Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

Dynamics of a SVEIR Epidemic Model with a Delay in Diagnosis in a Changing Environment

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Received: 1 May 2024; **Revised:** 12 June 2024; **Accepted:** 4 July 2024

Abstract: A SVEIR epidemic model with a delay in diagnosis is studied in a constant and variable environment. The mathematical analysis shows that the dynamics of the model in the constant environment are completely determined by the magnitude of the delay-induced reproduction number $\mathscr{R}_\alpha$. We established that if $\mathscr{R}_\alpha < 1$, the disease-free equilibrium is globally asymptotically stable, and when $\mathscr{R}_\alpha > 1$ the endemic equilibrium is globally asymptotically stable. In the variable environment, the model undergoes a transcritical bifurcation for $\mathscr{R}_\alpha = 1$ leading to changes in the stability of the equilibrium points. The analytical effect of the delays in epidemic diagnosis is investigated. A minimum diagnosis rate α_{min} has been determined to face or control the disease effectively. Finally, numerical illustrations were presented to support the theoretical results.

***Keywords*:** sensitivity, delays in diagnosis, changing environment, Lyapunov function

MSC: 34D20, 34D23, 34D45, 37C75, 92B05

1. Introduction

Infectious diseases include all diseases caused by the transmission of a pathogenic agent such as bacteria, viruses, parasites and champions. They are currently the leading cause of death worldwide. According to the WHO, infectious diseases are responsible for about 13 million deaths every year and are now the leading cause of death among children and young adults (see [1]). A number of measures have been taken to eradicate infectious diseases, including improved hygiene conditions, better nutrition, vaccination and isolation. Mathematical models are an essential tool for understanding the dynamics of infectious diseases. They are used to prevent and control the epidemic progression in a community (see [2]). Several mathematical models were proposed and studied in order to understand the dynamics of infectious diseases and to control them (see [3–21]). Among these mathematical models, the SVEIR type model is one of the models that is most often used. It is used in modeling diseases such as measles, tuberculosis, influenza, hepatitis B, Covid-19. For this type of model, the dynamics of the spread of the epidemic occurs according to five compartments (see [10, 12, 14, 15, 22–24]).

In [12], Miled and Amer proposed a SVEIR model of measles epidemic in which they examined the influence of the susceptible population that has been vaccinated and the rate of infection when susceptible individuals interact with infected individuals. Using an SVEIR model, Nkamba et al. (see [14]) studied the impact of vaccination in the control of

DOI: https://doi.org/10.37256/cm.5320244865

poliomyelitis disease. In [15], a SVEIR model for streptococcal pneumonia with a saturated incidence of infection was formulated and studied by Opara et al. The authors showed that vaccination coverage must be higher than the critical proportion of vaccination to eradicate streptococcal pneumonia disease in the community. A SVEIR model with two delays was formulated by Zhang et al. (see [23]). The authors analyzed the impact of these delay parameters on the dynamic behaviour of the system. Recently, several authors have used SVEIR models to assess the influence of incomplete vaccination on epidemics such as hepatitis B in [24], tuberculosis in [22], and HIV in [10].

During COVID-19 pandemic, it was established that more infected were undiagnosed due to the lacks of diagnostic reagents and the long waiting time for diagnosis. This situation had a great impact on the spreading of this disease. Therefore, it becomes important to diagnose and identify the infected in time and then treat the confirmed case.

In this work, we formulate a SVEIR epidemiological model with delay in diagnosis in a changing environment. This model treats diseases with total immunity, such as meningitis, chickenpox, etc. The infectious class I is divided into two classes, namely I_1 for individuals diagnosed in time and I_2 for those who received a delayed diagnosis. This work is organized as follows: in section 2, we formulate the model. Section 3, is devoted to the study of the dynamics of model without a changing environment. In section 4, we derive the analytical effects of the delay in diagnosis and we provide sensitivity analysis results. In section 5, the model in a variable environment is presented. We end by a conclusion in section 6.

2. Model formulation

The total population at time t denoted by N is subdivided into six compartments, namely, susceptible individuals (S), Vaccinated (V), undetected non-symptomatic (Latent) (E), infected individuals with timely diagnosis (I_1), infected individuals with delay in diagnosis (I_2) and recovered individuals (R). The flowchart of the model is given by Figure 1 below.

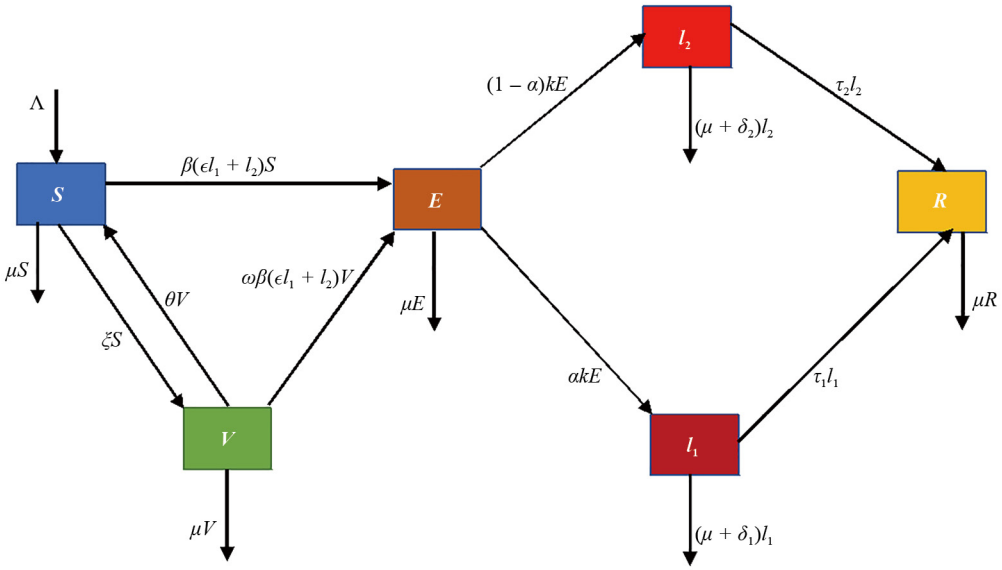

Figure 1. Transfer diagram

The state of infection is determined by the complex interaction between disease transmission, vaccinations, recoveries and different mortality rates. In fact, this model takes into account the impact of mortality without having an explicit variable for deaths, but individuals in each category (S, V, E, I_1, I_2, R) can die of natural causes, so we add a rate μ which

characterises natural mortality to each equation μS, μV, μE, μI_1, μI_2, and μR. In addition, infected individuals may die from the disease, with a rate δ_1 for those who received a rapid diagnosis and a rate δ_2 for those who received a delayed diagnosis. The dynamic of the model is given by following the differential equation system:

$$\begin{cases} \frac{dS}{dt} = \Lambda + \theta V - \beta(\varepsilon I_1 + I_2)S - (\xi + \mu)S, \\ \frac{dV}{dt} = \xi S - \omega\beta(\varepsilon I_1 + I_2)V - (\theta + \mu)V, \\ \frac{dE}{dt} = \beta(\varepsilon I_1 + I_2)S + \omega\beta(\varepsilon I_1 + I_2)V - (\mu + k)E, \\ \frac{dI_1}{dt} = \alpha k E - (\mu + \tau_1 + \delta_1)I_1, \\ \frac{dI_2}{dt} = (1-\alpha)kE - (\mu + \tau_2 + \delta_2)I_2, \\ \frac{dR}{dt} = \tau_1 I_1 + \tau_2 I_2 - \mu R, \\ \frac{d\alpha}{dt} = r, \end{cases} \tag{1}$$

where r is the speed of environmental change. From the last equation of system (1), we derive that $\alpha(t) = rt + \alpha_0$ (α_0 is the initial value), which represents the possible directional environmental change. The susceptible compartment is increased through the recruitment of individuals, either by immigration or birth into the population at a constant rate Λ and the natural mortality rate is μ. Susceptible individuals are vaccinated at a rate ξ, and the protection provided by the vaccine wanes over time at the rate θ. The disease transmission rate is β and ε $(0 < \varepsilon < 1)$ is the reduction in infection rate due to timely diagnosis. The vaccine offered to individuals is thought to be imperfect (i.e., it does not offer total protection against the disease), the vaccinated individuals may become infected again, but with a lower level than those in the susceptible compartment. Therefore, the transmission rate β is estimated by a scaling factor $(1-\omega)$, where ω $(0 \leq \omega \leq 1)$ is the viability of the vaccine and $\omega = 0$ implies that a vaccine provides 100% assurance against the disease, while $\omega = 1$ indicates that a vaccine does not secure individuals. Latent individuals become infected with rate k and a fraction of $\alpha(t)$ of these individuals receive timely diagnosis and the remaining fraction $(1-\alpha(t))$ are diagnosed with delay. Mortality rates due to disease in compartments I_1 and I_2 are δ_1 and δ_2 respectively, with $\delta_1 < \delta_2$. Infected individuals with timely diagnosis recover at rate τ_1 and infected individuals with delayed diagnosis recover at rate τ_2 , with $\tau_2 < \tau_1$.

