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Research Article

The Spectra of Multiplication Operators and Weighted Composition Operators on Iterated Weighted-Type Banach Spaces of Analytic Functions

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Abstract: This research aims to analyze the spectral of multiplication operators acting on weighted Banach spaces of analytic functions defined on the unit disk. These spaces, denoted by $S_n : n \in \mathbb{N}$, include well-known cases such as the Bloch space and the Zygmund space for $n = 1$ and $n = 2$, respectively. Additionally, we offer a description of the spectra of weighted composition operators within these spaces. The outline of this paper provides a structured framework for organizing the research, starting from the introduction to the conclusion and references. The main section is to investigate the spectrum of multiplication operators on S_n spaces, and it followed by a section that is designed to build upon the previous one, leading to characterize the spectrum of weighted composition operators on the same spaces.

Keywords: spectra, isometry, multiplication operators, weighted composition operators, iterated banach spaces of analytic functions

MSC: 47B38

1. Introduction

Let D represents the open unit disk in the complex plane, $H(D)$ the set of analytic functions on D, and $F(D)$ the set of analytic selfmaps of D. The nth weighted Banach spaces $\{S_n : n \in \mathbb{N}\}$ consists of all $f \in H(D)$ such that

$$\sup_{z\in\mathbb{D}} (1-|z|^2)|f^{(n)}(z)| < \infty,$$

where the norm of f in S_n is defined as follows

$$\|f\|_{S_n} = \sum_{j=0}^{n-1} |f^{(j)}(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f^{(n)}(z)|.$$

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In particular, S_1 is the Bloch space B_μ with $\mu = (1-|z|^2)$, which is the Banach space of analytic functions f on D such that $f' \in H_\mu^\infty$. Thus, $f \in B_\mu$ if and only if $f' \in H_\mu^\infty$ with norm

$$\|f\|_{B_\mu} = |f(0)| + \sup_{z\in} \mu(z)|f'(z)|.$$

And S_2 is the Zygmund space Z_μ with $\mu = (1-|z|^2)$, which is the Banach space of functions $f \in H(D)$ satisfying the condition $f' \in B_\mu$. The norm in Z_μ is defined by

$$\|f\|_{Z_\mu} = |f(0)| + |f'(0)| + \sup_{z\in D} \mu(z)|f''(z)|.$$

Recently, the iterated spaces S_n have been studied in several resources such as [1] and [2]. In particular, the authors of [1] show some interesting properties of such spaces. Indeed, (S_n) is a nested sequence which is contained in the disk algebra $H^\infty(D) \cap C(\overline{D})$ for all $n \geq 2$. Moreover, for $n \in \mathbb{N}$ and $f \in S_n$,

$$||f||_{S_{n-1}} \leq ||f||_{S_n}.$$

Also, for each $n \geq 3$, they find that S_n is an algebra.

The nth weighted Banach spaces find significant utility in approximation theory and numerical analysis [3]. These spaces play a crucial role in assessing the precision of various numerical techniques aimed at approximating functions possessing nth order derivatives, such as finite difference and finite element methods [4]. Additionally, they facilitate the establishment of convergence rates for diverse approximation strategies and aid in deriving error margins for numerical solutions of differential equations [5]. Beyond these, they serve in signal processing, enabling the analysis and manipulation of signals represented as analytic functions [6]. Moreover, they contribute to machine learning by providing a framework for modeling intricate data structures and generating predictions based on them. Further elaboration on these applications can be found in references [7].

For $\alpha \in H(D)$, the multiplication operator on S_n is the linear operator given by

$$M_\alpha f = \alpha f, \text{ for all } f \in S_n.$$

For $\beta \in F(D)$, the composition operator on S_n is the linear operator defined by

$$C_\beta f = f(\beta(z)), \text{ for all } f \in S_n,$$

the weighted composition operator is the linear operator defined as follows

$$W_{\alpha,\,\beta} f = (M_\alpha C_\beta) f = \alpha f(\beta(z)), \text{ for all } f \in S_n.$$

Given two normed vector spaces X and Y, a linear isometry is a linear map $T: X \to Y$ which preserves the norms,

$$||Tx||_Y = ||x||_X$$

for all $x \in X$.

Recall that the resolvent of a bounded linear operator T on a complex Banach space E is defined as

$$res(T) := \{r \in \mathbb{C} : T - rI \text{ is invertible}\},$$

where I is the identity operator. The spectrum of T is defined as

$$spec(T) := \mathbb{C} \setminus res(T).$$

It follows from the Neumann series expansion that the spectrum is a non-empty compact subset of the closed disk centered at the origin of radius $\|T\|$. In particular, the spectrum of an isometry is contained in $\overline{D}$ (see [8]). In fact, the spectrum of an operator on Banach spaces has various applications in several scientific fields. For instance, in quantum mechanics, the spectrum of an operator corresponds to the possible values of a physical observable. Also, in engineering, operators and spectra are often used in areas like signal processing, control systems, and optimization. More details can be found in [9] and [10].

