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Research Article

A φ-Contractivity Fixed Point Theory and Associated φ-Invariant Self-Similar Sets

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Abstract: In this paper, we apply a generalized variant of the concept of fixed point theory due to contraction mappings on metric spaces to construct a general class of iterated function systems relative to the so-called φ-contraction mappings on a metric space. In particular, we give a general framework to the Hutchinson method of constructing self-similar sets as fixed points of suitable mappings issued from the φ-contractions on the metric space. The results may open a new axis in the generalization of self-similar sets and associated self-similar functions. Moreover, our results may be extended to general metric spaces with suitable assumptions. The theoretical results are applied for the computation of the fractal dimension of a concrete example of the new self-similar sets.

***Keywords*:** self-similar sets, contraction mappings, fixed point, carathéodory dimension

MSC: 47H10, 47J26, 28A78

Notations

- $N \geq 1$ being an integer designating the dimension of the Euclidean real space $\mathbb{R}^N$.
- $\|.\|$ stands for the usual Euclidean norm on $\mathbb{R}^N$.
- $d(., .)$ stands for the Euclidean distance associated to the norm $\|.\|$.
- For $x \in \mathbb{R}^N$ and $r > 0$, $B(x, r)$ is the open ball of center x and radius r.
- $\overline{\mathbb{R}} = [-\infty, \infty]$.
- $\mathscr{C}^N$ the set of all compact subsets in $\mathbb{R}^N$.
- $\delta_N(., .)$ the Hausdorff distance defined on $\mathscr{C}^N$.

1. Introduction and motivations

One important question in fractal analysis and geometry is the fractality of sets. To conclude about the fractality of sets in the Taylor sense, we have to compare their Hausdorff and packing dimensions. This led researchers to construct examples for which these dimensions coincide or not. In the sense of Taylor, we say that a set is fractal if its Hausdorff and packing dimensions coincide. The fractal dimension is a tool to measure the roughness of sets in a non-smooth geometry.

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It therefore helps in exploring and exploiting the hidden properties of irregular and rough sets. The fractal dimension may be also seen as a single index informing us about such roughness and/or the irregularity of sets. Recently, the concept of fractal dimension has applications in many fields such as physics, biology, signal/image processing, computer vision, and so on. These reasons confirm that the computation of the fractal dimension of sets is essential. However, theoretically, such a computation is always difficult from the original definition of the Hausdorff and packing dimensions, or generally the Carathéodory approach, even in the simple case of linear constructions. This led researchers to develop alternative approaches to evaluate the fractal dimension of sets. One of the known concepts is the so-called self-similarity, which is a special property that permits us to evaluate the dimension of a set by exploiting the properties of the similarities used in its construction as an attractor of an iterated function system (IFS). A simple case is due to Cantor sets obtained as invariant sets (fixed points) of suitable contractions acting on the power set of metric spaces. A main original result in this subject, which relates the construction of the Cantor set to the dimension, is due to [1]. It states that the linear contractions used to construct the Cantor set are themselves used to compute its fractal dimension. However, not all self-similar constructions use linear IFS, there are indeed, nonlinear contractions, stochastic IFS, and so on. All these facts constitute reasonable motivations that lead to the study of the fractal dimension of sets ([1–7]).

Abraham et al. [8] considered a weak contraction assumption to construct a generalized IFS and showed the existence of a fixed point attractor of the system. Ben Mabrouk et al. in [9] showed that fractal dimensions may be good modelers for nanomaterials oscillations. Besides, these models are eventually nonlinear. The fractal models are combined with wavelets to investigate the anisotropic behavior of nanomaterials. El-Nabulsi [10, 11] applied the concept of fractal dimension for the investigation of the ocean Ekman transport dynamics based on product-like fractal measure and anisotropic continuum media. Fractal dimensions are shown to affect the velocity profile. In addition, slow/fast waves are characterized via the proposed model. (see also [12]. In [13], a nonlinear case dealing with spiral waves has been investigated via the fractal dimensions. The impact of fractal dimensions is also investigated in [14] for the breaking and instabilities of the foam drainage equation. The investigations showed a difference in the wave solution behavior from integer to fractal dimensions. In [15], the authors showed that the fractal dimension concept is useful in materials science by investigating the quantum effects in the metal oxide semiconductor. In [16], the concept of fractal dimensions was applied in fluid dynamics to investigate the effects on the Rayleigh problem, the Burger's Vortex, and the Kelvin-Helmholtz instability by using as previously a product-like fractal measure. El-Nabulsi [17] applied the concept of fractal dimension to model seismic waves by using the same concept of product-like fractal measure as previously. The model showed that effectively, the fractal dimension has an impact on wave propagation. In particular, eventual ranges for the fractal dimension are estimated to classify earthquakes into slow and fast waves. In [18], fractal dimensions are applied to study the thermal diffusion flames by using the concept of non-integer dimensional space for complex fractal media.