3. Analysis of the model in a constant environment

In a constant environment (i.e $r = 0$), system (1) is reduced to the following system:

$$\begin{cases} \frac{dS}{dt} = \Lambda + \theta V - \beta(\varepsilon I_1 + I_2)S - (\xi + \mu)S, \\ \frac{dV}{dt} = \xi S - \omega\beta(\varepsilon I_1 + I_2)V - (\theta + \mu)V, \\ \frac{dE}{dt} = \beta(\varepsilon I_1 + I_2)S + \omega\beta(\varepsilon I_1 + I_2)V - (\mu + k)E, \\ \frac{dI_1}{dt} = \alpha k E - (\mu + \tau_1 + \delta_1)I_1, \\ \frac{dI_2}{dt} = (1-\alpha)kE - (\mu + \tau_2 + \delta_2)I_2, \\ \frac{dR}{dt} = \tau_1 I_1 + \tau_2 I_2 - \mu R, \end{cases} \tag{2}$$

with initial conditions $(S(0), V(0), E(0), I_1(0), I_2(0), R(0)) \in \mathbb{R}^6_+$. For model (2) to be epidemiologically realistic, it is necessary to prove that all variables remain positive for all time. Clearly $\frac{dS}{dt}\Big|_{S=0} \geq 0$, $\frac{dV}{dt}\Big|_{V=0} \geq 0$, $\frac{dE}{dt}\Big|_{E=0} \geq 0$, $\frac{dI_1}{dt}\Big|_{I_1=0} \geq 0$, $\frac{dI_2}{dt}\Big|_{I_2=0} \geq 0$ and $\frac{dR}{dt}\Big|_{R=0} \geq 0$ within $\mathbb{R}^6_+$. Thus, Proposition 2.1 of [25] implies that $\mathbb{R}^6_+$ is positively invariant.

Let $N = S + V + E + I_1 + I_2$ be the total population. Thus, by summing all equations of (2), we obtain

$$\frac{dN}{dt} = \Lambda - \mu N - (\tau_1 + \delta_1)I_1 - (\tau_2 + \delta_2)I_2.$$

Thus,

$$\frac{dN}{dt} \leq \Lambda - \mu N,$$

and by the standard comparison theorem (see [26]) we derive that

$$N(t) \leq \frac{\Lambda}{\mu} - \left(\frac{\Lambda}{\mu} - N(0)\right)e^{-\mu t}.$$

In particular, if $N(0) \leq \frac{\Lambda}{\mu}$, then $N(t) \leq \frac{\Lambda}{\mu}$ for all $t > 0$.

Hence, the domain

$$\mathscr{D} = \left\{(S(t), V(t), E(t), I_1(t), I_2(t), R(t)) \in \mathbb{R}^6_+ : N(t) \leq \frac{\Lambda}{\mu}\right\}$$

is positively invariant.

Theorem 1 For every initial value in $\mathscr{D}$, solutions of system (2) exists for all time $t > 0$.

Proof. The right-hand-side of system (2) is locally Lipschitz because it is class $\mathscr{C}^1$, hence the local existence of solutions follows. The global existence of the solutions is due to the fact that $\mathscr{D}$ is positively invariant and attracting all the solutions. □

3.1 *Global dynamic of the disease-free equilibrium*

Since the recovered human population R does not appear in the remaining equations of system (2), it is then sufficient to consider the following system

$$\begin{cases} \frac{dS}{dt} = \Lambda + \theta V - \beta(\varepsilon I_1 + I_2)S - (\xi + \mu)S, \\ \frac{dV}{dt} = \xi S - \omega\beta(\varepsilon I_1 + I_2)V - (\theta + \mu)V, \\ \frac{dE}{dt} = \beta(\varepsilon I_1 + I_2)S + \omega\beta(\varepsilon I_1 + I_2)V - (\mu + k)E, \\ \frac{dI_1}{dt} = \alpha k E - (\mu + \tau_1 + \delta_1)I_1, \\ \frac{dI_2}{dt} = (1-\alpha)kE - (\mu + \tau_2 + \delta_2)I_2, \end{cases} \tag{3}$$

with initial conditions

$$(S(0), V(0), E(0), I_1(0), I_2(0)) \in \mathbb{R}^5_+.$$

The disease-free equilibrium $\mathscr{E}_0$ is given by

$$\mathscr{E}_0 = (S_0, V_0, 0, 0, 0),$$

where

$$S_0 = \frac{\Lambda(\theta + \mu)}{\mu(\mu + \theta + \xi)} \text{ and } V_0 = \frac{\xi\Lambda}{\mu(\mu + \theta + \xi)}.$$

Following the method of Driesche and Watmough (see [27]) the delay-induced reproduction number is given by

$$\mathscr{R}_\alpha = \frac{\beta\Lambda k\,(\theta + \mu + \omega\xi)\,[\varepsilon\alpha\,(\mu + \tau_2 + \delta_2) + (1-\alpha)\,(\mu + \tau_1 + \delta_1)]}{\mu\,(\mu + k)\,(\mu + \theta + \xi)\,(\mu + \tau_1 + \delta_1)\,(\mu + \tau_2 + \delta_2)}.$$

The following result hold.

Theorem 2 If $\mathscr{R}_\alpha < 1$, the disease-free equilibrium $\mathscr{E}_0$ is locally asymptotically stable.

Proof. Consider

$$g_1 = \frac{dS}{dt},\ g_2 = \frac{dV}{dt},\ g_3 = \frac{dE}{dt},\ g_4 = \frac{dI_1}{dt} \text{ and } g_5 = \frac{dI_2}{dt}.$$

By linearising system (3) around the equilibrium point $\mathscr{E}_0$, we obtain the system

$$\left(\frac{dS}{dt}, \frac{dV}{dt}, \frac{dE}{dt}, \frac{dI_1}{dt}, \frac{dI_2}{dt}\right)' = J(\mathscr{E}_0)(S, V, E, I_1, I_2)',$$

where $J(\mathscr{E}_0)$ is the Jacobian matrix evaluated at $\mathscr{E}_0$. Clearly, the matrix $J(S, V, E, I_1, I_2)$ is given by

$$J(S, V, E, I_1, I_2) = \begin{pmatrix} \frac{\partial g_1}{\partial S} & \frac{\partial g_1}{\partial V} & \frac{\partial g_1}{\partial E} & \frac{\partial g_1}{\partial I_1} & \frac{\partial g_1}{\partial I_2} \\ \frac{\partial g_2}{\partial S} & \frac{\partial g_2}{\partial V} & \frac{\partial g_2}{\partial E} & \frac{\partial g_2}{\partial I_1} & \frac{\partial g_2}{\partial I_2} \\ \frac{\partial g_3}{\partial S} & \frac{\partial g_3}{\partial V} & \frac{\partial g_3}{\partial E} & \frac{\partial g_3}{\partial I_1} & \frac{\partial g_3}{\partial I_2} \\ \frac{\partial g_4}{\partial S} & \frac{\partial g_4}{\partial V} & \frac{\partial g_4}{\partial E} & \frac{\partial g_4}{\partial I_1} & \frac{\partial q_4}{\partial I_2} \\ \frac{\partial g_5}{\partial S} & \frac{\partial g_5}{\partial V} & \frac{\partial g_5}{\partial E} & \frac{\partial g_5}{\partial I_1} & \frac{\partial g_5}{\partial I_2} \end{pmatrix},$$

and then, $J(\mathscr{E}_0)$ is given by:

$$J(\mathscr{E}_0) = \begin{pmatrix} -(\mu+\xi) & \theta & 0 & -\varepsilon\beta S_0 & -\beta S_0 \\ \xi & -(\mu+\theta) & 0 & -\omega\varepsilon\beta V_0 & -\omega\beta V_0 \\ 0 & 0 & -(\mu+k) & \varepsilon\beta Y_1 & \beta Y_1 \\ 0 & 0 & \alpha k & -Y_2 & 0 \\ 0 & 0 & (1-\alpha)k & 0 & -Y_3 \end{pmatrix},$$

where

$$Y_1 = S_0 + \omega V_0,\ Y_2 = \mu + \tau_1 + \delta_1 \text{ and } Y_3 = \mu + \tau_2 + \delta_2,$$

and this give us the characteristic polynomial

$$P(\lambda) = (\lambda+\mu)(\lambda+\mu+\theta+\xi)(\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3)$$

where

$$a_1 = \mu + k + \mu + \tau_1 + \delta_1 + \mu + \tau_2 + \delta_2,$$

$$a_2 = (\mu + k)(\mu + \tau_1 + \delta_1 + \mu + \tau_2 + \delta_2) + (\mu + \tau_1 + \delta_1)(\mu + \tau_2 + \delta_2)$$

$$- k\beta(S_0 + \omega V_0)(1 - \alpha + \varepsilon\alpha),$$

$$a_3 = (\mu + k)(\mu + \tau_1 + \delta_1)(\mu + \tau_2 + \delta_2)(1 - \mathscr{R}_\alpha).$$

The Hurwitz matrices associated with the polynomial $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$ are :

$$H_1 = a_1,\ H_2 = \begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix} \text{ and } H_3 = \begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ 0 & 0 & a_3 \end{pmatrix}.$$

Since $\mathscr{R}_\alpha < 1$, thus

$$\beta k(S_0 + \omega V_0) < \frac{(\mu + k)(\mu + \tau_1 + \delta_1)(\mu + \tau_2 + \delta_2)}{[(1 - \alpha)(\mu + \tau_1 + \delta_1) + \varepsilon\alpha(\mu + \tau_2 + \delta_2)]}, \tag{4}$$

that is

$$- k\beta(S_0 + \omega V_0)(1 - \alpha + \varepsilon\alpha) >$$

$$\frac{-(\mu + k)(\mu + \tau_1 + \delta_1)(\mu + \tau_2 + \delta_2)(1 - \alpha + \varepsilon\alpha)}{[(1 - \alpha)(\mu + \tau_1 + \delta_1) + \varepsilon\alpha(\mu + \tau_2 + \delta_2)]}.$$

We obtain

$$a_2 > \frac{1}{(1 - \alpha)(\mu + \tau_1 + \delta_1) + \varepsilon\alpha(\mu + \tau_2 + \delta_2)} \Big[(1 - \alpha)(2\mu + k$$

$$+ \tau_2 + \delta_2)(\mu + \tau_1 + \delta_1)^2 + \varepsilon\alpha(\mu + \tau_2 + \delta_2)^2(2\mu + k + \tau_1 + \delta_1)\Big],$$

that is $a_2 > 0$.

