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Research Article

Some Identities for the (*a*, *b*; *k*)-Nacci Sequences

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Abstract: In this paper, we introduce a generalization of the k-generalized Fibonacci sequence, called the (a, b; k)-*nacci sequence*, where a and b are real numbers and $k \geq 2$ is an integer. The (a, b; k)-nacci sequence $\{T_n(a,\ b)\}_{n=0}^{\infty}$ is defined recursively as follows:

$$T_n(a,\ b) = \begin{cases} a, & \text{if } 0 \leq n < k-1; \\ b, & \text{if } n = k-1; \\ T_{n-1}(a,\ b) + T_{n-2}(a,\ b) + \cdots + T_{n-k}(a,\ b), & \text{if } n \geq k. \end{cases}$$

We also provide some identities involving the sum of the (a, b; k)-nacci terms and investigate the sums of the squares of the (a, b; k)-nacci numbers.

Keywords: Fibonacci numbers, k-generalized Fibonacci sequence, recurrence relations

MSC: 11B37, 11B39, 11B50

1. Introduction

The Fibonacci sequence, a well-known sequence of integers in mathematics, has intriguing properties and applications that extend far beyond mathematics, influencing various scientific and artistic fields (e.g., [1-3]). It is defined recursively as follows:

$$F_0 = 0,\ F_1 = 1,\ \text{and } F_n = F_{n-1} + F_{n-2}\ \text{ for } n \geq 2.$$

We denote the nth Fibonacci number as F_n. The first few terms are 0, 1, 1, 2, 3, 5, 8, 13, ..., which can be found in Sloane [4] as sequences A000045.

Another intriguing sequence of integers, closely linked to the Fibonacci sequence, is the Lucas sequence. It is defined recursively as follows:

$$L_0 = 2,\ L_1 = 1,\ \text{and } L_n = L_{n-1} + L_{n-2}\ \text{ for } n \geq 2.$$

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We call L_n the nth Lucas number. The first few terms are 2, 1, 3, 4, 7, 11, 18, 29, …, which can be found in Sloane [4] as sequences A000032.

The Fibonacci and Lucas sequences have inspired a rich variety of generalizations, including Pell numbers, generalized Fibonacci numbers, generalized Lucas numbers, generalized Tribonacci sequence, and k-generalized Fibonacci sequences [5-15, 17].

Bueno [7] explored the generalized Tribonacci sequence, denoted by $\{S_n\}_{n=0}^{\infty}$, defined as $S_n = S_{n-1} + S_{n-2} + S_{n-3}$, for positive integer $n \geq 3$. Unlike the standard Fibonacci sequence, the initial values S_0, S_1 and S_2 are any arbitrary numbers, with not all zero. Bueno utilized limits to establish some properties of this sequence.

Howard and Cooper [16] investigated the k-generalized Fibonacci sequence. For positive integer k, the *k-generalized Fibonacci sequence*, say $\{G_n\}_{n=0}^{\infty}$, is defined as

$$G_n = \begin{cases} 0, & \text{if } 0 \leq n < k-1; \\ 1, & \text{if } n = k-1; \\ G_{n-1} + G_{n-2} + \cdots + G_{n-k}, & \text{if } n \geq k. \end{cases}$$

They established several identities, including a formula for the sum of the squares of the G_n terms and congruences involving these numbers. This k-generalized Fibonacci sequence serves as a prominent example among various Fibonacci number generalizations. It is also known as the *k-step Fibonacci sequence*, the *Fibonacci k-sequence*, or simply the *k-nacci sequence*. Notably, when k = 2, we recover the standard Fibonacci sequence; k = 3 gives the Tribonacci sequence; and k = 4 yields the Tetranacci sequence, and so on.

This paper introduces a novel generalization of the *k-generalized Fibonacci sequence*, known as the (a, b; k)-*nacci sequence*. Unlike the *k-generalized Fibonacci sequence*, the (a, b; k)-*nacci sequence* incorporates additional parameters for increased flexibility. We define the (a, b; k)-*nacci sequence* recursively. The first k terms are specified by the parameters a and b, and subsequent terms are obtained by summing the preceding k terms. Additionally, we explore some identities and derive a formula for the sums of the squares within the (a, b; k)-*nacci sequence*.

2. Main results

This section introduces a generalization of the k-generalized Fibonacci sequence. We extend the definition by incorporating additional parameters, allowing for increased flexibility. We then delve into some of the properties of this generalized sequence.

Definition 2.1 For any real numbers a, b and integer $k \geq 2$ the (a, b; k)-*nacci sequence* $\{T_n(a, b)\}_{n=0}^{\infty}$ is defined by

$$T_n(a, b) = \begin{cases} a, & \text{if } 0 \leq n < k-1; \\ b, & \text{if } n = k-1; \\ T_{n-1}(a, b) + T_{n-2}(a, b) + \cdots + T_{n-k}(a, b), & \text{if } n \geq k, \end{cases}$$

We call $T_n(a, b)$ the (a, b; k)-nacci number and shorten it as $T_n = T_n(a, b)$.

