_{131}^{(1)}-\sigma_{141}^{(1)}\right)$$

$$-\delta_2^{(1)}\left[\lambda^2\sigma_{42}^{(1)}+\lambda\sigma_{82}^{(1)}+\sigma_{41}^{(1)}+\sigma_{81}^{(1)}\right])=0, \tag{12}$$

and

$$k^2\lambda U_1{''}\left(a^{(2)}+10\kappa^3\delta_3^{(2)}\sigma_9^{(2)}-6\kappa^2\delta_2^{(2)}\sigma_3^{(2)}-3\kappa\delta_1^{(2)}\sigma_1^{(2)}\right)$$

$$+\lambda U_1\left(-\kappa^2a^{(2)}-\kappa^5\delta_3^{(2)}\sigma_9^{(2)}+\kappa^4\delta_2^{(2)}\sigma_3^{(2)}+\kappa^3\delta_1^{(2)}\sigma_1^{(2)}-\omega\right)+U_1{}^3(\lambda^3b_1^{(2)}$$

$$+\lambda b_2^{(2)}+\kappa^2[\kappa\delta_3^{(2)}(\lambda^3\sigma_{101}^{(2)}+\lambda^3\sigma_{131}^{(2)}-\lambda^3\sigma_{141}^{(2)}-\lambda^3\sigma_{151}^{(2)}+\lambda^2\sigma_{121}^{(2)}+\lambda\sigma_{102}^{(2)}$$

$$+\lambda\sigma_{132}^{(2)}-\lambda\sigma_{142}^{(2)}-\lambda\sigma_{152}^{(2)}+\sigma_{122}^{(2)})-\lambda\delta_2^{(2)}\{\lambda^2\sigma_{41}^{(2)}-\lambda^2\sigma_{61}^{(2)}+\lambda^2\sigma_{71}^{(2)}+\lambda^2\sigma_{81}^{(2)}$$

$$+\lambda\sigma_{82}^{(2)}+\sigma_{42}^{(2)}-\sigma_{62}^{(2)}+\sigma_{72}^{2}\}])+k^4\lambda U_1{}^{(4)}\left(\delta_2^{(2)}\sigma_3^{(2)}-5\kappa\delta_3^{(2)}\sigma_9^{(2)}\right)$$

$$+k^2U_1U_1{'}^2(\kappa\delta_3^{(2)}(2\lambda^2\sigma_{121}^{(2)}-\lambda(2\lambda^2\sigma_{131}^{(2)}+2\lambda^2\sigma_{141}^{(2)}+\lambda^2\sigma_{151}^{(2)}+2\sigma_{132}^{(2)}$$

$$+2\sigma_{142}^{(2)}+\sigma_{152}^{(2)})+2\sigma_{122}^{(2)})+\lambda\delta_2^{(2)}\left[\lambda^2\sigma_{61}^{(2)}+\lambda^2\sigma_{71}^{(2)}+\sigma_{62}^{(2)}+\sigma_{72}^{2}\right])$$

$$+k^2U_1{}^2U_1{''}(\lambda\delta_2^{(2)}\left(\lambda^2\sigma_{41}^{(2)}+\lambda\left(\lambda\sigma_{81}^{(2)}+\sigma_{82}^{(2)}\right)+\sigma_{42}^{(2)}\right)$$

$$-\kappa\delta_3^{(2)}\left[3\lambda^3\sigma_{101}^{(2)}+\lambda^3\sigma_{131}^{(2)}-\lambda^3\sigma_{141}^{(2)}+\lambda^2\sigma_{121}^{(2)}+3\lambda\sigma_{102}^{(2)}+\lambda\sigma_{132}^{(2)}-\lambda\sigma_{142}^{(2)}+\sigma_{122}^{(2)}\right])$$

$$+\lambda U_1{}^5\left(\delta_2^{(2)}\left(\lambda^4\sigma_{51}^{(2)}+\lambda^2\sigma_{52}^{(2)}+\sigma_{53}^{(2)}\right)-\kappa\delta_3^{(2)}\left(\lambda^4\sigma_{111}^{(2)}+\lambda^2\sigma_{112}^{(2)}+\sigma_{113}^{(2)}\right)\right)=0, \tag{13}$$

and Eqs. (10) and (11) become

$$kU_1{'}\left(2\kappa a^{(1)}+5\kappa^4\delta_3^{(1)}\sigma_9^{(1)}-4\kappa^3\delta_2^{(1)}\sigma_3^{(1)}-3\kappa^2\delta_1^{(1)}\sigma_1^{(1)}+\nu\right)$$

$$+U_1{}^2\left(k^3\lambda^2\delta_3^{(1)}\sigma_{102}^{(1)}U_1{}^{(3)}+k\lambda^2\delta_3^{(1)}\sigma_{112}^{(1)}U_1{}^2U_1{'}\right)+k^5\delta_3^{(1)}\sigma_9^{(1)}U_1{}^{(5)}$$

$$+k^3U_1{}^{(3)}\left(2\kappa\left(2\delta_2^{(1)}\sigma_3^{(1)}-5\kappa\delta_3^{(1)}\sigma_9^{(1)}\right)+\delta_1^{(1)}\sigma_1^{(1)}\right)+k^3\delta_3^{(1)}\left(\lambda^2\sigma_{152}^{(1)}+\sigma_{151}^{(1)}\right)U_1{'}^3$$

$$+k^3\delta_3^{(1)}U_1\left(\lambda^3\sigma_{122}^{(1)}+\lambda^2\sigma_{132}^{(1)}+\lambda^2\sigma_{142}^{(1)}+\lambda\sigma_{121}^{(1)}+\sigma_{131}^{(1)}+\sigma_{141}^{(1)}\right)U_1{'}U_1{''}$$

$$+k^3\delta_3^{(1)}\sigma_{101}^{(1)}U_1{}^2U_1{}^{(3)}+\kappa kU_1{}^2U_1{'}(2\delta_2^{(1)}\left\{\lambda^2\sigma_{42}^{(1)}+\lambda^2\sigma_{72}^{(1)}-\lambda\sigma_{82}^{(1)}+\sigma_{41}^{(1)}+\sigma_{71}^{(1)}-\sigma_{81}^{(1)}\right\}$$

$$-\kappa\delta_3^{(1)}[-\lambda^3\sigma_{122}^{(1)}+3\lambda^2\sigma_{102}^{(1)}+3\lambda^2\sigma_{132}^{(1)}-\lambda^2\sigma_{142}^{(1)}-\lambda^2\sigma_{152}^{(1)}-\lambda\sigma_{121}^{(1)}+3\sigma_{101}^{(1)}+3\sigma_{131}^{(1)}$$

$$-\sigma_{141}^{(1)}-\sigma_{151}^{(1)}])+k\delta_{3}^{(1)}U_{1}{}^{4}\left(\lambda^{4}\sigma_{113}^{(1)}+\sigma_{111}^{(1)}\right)U_{1}{}'+\delta_{1}^{(1)}U_{1}{}^{3}\left(\lambda^{2}\sigma_{22}^{(1)}+\sigma_{21}^{(1)}\right)=0, \tag{14}$$