Further,

$$
\begin{aligned}
a_1a_2 - a_3 =&(\mu+k)^2[\mu+\tau_1+\delta_1+\mu+\tau_2+\delta_2]+(2\mu+k+\tau_2\\
&+\delta_2)(\mu+\tau_1+\delta_1)^2+(\mu+\tau_2+\delta_2)^2(2\mu+k+\tau_1+\delta_1)\\
&+(\mu+k)(\mu+\tau_1+\delta_1)(\mu+\tau_2+\delta_2)(\mathscr{R}_\alpha+2)-(\mu+k\\
&+\mu+\tau_1+\delta_1+\mu+\tau_2+\delta_2)k\beta(S_0+\omega V_0)(1-\alpha+\varepsilon\alpha).
\end{aligned}
$$

Since $\mathscr{R}_\alpha < 1$, by using (4) we get

$$
\begin{aligned}
&-(\mu+k+\mu+\tau_1+\delta_1+\mu+\tau_2+\delta_2)k\beta(S_0+\omega V_0)(1-\alpha+\varepsilon\alpha) >\\
&\frac{1}{[(1-\alpha)(\mu+\tau_1+\delta_1)+\varepsilon\alpha(\mu+\tau_2+\delta_2)]}\Big[-(\mu+k+\mu+\tau_1+\delta_1\\
&+\mu+\tau_2+\delta_2)(\mu+k)(\mu+\tau_1+\delta_1)(\mu+\tau_2+\delta_2)(1-\alpha+\varepsilon\alpha)\Big].
\end{aligned}
$$

Then,

$$
\begin{aligned}
a_1a_2 - a_3 >&\frac{(1-\alpha)(\mu+\tau_1+\delta_1)^2}{(1-\alpha)(\mu+\tau_1+\delta_1)+\varepsilon\alpha(\mu+\tau_2+\delta_2)}\Big[(\mu+k)^2+(\mu+k)(\mu+\tau_1+\delta_1)\\
&+(\mu+\tau_1+\delta_1)(\mu+\tau_2+\delta_2)+(\mu+\tau_2+\delta_2)^2+(\mu+k)(\mu+\tau_2+\delta_2)(\mathscr{R}_\alpha+1)\Big]\\
&+\frac{\varepsilon\alpha(\mu+\tau_2+\delta_2)^2}{(1-\alpha)(\mu+\tau_1+\delta_1)+\varepsilon\alpha(\mu+\tau_2+\delta_2)}\Big[(\mu+k)^2+(\mu+\tau_1+\delta_1)(\mu+\tau_2+\delta_2)\\
&+(\mu+k)(\mu+\tau_2+\delta_2)+(\mu+\tau_1+\delta_1)^2+(\mu+k)(\mu+\tau_1+\delta_1)(\mathscr{R}_\alpha+1)\Big].
\end{aligned}
$$

Thus,

$$
a_1a_2 - a_3 > 0.
$$

Therefore, we have

$$\det H_1 = a_1 > 0,\ \det H_2 = a_1 a_2 - a_3 > 0 \text{ and}$$

$$\det H_3 = a_3(a_1 a_2 - a_3) > 0 \text{ if } \mathscr{R}_\alpha < 1.$$

Thus, according to the Routh-Hurwitz criterion (see [28]), the roots of the characteristic polynomial are negative if $\mathscr{R}_\alpha < 1$. Hence, the disease-free equilibrium $\mathscr{E}_0$ is locally asymptotically stable if $\mathscr{R}_\alpha < 1$. □

To establish the global stability of disease-free equilibrium $\mathscr{E}_0$, we follow the method discussed by Chavez et al. in [29] and details are reproduced below.

Suppose that a model can be written as following:

$$\begin{cases} \frac{dX}{dt} &= F(X, Z) \\ \\ \frac{dY}{dt} &= G(X, Z) \text{ with } G(X, 0) = 0, \end{cases} \tag{5}$$

where $X \in \mathbb{R}^{n_1}$ denotes the uninfected individuals and $Z \in \mathbb{R}^{n_2}$ denotes the infected individuals, n_1 and n_2 are positive integers. Let $\mathscr{E}_0 = (X_0, 0)$ be the disease free equilibrium of the system (5). Consider the following two assumptions as:

(C_1) For $\frac{dX}{dt} = F(X, 0)$, X_0 is globally asymptotically stable.

(C_2) $G(X, Z) = D_Y G(X_0, 0)Z - \widehat{G}(X, Z)$, $\widehat{G}(X, Z) \geq 0$ for $(X, Z) \in \mathscr{D}$, where $D_Z G(X_0, 0)$ is an M-matrix (stable matrix with non-negative off diagonal elements) and $\mathscr{D}$ is bounded invariant region.

If the model (5) satisfies the above two conditions then the following Lemma 1 holds.

Lemma 1 [29] The disease-free equilibrium $\mathscr{E}_0 = (X_0, 0)$ of the model (5) is globally asymptotically stable for $R_0 < 1$ provided the assumptions (C_1) and (C_2) are satisfied.

Following the method of Castillo-Chavez et al. (see [29]) we rewrite system (3) as follows.

$$\begin{cases} \frac{dX}{dt} = F(X, Z), \\ \\ \frac{dZ}{dt} = G(X, Z), \end{cases} \tag{6}$$

where $X = (S, V)^{'}$ and $Z = (E, I_1, I_2)^{'}$,

$$F(X, Z) = \Big(\Lambda + \theta V - \beta(\varepsilon I_1 + I_2)S - (\xi + \mu)S,\ \xi S - \omega\beta(\varepsilon I_1 + I_2)V - (\theta + \mu)V\Big)^{'},$$

$$G(X, Z) = \Big(\beta(\varepsilon I_1 + I_2)(S + \omega V) - (\mu + k)E,\ \alpha k E - (\mu + \tau_1 + \delta_1)I_1,\ (1 - \alpha)kE - (\mu + \tau_2 + \delta_2)I_2\Big)^{'}.$$

Here, $^{'}$ denotes the transpose. Clearly $G(X, 0) = (0, 0, 0)^{'}$ and the disease-free equilibrium of system (3) is given by $\mathscr{E}_0 = (X_0, 0)$ with $X_0 = (S_0, V_0)$. X_0 is globally asymptotically stable for $\frac{dX}{dt} = F(X, 0)$ as $X \to (S_0, V_0)$ whenever $t \to \infty$.

Further,

$$B = \frac{\partial G}{\partial Z}(X_0, 0) = \begin{pmatrix} -(\mu + k) & \varepsilon\beta(S_0 + \omega V_0) & \beta(S_0 + \omega V_0) \\ \alpha k & -(\mu + \tau_1 + \delta_1) & 0 \\ (1-\alpha)k & 0 & -(\mu + \tau_2 + \delta_2) \end{pmatrix},$$

is an M-matrix (the off-diagonal elements of B are non-negative). Then,

$$\begin{aligned} \hat{G}(X, Z) =& BZ - G(X, Z) \\ =& \begin{pmatrix} \beta(\varepsilon I_1 + I_2)\left(\frac{\Lambda}{\mu} - (S+V) + (1-\omega)V\right) \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Since $S + V \leq \frac{\Lambda}{\mu}$ and $1 - \omega \geq 0$, thus $\hat{G}(X, Z) \geq 0$. So, according to Lemma 1 we have the following result.

Theorem 3 The disease-free equilibrium $\mathscr{E}_0$ of system (3) is globally asymptotically stable in $\mathscr{D}$ if $\mathscr{R}_\alpha < 1$.

3.2 *Global dynamic of the endemic equilibrium*

In this subsection we show the stability result of the endemic equilibrium.

Theorem 4 System (3) has a unique endemic equilibrium whenever $\mathscr{R}_\alpha > 1$, and no endemic equilibrium otherwise.

Proof. Denote by $\mathscr{E}_* = (S^*, V^*, E^*, I_1^*, I_2^*)$ the endemic equilibrium of system (3). Thus, $\mathscr{E}_*$ solves the following system

$$\begin{cases} \Lambda + \theta V^* - \beta(\varepsilon I_1^* + I_2^*)S^* - (\xi + \mu)S^* = 0, \\ \xi S^* - \omega\beta(\varepsilon I_1^* + I_2^*)V^* - (\theta + \mu)V^* = 0, \\ \beta(\varepsilon I_1^* + I_2^*)S^* + \omega\beta(\varepsilon I_1^* + I_2^*)V^* - (\mu + k)E^* = 0, \\ \alpha k E^* - (\mu + \tau_1 + \delta_1)I_1^* = 0, \\ (1-\alpha)kE^* - (\mu + \tau_2 + \delta_2)I_2^* = 0, \end{cases} \tag{7}$$

that is,

$$\begin{cases} S^* = \frac{\Lambda+\theta V^*}{\mu+\xi+\beta(\varepsilon I_1^*+I_2^*)}, \\ V^* = \frac{\xi S^*}{\mu+\theta+\omega\beta(\varepsilon I_1^*+I_2^*)}, \\ \beta(\varepsilon I_1^*+I_2^*)S^* + \omega\beta(\varepsilon I_1^*+I_2^*)V^* - (\mu+k)E^* = 0, \\ I_1^* = \frac{\alpha k E^*}{\mu+\tau_1+\delta_1}, \\ I_2^* = \frac{(1-\alpha)kE^*}{\mu+\tau_2+\delta_2}. \end{cases} \tag{8}$$