Note that if $a = 0$ and $b = 1$, then the (a, b; k)-nacci sequence is the k-generalized Fibonacci sequence. In particular, the (0, 1; 2)-nacci sequence is the Fibonacci sequence, and the (2, 1; 2)-nacci sequence is the Lucas sequence.

Throughout this paper, let a, b be real numbers and let $k \geq 2$ be an integer. Now we present some identities for the (a, b; k)-nacci sequence $\{T_n\}_{n=0}^{\infty}$.

Theorem 2.2 Let $\{T_n\}_{n=0}^{\infty}$ be the (a, b; k)-nacci sequence. Then

(i) $T_n = 2T_{n-1} - T_{n-k-1}$, for all integer $n \geq k+1$,

(ii) $T_n = 2^{k-1}T_{n-k+1} - \sum_{m=1}^{k-1} 2^{m-1}T_{n-k-m}$, for all integer $n \geq 2k-1$.

Proof.

(i) Let $n \geq k + 1$ be an integer. Then $n-1 \geq k$ and by Definition 2.1 we have

$$T_{n-1} = T_{n-2} + \cdots + T_{n-k} + T_{n-k-1} \tag{1}$$

$$T_n = T_{n-1} + T_{n-2} + \cdots + T_{n-k} \tag{2}$$

Subtracting (1) from (2) we obtain $T_n - T_{n-1} = T_{n-1} - T_{n-k-1}$, and this implies that $T_n = 2T_{n-1} - T_{n-k-1}$.
(ii) Let $n \geq 2k-1$ be an integer. Using (i), we have

$$T_n = 2T_{n-1} - T_{n-k-1},$$

$$2T_{n-1} = 2^2 T_{n-2} - 2T_{n-k-2},$$

$$2^2 T_{n-2} = 2^3 T_{n-3} - 2^2 T_{n-k-3},$$

$$\vdots$$

$$2^{k-2} T_{n-k+2} = 2^{k-1} T_{n-k+1} - 2^{k-2} T_{n-k-(k-1)}.$$

Adding up these equations term by term, we have $T_n = 2^{k-1} T_{n-k+1} - \sum_{m=1}^{k-1} 2^{m-1} T_{n-k-m}$, which completes the proof.
The following corollary follow immediately.
Corollary 2.3 Let $\{T_n\}_{n=0}^{\infty}$ be the $(a, b; k)$-nacci sequence. Then, for all integer $n \geq 2k-1$,

$$T_n = 2^{k-1} T_{n-k} + \sum_{m=1}^{k-1} \left(2^{k-1} - 2^{m-1}\right) T_{n-k-m}.$$

Theorem 2.4 Let $\{T_n\}_{n=0}^{\infty}$ be the $(a, b; k)$-nacci sequence. Then, for $n = 0, 1, \ldots, k-1$,

$$T_{k+n} = \left(2^n (k-2) + 1\right) a + 2^n b.$$

Proof. We proceed by induction on n. Clearly, the result is true for $n = 0$. Now, assume the result is true for some integer r with $0 \leq r \leq k-2$. Using Theorem 2.2 (i), induction hypothesis, and Definition 2.1, then

$$\begin{aligned} T_{k+r+1} &= 2T_{k+r} - T_r \\ &= 2\left(\left(2^r (k-2) + 1\right) a + 2^r b\right) - a \\ &= \left(2^{r+1} (k-2) + 1\right) a + 2^{r+1} b \end{aligned}$$

Therefore, the result holds for $n = r + 1$ and the proof is complete.
Corollary 2.5 Let $\{T_n\}_{n=0}^{\infty}$ be the $(a, b; k)$-nacci sequence. Then,

$$T_{2k} = \left(2^k (k-2) + 2\right) a + \left(2^k - 1\right) b.$$

Proof. By Theorem 2.4, we have $T_{k+n} = \left(2^n(k-2)+1\right)a + 2^n b$. Then

$$\sum_{n=0}^{k-1} T_{k+n} = \sum_{n=0}^{k-1} \left\{\left(2^n(k-2)+1\right)a + 2^n b\right\}.$$

From the equality $\sum_{n=0}^{k-1} 2^n = 2^k - 1$ and using Definition 2.1, we get the result.

Theorem 2.6 Let $\{T_n\}_{n=0}^{\infty}$ be the $(a, b; k)$-nacci sequence. Then, for $n = 0, 1, \ldots, k-1$,

$$T_{2k+n} = \left(2^{n-1}(k-2)\left(2^{k+1}-n\right)+2^n+1\right)a + \left(2^{n+k} - 2^{n-1}(n+2)\right)b.$$

Proof. The proof is by induction on n. By Corollary 2.5, we have $T_{2k} = \left(2^k(k-2)+2\right)a + \left(2^k - 1\right)b$. Hence, the result is true for $n = 0$ Now, assume the result is true for some integer r with $0 \le r \le k-2$. Using Theorem 2.2 (i), induction hypothesis, and Theorem 2.4, then

$$\begin{aligned} T_{2k+(r+1)} &= 2T_{2k+r} - T_{k+r} \\ &= 2\left(\left(2^{r-1}(k-2)\left(2^{k+1}-r\right)+2^r+1\right)a + \left(2^{r+k} - 2^{r-1}(r+2)\right)b\right) - \left(2^r(k-2)+1\right)a - 2^r b \\ &= \left(2^r(k-2)\left(2^{k+1}-(r+1)\right)+2^{r+1}+1\right)a + \left(2^{(r+1)+k} - 2^r((r+1)+2)\right)b. \end{aligned}$$

Therefore, the result holds for $n = r + 1$, and the proof is complete.