and

$$\begin{aligned}
&-k\lambda U_{1}{}'\left(2\kappa a^{(2)}+5\kappa^{4}\delta_{3}^{(2)}\sigma_{9}^{(2)}-4\kappa^{3}\delta_{2}^{(2)}\sigma_{3}^{(2)}-3\kappa^{2}\delta_{1}^{(2)}\sigma_{1}^{(2)}+v\right)\\
&-k^{5}\lambda\delta_{3}^{(2)}\sigma_{9}^{(2)}U_{1}{}^{(5)}-k^{3}\lambda U_{1}{}^{(3)}\left(2\kappa\left(2\delta_{2}^{(2)}\sigma_{3}^{(2)}-5\kappa\delta_{3}^{(2)}\sigma_{9}^{(2)}\right)+\delta_{1}^{(2)}\sigma_{1}^{(2)}\right)-k\lambda\delta_{3}^{(2)}U_{1}{}^{4}\left(\lambda^{4}\sigma_{111}^{(2)}+\sigma_{113}^{(2)}\right)U_{1}{}'\\
&-k^{3}\delta_{3}^{(2)}U_{1}\left(\lambda^{2}\sigma_{121}^{(2)}+\lambda\left(\lambda^{2}\sigma_{131}^{(2)}+\lambda^{2}\sigma_{141}^{(2)}+\sigma_{132}^{(2)}+\sigma_{142}^{(2)}\right)+\sigma_{122}^{(2)}\right)U_{1}{}'U_{1}{}''-k^{3}\lambda\delta_{3}^{(2)}\left(\lambda^{2}\sigma_{151}^{(2)}+\sigma_{152}^{(2)}\right)U_{1}{}'^{3}\\
&+U_{1}{}^{2}\left(-k^{3}\lambda\delta_{3}^{(2)}U_{1}{}^{(3)}\left(\lambda^{2}\sigma_{101}^{(2)}+\sigma_{102}^{(2)}\right)-k\lambda^{3}\delta_{3}^{(2)}\sigma_{112}^{(2)}U_{1}{}^{2}U_{1}{}'\right)-\lambda\delta_{1}^{(2)}U_{1}{}^{3}\left(\lambda^{2}\sigma_{21}^{(2)}+\sigma_{22}^{(2)}\right)\\
&+\kappa kU_{1}{}^{2}U_{1}{}'(-\kappa\delta_{3}^{(2)}\{-3\lambda^{3}\sigma_{101}^{(2)}-3\lambda^{3}\sigma_{131}^{(2)}+\lambda^{3}\sigma_{141}^{(2)}+\lambda^{3}\sigma_{151}^{(2)}+\lambda^{2}\sigma_{121}^{(2)}-3\lambda\sigma_{102}^{(2)}-3\lambda\sigma_{132}^{(2)}\\
&+\lambda\sigma_{142}^{(2)}+\lambda\sigma_{152}^{(2)}+\sigma_{122}^{(2)}\}-2\lambda\delta_{2}^{(2)}\left[\lambda^{2}\sigma_{41}^{(2)}+\lambda^{2}\sigma_{71}^{(2)}-\lambda^{2}\sigma_{81}^{(2)}-\lambda\sigma_{82}^{(2)}+\sigma_{42}^{(2)}+\sigma_{72}^{2}\right])=0.
\end{aligned} \tag{15}$$

Through a meticulous comparison of the coefficients belonging to the linearly independent functions within equations (14) and (15), we can equate them to zero. This procedure allows us to derive the velocities associated with the two components as well as the parametric constraints, leading to the following outcomes.

When the coefficients of U_1 are set to zero, the velocities along the two components are determined to be:

$$v=-\kappa\left(2a^{(1)}+5\kappa^{3}\delta_{3}^{(1)}\sigma_{9}^{(1)}-4\kappa^{2}\delta_{2}^{(1)}\sigma_{3}^{(1)}-3\kappa\delta_{1}^{(1)}\sigma_{1}^{(1)}\right), \tag{16}$$

and

$$v=-\kappa\left(2a^{(2)}+5\kappa^{3}\delta_{3}^{(2)}\sigma_{9}^{(2)}-4\kappa^{2}\delta_{2}^{(2)}\sigma_{3}^{(2)}-3\kappa\delta_{1}^{(2)}\sigma_{1}^{(2)}\right). \tag{17}$$

By equating the velocities of the solitons along the two components as expressed in equations (16) and (17), we obtain the parameter constraint:

$$2a^{(1)}+5\kappa^{3}\delta_{3}^{(1)}\sigma_{9}^{(1)}-4\kappa^{2}\delta_{2}^{(1)}\sigma_{3}^{(1)}-3\kappa\delta_{1}^{(1)}\sigma_{1}^{(1)}=2a^{(2)}+5\kappa^{3}\delta_{3}^{(2)}\sigma_{9}^{(2)}-4\kappa^{2}\delta_{2}^{(2)}\sigma_{3}^{(2)}-3\kappa\delta_{1}^{(2)}\sigma_{1}^{(2)}. \tag{18}$$

To ensure integrability, it is necessary to set the coefficients of certain functions to zero. This action leads to the emergence of additional parameter constraints, the specifics of which are outlined as follows:

When the coefficients of $U_1{}^3$ are set to zero, the resulting outcome is:

$$\lambda^2\sigma_{22}^{(1)}+\sigma_{21}^{(1)}=\lambda^2\sigma_{21}^{(2)}+\sigma_{22}^{(2)}=0. \tag{19}$$

When the coefficients of ${U_1}'{U_1}^2$ are set to zero, the consequence is:

$$-\kappa\delta_3^{(1)}\left(-\lambda^3\sigma_{122}^{(1)}+3\lambda^2\sigma_{102}^{(1)}+3\lambda^2\sigma_{132}^{(1)}-\lambda^2\sigma_{142}^{(1)}-\lambda^2\sigma_{152}^{(1)}-\lambda\sigma_{121}^{(1)}+3\sigma_{101}^{(1)}+3\sigma_{131}^{(1)}-\sigma_{141}^{(1)}-\sigma_{151}^{(1)}\right)$$

$$+2\delta_2^{(1)}\left(\lambda^2\sigma_{42}^{(1)}+\lambda^2\sigma_{72}^{(1)}-\lambda\sigma_{82}^{(1)}+\sigma_{41}^{(1)}+\sigma_{71}^{(1)}-\sigma_{81}^{(1)}\right)=0, \tag{20}$$

and

$$-\kappa\delta_3^{(2)}\left(-3\lambda^3\sigma_{101}^{(2)}-3\lambda^3\sigma_{131}^{(2)}+\lambda^3\sigma_{141}^{(2)}+\lambda^3\sigma_{151}^{(2)}+\lambda^2\sigma_{121}^{(2)}-3\lambda\sigma_{102}^{(2)}-3\lambda\sigma_{132}^{(2)}+\lambda\sigma_{142}^{(2)}+\lambda\sigma_{152}^{(2)}+\sigma_{122}^{(2)}\right)$$

$$-2\lambda\delta_2^{(2)}\left(\lambda^2\sigma_{41}^{(2)}+\lambda^2\sigma_{71}^{(2)}-\lambda^2\sigma_{81}^{(2)}-\lambda\sigma_{82}^{(2)}+\sigma_{42}^{(2)}+\sigma_{72}^{2}\right)=0. \tag{21}$$

When the coefficients of ${U_1}^4U_1$ are set to zero, the outcome is:

$$\lambda^4\sigma_{113}^{(1)}+\sigma_{111}^{(1)}+\lambda^2\sigma_{112}^{(1)}=\lambda^4\sigma_{111}^{(2)}+\sigma_{113}^{(2)}+\lambda^2\sigma_{112}^{(2)}=0. \tag{22}$$

When the coefficients of ${U_1}'^3$ are set to zero, the outcome is:

$$\lambda^2\sigma_{152}^{(1)}+\sigma_{151}^{(1)}=\lambda^2\sigma_{151}^{(2)}+\sigma_{152}^{(2)}=0. \tag{23}$$

When the coefficients of $U_1{U_1}'{U_1}''$ are set to zero, the result is:

$$\lambda^3\sigma_{122}^{(1)}+\lambda^2\sigma_{132}^{(1)}+\lambda^2\sigma_{142}^{(1)}+\lambda\sigma_{121}^{(1)}+\sigma_{131}^{(1)}+\sigma_{141}^{(1)}=0, \tag{24}$$

and

$$\lambda^2\sigma_{121}^{(2)}+\lambda\left(\lambda^2\sigma_{131}^{(2)}+\lambda^2\sigma_{141}^{(2)}+\sigma_{132}^{(2)}+\sigma_{142}^{(2)}\right)+\sigma_{122}^{(2)}=0. \tag{25}$$

When the coefficients of $U_1^{(3)}$ are set to zero, the outcome is:

$$2\kappa\left(2\delta_2^{(1)}\sigma_3^{(1)}-5\kappa\delta_3^{(1)}\sigma_9^{(1)}\right)+\delta_1^{(1)}\sigma_1^{(1)}=0, \tag{26}$$

and

$$2\kappa\left(2\delta_2^{(2)}\sigma_3^{(2)}-5\kappa\delta_3^{(2)}\sigma_9^{(2)}\right)+\delta_1^{(2)}\sigma_1^{(2)}=0. \tag{27}$$

When the coefficients of $U_1^2U_1^{(3)}$ are set to zero, the outcome is:

$$\lambda^2\sigma_{102}^{(1)}+\sigma_{101}^{(1)}=\lambda^2\sigma_{101}^{(2)}+\sigma_{102}^{(2)}=0. \tag{28}$$

When the coefficients of $U_1^{(5)}$ are set to zero, the result is:

$$\sigma_9^{(1)}=\sigma_9^{(2)}=0. \tag{29}$$

Through a comparative analysis of equations (14) and (15) and their consolidation into a unified equation, we can deduce the following parametric limitations.