Setting $d_1 = \mu+\tau_1+\delta_1$, $d_2 = \mu+\tau_2+\delta_2$ and $A = \frac{k(\varepsilon\alpha d_2+(1-\alpha)d_1)}{d_1 d_2}$ we get

$$\begin{cases} S^* = \frac{\Lambda(\mu+\theta+\omega\beta AE^*)}{\mu(\mu+\xi+\theta)+\beta(\mu+\theta+\omega(\mu+\xi))AE^*+\omega\beta^2A^2(E^*)^2}, \\ V^* = \frac{\xi\Lambda}{\mu(\mu+\xi+\theta)+\beta(\mu+\theta+\omega(\mu+\xi))AE^*+\omega\beta^2A^2(E^*)^2}, \\ \beta(\varepsilon I_1^*+I_2^*)S^* + \omega\beta(\varepsilon I_1^*+I_2^*)V^* - (\mu+k)E^* = 0, \\ I_1^* = \frac{\alpha k E^*}{d_1}, \\ I_2^* = \frac{(1-\alpha)kE^*}{d_2}. \end{cases} \tag{9}$$

It follows from the third equation of (9) that E^* solves the equation

$$m_2(E^*)^3 + m_1(E^*)^2 + m_0E^* = 0. \tag{10}$$

where,

$$\begin{aligned} m_2 =& -\omega\beta^2A^2(\mu+k), \\ m_1 =& \omega\Lambda\beta^2A^2 - \beta A(\mu+k)(\mu+\theta+\omega(\mu+\xi), \\ m_0 =& \beta A\Lambda(\mu+\theta+\omega\xi) - \mu(\mu+k)(\mu+\xi+\theta) \\ =& \mu(\mu+k)(\mu+\xi+\theta)(\mathscr{R}_\alpha - 1). \end{aligned}$$

Equation (10) give us $E^* = 0$ or

$$m_2(E^*)^2 + m_1(E^*) + m_0 = 0. \tag{11}$$

The solution $E^* = 0$ gives the disease-free equilibrium $\mathscr{E}_0$ defined above. Since, $\mathscr{R}_\alpha > 1$, thus $m_0 > 0$ and equation (11) has a unique positive solution given by

$$E^* = \frac{-m_1 - \sqrt{m_1^2 - 4m_2m_0}}{2m_2}.$$

If $\mathscr{R}_\alpha \leq 1$ then $m_0 \leq 0$, so the model has no positive solution. □

In order to establish the local stability of the endemic equilibrium $\mathscr{E}_*$ we use the theory of the central variety as discussed by Chavez and Song [22], and their result is given as follows

Lemma 2 [22] Consider the following system of ODEs with a parameter β

$$\frac{dx}{dt} = q(x, \beta), \quad q: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \text{ and } q \in \mathbb{C}^2(\mathbb{R}^n \times \mathbb{R}), \tag{12}$$

with 0 is an equilibrium of this system and $q(0, \beta) = 0$ for all β. Assume

P_1: $D_xq(0, 0) = \left(\frac{\partial q_p}{\partial x_j}(0, 0)\right)$ is the linearization matrix of the system (12) around the equilibrium 0 with β evaluated at 0. Zero is a simple eigenvalue of $D_xq(0, 0)$ and all other eigenvalues of $D_xq(0, 0)$ have negative real parts.

P_2: Matrix $D_xq(0, 0)$ has a non-negative right eigenvector l and a left eigenvector v corresponding to the zero eigenvalue.

Let q_p be the p-th component of q and

$$a = \sum_{p,\, i,\, j=1}^{n} v_p l_i l_j \frac{\partial^2 q_p}{\partial x_i \partial x_j}(0, 0)$$

$$b = \sum_{p,\, i=1}^{n} q_p l_i \frac{\partial^2 q_p}{\partial x_i \partial \beta}(0, 0).$$

The local dynamics of the system (12) around 0 is totally determined by a and b.

(i) $a > 0$, $b > 0$. When $\beta < 0$ with $|\beta| \ll 1$, 0 is locally asymptotically stable, and there exists a positive unstable equilibrium; when $0 < \beta \ll 1$, 0 is unstable and there exists a negative and locally asymptotically stable equilibrium.

(ii) $a < 0$, $b < 0$. When $\beta < 0$ with $|\beta| \ll 1$, 0 is unstable; when $0 < \beta \ll 1$, 0 is locally asymptotically stable, and there exists a positive unstable equilibrium.

(iii) $a > 0$, $b < 0$. When $\beta < 0$ with $|\beta| \ll 1$, 0 is unstable, and there exists a locally asymptotically stable negative equilibrium; when $0 < \beta \ll 1$, 0 is stable, and a positive unstable equilibrium appears.

(iv) $a < 0$, $b > 0$. When β changes from negative to positive, 0 changes its stability from stable to unstable. Corresponding a negative unstable equilibrium becomes positive and locally asymptotically stable.

We set $S = x_1$, $V = x_2$, $E = x_3$, $I_1 = x_4$ and $I_2 = x_5$. Using vector notation $W = (S, V, E, I_1, I_2)'$, system (3) can be re-written in the form

$$\frac{dW}{dt} = q_p = (q_1, q_2, q_3, q_4, q_5)^{'},$$

that is

$$\begin{cases} \frac{dx_1}{dt} = \Lambda + \theta x_2 - \beta(\varepsilon x_4 + x_5)x_1 := q_1, \\ \frac{dx_2}{dt} = \xi x_1 - \omega\beta(\varepsilon x_1 + x_5)x_2 - (\theta + \mu)x_2 := q_2, \\ \frac{dx_3}{dt} = \beta(\varepsilon x_4 + x_5)x_1 + \omega\beta(\varepsilon x_4 + x_5)x_2 - (\mu + k)x_3 := q_3, \\ \frac{dx_4}{dt} = \alpha k x_3 - (\mu + \tau_1 + \delta_1)x_4 := q_4, \\ \frac{dx_5}{dt} = (1-\alpha)k x_3 - (\mu + \tau_2 + \delta_2)x_5 := q_5. \end{cases} \tag{13}$$

By linearising system (13) around the equilibrium point $\mathscr{E}_0$, we obtain the system

$$\left(\frac{dS}{dt}, \frac{dV}{dt}, \frac{dE}{dt}, \frac{dI_1}{dt}, \frac{dI_2}{dt}\right)^{'} = J(\mathscr{E}_0)(S, V, E, I_1, I_2)^{'},$$

where $J(\mathscr{E}_0)$ is the Jacobian matrix evaluated at $\mathscr{E}_0$. Clearly, the matrix $J(S, V, E, I_1, I_2)$ is given by

$$J(S, V, E, I_1, I_2) = \begin{pmatrix} \frac{\partial q_1}{\partial S} & \frac{\partial q_1}{\partial V} & \frac{\partial q_1}{\partial E} & \frac{\partial q_1}{\partial I_1} & \frac{\partial q_1}{\partial I_2} \\ \frac{\partial q_2}{\partial S} & \frac{\partial q_2}{\partial V} & \frac{\partial q_2}{\partial E} & \frac{\partial q_2}{\partial I_1} & \frac{\partial q_2}{\partial I_2} \\ \frac{\partial q_3}{\partial S} & \frac{\partial q_3}{\partial V} & \frac{\partial q_3}{\partial E} & \frac{\partial q_3}{\partial I_1} & \frac{\partial q_3}{\partial I_2} \\ \frac{\partial q_4}{\partial S} & \frac{\partial q_4}{\partial V} & \frac{\partial q_4}{\partial E} & \frac{\partial q_4}{\partial I_1} & \frac{\partial q_4}{\partial I_2} \\ \frac{\partial q_5}{\partial S} & \frac{\partial q_5}{\partial V} & \frac{\partial q_5}{\partial E} & \frac{\partial q_5}{\partial I_1} & \frac{\partial q_5}{\partial I_2} \end{pmatrix},$$

and then, $J(\mathscr{E}_0)$ is given by:

$$J(\mathscr{E}_0) = \begin{pmatrix} -(\mu+\xi) & \theta & 0 & -\varepsilon\beta S_0 & -\beta S_0 \\ \xi & -(\mu+\theta) & 0 & -\omega\varepsilon\beta V_0 & -\omega\beta V_0 \\ 0 & 0 & -(\mu+k) & \varepsilon\beta Y_1 & \beta Y_1 \\ 0 & 0 & \alpha k & -Y_2 & 0 \\ 0 & 0 & (1-\alpha)k & 0 & -Y_3 \end{pmatrix},$$

where

$$Y_1 = S_0 + \omega V_0,\ Y_2 = \mu + \tau_1 + \delta_1 \text{ and } Y_3 = \mu + \tau_2 + \delta_2.$$

If β is the bifurcation point and if we consider the case when $\mathscr{R}_\alpha = 1$ and then solve for β, we obtain

$$\beta = \beta^* = \frac{(\mu+k)(\mu+\tau_1+\delta_1)(\mu+\tau_2+\delta_2)}{k(S_0+\omega V_0)\left[\varepsilon\alpha(\mu+\tau_2+\delta_2)+(1-\alpha)(\mu+\tau_1+\delta_1)\right]}.$$

The Jacobian matrix $J(\mathscr{E}_0)$ with $\beta = \beta^*$ has a simple zero eigenvalue, hence we can use the center manifold theory in the analysis of the dynamics of system (13) near $\beta = \beta^*$. The Jacobian matrix $J(\mathscr{E}_0)$ near $\beta = \beta^*$ has a right eigenvector associated with the zero eigenvalue given by $l = (l_1, l_2, l_3, l_4, l_5)^{'}$.