Navascues [19] investigated a partial contractivity concept to show the validity of fixed points theory in some b-metric spaces. The new weak concept is applied to the approximation of fractal functions corresponding to contractive and non-contractive operators. In [20, 21], the concept of Rakotch contraction is used to derive an iterated function system provided with the computation of dimensional results of the associated graph to the IFS. Verma et al. [22] applied a weak contraction theory to obtain a generalized form of IFS already with the evaluation of its fractal dimension. Dalla, Drakopoulos, and Prodromou [23], studied the fractal dimension in the sense of box-counting for a nonaffine fractal functions system generalizing thus some known results for the affine case. In [24], the authors studied the quantization dimension due to an infinite number of contracting similarities and an infinite probability vector. The authors showed that for the associated fractal measure, its fractal dimension in the sense of Hausdorff coincides with its quantization dimension. Furthermore, the stability of the infinite IFS is investigated. Verma [25] studied the Hausdorff dimension for some IFS associated with infinitesimal similitudes and showed that the Hausdorff dimension of the self-similar attractor coincides with the box dimension of the attractor. Verma et al. [26] investigated the problem of fractal interpolation functions and fractal dimension of their graphs Bounds of the Hausdorff dimension of the graph are established with eventual relation to the contractions used for the interpolation. In [27, 28], the concept of fractal dimension of the graph of fractal measures in the sense of the quantization dimension is studied in the case of weighted iterated function systems and bi-Lipschitz mappings, and recurrent IFS. Additionally, in [29], the case of inhomogeneous bi-Lipshitz IFS is investigated for the quantization dimension. Dubey and Verma studied an inhomogeneous case of directed attractors corresponding to Mauldin-Williams graph dimension. Upper and lower bounds for the fractal dimension have been established.

From a simple geometric point of view, a definition of self-similar objects that may be familiar to the whole community (even non-scientific) may be those objects that appear to own the 'same' or 'similar' shape or form when zoomed out or in. Therefore, an alternative, simplified, and intuitive definition could be that self-similar objects are those retaining their shapes independently of the zooming scale.

These 'even resembling' definitions show that mathematically, a self-similar object (set, function, measure, ...) to be defined rigorously should be linked to its context or framework. Therefore, to be more precise mathematically, a self-similar object may be defined as an object, which is exactly or approximately looking like a part of itself.

In nature, we recognize aspects of self-similarity around us. Trees show self-similarity, coastlines present statistical self-similarity, the clouds and fog in the sky are types of self-similar objects, the pieces of ice, including crystals and their microscopic structure, we, ourselves, grow older while maintaining a kind of self-similarity, and so on ([2, 4–6, 9]).

In mathematics, a special kind of self-similarity is related to fractals. In this context, many geometrical sets have been investigated and understood, and thus applied by the next in many fields. We may cite the Koch snowflake, the Sierpinski triangle and gasket, the Julia set, the Mandelbrot set, and the original Cantor set which may be considered as the basic example. However, self-similar objects are generally different from fractals and may not be necessary fractals, such as the line, the plane, or generally the usual Euclidean space which are self-similar but not fractal. One mathematical example is the Mandelbrot set due to the father of fractals Benoit Mandelbrot. It is shown that such a set is locally self-similar on a dense subset of points situated on its frontier, precisely, around the so-called Misiurewicz points. Such self-similarities are surely non-linear. Moreover, many spaces may not be metric but instead have some weak structures such as pseudometric, quasi-metric, or quasi-pseudometric. In these cases, we have to apply some weak forms of contractivity as in the present case ([30]).