Upon comparing the coefficients of $U_1^{(4)}$, we derive:

$$\delta_2^{(1)}\sigma_3^{(1)}-5\kappa\delta_3^{(1)}\sigma_9^{(1)}=\lambda\left(\delta_2^{(2)}\sigma_3^{(2)}-5\kappa\delta_3^{(2)}\sigma_9^{(2)}\right). \tag{30}$$

Through a comparison of the coefficients of $U_1^2U_1''$, we arrive at:

$$\begin{aligned}&\kappa\delta_3^{(1)}\left(\lambda^3\sigma_{122}^{(1)}+3\lambda^2\sigma_{102}^{(1)}+\lambda^2\sigma_{132}^{(1)}-\lambda^2\sigma_{142}^{(1)}+\lambda\sigma_{121}^{(1)}+3\sigma_{101}^{(1)}+\sigma_{131}^{(1)}-\sigma_{141}^{(1)}\right)\\&-\delta_2^{(1)}\left(\lambda^2\sigma_{42}^{(1)}+\lambda\sigma_{82}^{(1)}+\sigma_{41}^{(1)}+\sigma_{81}^{(1)}\right)=\lambda\delta_2^{(2)}\left(\lambda^2\sigma_{41}^{(2)}+\lambda\left(\lambda\sigma_{81}^{(2)}+\sigma_{82}^{(2)}\right)+\sigma_{42}^{(2)}\right)\\&-\kappa\delta_3^{(2)}\left(3\lambda^3\sigma_{101}^{(2)}+\lambda^3\sigma_{131}^{(2)}-\lambda^3\sigma_{141}^{(2)}+\lambda^2\sigma_{121}^{(2)}+3\lambda\sigma_{102}^{(2)}+\lambda\sigma_{132}^{(2)}-\lambda\sigma_{142}^{(2)}+\sigma_{122}^{(2)}\right).\end{aligned} \tag{31}$$

Through a comparison of the coefficients of U_1'', we deduce:

$$\begin{aligned}&a^{(1)}+10\kappa^3\delta_3^{(1)}\sigma_9^{(1)}-6\kappa^2\delta_2^{(1)}\sigma_3^{(1)}-3\kappa\delta_1^{(1)}\sigma_1^{(1)}\\&=-\lambda\left(a^{(2)}+10\kappa^3\delta_3^{(2)}\sigma_9^{(2)}-6\kappa^2\delta_2^{(2)}\sigma_3^{(2)}-3\kappa\delta_1^{(2)}\sigma_1^{(2)}\right).\end{aligned} \tag{32}$$

Through a comparison of the coefficients of $U_1U_1'^2$ we derive:

$$\kappa\delta_3^{(1)}\left(-2\lambda^3\sigma_{122}^{(1)}+2\lambda^2\sigma_{132}^{(1)}+2\lambda^2\sigma_{142}^{(1)}+\lambda^2\sigma_{152}^{(1)}-2\lambda\sigma_{121}^{(1)}+2\sigma_{131}^{(1)}+2\sigma_{141}^{(1)}+\sigma_{151}^{(1)}\right)$$

$$-\delta_2^{(1)}\left(\lambda^2\sigma_{62}^{(1)}+\lambda^2\sigma_{72}^{(1)}+\sigma_{61}^{(1)}+\sigma_{71}^{(1)}\right)$$

$$=\kappa\delta_3^{(2)}\left(2\lambda^2\sigma_{121}^{(2)}-\lambda\left(2\lambda^2\sigma_{131}^{(2)}+2\lambda^2\sigma_{141}^{(2)}+\lambda^2\sigma_{151}^{(2)}+2\sigma_{132}^{(2)}+2\sigma_{142}^{(2)}+\sigma_{152}^{(2)}\right)+2\sigma_{122}^{(2)}\right)$$

$$+\lambda\delta_2^{(2)}\left(\lambda^2\sigma_{61}^{(2)}+\lambda^2\sigma_{71}^{(2)}+\sigma_{62}^{(2)}+\sigma_{72}^2\right). \tag{33}$$

Upon comparing the coefficients of U_1^5, we obtain:

$$\kappa\delta_3^{(1)}\left(\lambda^4\sigma_{113}^{(1)}+\lambda^2\sigma_{112}^{(1)}+\sigma_{111}^{(1)}\right)-\delta_2^{(1)}\left(\lambda^4\sigma_{53}^{(1)}+\lambda^2\sigma_{52}^{(1)}+\sigma_{51}^{(1)}\right)$$

$$=\lambda\left(\delta_2^{(2)}\left(\lambda^4\sigma_{51}^{(2)}+\lambda^2\sigma_{52}^{(2)}+\sigma_{53}^{(2)}\right)-\kappa\delta_3^{(2)}\left(\lambda^4\sigma_{111}^{(2)}+\lambda^2\sigma_{112}^{(2)}+\sigma_{113}^{(2)}\right)\right). \tag{34}$$

Through a comparison of the coefficients of U_1, we derive:

$$\kappa^2a^{(1)}+\kappa^5\delta_3^{(1)}\sigma_9^{(1)}-\kappa^4\delta_2^{(1)}\sigma_3^{(1)}-\kappa^3\delta_1^{(1)}\sigma_1^{(1)}+\omega$$

$$=\lambda\left(-\kappa^2a^{(2)}-\kappa^5\delta_3^{(2)}\sigma_9^{(2)}+\kappa^4\delta_2^{(2)}\sigma_3^{(2)}+\kappa^3\delta_1^{(2)}\sigma_1^{(2)}-\omega\right). \tag{35}$$

Upon comparing the coefficients of U_1^3, we derive:

$$-\lambda^2b_2^{(1)}-b_1^{(1)}+\kappa^2(\delta_2^{(1)}\left(\lambda^2\sigma_{42}^{(1)}-\lambda^2\sigma_{62}^{(1)}+\lambda^2\sigma_{72}^{(1)}+\lambda\sigma_{82}^{(1)}+\sigma_{41}^{(1)}-\sigma_{61}^{(1)}+\sigma_{71}^{(1)}+\sigma_{81}^{(1)}\right)$$

$$-\kappa\delta_3^{(1)}\left[\lambda^3\sigma_{122}^{(1)}+\lambda^2\sigma_{102}^{(1)}+\lambda^2\sigma_{132}^{(1)}-\lambda^2\sigma_{142}^{(1)}-\lambda^2\sigma_{152}^{(1)}+\lambda\sigma_{121}^{(1)}+\sigma_{101}^{(1)}+\sigma_{131}^{(1)}-\sigma_{141}^{(1)}-\sigma_{151}^{(1)}\right])$$

$$=\lambda^3b_1^{(2)}+\lambda b_2^{(2)}+\kappa^2(\kappa\delta_3^{(2)}\{\lambda^3\sigma_{101}^{(2)}+\lambda^3\sigma_{131}^{(2)}-\lambda^3\sigma_{141}^{(2)}-\lambda^3\sigma_{151}^{(2)}+\lambda^2\sigma_{121}^{(2)}+\lambda\sigma_{102}^{(2)}+\lambda\sigma_{132}^{(2)}$$

$$-\lambda\sigma_{142}^{(2)}-\lambda\sigma_{152}^{(2)}+\sigma_{122}^{(2)}\}-\lambda\delta_2^{(2)}\left[\lambda^2\sigma_{41}^{(2)}-\lambda^2\sigma_{61}^{(2)}+\lambda^2\sigma_{71}^{(2)}+\lambda^2\sigma_{81}^{(2)}+\lambda\sigma_{82}^{(2)}+\sigma_{42}^{(2)}-\sigma_{62}^{(2)}+\sigma_{72}^2\right]). \tag{36}$$

The nonlinear ordinary differential equation that has been transformed and will be analyzed using the enhanced direct algebraic technique is:

$$
-k^2U_1{}'' \left(a^{(1)} - 6\kappa^2\delta_2^{(1)}\sigma_3^{(1)} - 3\kappa\delta_1^{(1)}\sigma_1^{(1)}\right) + U_1\left(\kappa^2 a^{(1)} - \kappa^4\delta_2^{(1)}\sigma_3^{(1)} - \kappa^3\delta_1^{(1)}\sigma_1^{(1)} + \omega\right)
$$

$$
+U_1{}^5\left(\kappa\delta_3^{(1)}\left(\lambda^4\sigma_{113}^{(1)} + \lambda^2\sigma_{112}^{(1)} + \sigma_{111}^{(1)}\right) - \delta_2^{(1)}\left(\lambda^4\sigma_{53}^{(1)} + \lambda^2\sigma_{52}^{(1)} + \sigma_{51}^{(1)}\right)\right)
$$

$$
+U_1{}^3(-\lambda^2 b_2^{(1)} - b_1^{(1)} + \kappa^2[\delta_2^{(1)}\left(\lambda^2\sigma_{42}^{(1)} - \lambda^2\sigma_{62}^{(1)} + \lambda^2\sigma_{72}^{(1)} + \lambda\sigma_{82}^{(1)} + \sigma_{41}^{(1)} - \sigma_{61}^{(1)} + \sigma_{71}^{(1)} + \sigma_{81}^{(1)}\right)
$$