2. The spectra of the multiplication operators on S_n

In [11], the authors characterize the spectrum of the multiplication operators on S_1 which is the Bloch space. In this section, we show a similar result for all spaces S_n.

The following findings are needed to prove the primary result in this section.

Proposition 2.1 ([1], Proposition 3.1) The multiplication operator M_α on S_n is bounded if and only if $\alpha \in S_n$, for all $n \geq 3$.

Similar results are proven for $n = 1, 2$. In particular, M_α on S_n is bounded if and only if $\alpha \in S_n \cap B_{n\log}$, for $n = 1, 2$, where $B_{\log}$ is the logarithmic Bloch space, see ([12], Corollary 2.5) and ([13], Theorem 3.1).

Theorem 2.2 ([1], Theorem 5.2) For $n \in \mathbb{N}$, if the weighted composition operator $W_{\alpha,\,\beta}$ on S_n is bounded, then it is invertible if and only if α is bounded away from zero and β is automorphism of . The inverse is also a weighted composition operator on S_n such that

$$W_{\alpha,\,\beta}^{-1} = W_{\frac{1}{\alpha\beta^{-1}},\,\beta^{-1}}.$$

In particular, the composition operator C_β is invertible on S_n if and only if β is an automorphism on , and the multiplication operator M_α on S_n is invertible if and only if α is bounded away from zero such that

$$C_\beta^{-1} = C_{\beta^{-1}},$$

and

$$M_\alpha^{-1} = M_{1/\alpha}.$$

Theorem 2.3 ([14], Theorem 2.4) For $n \in \mathbb{N}$, the surjective bounded multiplication operator M_α acting on S_n is isometric if and only if α is a constant of modulus 1.

The following result is to extend Theorem 4.1 in [11].

Theorem 2.4 Let M_α be a bounded multiplication operator on S_n, $n \in \mathbb{N}$. Then

$$spec(M_\alpha) = \overline{\alpha(D)},$$

where $\alpha\,(D)$ is the set of values of α.

Proof. Clear that the operator $M_{\alpha-r}$ is equivalent to $M_\alpha - rI$ for any constant r. Thus $r \in spec(M_\alpha)$ if and only if $M_{\alpha-r}$ is not invertible.

Suppose that $M_{\alpha-r}$ is invertible, then by Theorem 2.2

$$(M_{\alpha-r})^{-1} = M_{\frac{1}{\alpha-r}}.$$

Let $r \in \alpha(\mathbb{D})$, then $\alpha(r_0) = r$ for some $r_0 \in \mathbb{D}$. It means that $(M_{\alpha-r})^{-1}$ is not well defined since $\dfrac{1}{\alpha-r}$ has a pole at r_0. Hence, $\overline{\alpha(D)} \subset spec(M_\alpha)$, since $\partial(\overline{spec(M_\alpha)}) \subset spec(M_\alpha)$ ([13], Proposition 6.7).

On the other hand, suppose that $r \notin \overline{\alpha(D)}$, then $\alpha(z) - r \neq 0$ for all $z \in D$. Thus, $|\alpha - r| > c$ for some constant $c > 0$.

Let $G(z) = \dfrac{1}{\alpha(z)-r}$, then $G(z) \in H^\infty(D)$. For $z \neq r$, let $H(z) = \dfrac{1}{z-r}$, then by applying Faà di Bruno's formula on $(H \circ \alpha)$, we get

$$G^{(n)}(z) = (H \circ \alpha)^{(n)}(z) = \left(\frac{1}{\alpha(z)-r}\right)^{(n)} = \sum_{\lambda \vdash n} \frac{n!}{\lambda_1!(1!)^{\lambda_1} \ldots \lambda_n!(n!)^{\lambda_n}} \left(\frac{(-1)^{|\lambda|}|\lambda|!}{(\alpha(z)-r)^{|\lambda|+1}}\right) \prod_{j=1}^{n} \left(\alpha^{(n)}(z)\right)^{\lambda_j},$$

where $\lambda \vdash n$ satisfies $\sum_{j=1}^{n} j\lambda_j = n$, and $|\lambda| = \sum_{j=1}^{n} \lambda_j$.

Since M_α is bounded on S_n, then by Proposition 2.1, $\alpha \in S_n$, which means that $|\alpha^{(n)}(z)| < \infty$. Thus,

$$|G^{(n)}(z)| \le \sum_{\lambda \vdash n} \frac{n!}{\lambda_1!(1!)^{\lambda_1} \ldots \lambda_n!(n!)^{\lambda_n}} \left(\frac{|\lambda|!}{|\alpha(z)-r|^{|\lambda|+1}}\right) \prod_{j=1}^{n} \left|\alpha^{(n)}(z)\right|^{\lambda_j}$$

$$\le \sum_{\lambda \vdash n} \frac{n!}{\lambda_1!(1!)^{\lambda_1} \ldots \lambda_n!(n!)^{\lambda_n}} \left(\frac{|\lambda|!}{c^{(|\lambda|+1)}}\right) \prod_{j=1}^{n} \left|\psi^{(n)}(z)\right|^{\lambda_j} < \infty.$$

That means thar G belongs to S_n, therefore, $M_G = M_{\frac{1}{\alpha-r}}$ is bounded on S_n. Thus, $r \notin spec(M_\alpha)$. Hence

$$spec(M_\alpha) = \overline{\alpha(D)}.$$

□

The next is an immediate consequence of Theorem 2.3 and Theorem 2.4.