Cohen [31] used the concept of self-similar sets and their fractal dimension to establish an uncertainty principle on Cantor-type sets in the sense of Bourgain-Dyatlov fractal uncertainty principle. In [32], the fractal uncertainty principle is extended for special cases of fractal sets involving porosity lines. Betti [33] showed that the concepts of self-similarity and fractal dimensions may be applied to the understanding of the famous Riemann Hypothesis on the distribution of prime numbers by exploiting fractal geometry and algebraic geometry. In [34], the author introduced the notion of Fractal Verse Theory as a novel conceptual framework combining many concepts from mathematics, physics, nature, and philosophy to constitute a large look perceiving the whole universe. Moreover, in [35], a general framework of fractal mathematics was introduced. Hohlfeld and Cohen [36] applied the concept of self-similar sets to study the frequency properties of antennae. Besides, the authors concluded that some new families of practical designs may result from the self-similar geometric insights. Gorman et al. [37] studied the concept of Chaudhuri, Sankaranarayanan, and Vardi regularity of functions, and applied the fractal analysis techniques to establish a necessary and sufficient condition for the regularity of some classes of functions according to their local affinity. Gorman and Schulz applied in [38, 39] the concept of fractal dimensions such as Box-counting, Hausdorff to investigate the behavior of the fractal geometry of some automatic sets known as Buchi automata.

In [40], Diatseris discussed many variants of fractal dimensions and numerical estimators such as generalized entropy, correlation sum, and extreme value theory. The proposed estimators are compared in different practical frameworks such as dynamical systems, financial time series, and noised data. Mayor et al. [41] discussed the utility of fractal concepts in the interpretation of Big Data such as the Complexity and Entropy in Physiological Signals. They found that fractal dimensions may be good tools in the resampling data.

Kameyama [42] considered a pseudometric case to investigate the fractal geometry problems allowing the existence of topological self-similar set structures. It is shown the possibility of topological self-similar sets without self-metrics and with special contractions. Lapidus et al. [43] discussed many fractal dimensions and defined the concept of box-counting zeta functions of sets allowing the development of a concept of complex dimensions of sets. In [44], pointwise and distributional fractal tube formulas are established for fractal drums. The associated complex dimensions are defined as poles of a fractal zeta function. The results are shown to include many self-similar classical cases such as the Sierpinski gasket and self-similar fractal sprays. In [45], complex dimensions of nonlattice self-similar fractal strings are investigated in the framework of quasicrystals via Lagarias formula. Lapidus and Frankenhuijsen studied in [46] the fractality, self-similarity, and complex dimensions of strings by comparison to the theory of varieties over finite fields, and from a dynamical and geometric point of view. See also [47].

Having introduced briefly the concept of self-similarity, it is worth recalling that, in the majority of the classical existing examples, the essential concept related to the construction of self-similar sets is the fixed point theory. It holds, indeed, that self-similar sets are fixed points of suitable set-valued contractions defined on suitable metric spaces.

Details and backgrounds on these concepts, and some beautiful, amazing, and exciting applications of both concepts of fractals and self-similar sets may be found in [1–5].

The next section is concerned with the preliminaries and general definitions to be applied in the rest of the paper. Section 4 is devoted to our main results with their proofs. Section 5 concludes the paper.

2. Preliminaries

In this section, we recall the classical definition and/or construction of self-similar sets. The original construction is due to Hutchinson [1], and is based on the application of the fixed point theory. It is next applied, especially, in the study of fractal sets and fractal functions such as self-similar ones. An exhaustive list of references may be consulted in this subject [3, 4].

Meanwhile, we recall before going on introducing and reviewing the basic preliminaries for our study, that the results exposed in this paper may be extended naturally to metric spaces with suitable assumptions and modifications, such as complete metric spaces.

Definition 1 A function $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is said to be contractive or a Contraction on $\mathbb{R}^N$ if there exists a constant c, $0 < c < 1$ (known as the contraction ratio, or sometimes the Lipschitz constant), such that

$$\|f(x) - f(y)\| \leq c\|x - y\|; \ \forall x, y \in \mathbb{R}^N.$$

Definition 2 For a finite set of self-maps $\mathbb{S} = (S_i)_{1 \leq m}$ ($m \in \mathbb{N}$ fixed) on $\mathbb{R}^N$, a subset $X \subset \mathbb{R}^N$ is said to be $\mathbb{S}$-invariant if it satisfies $X = \bigcup_{i=1}^{n} S_i(X)$. The set X is called the self-similar set associated with $\mathbb{S}$. The couple $(X, \mathbb{S})$ is called an iterated function system.

The following result due essentially to Hutchinson [1] deals with the existence, and uniqueness of $\mathbb{S}$-invariant sets, and is based on the application of the fixed point theory.

Proposition 1 [1] For any finite set of contractions $\mathbb{S} = (S_i)_{1 \leq i \leq m}$ on $\mathbb{R}^N$, there exists a unique $\mathbb{S}$-invariant subset $X \subset \mathbb{R}^N$.