$$
-\kappa\delta_3^{(1)}\left\{\lambda^3\sigma_{122}^{(1)} + \lambda^2\sigma_{102}^{(1)} + \lambda^2\sigma_{132}^{(1)} - \lambda^2\sigma_{142}^{(1)} - \lambda^2\sigma_{152}^{(1)} + \lambda\sigma_{121}^{(1)} + \sigma_{101}^{(1)} + \sigma_{131}^{(1)} - \sigma_{141}^{(1)} - \sigma_{151}^{(1)}\right\}])
$$

$$
-k^4U_1{}^{(4)}\left(\delta_2^{(1)}\sigma_3^{(1)}\right) + k^2U_1U_1{}'^2(\kappa\delta_3^{(1)}(-2\lambda^3\sigma_{122}^{(1)} + 2\lambda^2\sigma_{132}^{(1)} + 2\lambda^2\sigma_{142}^{(1)} + \lambda^2\sigma_{152}^{(1)}
$$

$$
-2\lambda\sigma_{121}^{(1)} + 2\sigma_{131}^{(1)} + 2\sigma_{141}^{(1)} + \sigma_{151}^{(1)}) - \delta_2^{(1)}\left[\lambda^2\sigma_{62}^{(1)} + \lambda^2\sigma_{72}^{(1)} + \sigma_{61}^{(1)} + \sigma_{71}^{(1)}\right]
$$

$$
+k^2U_1{}^2U_1{}''(\kappa\delta_3^{(1)}\left(\lambda^3\sigma_{122}^{(1)} + 3\lambda^2\sigma_{102}^{(1)} + \lambda^2\sigma_{132}^{(1)} - \lambda^2\sigma_{142}^{(1)} + \lambda\sigma_{121}^{(1)} + 3\sigma_{101}^{(1)} + \sigma_{131}^{(1)} - \sigma_{141}^{(1)}\right)
$$

$$
-\delta_2^{(1)}\left[\lambda^2\sigma_{42}^{(1)} + \lambda\sigma_{82}^{(1)} + \sigma_{41}^{(1)} + \sigma_{81}^{(1)}\right]) = 0. \tag{37}
$$

This equation can be reformulated in a more concise structure as:

$$
B_4U_1U_1{}'^2 + B_6U_1^2U_1{}'' + B_5U_1{}'' + B_3U_1^5 + B_2U_1^3 + B_1U_1 + k^2U_1^{(4)} = 0, \tag{38}
$$

where

$$
B_1 = -\frac{\kappa^2 a^{(1)} - \gamma^2 - \kappa^4\delta_2^{(1)}\sigma_3^{(1)} - \kappa^3\delta_1^{(1)}\sigma_1^{(1)} + \omega}{k^2\delta_2^{(1)}\sigma_3^{(1)}},
$$

$$
B_2 = -\frac{1}{k^2\delta_2^{(1)}\sigma_3^{(1)}}, (\lambda^2 b_2^{(1)} + b_1^{(1)} + \kappa^2[\kappa\delta_3^{(1)}(\lambda^3\sigma_{122}^{(1)} + \lambda^2\sigma_{102}^{(1)} + \lambda^2\sigma_{132}^{(1)} - \lambda^2\sigma_{142}^{(1)} - \lambda^2\sigma_{152}^{(1)} + \lambda\sigma_{121}^{(1)}
$$

$$
+\sigma_{101}^{(1)} + \sigma_{131}^{(1)} - \sigma_{141}^{(1)} - \sigma_{151}^{(1)}) - \delta_2^{(1)}\left(\lambda^2\sigma_{42}^{(1)} - \lambda^2\sigma_{62}^{(1)} + \lambda^2\sigma_{72}^{(1)} + \lambda\sigma_{82}^{(1)} + \sigma_{41}^{(1)} - \sigma_{61}^{(1)} + \sigma_{71}^{(1)} + \sigma_{81}^{(1)}\right)]),
$$

$$
B_3 = -\frac{\kappa\delta_3^{(1)}\left(\lambda^4\sigma_{113}^{(1)} + \lambda^2\sigma_{112}^{(1)} + \sigma_{111}^{(1)}\right) - \delta_2^{(1)}\left(\lambda^4\sigma_{53}^{(1)} + \lambda^2\sigma_{52}^{(1)} + \sigma_{51}^{(1)}\right)}{k^2\delta_2^{(1)}\sigma_3^{(1)}},
$$

$$B_4 = -\frac{1}{\delta_2^{(1)}\sigma_3^{(1)}}(\kappa\delta_3^{(1)}\left(-2\lambda^3\sigma_{122}^{(1)} + 2\lambda^2\sigma_{132}^{(1)} + 2\lambda^2\sigma_{142}^{(1)} + \lambda^2\sigma_{152}^{(1)} - 2\lambda\sigma_{121}^{(1)} + 2\sigma_{131}^{(1)}\right.$$

$$\left. + 2\sigma_{141}^{(1)} + \sigma_{151}^{(1)} - \delta_2^{(1)}\left(\lambda^2\sigma_{62}^{(1)} + \lambda^2\sigma_{72}^{(1)} + \sigma_{61}^{(1)} + \sigma_{71}^{(1)}\right)\right),$$

$$B_5 = \frac{a^{(1)} - 6\kappa^2\delta_2^{(1)}\sigma_3^{(1)} - 3\kappa\delta_1^{(1)}\sigma_1^{(1)}}{\delta_2^{(1)}\sigma_3^{(1)}},$$

$$B_6 = -\frac{\kappa\delta_3^{(1)}\left(\lambda^3\sigma_{122}^{(1)} + 3\lambda^2\sigma_{102}^{(1)} + \lambda^2\sigma_{132}^{(1)} - \lambda^2\sigma_{142}^{(1)} + \lambda\sigma_{121}^{(1)} + 3\sigma_{101}^{(1)} + \sigma_{131}^{(1)} - \sigma_{141}^{(1)}\right) - \delta_2^{(1)}\left(\lambda^2\sigma_{42}^{(1)} + \lambda\sigma_{82}^{(1)} + \sigma_{41}^{(1)} + \sigma_{81}^{(1)}\right)}{\delta_2^{(1)}\sigma_3^{(1)}}, \tag{39}$$

provided $\delta_2^{(1)}\sigma_3^{(1)} \neq 0$.

2. Sardar sub-equation method (SSEM)

In this method, to tackle Eq. (38), we adopt the assumption that the solution takes the following form:

$$\$(\xi) = \sum_{n=0}^{\mathrm{N}} \lambda_n \Psi^n(\xi), \qquad \lambda_{\mathrm{N}} \neq 0. \tag{40}$$

Here, λ_n (where $n = 0, 1, \ldots, N$) represents a constant to be determined subsequently. The integer N is established utilizing the principle of homogeneous balance method, balancing the nonlinear term with the highest-order derivative in Eq. (38). Additionally, the function $\Psi^n(\xi)$ in Eq. (40) must fulfill the following equation:

$$\Psi'(\xi) = \sqrt{\eta_2\Psi(\xi)^4 + \eta_1\Psi(\xi)^2 + \eta_0}, \tag{41}$$

where η_l (where $l = 0, 1, 2$) are constants.

Accordingly, depending on the specific values of the parameters η_l, Eq. (41) admits various known solutions, outlined as follows:

Case I $\eta_0 = 0$.