Thus,

$$\begin{pmatrix} -(\mu+\xi) & \theta & 0 & -\varepsilon\beta^* S_0 & -\beta^* S_0 \\ \xi & -(\mu+\theta) & 0 & -\omega\varepsilon\beta^* V_0 & -\omega\beta^* V_0 \\ 0 & 0 & -(\mu+k) & \varepsilon\beta^* Y_1 & \beta^* Y_1 \\ 0 & 0 & \alpha k & -Y_2 & 0 \\ 0 & 0 & (1-\alpha)k & 0 & -Y_3 \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \\ l_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which gives

$$\begin{cases} -(\mu+\xi)l_1+\theta l_2-\varepsilon\beta^* S_0 l_4-\beta^* S_0 l_5=0,\\ \xi l_1-(\mu+\theta)l_2-\varepsilon\omega\beta^* V_0 l_4-\omega\beta^* V_0 l_5=0,\\ -(\mu+k)l_3+\varepsilon\beta^*(S_0+\omega V_0)l_4+\beta^*(S_0+\omega V_0)l_5=0,\\ \alpha k l_3-(\mu+\tau_1+\delta_1)l_4=0,\\ (1-\alpha)k l_3-(\mu+\tau_2+\delta_2)l_5=0. \end{cases} \tag{14}$$

After solving system (14) we get

$$l_1=-\left(\frac{\varepsilon\alpha k}{\mu+\tau_1+\delta_1}+\frac{(1-\alpha)k}{\mu+\tau_2+\delta_2}\right)\left(\frac{(\mu+\theta)\beta^* S_0+\omega\theta\beta^* V_0}{\mu(\mu+\theta+\xi)}\right)<0,$$

$$l_2=-\left(\frac{\varepsilon\alpha k}{\mu+\tau_1+\delta_1}+\frac{(1-\alpha)k}{\mu+\tau_2+\delta_2}\right)\left(\frac{\omega\beta^* V_0(\mu+\xi)+\beta^*\xi S_0}{\mu(\mu+\theta+\xi)}\right)<0,$$

$$l_3>0,\ l_4=\frac{\alpha k}{\mu+\tau_1+\delta_1}l_3>0,\ l_5=\frac{(1-\alpha)k}{\mu+\tau_2+\delta_2}l_3>0.$$

Let $v=(v_1, v_2, v_3, v_4, v_5)^{'}$ be the left eigenvector of the Jacobian matrix $J(\mathscr{E}_0)$ associated with the zero eigenvalue at $\beta=\beta^*$.

Thus,

$$\begin{pmatrix} -(\mu+\xi) & \xi & 0 & 0 & 0\\ \theta & -(\mu+\theta) & 0 & 0 & 0\\ 0 & 0 & -(\mu+k) & \alpha k & (1-\alpha)k\\ -\varepsilon\beta^* S_0 & -\varepsilon\omega\beta^* V_0 & \varepsilon\beta^* Y_1 & -Y_2 & 0\\ -\beta^* S_0 & -\omega\beta^* V_0 & \beta^* Y_1 & 0 & -Y_3 \end{pmatrix}\begin{pmatrix} v_1\\ v_2\\ v_3\\ v_4\\ v_5 \end{pmatrix}=\begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 0 \end{pmatrix},$$

which gives

$$\begin{cases} -(\mu+\xi)v_1+\xi v_2=0, \\ \theta v_1-(\mu+\theta)v_2=0, \\ -(\mu+k)v_3+\alpha k v_4+(1-\alpha)k v_5=0, \\ -\varepsilon\beta^* S_0 v_1-\varepsilon\omega\beta^* V_0 v_2+\varepsilon\beta^*(S_0+\omega V_0)v_3-(\mu+\tau_1+\delta_1)v_4=0, \\ -\beta^* S_0 v_1-\omega\beta^* V_0 v_2+\beta^*(S_0+\omega V_0)v_3-(\mu+\tau_2+\delta_2)v_5=0. \end{cases} \tag{15}$$

After solving system (15) we get

$$v_1=v_2=0,\ \ v_3>0,$$

$$v_4=\frac{\varepsilon\beta^*(S_0+\omega V_0)}{\mu+\tau_1+\delta_1}v_3>0,$$

$$v_5=\frac{\beta^*(S_0+\omega V_0)}{\mu+\tau_2+\delta_2}v_3>0.$$

As in [22], we now calculate the values of a and b defined by

$$a=\sum_{p,\,i,\,j=1}^{5} v_p l_i l_j \frac{\partial^2 q_p}{\partial x_i \partial x_j}(\mathscr{E}_0),\ \ b=\sum_{p,\,i=1}^{5} v_p l_i \frac{\partial^2 q_p}{\partial x_i \partial \beta^*}(\mathscr{E}_0),$$

where q_p the p-th component of q.

From system (13), we have

$$\frac{\partial^2 q_3}{\partial x_1 \partial x_4}(\mathscr{E}_0)=\frac{\partial^2 q_3}{\partial x_4 \partial x_1}(\mathscr{E}_0)=\varepsilon\beta^*,$$

$$\frac{\partial^2 q_3}{\partial x_1 \partial x_5}(\mathscr{E}_0)=\frac{\partial^2 q_3}{\partial x_5 \partial x_1}(\mathscr{E}_0)=\beta^*,$$

$$\frac{\partial^2 q_3}{\partial x_2 \partial x_4}(\mathscr{E}_0)=\frac{\partial^2 q_3}{\partial x_4 \partial x_2}(\mathscr{E}_0)=\varepsilon\omega\beta^*,$$

$$\frac{\partial^2 q_3}{\partial x_2 \partial x_5}(\mathscr{E}_0)=\frac{\partial^2 q_3}{\partial x_5 \partial x_2}(\mathscr{E}_0)=\omega\beta^*.$$

It follows that

$$a = v_3 \sum_{p,\,i,\,j=1}^{5} l_i l_j \frac{\partial^2 q_3}{\partial x_i \partial x_j}(\mathscr{E}_0)$$

$$= v_3 \left(2l_1 l_4 \beta^* \varepsilon + 2l_1 l_5 \beta^* + 2l_2 l_4 \omega \beta^* \varepsilon + 2l_2 l_5 \omega \beta^*\right)$$

$$= 2\beta^* v_3 l_1 \left(\varepsilon l_4 + l_5\right) + 2\beta^* v_3 \omega l_2 \left(\varepsilon l_4 + l_5\right)$$

$$= 2v_3 \beta^* \left(l_1 + \omega l_2\right)\left(\varepsilon l_4 + l_5\right).$$

Further,

$$\frac{\partial^2 q_3}{\partial x_4 \partial \beta^*}(\mathscr{E}_0) = \varepsilon(S_0 + \omega V_0), \;\; \frac{\partial^2 q_3}{\partial x_5 \partial \beta^*}(\mathscr{E}_0) = S_0 + \omega V_0.$$

Hence,

$$b = v_3 \sum_{p,\,i=1}^{5} l_i \frac{\partial^2 q_3}{\partial x_i \partial \beta^*}(\mathscr{E}_0)$$

$$= v_3 \left(\varepsilon l_4 (S_0 + \omega V_0) + l_5 (S_0 + \omega V_0)\right)$$

$$= v_3 \left(\varepsilon l_4 + l_5\right)\left(S_0 + \omega V_0\right).$$

Clearly $a < 0$ and $b > 0$. Now by applying condition (iv) of Lemma 2 we have the following result.

Theorem 5 The endemic equilibrium $\mathscr{E}_* = (S^*, V^*, E^*, I_1^*, I_2^*)$ is locally asymptotically stable for $\mathscr{R}_\alpha \geq 1$.

We now demonstrate the global stability of the endemic equilibrium $\mathscr{E}_*$ of model (3).

Theorem 6 The unique endemic equilibrium $\mathscr{E}_*$ is globaly asymptoticaly stable whenever $\mathscr{R}_\alpha > 1$.

Proof. Consider the following Lyapunov function

$$\mathscr{L} = \left(S - S^* - S^* \ln \frac{S}{S^*}\right) + \left(V - V^* - V^* \ln \frac{V}{V^*}\right) + \left(E - E^* - E^* \ln \frac{E}{E^*}\right)$$

$$+ c_1 \left(I_1 - I_1^* - I_1^* \ln \frac{I_1}{I_1^*}\right) + c_2 \left(I_2 - I_2^* - I_2^* \ln \frac{I_2}{I_2^*}\right),$$

where c_1 and c_2 are positives. The positive equilibrium $\mathscr{E}_*$ satisfies the following equations

$$\Lambda = (\mu+\xi)S^* + \beta(\varepsilon I_1^* + I_2^*)S^* - \theta V^*,$$

$$(\mu+\theta)V^* = \xi S^* - \omega\beta(\varepsilon I_1^* + I_2^*)V^*,$$

$$(\mu+k)E^* = \beta(\varepsilon I_1^* + I_2^*)S^* + \omega\beta(\varepsilon I_1^* + I_2^*)V^*,$$

$$(\mu+\tau_1+\delta_1)I_1^* = \alpha k E^*,$$

$$(\mu+\tau_2+\delta_2)I_2^* = (1-\alpha)kE^*. \tag{16}$$

Differentiating $\mathscr{L}$ with respect to time yields

$$\begin{aligned}
\frac{d\mathscr{L}}{dt} =& \left(1-\frac{S^*}{S}\right)\dot{S} + \left(1-\frac{V^*}{V}\right)\dot{V} + \left(1-\frac{E^*}{E}\right)\dot{E} + c_1\left(1-\frac{I_1^*}{I_1}\right)\dot{I}_1 \\
&+ c_2\left(1-\frac{I_2^*}{I_2}\right)\dot{I}_2 \\
=& \left(1-\frac{S^*}{S}\right)\Big((\mu+\xi)S^* + \beta(\varepsilon I_1^* + I_2^*)S^* - \theta V^* + \theta V \\
&- \beta(\varepsilon I_1 + I_2)S - (\mu+\xi)S\Big) + \left(1-\frac{V^*}{V}\right)\Big(\xi S - \omega\beta(\varepsilon I_1 + I_2)V \\
&- \xi\frac{S^*V}{V^*} + \omega\beta(\varepsilon I_1^* + I_2^*)V\Big) + \left(1-\frac{E^*}{E}\right)\Big(\beta(\varepsilon I_1 + I_2)(S+\omega V) \\
&- \beta(\varepsilon I_1^* + I_2^*)\frac{S^*E}{E^*} - \omega\beta(\varepsilon I_1^* + I_2^*)\frac{V^*E}{E^*}\Big) + c_1\left(1-\frac{I_1^*}{I_1}\right)\Big(\alpha k E \\
&- \alpha k\frac{E^*I_1}{I_1^*}\Big) + c_2\left(1-\frac{I_2^*}{I_2}\right)\Big((1-\alpha)kE - (1-\alpha)k\frac{E^*I_2}{I_2^*}\Big).
\end{aligned}$$