Definition 3 A function $\varphi\colon [0, \infty) \longrightarrow [0, \infty)$ is said to be admissible iff

• it is nondecreasing,

• it is continuous on the right,

• $\varphi(x) < x$ on $[0, \infty)$.

We now introduce the concept of φ-contractions used in [48], where a general form of the classical Picard-Banach contraction principle was established.

Definition 4 A function $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is said to be a φ-contraction iff

$$d(f(x), f(y)) \leq \varphi(d(x, y)), \ \forall x, y \in \mathbb{R}^N.$$

The following result is due to Browder [48], reformulated in the simple case of the Euclidean space $\mathbb{R}^N$.

Proposition 2 [48] Let $M \subset \mathbb{R}^N$ be non-empty compact, and $T\colon M \longrightarrow M$ be φ-contractive. Then, T has a unique fixed point.

Definition 5 Let $E \subset \mathbb{R}^N$ and $\eta > 0$. We call an η-covering of E, any countable collection of balls $\{B(x_i,\, r_i)\}_i$ satisfying

$$E \subset \bigcup_i B(x_i,\, r_i); \;\; x_i \in E \;\text{ and }\; 0 < r_i < \eta,\, \forall i.$$

We call an η-packing of E any countable collection of balls $(B(x_i,\, r_i))_i$ satisfying

$$\forall i,\, x_i \in E,\, 0 < r_i < \eta \quad \text{and} \quad \forall i \neq j,\; B(x_i,\, r_i) \cap B(x_j,\, r_j) = \emptyset.$$

The first versions of fractal dimensions are known as Hausdorff and packing dimensions. We recall in brief their constructions. Let $E \subset \mathbb{R}^N$, and $\eta > 0$. For $\alpha >$ positive, let

$$\mathscr{H}_\eta^\alpha(E) = \inf \sum_i (2r_i)^\alpha,$$

where the inf is taken over all η-covering $\{B(x_j,\, r_j)\}_j$ of E. It is a monotone function of the variable η. So it has a limit as $\eta \downarrow 0$,

$$\mathscr{H}^\alpha(E) = \lim_{\eta \downarrow 00} \mathscr{H}_\eta^\alpha(E) = \sup_{\eta > 0} \mathscr{H}_\eta^\alpha(E).$$

We know in fractal analysis or generally in measure theory that $\mathscr{H}^\alpha$ is an outer metric measure on $\mathbb{R}^N$. Its restriction on the Borel sets ($\mathscr{H}^\alpha$-measurable sets) is called the Hausdorff measure with index α. It holds that for any set E, there exists a critical value called the Hausdorff dimension of E, denoted by $\dim E$, satisfying

$$\mathscr{H}^\alpha(E) = 0 \text{ if } \alpha > \dim E, \;\text{ and }\; \mathscr{H}^\alpha(E) = \infty \text{ if } \alpha < \dim E.$$

Similarly, the packing dimension is related to the packing measure. For $\alpha \geq 0$, let

$$\overline{\mathscr{P}}_\eta^\alpha(E) = \sup \sum_i (2r_i)^\alpha,$$

where the sup is taken over all the η-packings of E. Let also

$$\overline{\mathscr{P}}^\alpha(E) = \lim_{\eta \downarrow 0} \overline{\mathscr{P}}_\eta^\alpha(E) = \inf_{\eta > 0} \overline{\mathscr{P}}_\eta^\alpha(E),$$

and

$$\mathscr{P}^{\alpha}(E) = \inf\{\sum_i \overline{\mathscr{P}}^{\alpha}(E_i),\ E \subseteq \cup_i E_i\}.$$

As in the Hausdorff case, $\mathscr{P}^{\alpha}$ is an outer metric measure, which induces a measure on the Borel sets called the packing measure. For any set E, $\mathscr{P}^{\alpha}(E)$ has a cut-off value called the packing dimension of E, written $\mathrm{Dim}E$, satisfying

$$\mathscr{P}^{\alpha}(E) = \infty \text{ if } \alpha < \mathrm{Dim}E \quad \text{and} \quad \mathscr{P}^{\alpha}(E) = 0 \text{ if } \alpha > \mathrm{Dim}E.$$

Moreover, there exists a cut-off value $\Delta(E)$ (called the logarithmic index of E) satisfying

$$\overline{\mathscr{P}}^{\alpha}(E) = \infty \text{ if } \alpha < \Delta(E) \quad \text{and} \quad \overline{\mathscr{P}}^{\alpha}(E) = 0 \text{ if } \alpha > \Delta(E).$$

3. Main results

In the present work, we propose to develop some new classes of self-similar sets relative to the concept of φ-contractions. To do this, consider a finite set of φ-contractions $\mathbb{S} = (S_i)_{1\leq m}$ on $\mathbb{R}^N$ as in Definition 4 above. We define as in the classical case a sequence of set-valued maps S^k, $k \geq 0$ by induction on k by setting for $E \subset \mathbb{R}^N$,

$$S^0(E) = E,\ S^1(E) = S(E) = \bigcup_{i=1}^{n} S_i(E),\ \text{and } S^{k+1}(E) = S(S^k(E)),\ \forall k. \tag{1}$$

We thus obtain the new or the extended definition of self-similar sets to the φ-$\mathbb{S}$-self-similar sets as follows.