Corollary 2.7 Let $\{T_n\}_{n=0}^{\infty}$ be the $(a, b; k)$-nacci sequence. Then,

$$T_{3k} = \left(2^{k-1}\left(2^{k+1}-k\right)(k-2)+2^k+1\right)a + 2^{k-1}\left(2^{k+1}-k-2\right)b.$$

Proof. The proof is straightforward by using Theorem 2.2 (i), Theorem 2.6 and Theorem 2.4.

Theorem 2.8 Let $\{T_n\}_{n=0}^{\infty}$ be the $(a, b; k)$-nacci sequence and $n \ge k-1$ be an integer.

(i) If $k = 2$ then $\sum_{j=0}^{n} T_j^2 = a^2 - ab + T_n T_{n+1}$.

(ii) If $k \ge 3$ then $\sum_{j=0}^{n} T_j^2 = \dfrac{k^2-3k+4}{2}a^2 - ab + T_n T_{n+1} - \sum_{j=2}^{k-1}\sum_{i=0}^{n-j} T_i T_{i+j}$.

Proof.

(i) Let $k = 2$. For $j = 1, 2, \ldots, n$, we have $T_j = T_{j+1} - T_{j-1}$, and multiplying both sides by T_j, we get $T_j^2 = T_j T_{j+1} - T_j T_{j-1}$. Thus, $\sum_{j=1}^{n} T_j^2 = T_n T_{n+1} - ab$.

Then, $\sum_{j=0}^{n} T_j^2 = T_0^2 + T_n T_{n+1} - ab$

(ii) Let $k = 3$ For $j = n, n-1, \ldots, k-1$, we have $T_{j+1} = T_j + T_{j-1} + \cdots + T_{j-k+1}$, then $T_j = T_{j+1} - \sum_{i=1}^{k-1} T_{j-i}$.

Multiplying both sides by T_j, we get $T_j^2 = T_j T_{j+1} - \sum_{i=1}^{k-1} T_j T_{j-i}$.

Thus,

$$\sum_{j=k-1}^{n} T_j^2 = T_n T_{n+1} - ab - \sum_{j=k-1}^{n}\sum_{i=2}^{k-1} T_j T_{j-i}.$$

Now we have

$$\sum_{j=k-1}^{n}\sum_{i=2}^{k-1}T_jT_{j-i} = \sum_{j=2}^{k-1}\sum_{i=0}^{n-j}T_iT_{i+j} - \sum_{j=0}^{k-4}\sum_{i=0}^{j}T_iT_{i+k-2-j}$$

$$= \sum_{j=2}^{k-1}\sum_{i=0}^{n-j}T_iT_{i+j} - \frac{(k-2)(k-3)}{2}a^2.$$

Note that in the last equality we use the fact that $T_i = a$ for all $0 \le i \le k-2$, see Definition 2.1.

Then

$$\sum_{j=0}^{n}T_j^2 = \sum_{j=0}^{k-2}T_j^2 + \sum_{j=k-1}^{n}T_j^2$$

$$= (k-1)a^2 + T_nT_{n+1} - ab - \sum_{j=2}^{k-1}\sum_{i=0}^{n-j}T_iT_{i+j} + \frac{(k-2)(k-3)}{2}a^2$$

$$= \frac{k^2-3k+4}{2}a^2 - ab + T_nT_{n+1} - \sum_{j=2}^{k-1}\sum_{i=0}^{n-j}T_iT_{i+j}\sum_{j=0}^{n}T_j^2.$$

The proof is complete.

Note that if $a = 0$ and $b = 1$, then we obtain identity for the k-generalized Fibonacci sequence, for any positive integer $n \ge k-1$,

$$\sum_{j=0}^{n}T_j^2 = T_nT_{n+1} - \sum_{j=2}^{k-1}\sum_{i=0}^{n-j}T_iT_{i+j}.$$

If $k = 3$, $a = 1$ and $b = 2$, then we obtain known equality for the Tribonacci sequence, for any positive integer $n \ge 2$

$$\sum_{j=0}^{n}T_j^2 = T_nT_{n+1} - \sum_{i=0}^{n-2}T_iT_{i+2}.$$

If $k = 2$, $a = 0$ and $b = 1$, then we obtain known equality for the Fibonacci sequence, for any positive integer n,

$$\sum_{i=0}^{n}F_i^2 = F_nF_{n+1}.$$

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Conflict of interest

There is no conflict of interest for this study.

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