If $\eta_1 > 0$ and $\eta_2 \neq 0$, then we get the bright and singular soliton solutions:

$$\Psi_1^{\pm}(\xi) = \pm\sqrt{-pq\eta_1/\eta_2}\,\mathrm{sech}_{pq}(\sqrt{\eta_1}\xi), \quad \eta_2 < 0, \tag{42}$$

and

$$\Psi_2^{\pm}(\xi) = \pm\sqrt{pq^{\eta_1}/_{\eta_2}}\operatorname{csch}_{pq}(\sqrt{\eta_1}\xi), \quad \eta_2 > 0, \tag{43}$$

where

$$\operatorname{sech}_{pq}(\sqrt{\eta_1}\xi) = \frac{2}{p\, e^{\sqrt{\eta_1}\xi} + q\, e^{-\sqrt{\eta_1}\xi}} \quad \text{and} \quad \operatorname{csch}_{pq}(\sqrt{\eta_1}\xi) = \frac{2}{p\, e^{\sqrt{\eta_1}\xi} - q\, e^{-\sqrt{\eta_1}\xi}}. \tag{44}$$

Case II $\eta_0 = \frac{1}{4}\frac{\eta_1^2}{\eta_2}$ and $\eta_2 > 0$. If $\eta_1 < 0$, then one obtains the dark and singular soliton solutions:

$$\Psi_3^{\pm}(\xi) = \pm\sqrt{-^{\eta_1}/_{2\eta_2}}\tanh_{pq}\left(\sqrt{-\frac{\eta_1}{2}}\xi\right), \tag{45}$$

and

$$\Psi_4^{\pm}(\xi) = \pm\sqrt{-^{\eta_1}/_{2\eta_2}}\coth_{pq}\left(\sqrt{-\frac{\eta_1}{2}}\xi\right), \tag{46}$$

where

$$\tanh_{pq}(\sqrt{\eta_1}\xi) = \frac{p\, e^{\sqrt{\eta_1}\xi} - q\, e^{-\sqrt{\eta_1}\xi}}{p\, e^{\sqrt{\eta_1}\xi} + q\, e^{-\sqrt{\eta_1}\xi}} \quad \text{and} \quad \coth_{pq}(\sqrt{\eta_1}\xi) = \frac{p\, e^{\sqrt{\eta_1}\xi} + q\, e^{-\sqrt{\eta_1}\xi}}{p\, e^{\sqrt{\eta_1}\xi} - q\, e^{-\sqrt{\eta_1}\xi}}. \tag{47}$$

2.1 *Application of the modified Sardar sub-equation method*

Our analysis commenced with the application of the homogeneous balance method principle, balancing the nonlinear term $u^{(4)}$ with the linear term u^5 from Eq. (38). This yields $N+4 = 5N$, resulting in $N = 1$. Consequently, Eq. (40) transforms into:

$$U_1(\xi) = (\lambda_0 + \lambda_1\Psi), \tag{48}$$

$$U_1{}'(\xi) = \lambda_1\sqrt{(\eta_2\Psi^4 + \eta_1\Psi^2 + \eta_0)}, \tag{49}$$

$$U_1{}''(\xi) = \lambda_1\left(2\,\eta_2\Psi^3 + \eta_1\Psi\right), \tag{50}$$

$$U_1{}^{(3)} = \lambda_1\left(6\,\eta_2\Psi^2 + \eta_1\right)\sqrt{\eta_2\Psi(\xi)^4 + \eta_1\Psi(\xi)^2 + \eta_0}, \tag{51}$$

$$U_1{}^{(4)} = \lambda_1\,\Psi\left[12\,\eta_2(1+\eta_2)\Psi^4 + 2\eta_1(9\,\eta_2 + 1)\Psi^2 + \eta_1{}^2 + 12\,\eta_2\eta_0\right]. \tag{52}$$

After substituting Eqs. (48-52) into Eq. (38) and considering Eq. (41), we arrive at:

$$B_4\,\lambda_1{}^2\;\left(\eta_2\left(\lambda_0\Psi^4+\lambda_1\Psi^5\right)+\eta_1\left(\lambda_0\Psi^2+\lambda_1\Psi^3\right)+\eta_0\left(\lambda_0+\lambda_1\Psi\right)\right)$$

$$+\;B_6\lambda_1\;\left(\lambda_0{}^2\left(2\;\eta_2\Psi^3+\;\eta_1\Psi\right)+2\lambda_0\lambda_1\left(2\;\eta_2\Psi^4+\;\eta_1\Psi^2\right)+\lambda_1{}^2\left(2\;\eta_2\Psi^5+\;\eta_1\Psi^3\right)\right)$$

$$+\;B_5\;\lambda_1\;\left(2\;\eta_2\Psi^3+\;\eta_1\Psi\right)+B_3\;\left(\lambda_0{}^5+5\;\lambda_0{}^4\lambda_1\Psi+10\;\lambda_0{}^3\lambda_1{}^2\Psi^2+10\lambda_0{}^2\lambda_1{}^3\Psi^3+5\;\lambda_0\lambda_1{}^4\Psi^4+\lambda_1{}^5\Psi^5\right)$$

$$+B_2\;\left(\lambda_0{}^3+3\lambda_0{}^2\lambda_1\Psi+3\lambda_0\lambda_1{}^2\Psi^2+\lambda_1{}^3\Psi^3\right)+B_1\;\left(\lambda_0+\lambda_1\Psi\right)$$

$$+k^2\;\lambda_1\left[12\;\eta_2\left(1+\eta_2\right)\Psi^5+\;2\eta_1\left(9\;\eta_2+1\right)\;\Psi^3+\left(\eta_1{}^2+12\;\eta_2\eta_0\right)\Psi\;\right]=0. \tag{53}$$

By collecting and setting the coefficients of the independent functions $\Psi^j(\xi)$ to zero, we establish the following system of algebraic equations (SAE) for the specified cases:

Case I $\eta_0=0$ and $\lambda_0=0$.

Equation (53) simplifies to the following form:

$$B_4\,\lambda_1{}^2\;\left(\eta_2\lambda_1\Psi^5+\eta_1\lambda_1\Psi^3\right)+\;B_6\lambda_1{}^3\left(2\;\eta_2\Psi^5+\;\eta_1\Psi^3\right)\;+\;B_5\;\lambda_1\;\left(2\;\eta_2\Psi^3+\;\eta_1\Psi\right)$$

$$+B_3\;\lambda_1{}^5\Psi^5+B_2\;\lambda_1{}^3\Psi^3+B_1\;\lambda_1\Psi+k^2\;\lambda_1\left[12\;\eta_2\left(1+\eta_2\right)\Psi^5+\;2\eta_1\left(9\;\eta_2+1\right)\;\Psi^3+\eta_1{}^2\Psi\;\right]=0. \tag{54}$$

We derive the following SAE for the same Ψ^j, where $j=1,\,2,\,3,\,4,\,5$:

$$\Psi^5:\quad \left(B_4+2\,B_6\right)\eta_2\;\lambda_1{}^2\;+B_3\;\lambda_1{}^4+k^2\left[12\;\eta_2\left(1+\eta_2\right)\right]=0,$$

$$\Psi^4:\quad B_4\;+4\,B_6\eta_2\;+B_3\;5\;\lambda_1{}^2=0,$$

$$\Psi^3:\quad B_4\;\lambda_1{}^2\;\;\eta_1+\;B_6\eta_1\lambda_1{}^2\;+2\;\eta_2\;B_5\;+\lambda_1{}^2B_2\;+k^2\;\left[2\eta_1\left(9\;\eta_2+1\right)\right]=0,$$

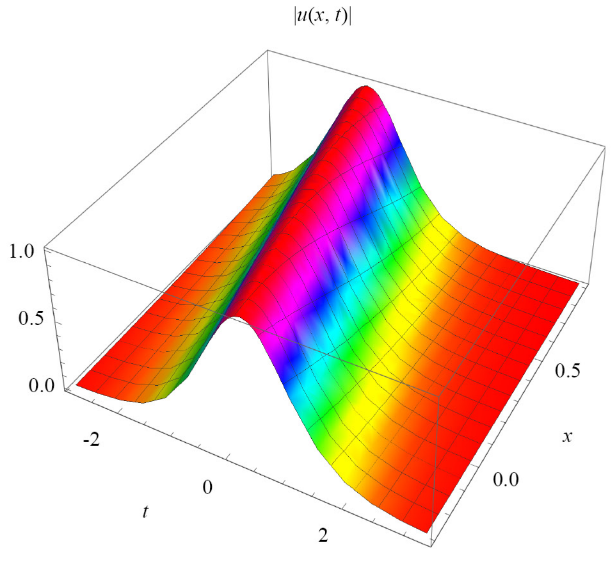
$$\Psi^2:\quad B_4\;\lambda_1{}^2\eta_1+2\;B_6\lambda_1{}^2\;\eta_1\;+3\lambda_1{}^2B_2\;=0,$$

$$\Psi:\quad B_5\eta_1+k^2\;\eta_1{}^2=0. \tag{55}$$

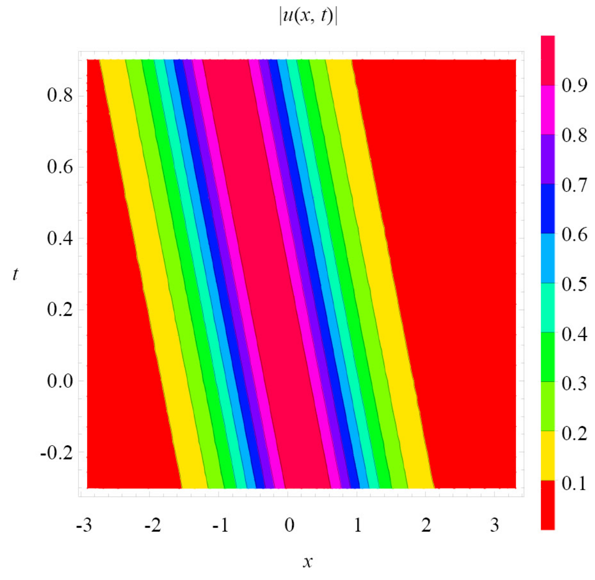
Solving the SAE (55) yields:

$$\eta_1=-\frac{B_5}{k^2},\quad \lambda_0=0,\quad B_1=0,\quad B_2\;=\frac{B_5\,B_4+2\,B_6\,B_5}{3k^2},$$