Definition 6 A subset $F \subset \mathbb{R}^N$ is said to be φ-$\mathbb{S}$-invariant if it satisfies $S(F) = F$.

Our first result deals with the existence, and uniqueness of φ-$\mathbb{S}$-invariant sets and generalizes the results of Proposition 1.

Theorem 1 Let $S = (S_i)_{1\leq i\leq m}$ be a finite set of φ-contractions on $\mathbb{R}^N$. There exists a unique non-empty compact F in $\mathbb{R}^N$ which is φ-$\mathbb{S}$-invariant. Moreover,

i. $F \neq \emptyset$.

ii. F is compact.

iii. $F = \bigcap_{k\geq 0} S^k(A)$, $\forall A \subset \mathbb{R}^N$, $A \neq \emptyset$ and compact, with $S(A) \subset A$.

Proof. Consider the set $\mathscr{C}^N$ equipped with the well-known Hausdorff distance defined by

$$H(A, B) = \inf\Big\{\eta > 0;\ A \subset B(\eta),\ \text{and } B \subset A(\eta)\Big\},$$

where, for $\eta > 0$, $A(\eta)$ is the η-neighborhood of A defined by

$$A(\eta) = \Big\{x \in \mathbb{R}^N;\ d(x, A) = \inf_{y\in A} d(x, y) \leq \eta\Big\},$$

and similarly $B(\eta)$. It is well-known that H is a metric on $\mathscr{C}^N$, and that $(\mathscr{C}^N, H)$ is a complete metric space. Consider next the set-valued function S: $\mathscr{C}^N \longrightarrow \mathscr{C}^N$ defined as in (1). Let next $\varepsilon > \max_i H(S_i(A), S_i(B))$. We may write

$$\forall i, \;\; S_i(A) \subset S_i(B)(\varepsilon), \quad \text{and} \quad S_i(B) \subset S_i(A)(\varepsilon).$$

Therefore,

$$\forall i, \;\; S_i(A) \subset \left(\underset{j}{\cup} S_j(B)\right)(\varepsilon), \quad \text{and} \quad S_i(B) \subset \left(\underset{j}{\cup} S_j(A)\right)(\varepsilon),$$

which leads to

$$\underset{i}{\cup} S_i(A) \subset \left(\underset{j}{\cup} S_j(B)\right)(\varepsilon), \quad \text{and} \quad \underset{i}{\cup} S_i(B) \subset \left(\underset{j}{\cup} S_j(A)\right)(\varepsilon),$$

or equivalently,

$$S(A) \subset S(B)(\varepsilon), \;\; \text{and} \;\; S(B) \subset S(A)(\varepsilon).$$

As a consequence

$$H(S(A), S(B)) \leq \varepsilon, \;\; \forall \varepsilon > \max_i H(S_i(A), S_i(B)).$$

As a result,

$$H(S(A), S(B)) \leq \max_i H(S_i(A), S_i(B)).$$

Let now $\eta > \varphi(H(A, B))$ and $t \in S_i(A)$. Whenever $A \subset B(\eta)$, we get $t \in S_i(B(\eta))$. Therefore, there exists $y_t \in B(\eta)$ such that $t = S_i(y_t)$. For y_t, there exists $y_0 \in B$ such that $d(y_t, y_0) \leq \eta$. As a result,

$$d(t, S_i(y_0)) = d(S_i(y_t), S_i(y_0)) \leq \varphi(d(y_t, y_0)) \leq \eta.$$

Similarly, for $t \in S_i(B)$. Consequently,

$$H(S_i(A), S_i(B)) \leq \eta, \;\; \forall \eta > \varphi(H(A, B)),$$

which reads that

$$H(S_i(A), S_i(B)) \leq \varphi(H(A, B)), \ \forall i.$$

As a result,

$$\max_i H(S_i(A), S_i(B)) \leq \varphi(H(A, B)).$$

Consequently,

$$H(S(A), S(B)) \leq \varphi(H(A, B)),$$

which reads finally that S is φ-contractive on $(\mathscr{C}^N, H)$. So, there exists a unique compact subset F of $\mathbb{R}^N$, such that,