After a bit of algebra, we obtain

$$\frac{d\mathscr{L}}{dt} = -(\mu+\xi)\frac{(S-S^*)^2}{S} + \varepsilon\beta I_1^* S^*\Big(1 - \frac{I_1 S}{I_1^* S^*} - \frac{S^*}{S} + \frac{I_1}{I_1^*}\Big)$$

$$+ \beta I_2^* S^*\Big(1 - \frac{I_2 S}{I_2^* S^*} - \frac{S^*}{S} + \frac{I_2}{I_2^*}\Big) + \theta V^*\Big(\frac{V}{V^*} - 1 - \frac{S^* V}{S V^*}$$

$$+ \frac{S^*}{S}\Big) + \xi S^*\Big(1 + \frac{S}{S^*} - \frac{V}{V^*} - \frac{S V^*}{S^* V}\Big) + \varepsilon\omega\beta I_1^* V^*\Big(\frac{V}{V^*}$$

$$- \frac{I_1 V}{I_1^* V^*} - 1 + \frac{I_1}{I_1^*}\Big) + \omega\beta I_2^* V^*\Big(\frac{V}{V^*} - \frac{I_2 V}{I_2^* V^*} - 1 + \frac{I_2}{I_2^*}\Big)$$

$$+ \varepsilon\beta I_1^* S^*\Big(1 + \frac{I_1 S}{I_1^* S^*} - \frac{E}{E^*} - \frac{I_1 S E^*}{I_1^* S^* E}\Big) + \beta I_2^* S^*\Big(1 + \frac{I_2 S}{I_2^* S^*}$$

$$- \frac{E}{E^*} - \frac{I_2 S E^*}{I_2^* S^* E}\Big) + \varepsilon\omega\beta I_1^* V^*\Big(1 + \frac{I_1 V}{I_1^* V^*} - \frac{E}{E^*} - \frac{I_1 E^* V}{I_1^* E V^*}\Big)$$

$$+ \omega\beta I_2^* V^*\Big(1 + \frac{I_2 V}{I_2^* V^*} - \frac{E}{E^*} - \frac{I_2 E^* V}{I_2^* E V^*}\Big) + c_1\alpha k E^*\Big(1 + \frac{E}{E^*}$$

$$- \frac{I_1}{I_1^*} - \frac{E I_1^*}{E^* I_1}\Big) + c_2(1-\alpha)k E^*\Big(1 + \frac{E}{E^*} - \frac{I_2}{I_2^*} - \frac{E I_2^*}{E^* I_2}\Big). \tag{17}$$

We now introduce the new variables

$$x = \frac{S}{S^*},\ y = \frac{V}{V^*},\ z = \frac{E}{E^*},\ u = \frac{I_1}{I_1^*} \text{ and } n = \frac{I_2}{I_2^*}.$$

Thus,

$$\frac{d\mathscr{L}}{dt} = (\mu+\xi)\Big(2 - x - \frac{1}{x}\Big)S^* + \varepsilon\beta I_1^* S^*\Big(1 - ux - \frac{1}{x} + u\Big)$$

$$+ \beta I_2^* S^*\Big(1 - xn - \frac{1}{x} + n\Big) + \theta V^*\Big(y - 1 - \frac{y}{x} + \frac{1}{x}\Big)$$

$$+ \xi S^*\Big(1 + x - y - \frac{x}{y}\Big) + \varepsilon\omega\beta I_1^* V^*\Big(y - uy - 1 + u\Big)$$

$$+ \omega\beta I_2^* V^*\Big(y - ny - 1 + n\Big) + \beta\varepsilon I_1 S^*\Big(1 + xu - z - \frac{xu}{z}\Big)$$

$$
\begin{aligned}
&+\beta I_2 S^*\left(1+xn-z-\frac{xn}{z}\right)+\omega\beta\varepsilon I_1^* V^*\left(1+yu-z-\frac{yu}{z}\right)\\
&+\omega\beta I_2^* V^*\left(1+yn-z-\frac{yn}{z}\right)+c_1\alpha k E^*\left(1+z-u-\frac{z}{u}\right)\\
&+c_2(1-\alpha)kE^*\left(1+z-n-\frac{z}{n}\right)\\
=&(\mu+\xi)S^*\left(2-x-\frac{1}{x}\right)+\left(\varepsilon\beta I_1^* S^*+\beta I_2^* S^*-\theta V^*+\xi S^*\right.\\
&-\varepsilon\omega\beta I_1^* V^*-\omega\beta I_2^* V^*+\beta\varepsilon I_1^* S^*+\beta I_2^* S^*+\omega\beta\varepsilon I_1^* V^*\\
&\left.+\omega\beta I_2^* V^*+c_1\alpha k E^*+c_2(1-\alpha)kE^*\right)+u\left(\varepsilon\beta I_1^* S^*\right.\\
&\left.+\varepsilon\omega\beta I_1^* V^*-c_1\alpha k E^*\right)+n\left(\beta I_2^* S^*+\omega\beta I_2^* V^*\right.\\
&\left.-c_1(1-\alpha)kE^*\right)+y\left(\theta V^*-\xi S^*+\omega\beta\varepsilon I_1^* V^*+\omega\beta I_2^* V^*\right)\\
&-\beta I_1^* S^*-\beta I_2^* S^*-\omega\beta\varepsilon I_1^* V^*-\omega\beta I_2^* V^*+c_1\alpha k E^*\\
&+c_2(1-\alpha)kE^*-\frac{1}{x}\beta\varepsilon I_1^* S^*-\frac{1}{x}\beta I_2^* S^*-\frac{y}{x}\theta V^*-\frac{x}{y}\xi S^*\\
&+x\xi S^*+\frac{1}{x}\theta V^*-\frac{xu}{z}\beta\varepsilon I_1^* S^*-\frac{xn}{z}\beta I_2^* S^*-\frac{yu}{z}\omega\beta\varepsilon I_1^* V^*\\
&-\frac{yn}{z}\omega\beta I_2^* V^*-c_1\frac{z}{u}\alpha k E^*-c_2(1-\alpha)\frac{z}{n}kE^*. \qquad (18)
\end{aligned}
$$

We set in equation (18)

$$
c_1=\frac{\varepsilon\beta I_1^*(S^*+\omega V^*)}{\alpha k E^*}\text{ and }c_2=\frac{\beta I_2^*(S^*+\omega V^*)}{(1-\alpha)kE^*}.
$$

Using (16) and replacing c_1 and c_2 by their expressions in (18), we obtain

$$\frac{d\mathscr{L}}{dt} = \mu S^*\left(2-x-\frac{1}{x}\right)+\left(2-x-\frac{1}{x}\right)\left(\theta V^*+\omega\beta\varepsilon I_1^*V^*+\omega\beta I_2^*V^*\right)$$

$$-\frac{x}{y}\left(\theta V^*+\omega\beta\varepsilon I_1^*V^*+\omega\beta I_2^*V^*\right)+x\left(\theta V^*+\omega\beta\varepsilon I_1^*V^*\right.$$

$$\left.+\omega\beta I_2^*V^*\right)+2\omega\beta\varepsilon I_1^*V^*+2\omega\beta I_2^*V^*-\frac{y}{x}\theta V^*+\frac{1}{x}\theta V^*$$

$$-\frac{yu}{z}\omega\beta\varepsilon I_1^*V^*-\frac{yn}{z}\omega\beta I_2^*V^*-\frac{z}{u}\omega\beta\varepsilon I_1^*V^*-\frac{z}{n}\omega\beta I_2^*V^*-\mu V^*y,$$

and with a rearrangement we get

$$\frac{d\mathscr{L}}{dt} = \mu S^*\left(2-x-\frac{1}{x}\right)+\theta V^*\left(2-\frac{y}{x}-\frac{x}{y}\right)+\varepsilon\beta I_1^*S^*\left(3-\frac{1}{x}-\frac{xu}{z}\right.$$

$$\left.-\frac{z}{u}\right)+\beta I_2^*S^*\left(3-\frac{1}{x}-\frac{xn}{z}-\frac{z}{n}\right)+\omega\beta\varepsilon I_1^*V^*\left(4-\frac{1}{x}-\frac{x}{y}\right.$$

$$\left.-\frac{yu}{z}-\frac{z}{u}\right)+\omega\beta I_2^*V^*\left(4-\frac{1}{x}-\frac{x}{y}-\frac{yn}{z}-\frac{z}{n}\right)-\mu V^*y.$$

Since the arithmetic mean exceeds the geometric mean, the following inequalities hold:

$$\left(2-x-\frac{1}{x}\right)\le 0,\ \left(2-\frac{y}{x}-\frac{x}{y}\right)\le 0,\ \left(3-\frac{1}{x}-\frac{xu}{z}-\frac{z}{u}\right)\le 0,$$

$$\left(3-\frac{1}{x}-\frac{xn}{z}-\frac{z}{n}\right)\le 0,\ \left(4-\frac{1}{x}-\frac{x}{y}-\frac{yu}{z}-\frac{z}{u}\right)\le 0 \text{ and}$$

$$\left(4-\frac{1}{x}-\frac{x}{y}-\frac{yn}{z}-\frac{z}{n}\right)\le 0.$$

That is,

$$\frac{d\mathscr{L}}{dt}\le 0.$$

Moreover, we observe that $\frac{d\mathscr{L}}{dt}=0$ if and only if $(S, V, E, I_1, I_2)=(S^*, V^*, E^*, I_1^*, I_2^*)$. Therefore, $\{\mathscr{E}_*\}$ represent the largest compact invariant set in $\{(S, V, E, I_1, I_2)\in\mathbb{R}_+^5 \mid \frac{d\mathscr{L}}{dt}=0\}$. Thus, by LaSalle's invariant principle (see [30]), $\mathscr{E}_*$ is globally asymptotically stable for $\mathscr{R}_\alpha>1$. □

4. Effect of delays in diagnosis and sensitivity analysis results

4.1 *Analytical effect of delays in diagnosis*

When the disease is timely diagnosed, thus the basic reproduction number is given by

$$\mathscr{R}_0 = \frac{\beta \Lambda k \varepsilon (\theta + \mu + \omega\xi)}{\mu (\mu + k)(\mu + \theta + \xi)(\mu + \tau_1 + \delta_1)}. \tag{19}$$

Assume that $\mathscr{R}_0 > 1$ and we want to establish the influence of the delay in the diagnostic by following the idea of Hsu Schmitz (see [31]).