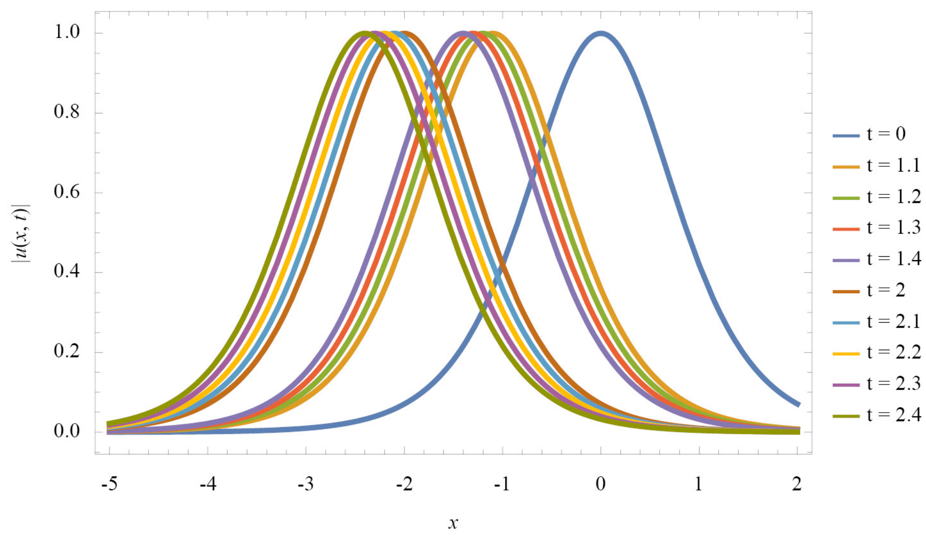
$$\eta_2 = \frac{-\left(5\,B_4{}^2 - 2\,B_4B_6 - 300\,B_3k^2\right) \mp \sqrt{\left(5\,B_4{}^2 - 2\,B_4B_6 - 300\,B_3k^2\right)^2 - 4\left(300\,B_3k^2\ - 24\,B_6{}^2 - 20B_6B_4\right)B_4{}^2}}{2\left(300\,B_3k^2\ - 24\,B_6{}^2 - 20B_6B_4\right)}. \tag{56}$$



(a) Surface plot



(b) Contour plot



(c) 2D plot

Figure 1. Profile of a bright soliton solution

Family 1 $\lambda_1 = \mp\sqrt{-\frac{(B_4+4\,B_6\,\eta_2)}{5B_3}}$ and $B_5 < 0$.

Thus, we arrive at the bright and singular soliton solutions:

$$u(x, t) = \mp \lambda_1 \sqrt{-pq^{\eta_1}/_{\eta_2}} \text{sech}_{pq}\left(\sqrt{\eta_1}\, k(x - vt)\right) exp\left[i(-\kappa x + \omega t + \theta_0)\right], \quad \eta_2 < 0, \tag{57}$$

$$v(x, t) = \mp\lambda\ \lambda_1 \sqrt{-pq^{\eta_1}/_{\eta_2}} \text{sech}_{pq}\left(\sqrt{\eta_1}\, k(x - vt)\right) exp\left[i(-\kappa x + \omega t + \theta_0)\right], \quad \eta_2 < 0, \tag{58}$$

and

$$u(x, t) = \mp\lambda_1 \sqrt{-pq^{\eta_1}/_{\eta_2}} \text{csch}_{pq}\left(\sqrt{\eta_1} k(x - vt)\right) exp\left[i(-\kappa x + \omega t + \theta_0)\right], \quad \eta_2 > 0, \tag{59}$$

$$v(x, t) = \mp\lambda\lambda_1 \sqrt{-pq^{\eta_1}/_{\eta_2}} \text{csch}_{pq}\left(\sqrt{\eta_1} k(x - vt)\right) exp\left[i(-\kappa x + \omega t + \theta_0)\right], \quad \eta_2 > 0. \tag{60}$$

Family 2 $\lambda_1 = \mp\sqrt{-\frac{6k^2(8\,\eta_2+1)}{(2\,B_4 + B_6)}}$ and $B_5 < 0$.
Therefore, we obtain the bright and singular soliton solutions:

$$u(x, t) = \mp \lambda_1 \sqrt{-pq^{\eta_1}/_{\eta_2}} \text{sech}_{pq}\left(\sqrt{\eta_1} k(x - vt)\right) exp\left[i(-\kappa x + \omega t + \theta_0)\right], \quad \eta_2 < 0, \tag{61}$$

$$v(x, t) = \mp\lambda\ \lambda_1 \sqrt{-pq^{\eta_1}/_{\eta_2}} \text{sech}_{pq}\left(\sqrt{\eta_1} k(x - vt)\right) exp\left[i(-\kappa x + \omega t + \theta_0)\right], \quad \eta_2 < 0, \tag{62}$$

and

$$u(x, t) = \mp\lambda_1 \sqrt{-pq^{\eta_1}/_{\eta_2}} \text{csch}_{pq}\left(\sqrt{\eta_1} k(x - vt)\right) exp\left[i(-\kappa x + \omega t + \theta_0)\right], \quad \eta_2 > 0, \tag{63}$$

$$v(x, t) = \mp\lambda\lambda_1 \sqrt{-pq^{\eta_1}/_{\eta_2}} \text{csch}_{pq}\left(\sqrt{\eta_1} k(x - vt)\right) exp\left[i(-\kappa x + \omega t + \theta_0)\right], \quad \eta_2 > 0. \tag{64}$$

Case II $\eta_0 = \frac{1}{4}\frac{\eta_1^2}{\eta_2}$ and $\lambda_0 = 0$.
Equation (53) simplifies to the following form:

$$B_3\, \lambda_1{}^4\Psi^5 + [B_4\left(\eta_2\Psi^5 + \eta_1\Psi^3 + \frac{1}{4}\frac{\eta_1^2}{\eta_2}\Psi\right) + B_6\left(2\,\eta_2\Psi^5 + \eta_1\Psi^3\right) + B_2\,\Psi^3\,]\lambda_1{}^2$$