$$S(F) = F = \bigcup_{i=1}^{m} S_i(F).$$

The set F is non-empty as it contains the fixed points of any finite composition $S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_k}$ of k $(\in \mathbb{N})$ contractions from the set $\mathbb{S} = (S_i)_{1 \leq m}$. It remains now to prove the last point. So, let $A \in \mathscr{C}^N$ be as stated in the theorem. Consider the (decreasing) sequence of sets $(A_k = S^k(A))_k$, and write

$$E = \bigcap_k A_k \ (= \lim_{k \to \infty} A_k).$$

It is straightforward that the real-valued sequence $(H(A_k, E))_k$ is decreasing to 0, which means that $(A_k)_k$ converges to E in $(\mathscr{C}^N, H)$. On the other hand, observe that $A_{k+1} = S(A_k)$. Therefore, $E = S(E)$. Hence,

$$F = E = \bigcap_{k \geq 0} S^k(A).$$

□

Definition 7 The φ-$\mathbb{S}$-invariant set F in Definition 6 and Theorem 1 is said to the φ-$\mathbb{S}$-self-similar set associated to the set of φ-contractions $S = (S_i)_{1 \leq m}$. The pair (S, F) is called a φ-iterated function system.

The next result deals with a type of Carathéodory dimension of the φ-$\mathbb{S}$-self-similar set F. To investigate such a problem, we will introduce firstly the φ-Carathéodory measure of sets. Original definitions and backgrounds on such a measure and dimension may be found in [7].

For a subset $E \subset \mathbb{R}^N$, and $\alpha \in \mathbb{R}$, write

$$\Sigma_{\varphi}^{\alpha}(E) = \lim_{\eta \to 0} \left(\inf \left\{ \sum_i (\varphi(r_i))^{\alpha}; \ (B(x_i, r_i))_i \text{ is an } \eta\text{-covering of } E \right\} \right).$$

It is straightforward that Σ_φ^α is an outer measure on $\mathbb{R}^N$. For the simple choice $\varphi(x) = ax$, with some $a > 0$, we come back to the classical case of the Hausdorff measure.

By the same way, we introduce a packing measure as follows. For $\alpha \geq 0$, write

$$\overline{\Theta}_\varphi^\alpha(E) = \lim_{\eta \to 0} \left(\left\{ \sup \sum_i (2r_i)^\alpha;\ (B(x_i,\, r_i))_i \text{ is an } \eta\text{-packing of } E \right\} \right).$$

This is a pre-measure as it lacks the sub-additivity property. To get a measure, we set

$$\Theta_\varphi^\alpha(E) = \inf \left\{ \sum_i \overline{\Theta}_\varphi^\alpha(E_i);\ E \subseteq \bigcup_i E_i \right\}.$$

We get here a metric outer measure on $\mathbb{R}^N$. As previously, for the simple choice $\varphi(x) = ax$, with some $a > 0$, we come back to the classical case of the packing measure.

The following proposition gives a general variant of the well-known Hausdorff and packing dimensions of sets, as well as the logarithmic index.

Lemma 1 For any admissible function φ, and any subset $E \subset \mathbb{R}^N$, there exists unique cut-off values $\alpha_E,\, \beta_E \in \overline{\mathbb{R}}$ such that

1. $\Sigma_\varphi^\alpha(E) = 0,\ \forall \alpha > \alpha_E$, and $\Sigma_\varphi^\alpha(E) = \infty,\ \forall \alpha < \alpha_E$.
2. $\Theta_\varphi^\alpha(E) = 0,\ \forall \alpha > \beta_E$, and $\Theta_\varphi^\alpha(E) = \infty,\ \forall \alpha < \beta_E$.

The proof is easy and follows similar techniques as in [1, 4].

Definition 8 For any admissible function φ, and any subset $E \subset \mathbb{R}^N$, we call the cut-off values

1. α_E: the φ-Hausdorff-dimension of the set E, and denoted $dim_\varphi(E)$.
2. β_E: the φ-packing-dimension of the set E, and denoted $Dim_\varphi(E)$.

We may see easily that such a dimension satisfies some common properties with the concept of dimension in general, especially Carathéodory, Hausdorff, packing, and capacity dimensions of sets. More precisely, we have the following lemma.

Proposition 3 For any admissible function φ, the following assertions hold.