Differentiating $\mathscr{R}_\alpha$ with respect to α, gives us

$$\frac{\partial \mathscr{R}_\alpha}{\partial \alpha} = \frac{-k\beta\Lambda(\theta + \mu + \omega\xi)[\mu(1-\varepsilon) + (\tau_1 - \varepsilon\tau_2) + (\delta_1 - \varepsilon\delta_2)]}{\mu(\mu + k)(\mu + \theta + \xi)(\mu + \tau_1 + \delta_1)(\mu + \tau_2 + \delta_2)}.$$

Since $\tau_1 > \tau_2$, thus $\frac{\partial \mathscr{R}_\alpha}{\partial \alpha} < 0$ if $\varepsilon < \varepsilon^* = \frac{\delta_1 + \tau_1 + \mu}{\delta_2 + \tau_2 + \mu}$.

This indicates that improving the proportion of timely diagnosis of the disease can decrease the delay-induced reproduction number if $\varepsilon < \varepsilon^*$.

Setting $\mathscr{R}_\alpha = 1$ and solving for α gives us the threshold of timely diagnosis by

$$\alpha_{min} = \frac{(\mu + \tau_1 + \delta_1)\left[\beta\Lambda k(\theta + \mu + \omega\xi) - \mu(\mu + k)(\mu + \tau_2 + \delta_2)\right]}{\beta\Lambda k(\mu + \theta + \omega\xi)[\mu + \tau_1 + \delta_1 - \varepsilon(\mu + \tau_2 + \delta_2)]}.$$

If the proportion of timely diagnosis is larger than α_{min}, then the final size of infected components will decrease.

4.2 *Sensitivity analysis results*

Here, we provide sensitivity analysis of model (3). This is done to identify the most influential parameters of model. In this subsection, the values of the parameters chosen for the numerical simulations are presented in Table 1.

Table 1. Parameter values

Symbol	Value (per year)	Ref.
Λ	5	[32]
ω	0.90	[33]
θ	0.067	[34]
ξ	0.95	[35]
β	0.2	Assumed
μ	0.15	[36]
α	0.5	Assumed
δ_1	0.5	Assumed
δ_2	0.2	Assumed
τ_1	0.5	Assumed
τ_2	0.3	Assumed
k	0.5	Assumed
ε	0.3	Assumed

The scatter plots of each parameter against the delay-induced reproduction number. Figure 2 shows a scatter plot of the sensitivity analysis for each parameter. The plots describe qualitatively the influence of the corresponding parameters on $\mathscr{R}_\alpha$. We observe that the delay-induced reproduction number increase significantly with the increase of the rate of infection reduction due to timely diagnosis ε. It also increase with the rate of infection of latent individuals k and the contact rate β.

It decreases slightly with the increase of the other parameters.

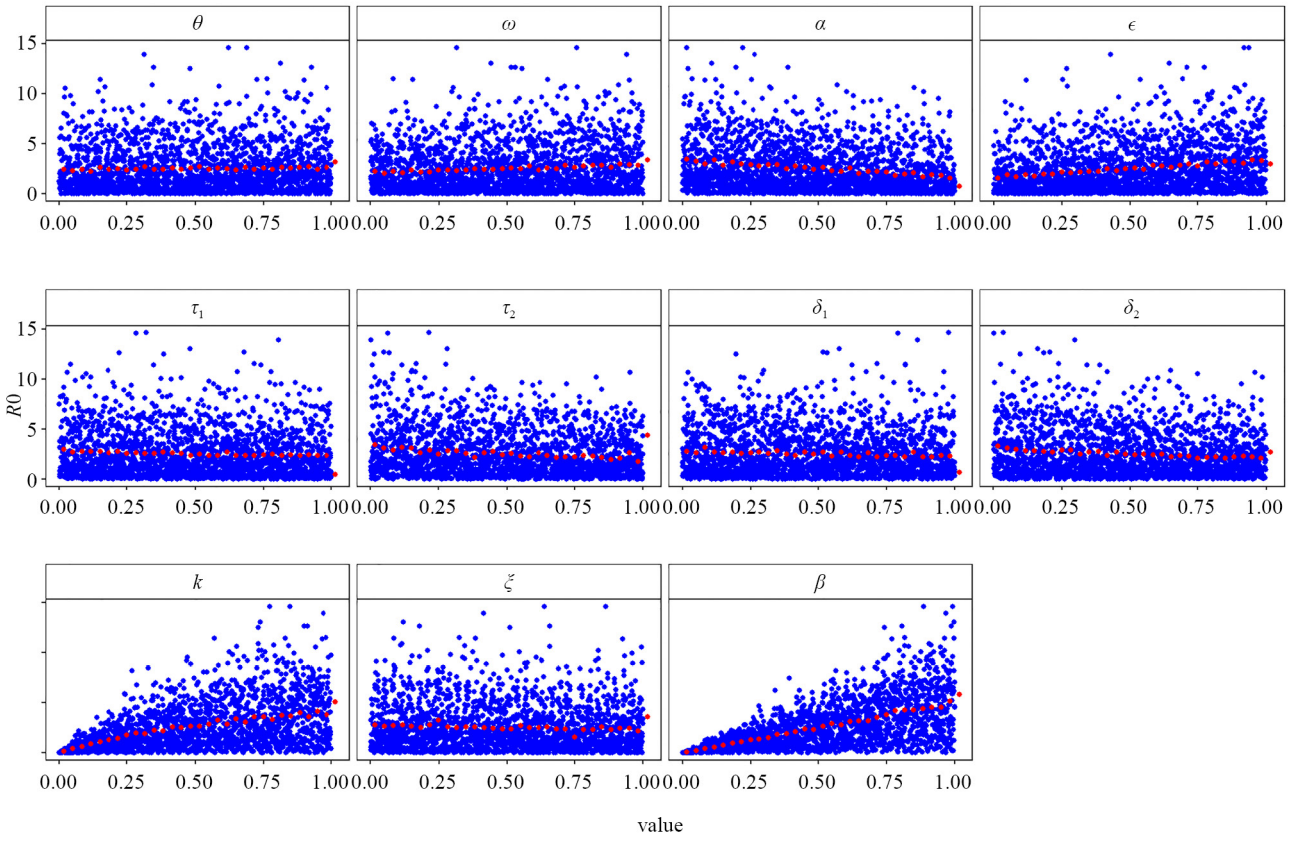

Figure 2. Scatter plots sensitivity analysis of parameters

Figure 3 is a representation of the partial correlation coefficients for each parameter of model (3). The PCC is provided to illustrate the correlation between the parameters and the delay-induced reproduction number. A positive value indicates a positive correlation, and a negative value indicates a negative correlation.

We observe that the parameters ε, k and β are positively correlated and the others negatively correlated.

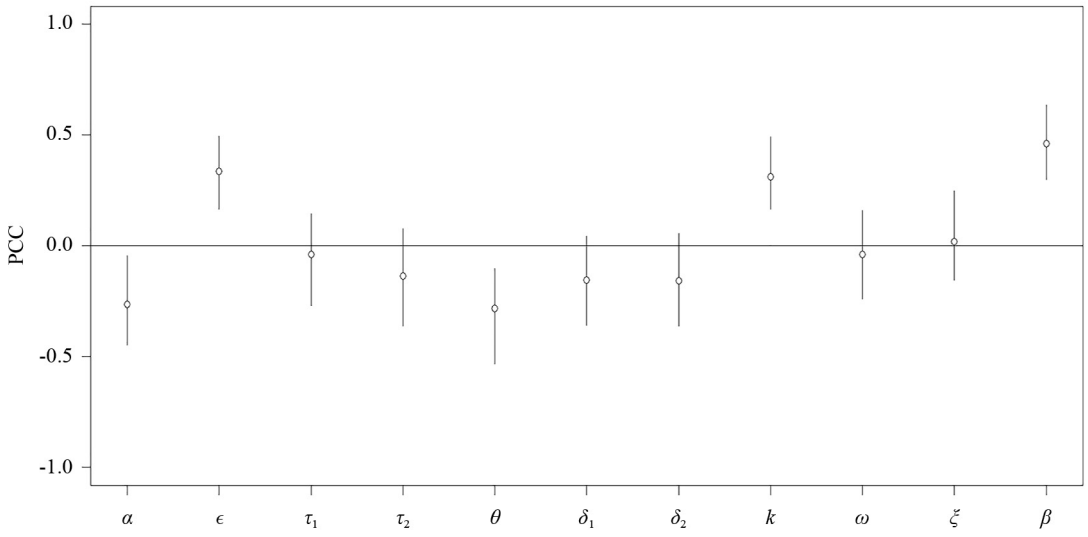

Figure 3. Partial correlation coefficient plot of model parameters

5. Changing environment: Existence of bifurcations

We establish how the speed of environmental change can regulate the dynamics of model (1). We use the bifurcation software Matcont to plot one-parameter bifurcation diagram in the planes $\alpha - I_1$ and $\alpha - I_2$ for model (1). Clearly, in model (1), $\alpha(t)$ increases continuously over time when $r > 0$, and continuously decreases when $r < 0$.