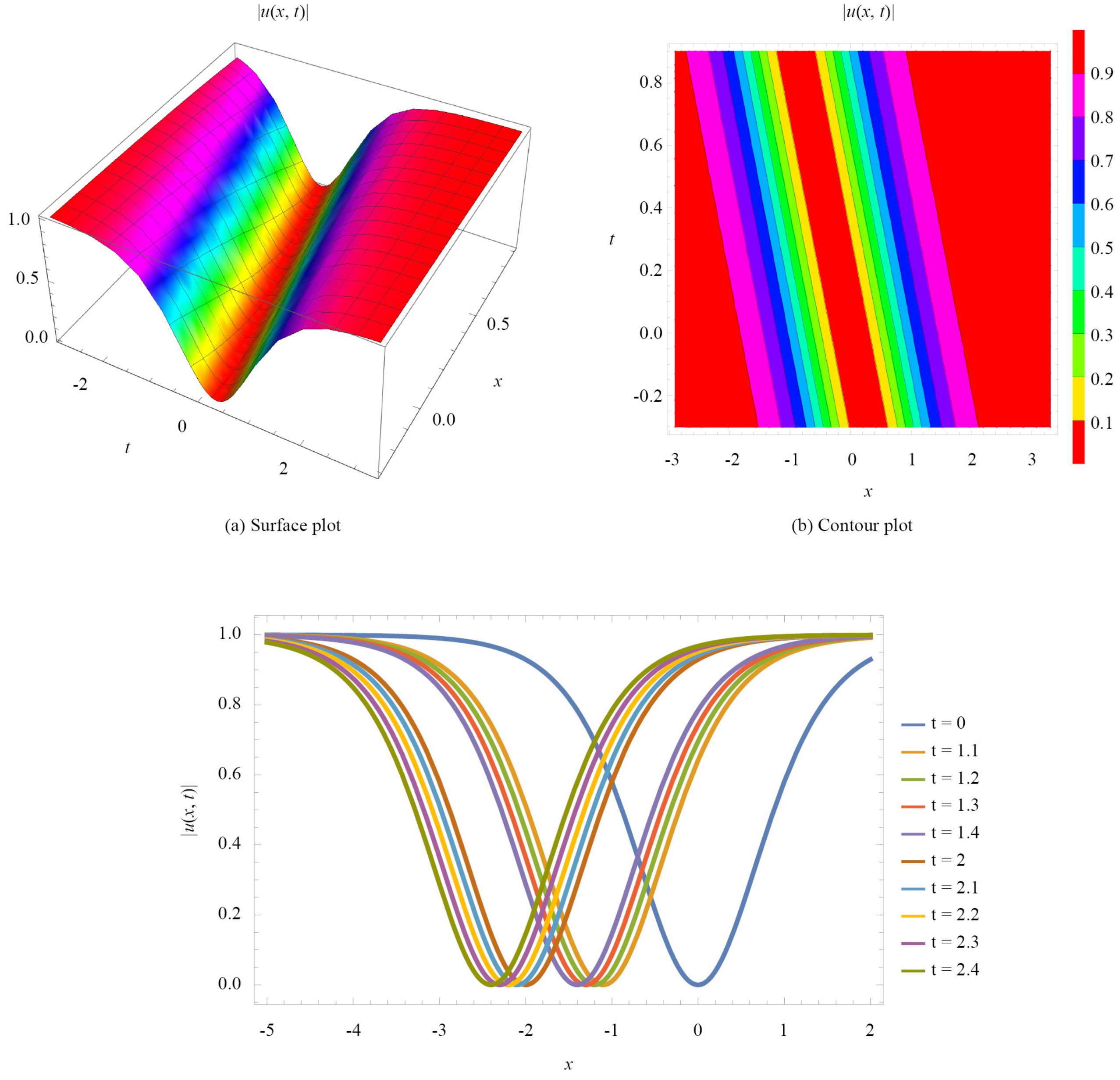
$$+ B_5\left(2\,\eta_2\Psi^3 + \eta_1\Psi\right) + B_1\,\Psi + k^2[\,12\,\eta_2(1+\eta_2)\Psi^5 + 2\eta_1(9\,\eta_2+1)\,\Psi^3 + 4\eta_1{}^2\Psi\,] = 0. \tag{65}$$

We derive the following SAE for the same Ψ^j, where $j = 1, 3, 5$:

$$\Psi^5: \quad B_3\,\lambda_1{}^4 + [B_4 + 2B_6]\,\eta_2\,\lambda_1{}^2 + 12\,\eta_2\,(1+\eta_2)\,k^2 = 0,$$

$$\Psi^3: \quad [B_4\,\eta_1 + B_6\,\eta_1 + B_2]\ \lambda_1{}^2 + 2\left(B_5\,\eta_2 + k^2\,\eta_1\,(9\,\eta_2+1)\right) = 0,$$

$$\Psi\ : \quad B_4\,\eta_1^2\,\lambda_1{}^2 + 4\eta_2\left(B_5\,\eta_1 + B_1 + k^2\,4\eta_1{}^2\right) = 0. \tag{66}$$

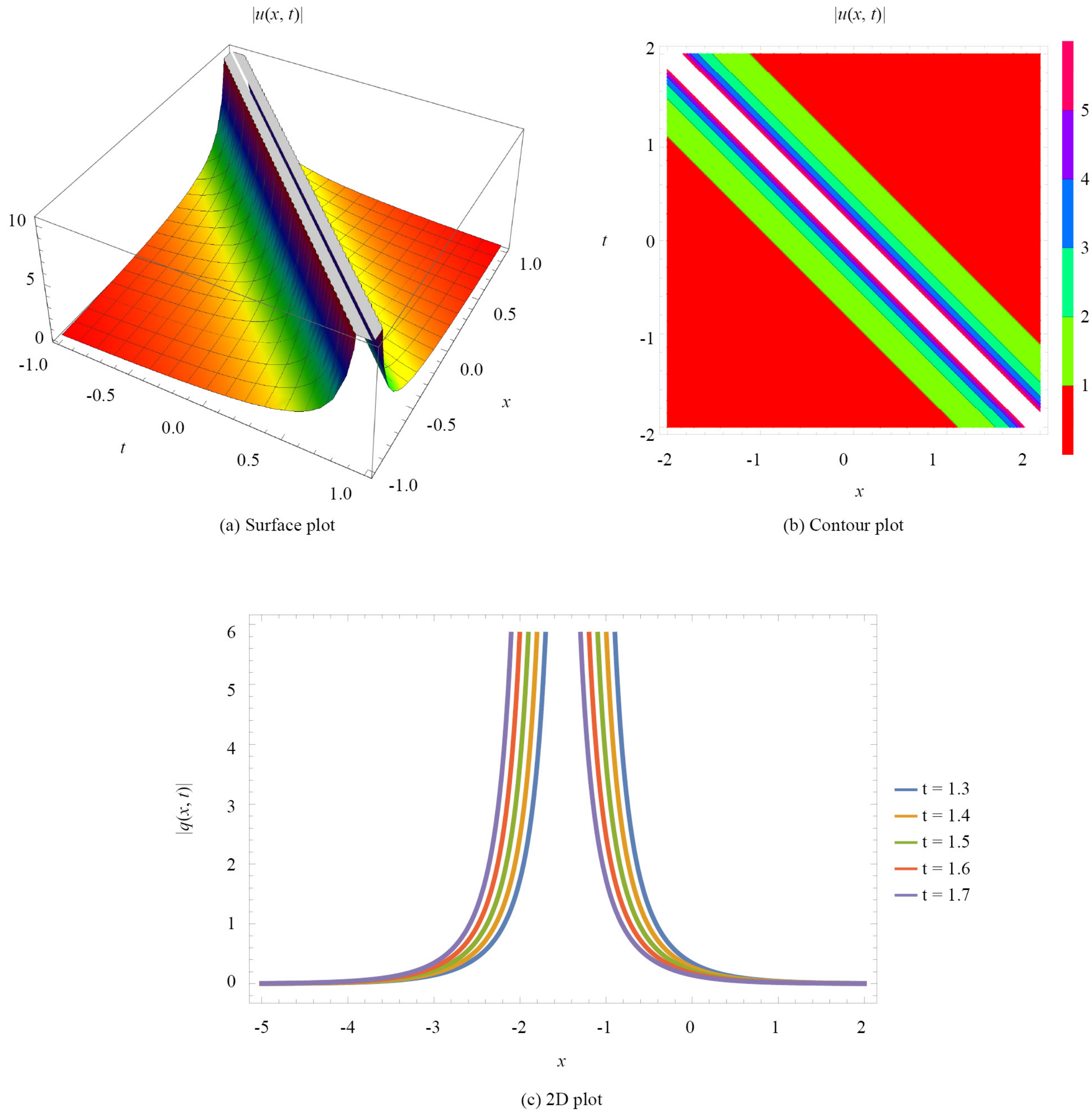


(a) Surface plot

(b) Contour plot

(c) 2D plot

Figure 2. Profile of a dark soliton solution



(a) Surface plot (b) Contour plot

(c) 2D plot

Figure 3. Profile of a singular soliton solution

Solving the SAE (66) yields:

Family 1 $\lambda_0 = 0$, $\lambda_1 = \mp\sqrt{-\frac{2\left(B_5\,\eta_2 + k^2\,\eta_1(9\,\eta_2+1)\right)}{B_4\,\eta_1 + B_6\,\eta_1 + B_2}}$, $\eta_0 = \frac{1}{4}\frac{\eta_1^2}{\eta_2}$, $\eta_1 < 0$ and $\eta_2 > 0$.