1. $0 \leq dim_\varphi(A) \leq N$ and $0 \leq Dim_\varphi(A) \leq N$, $\forall A \subset \mathbb{R}^N$.
2. $dim_\varphi(A) \leq dim_\varphi(B)$ and $Dim_\varphi(A) \leq Dim_\varphi(B)$, $\forall A \subset B \subset \mathbb{R}^N$.
3. $dim_\varphi(\bigcup_n A_n) = \max_n(dim_\varphi(A_n))$ and $Dim_\varphi(\bigcup_n A_n) = \max_n(Dim_\varphi(A_n))$, $\forall (A_n)_n \subset \mathbb{R}^N$.
4. $dim_\varphi(A) \leq Dim_\varphi(A)$, $\forall A \subset \mathbb{R}^N$.

Proof. We will develop the proofs for the case of dim_φ. The remaining case may be deduced by similar techniques.

1. Is obvious.

2. For $\eta > 0$ and any η-covering $(B(x_i,\, r_i))_i$ of B, with $0 < r_i < \eta$, this is obviously an η-covering of $A \subset B$. Therefore,

$$\Sigma_{\varphi,\,\eta}^\alpha(A) \geq \sum_i (\varphi(r_i))^\alpha.$$

Taking the infimum on all these coverings, we get

$$\Sigma_{\varphi,\,\eta}^\alpha(A) \geq \Sigma_{\varphi,\,\eta}^\alpha(B).$$

$$\Sigma_{\varphi}^{\alpha}(A) \geq \Sigma_{\varphi}^{\alpha}(B).$$

Consequently,

$$\Sigma_{\varphi}^{\alpha}(A) = 0, \;\; \forall\, \alpha > dim_{\varphi}(B),$$

which reads that

$$dim_{\varphi}(A) \leq \alpha, \;\; \forall\, \alpha > dim_{\varphi}(B),$$

which is equivalent to

$$dim_{\varphi}(A) \leq dim_{\varphi}(B),$$

3. On one side, observe that

$$A_j \subset \underset{n}{\cup} A_n), \;\; \forall\, j.$$

Hence, Assertion 2 yields that

$$dim_{\varphi}(A_j) \leq dim_{\varphi}(\underset{n}{\cup} A_n), \;\; \forall\, j.$$

As a result,

$$\max_{j}(dim_{\varphi}(A_j)) \leq dim_{\varphi}(\underset{n}{\cup} A_n).$$

On the other side, consider, for each n, an η-covering $(B(x_{i,\,n},\, r_{i,\,n}))_i$ of A_n satisfying

$$\Sigma_{\varphi,\,\eta}^{\alpha}(A_n) \leq \sum_{i} (\varphi(r_{i,\,n}))^{\alpha} \leq \Sigma_{\varphi,\,\eta}^{\alpha}(A_n) + \frac{\eta}{2^n}.$$

Then, the whole set $(B(x_{i,\,n},\, r_{i,\,n}))_{i,\,n}$ is an η-covering for the whole union $\underset{n}{\cup} A_n$. Therefore, for $\eta > 0$ such that $r_{i,\,n} < \eta$, $\forall i$, n, we get

$$\Sigma_{\varphi,\,\eta}^{\alpha}\left(\underset{n}{\cup} A_n\right) \leq \sum_{n}\sum_{i} (\varphi(r_{i,\,n}))^{\alpha} \leq \sum_{n} \left(\Sigma_{\varphi,\,\eta}^{\alpha}(A_n) + \eta\right).$$

Letting $\eta \to 0$, we get

$$\Sigma_{\varphi}^{\alpha}\left(\bigcup_{n} A_n\right) \leq \sum_{n} \Sigma_{\varphi}^{\alpha}(A_n).$$

Consequently, for $\alpha > \max_{n}(dim_{\varphi}(A_n))$, we get

$$\Sigma_{\varphi}^{\alpha}\left(\bigcup_{n} A_n\right) = 0,$$

which reads that

$$dim_{\varphi}\left(\bigcup_{n} A_n\right) \leq \alpha, \ \ \forall \alpha > \max_{n}(dim_{\varphi}(A_n)),$$

which is equivalent to

$$dim_{\varphi}\left(\bigcup_{n} A_n\right) \leq \max_{n}(dim_{\varphi}(A_n)).$$

4. Using the well-known Besicovitch covering theorem, we show that there exists a constant $C \in]0, +\infty[$ satisfying for any $E \subseteq \mathbb{R}^N$,

$$\Sigma_{\varphi}^{\alpha}(E) \leq C\Theta_{\varphi}^{\alpha}(E).$$

As a result,

$$\Sigma_{\varphi}^{\alpha}(E) = 0, \ \forall \alpha > Dim_{\varphi}(E),$$

which reads that

$$dim_{\varphi}(E) \leq \alpha, \ \forall \alpha > Dim_{\varphi}(E),$$

or equivalently,

$$dim_{\varphi}(E) \leq Dim_{\varphi}(E).$$

□

The second main result of this paper deals with the characterization of the φ-S-self-similar sets introduced previously by means of their φ-dimension. This gives a general form of the Hutchinson theorem about classical self-similar sets.