The delay-induced reproduction number $\mathscr{R}_\alpha = 1$ is a threshold in the sens that disease is persistent if $\mathscr{R}_\alpha > 1$ and extinct if $\mathscr{R}_\alpha < 1$ in model (1). In Figure 4, we choose $\Lambda = 2$, $\mu = 0.2$, $\varepsilon = 0.3$, $\tau_1 = 0.5$, $\tau_2 = 0.3$, $\theta = 0.3$, $\delta_1 = 0.5$, $\delta_2 = 0.2$, $k = 0.5$, $\omega = 0.03$, $\xi = 0.03$, $\beta = 0.2$. We obtain a bifurcation at point $\alpha_{BP} = 0.58$. This provide that if $0 < \alpha < \alpha_{BP}$, then system (1) has a stable disease-free equilibrium point $\mathscr{E}_0$. When $\alpha = \alpha_{BP}$ the system (1) undergoes a transcritical bifurcation and the disease-free equilibrium $\mathscr{E}_0$ becomes unstable and a stable endemic equilibrium occurs if $\alpha_{BP} < \alpha$.

In Figure 5, we choose $\Lambda = 2$, $\mu = 0.2$, $\varepsilon = 0.3$, $\tau_1 = 0.5$, $\tau_2 = 0.3$, $\theta = 0.3$, $\delta_1 = 0.5$, $\delta_2 = 0.2$, $k = 0.5$, $\omega = 0.3$, $\xi = 0.3$, $\beta = 0.2$. We obtain a bifurcation at $\alpha_{BP} = 0.41$. This provide that if $0 < \alpha < \alpha_{BP}$, then system (1) has a stable disease-free equilibrium point $\mathscr{E}_0$. When $\alpha = \alpha_{BP}$ the system (1) undergoes a transcritical bifurcation and the disease-free equilibrium $\mathscr{E}_0$ becomes unstable, and a stable endemic equilibrium occurs if $\alpha_{BP} < \alpha$.

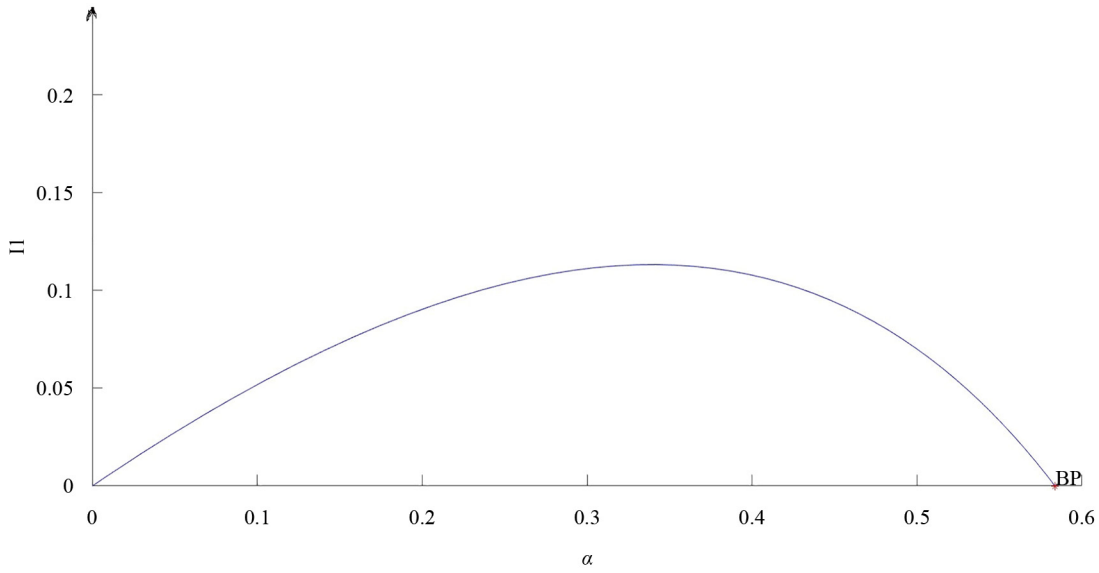

Figure 4. Bifurcation diagram of model (1) in the plane $\alpha - I_1$ obtained using matcont. For $\alpha_{BP} = 0.58$, there is a *BP* branching point or bifurcation indicating a change in the stability of the equilibrium points

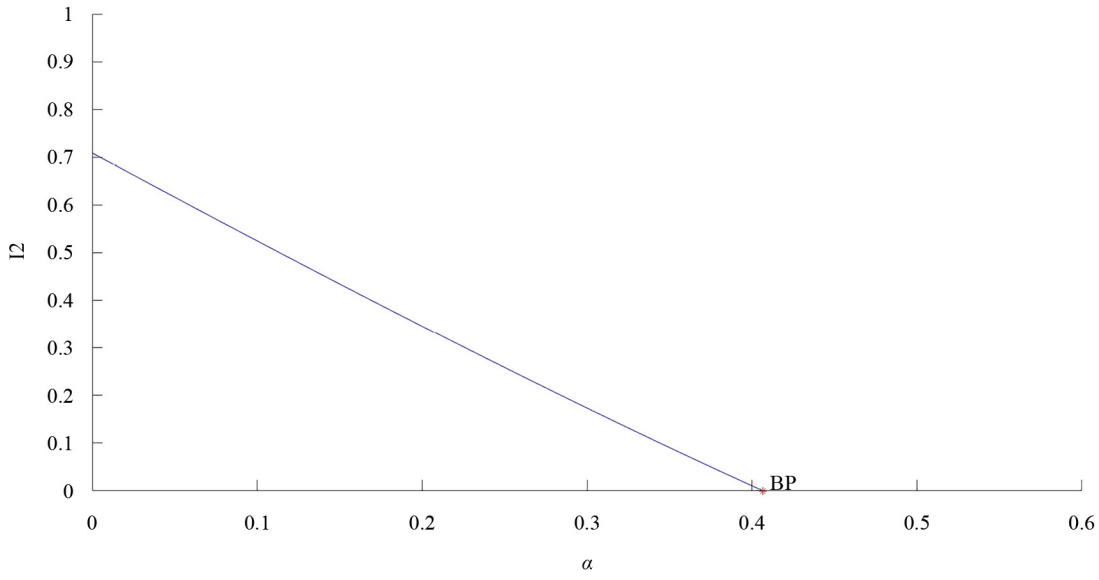

Figure 5. Bifurcation diagram of model (1) in the plane $\alpha - I_2$ obtained using matcont. For $\alpha_{BP} = 0.41$, there is a *BP* branching point or bifurcation indicating a change in the stability of the equilibrium points

6. Conclusion

In this in-depth study, we explored the dynamics of an SVEIR model to understand the impact of diagnostic delays on the spread of an epidemic in constant and variable environments. Our results highlight several key aspects of infectious disease behavior and control under different diagnostic and environmental conditions.

Firstly, in a constant environment, we established the conditions for stable disease-free equilibrium and endemic equilibrium. We have shown that the global behavior of the epidemic is intrinsically linked to the size of the reproduction number induced by the delay $\mathscr{R}_\alpha$. If $\mathscr{R}_\alpha < 1$, the disease-free equilibrium is globally asymptotically stable and the disease tends to disappear from the population, and if $\mathscr{R}_\alpha > 1$, the endemic equilibrium is globally asymptotically stable and the disease persists in the population. The minimum diagnostic effort noted α_{min} required to control the disease is calculated and is equal to

$$\alpha_{min} = \frac{(\mu+\tau_1+\delta_1)\left[\beta\Lambda k(\theta+\mu+\omega\xi)-\mu(\mu+k)(\mu+\tau_2+\delta_2)\right]}{\beta\Lambda k(\mu+\theta+\omega\xi)\left[\mu+\tau_1+\delta_1-\varepsilon(\mu+\tau_2+\delta_2)\right]}.$$

This threshold is an important measure for public health interventions, as it provides a clear target for the level of diagnostic effort required to effectively manage and eradicate the disease.

The sensitivity analysis revealed that the parameters with the most significant impact on the model are the reduction rate of infection due to timely diagnosis ε, the infection rate of latent individuals k and the contact rate β. Knowledge of the most influential parameters is essential for prioritizing resources and efforts in the epidemic control, as it allows focus on the factors that have the greatest influence on the spread of the disease.

In a variable environment, the existence of bifurcations is carried out in the planes $\alpha - I_1$ and $\alpha - I_2$ with α as the bifurcation parameter. It is shown that the model undergoes a *BP* transcritical bifurcation for values $\alpha = 0.58$ and $\alpha = 0.41$ in the planes $\alpha - I_1$ and $\alpha - I_2$ respectively. Identifying these bifurcation points is important because it indicates the critical thresholds at which the system undergoes qualitative changes in its behavior, moving from one epidemic regime to another.

This work provides a detailed mathematical framework for understanding the effects of diagnostic delays on epidemic dynamics. The conclusions drawn from this study have important implications for public health policies, particularly in optimizing diagnostic efforts and resource allocation to control and prevent the spread of infectious diseases.

Future research could build on these findings by incorporating more complex environmental factors and exploring the impact of other types of delays and interventions in epidemic modeling.

Acknowledgement

The authors express their deepest thanks to two anonymous referees for their valuables comments which contributed to improve the paper.

Conflict of interest

The authors declare no competing financial interest.

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