Corollary 2.5 Let M_α be a surjective isometric multiplication operator on S_n, $n \in \mathbb{N}$. Then

$$spec(M_\alpha) = \{k\},$$

where k is the unimodular constant value of α.

3. The spectrum of weighted composition operators on S_n

The authors of [11] characterize the spectrum of weighted composition operators on S_1. We provide a more specific description of the spectrum of weighted composition operators on S_n, for $n \geq 2$.

We state the following findings that are needed for the main result of this section.

Proposition 3.1 ([11], Proposition 5.1) Let A be a complex Banach space and suppose $T : A \to A$ is an isometry. If T is invertible, then $spec(T) \subset \partial D$. If T is not invertible, then $spec(T) = \overline{D}$.

Let $w \in \mathbb{C}$ with $|w| = 1$. The order of w denoted by $ord(w)$ is defined to be the smallest $m \in \mathbb{N}$ such that $w^m = 1$. We say that w has infinite order if there is no such m, and then we write $ord(w) = \infty$.

The following lemma is a modified generlization of Theorem 5.1 in [11].

Lemma 3.2 Let C_β be a composition operator on S_n for $n \in \mathbb{N}$. If $\beta(z) = wz$ with $|w| = 1$, then

$$spec(C_\beta) = \begin{cases} \partial D, \text{ if } ord(w) = \infty, \\ \langle w \rangle, \text{ if } ord(w) = m < \infty, \end{cases}$$

where $\langle w \rangle$ is the cyclic group generated by w.

Proof. Note that C_β is invertible since $C_\beta^{-1} = C_{\beta^{-1}} = C_{\frac{1}{w}}$. By proposition 3.1, $spec(C_\beta) \subset \partial D$.

Since $|w| = 1$, then $\langle w \rangle = \{w^j : j \in \mathbb{N} \cup \{0\}\} \subset \partial D$. Let $f(z) = z^j$ for $j \in \mathbb{N} \cup \{0\}$, then $f \in S_n$ for all $n \in \mathbb{N}$. Thus

$$(C_\beta - w^j I)(f(z)) = f(\beta(z)) - w^j f(z) = w^j z^j - w^j z^j = 0,$$

which means that $(C_\beta - w^j I)$ is not invertible since it is not injective. Thus $w^j \in spec(C_\beta)$. So

$$\langle w \rangle \subset spec(C_\beta) \subset \partial D$$

Since $spec(C_\beta)$ is closed, then $\overline{\langle w \rangle} \subset spec(C_\beta)$.

If $ord(w) = \infty$, then $\overline{\langle w \rangle} = \partial$ since $\langle w \rangle$ is dense in ∂D. Hence $spec(C_\beta) = \partial$.

If $ord(w) = m < \infty$, then $\langle w \rangle = \{w^j : j = 1, \ldots, m\}$. Let $a \in \partial \setminus \langle w \rangle$, then by the same argument of the proof of Theorem 5.1 in [9], we see that $(C_\beta - aI)$ is invertible. Thus $a \notin spec(C_\beta)$. Hence $spec(C_\beta) = \langle w \rangle$. □

Theorem 3.3 Let $W_{\alpha,\, \beta}$ be a bounded weighted composition operator on S_n, $n \in \mathbb{N}$. If $\beta(z) = wz$ with $|w| = 1$ and β is a constant of modulus 1. Then

$$spec(W_{\alpha,\,\beta}) = \begin{cases} \partial D, \text{ if } ord(w) = \infty, \\ \overline{k}\langle w\rangle, \text{ if } ord(w) = m < \infty, \end{cases}$$

where k is the unimodular constant value of α, and $k\langle w\rangle = \{kw^j : j = 1, \ldots, m\}$.

Proof. Note that $W_{\alpha,\,\beta} = kC_\beta$. For $r \in \mathbb{C}$, since $|k| = 1$, then $W_{\alpha,\,\beta} - rI = kC_\beta - rI = k(C_\beta - \overline{k}rI)$. Thus $W_{\alpha,\,\beta} - rI$ is not invertible if and only if $C_\beta - \overline{k}rI$ is not invertible. Thus, $r \in spec(W_{\alpha,\,\beta})$ if and only if $\overline{k}r \in spec(C_\beta)$. Hence, the desired result follows from the previous lemma. □

4. Coclusion

In conclusion, this study delves into the spectral properties of multiplication operators on weighted Banach spaces of analytic functions over the unit disk. Through our investigation, we provide comprehensive insights into the spectra of weighted composition operators within these designated spaces. By shedding light on the spectral behavior of these operators, our research contributes to a deeper understanding of the structural nuances within function spaces, thereby facilitating advancements in various fields reliant on analytic function theory.

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