Theorem 2 Let φ be an admissible function, $S = (S_i)_{1 \leq m}$ a set of φ-contractions on $\mathbb{R}^N$, and F the φ-$\mathbb{S}$-self-similar set associated to S. Denote also $\Lambda(\varphi) = \sup\limits_{t \neq 0} \frac{\varphi(t)}{t}$. Then,

$$dim_\varphi(F) = Dim_\varphi(F) = -\frac{\log(m)}{\log(\Lambda(\varphi))}.$$

Proof. Denote $s = -\frac{\log(m)}{\log(\Lambda(\varphi))}$. We shall prove that

$$0 < \Sigma_\varphi^s(F) < \infty.$$

To do this, write for $k \in \mathbb{N}$, $I_k = \{1, 2, ..., m\}^k$, and for a multi-index $i = (i_1, ..., i_k) \in I_k$, we extend the notation of a single index to the multi-one by writing $F_i = S_{i_1} \circ ... \circ S_{i_k}(F)$. We immediately observe that

$$F = \bigcup_{(i_1, ..., i_k) \in J_k} F_{i_1 ... i_k}.$$

By letting $r_i = |F_i|$ the diameter of F_i, we may write $|F_i| \leq \varphi(|F|)$. Therefore,

$$\sum_i \varphi(r_i)^s \leq \sum_i \varphi(|F|)^s,$$

which leads to $\Sigma_\varphi^s(F) < \infty$. Similarly, we prove that $\Sigma_\varphi^s(F) > 0$.
The case dealing with Dim_φ may be treated by similar techniques. □

Example 1 Take $\varphi(x) = \dfrac{1}{3}x$ and $S_1(x) = \dfrac{1}{3}x$, and $S_2(x) = \dfrac{1}{3}x + \dfrac{2}{3}$, we come back to the well-known triadic Cantor set as the self-similar (invariant) set associated to (S_1, S_2). We see easily that for $i = 1, 2$,

$$|S_i(x) - S_i(y)| = \varphi(|x-y|), \quad \forall x, y.$$

Furthermore, we have here $m = 2$ and $\Lambda(\varphi) = \dfrac{1}{3}$. Moreover, $\Sigma_\varphi^\alpha = H^\alpha$ the well-known Hausdorff measure, while $\Theta_\varphi^\alpha = P^\alpha$ the well-known packing measure. From Theorem 2, we deduce that

$$dim_\varphi(F) = Dim_\varphi(F) = \frac{\log 2}{\log 3},$$

which is effectively the Hausdorff dimension of the triadic Cantor set.

Example 2 Take $\varphi(x) = \dfrac{2}{3}x$ and $S_1(x) = \dfrac{1}{3}x^2$, and $S_2(x) = \dfrac{1}{3}x^2 + \dfrac{2}{3}$. We may write easily that for $i = 1, 2$,

$$|S_i(x) - S_i(y)| = \varphi(|x-y|), \quad \forall x, y.$$

It holds that the pair of φ-contractions (S_1, S_2) possesses an invariant self-similar set F (which resembles the Cantor set). Furthermore, we have for this example, $m = 2$ and $\Lambda(\varphi) = \frac{2}{3}$. From Theorem 2, we deduce that

$$dim_\varphi(F) = Dim_\varphi(F) = \frac{\log 2}{\log 3 - \log 2}.$$

4. Conclusion

In this paper, we used a general concept of contractivity relative to a gauge function leading to some general form of invariant self-similar sets generalizing the result of Hutchinson [1]. This concept is then applied to a general variant of Carathéodory's measure and dimension. Our results fall with the classic notions of Hausdorff measure and dimension for the special choice of the gauge function. Furthermore, the new variant is more adaptable to nonlinear cases, where the classical variants are difficult to handle.

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Data availability statement

No data is used in this paper.

Conflict of interest

I declare no conflict of interest regarding this work.

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