As a result, one gets the dark and singular soliton solutions:

$$u(x,t) = \mp \lambda_1 \sqrt{-\eta_1/2\eta_2} \tanh_{pq}\left(\sqrt{-\frac{\eta_1}{2}} k(x - vt)\right) exp[i(-\kappa x + \omega t + \theta_0)], \tag{67}$$

$$v(x,t) = \mp \lambda \lambda_1 \sqrt{-\eta_1/2\eta_2} \tanh_{pq}\left(\sqrt{-\frac{\eta_1}{2}} k(x - vt)\right) exp[i(-\kappa x + \omega t + \theta_0)], \tag{68}$$

and

$$u(x,t) = \mp \lambda_1 \sqrt{-\eta_1/2\eta_2} \coth_{pq}\left(\sqrt{-\frac{\eta_1}{2}} k(x - vt)\right) exp[i(-\kappa x + \omega t + \theta_0)], \tag{69}$$

$$v(x,t) = \mp \lambda \lambda_1 \sqrt{-\eta_1/2\eta_2} \coth_{pq}\left(\sqrt{-\frac{\eta_1}{2}} k(x - vt)\right) exp[i(-\kappa x + \omega t + \theta_0)]. \tag{70}$$

Family 2 $\lambda_0 = 0$, $\lambda_1 = \mp\sqrt{-\frac{4\eta_2\left(B_5\,\eta_1 + B_1 + k^2\,4{\eta_1}^2\right)}{B_4\,\eta_1^2}}$, $\eta_0 = \frac{1}{4}\frac{\eta_1^2}{\eta_2}$, $\eta_1 < 0$ and $\eta_2 > 0$.
In this case, the dark and singular soliton solutions appear as:

$$u(x,t) = \mp \lambda_1 \sqrt{-\eta_1/2\eta_2} \tanh_{pq}\left(\sqrt{-\frac{\eta_1}{2}} k(x - vt)\right) exp[i(-\kappa x + \omega t + \theta_0)], \tag{71}$$

$$v(x,t) = \mp \lambda \lambda_1 \sqrt{-\eta_1/2\eta_2} \tanh_{pq}\left(\sqrt{-\frac{\eta_1}{2}} k(x - vt)\right) exp[i(-\kappa x + \omega t + \theta_0)], \tag{72}$$

and

$$u(x,t) = \mp \lambda_1 \sqrt{-\eta_1/2\eta_2} \coth_{pq}\left(\sqrt{-\frac{\eta_1}{2}} k(x - vt)\right) exp[i(-\kappa x + \omega t + \theta_0)], \tag{73}$$

$$v(x,t) = \mp \lambda \lambda_1 \sqrt{-\eta_1/2\eta_2} \coth_{pq}\left(\sqrt{-\frac{\eta_1}{2}} k(x - vt)\right) exp[i(-\kappa x + \omega t + \theta_0)]. \tag{74}$$

Family 3 $\lambda_0 = 0$, $\lambda_1 = \mp\sqrt{\frac{-B_4 + 2B_6\,\eta_2 \mp \sqrt{[B_4 + 2B_6]^2{\eta_2}^2 - 48B_3\eta_2(1+\eta_2)k^2}}{2\,B_3}}$, $\eta_0 = \frac{1}{4}\frac{\eta_1^2}{\eta_2}$, $\eta_1 < 0$ and $\eta_2 > 0$.
Consequently, the dark and singular soliton solutions shape up as:

$$u(x,t) = \mp \lambda_1 \sqrt{-\eta_1/2\eta_2} \tanh_{pq}\left(\sqrt{-\frac{\eta_1}{2}}k(x-vt)\right) exp[i(-\kappa x+\omega t+\theta_0)], \tag{75}$$

$$v(x,t) = \mp\lambda\, \lambda_1 \sqrt{-\eta_1/2\eta_2} \tanh_{pq}\left(\sqrt{-\frac{\eta_1}{2}}k(x-vt)\right) exp[i(-\kappa x+\omega t+\theta_0)], \tag{76}$$

and

$$u(x,t) = \mp \lambda_1 \sqrt{-\eta_1/2\eta_2} \coth_{pq}\left(\sqrt{-\frac{\eta_1}{2}}k(x-vt)\right) exp[i(-\kappa x+\omega t+\theta_0)], \tag{77}$$

$$v(x,t) = \mp\lambda\, \lambda_1 \sqrt{-\eta_1/2\eta_2} \coth_{pq}\left(\sqrt{-\frac{\eta_1}{2}}k(x-vt)\right) exp[i(-\kappa x+\omega t+\theta_0)]. \tag{78}$$

3. Results and discussion

In this section, we present and discuss the evolution of bright, dark, and singular soliton solutions for the complex-valued solutions described by equations (57), (67), and (77), respectively. The analysis is based on the surface plots, contour plots, and 2D plots provided in Figures 1-3, with the time variable t set at various intervals. The parameters used for these simulations are $\eta_2 = -1$, $\lambda_1 = 1$, $\eta_1 = 1$, $k = 1$, $p = 1$, $q = 1$, $\kappa = 1$, $a^{(1)} = 1$, $a^{(2)} = 1$, $\delta_1^{(1)} = 1$, $\delta_1^{(2)} = 1$, $\delta_2^{(1)} = 1$, $\delta_2^{(2)} = 1$, $\delta_3^{(1)} = 1$, $\delta_3^{(2)} = 1$, $\sigma_1^{(1)} = 1$, $\sigma_1^{(2)} = 1$, $\sigma_3^{(1)} = 1$ and $\sigma_3^{(2)} = 1$. Figure 1(a) presents the surface plot of the bright soliton solution $u(x,t)$ as described by equation (57). The plot shows the amplitude of the soliton as a function of both space x and time t. The soliton maintains a well-defined peak that travels through space over time, illustrating the stable and localized nature of the bright soliton. The amplitude remains significant and does not disperse, which is characteristic of a bright soliton. Figure 1(b) provides the contour plot of the same solution. This visualization allows us to track the soliton's trajectory more clearly, showing the soliton's position and amplitude at various times. The contours indicate the peak regions and how they evolve, confirming the persistence and non-dispersive properties of the bright soliton. Figure 1(c) displays the 2D plot of the bright soliton solution at specific time intervals. This plot highlights the soliton's shape and amplitude at each selected time point. The 2D cross-sectional views confirm the soliton's stability and the consistent amplitude of the peak as it propagates through space. Figure 2(a) illustrates the surface plot of the dark soliton solution $u(x,t)$ as given by equation (67). Unlike the bright soliton, the dark soliton is characterized by a localized drop in amplitude (a "dip") against a continuous wave background. The surface plot shows the evolution of this dip as it moves through space over time, maintaining its shape and depth. Figure 2(b) shows the contour plot of the dark soliton solution. The contours clearly depict the soliton's path and the depth of the dip. The plot confirms the soliton's stability, showing that the dip's position and depth remain consistent over time, which is a hallmark of dark solitons. Figure 2(c) presents the 2D plot of the dark soliton at different times. These cross-sectional views illustrate the characteristic dip in the amplitude and its progression through space at the given time intervals. The dark soliton retains its shape and depth consistently, verifying its non-dispersive nature. Figure 3(a) shows the surface plot of the singular soliton solution $u(x,t)$ as described by equation (77). The singular soliton is marked by a singularity or a sharp peak, which can be seen clearly in the surface plot. The evolution of this singularity over time indicates that the soliton maintains its intensity and sharpness, unlike typical solitons which have a smooth peak or dip. Figure 3(b) provides the contour plot of the singular soliton solution. The contours highlight the extreme peak of the soliton and its movement through space. The plot suggests that the singularity does not disperse and continues to propagate, illustrating the unique characteristics of singular solitons. Figure 3(c) presents the 2D plot of the singular soliton solution at selected time intervals. These plots show the sharp, singular peak and its progression over time. The persistence of the singularity at each time point confirms the soliton's stable and non-dispersive nature despite its singularity. The figures and their respective subplots effectively demonstrate the unique characteristics of bright, dark, and singular soliton solutions. Bright solitons exhibit a stable, localized peak that propagates without dispersing. Dark solitons show a consistent, localized dip against a continuous wave background, maintaining their shape and depth. Singular solitons are characterized by sharp, intense peaks that remain stable and propagate through space. Each type of soliton solution exhibits its distinct properties, highlighting the diverse nature of soliton behaviors in different contexts.

4. Conclusions

The current study delves into the exploration of soliton solutions within the dispersive concatenation model, particularly focusing on the incorporation of the Sardar sub-equation approach. Through this method, the paper successfully unveils a spectrum of soliton solutions, including bright, dark, and singular 1-solitons. These findings are meticulously documented and illustrated within the paper, providing a comprehensive understanding of their characteristics. Moreover, the paper goes beyond mere identification, presenting parameter constraints crucial for the existence of these solitons. Such constraints offer valuable insights into the conditions under which these soliton phenomena manifest, enriching our comprehension of the model's behavior. However, this work serves as a stepping stone for further investigation. It beckons the exploration of various avenues, such as the elucidation of conservation laws and the identification of conserved quantities inherent within the model. Additionally, the model's applicability extends to diverse realms, including the examination of gap solitons and its relevance in magneto-optic waveguides and a plethora of optoelectronic devices such as optical couplers, optical metamaterials, and photonic crystal fibers (PCF). Crucially, future endeavors are directed towards adapting the model to dispersion-flattened fibers, unraveling new sets of solutions and understanding their properties in this specific context. These forthcoming results hold significant promise and are eagerly anticipated. Once attained, they will be seamlessly integrated with existing literature, enriching our understanding of the model's implications and applications [15–24].

Conflict of interest

The authors claim that there is no conflict of interest.

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