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Research Article

Two Iterates of Symmetric Generalized Skew 3-Derivation

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Abstract: Our goal in this study is to validate the following finding: Assume that a prime ring R having $char(R) \neq 2, 3, 5$, D is a symmetric skew 3-derivation of R with automorphism α. If ∇_1, ∇_2: $R^3 \to R$ are symmetric generalized skew 3-derivations with α and associated skew 3-derivations D_1, D_2 respectively such that $\partial_1(\tau)\partial_2(\tau) = 0$ for every $\tau \in R$, then either $\nabla_1 = 0$ or $\nabla_2 = 0$ on R, where ∂_1 and ∂_2 stands for the traces of ∇_1 and ∇_2 respectively.

Keywords: semi(prime) ring, generalized skew 3-derivation, automorphism

MSC: 16N60, 16R50, 16W25

1. Introduction

An associative ring R with a non-zero center $Z(R)$ is taken throughout this work. Q denotes Martindale right quotient ring of R. The center of Q is shown by C, which is frequently thought of to represent an extended centroid of R. Just in case, C is a field if R assumes to be a prime ring. We encourage the reader to check [1, 2] for additional facts. If $tr = 0$ implies $r = 0$ for each r in R and $t > 1$, an integer, then R is a t-torsion free ring. A ring R is classified as semiprime if it satisfies the criteria that $mRm = \{0\}$ gives $m = 0$. R becomes prime if $rRs = \{0\}$ gives that either $r = 0$ or $s = 0$. The meaning of iterates here is the repetition of n-successive terms. In this research, we deals with the two iterates ($n = 2$) (or product of two) of generalized skew-3-derivations on prime rings.

A number of scholars have examined the relationship between particular distinctive types of mappings on a ring R and R's commutativity. The first achievement in this area was made possible by Divinsky [3], who proved that if an automorphism of an Artinian ring R is nontrivial and commutative, then R must also be commutative. Luh [4] expanded Divinsky's argument to prime rings. Mayne [5] proved that if a prime ring has an automorphism (non-identity and centralizing), then R must be a commutative ring. These results have now been applied to additional algebraic structures. Posner [6] confirmed that once a derivation takes place on a prime ring that is centralizing and nonzero, the commutative structure in the prime ring must exist. Over the last few decades, a number of scholars, including Martindale [2], Vukman [7], etc., have changed and improved these findings in various ways (see, for instance, [8–12], and [13] for further references).

Let us say a ring possesses an automorphism β. If $\mathrm{h}(bd) = \mathrm{h}(b)\beta(d) + b\mathrm{h}(d)$ fulfills for all b, d in R and exhibits additivity, then the map h on R will be known as β-derivation (or skew derivation). Denote identity mapping by $\mathscr{I}$ on R,

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then h $= \beta - I$ functioned as β-derivation. A map $\mathscr{F}$ defined on R, that is additive and is referred to a generalized skew derivation, with a related skew derivation d and an automorphism α if it fulfills the following condition for all $t, m \in R$:

$$\mathscr{F}(mt) = \mathscr{F}(m)t + \alpha(m)d(t) = \mathscr{F}(m)\alpha(t) + md(t).$$

For example: Specify the maps $\mathscr{F}$, d, α: $\mathscr{M}_{\mathscr{R}} \longrightarrow \mathscr{M}_{\mathscr{R}}$ such as

$$\mathscr{F}\left[\begin{pmatrix} 0 & 0 & \kappa \\ 0 & 0 & \omega \\ 0 & 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & 0 & 0 \end{pmatrix},$$

$$d\left[\begin{pmatrix} 0 & 0 & \kappa \\ 0 & 0 & \omega \\ 0 & 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 & \kappa \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\alpha\left[\begin{pmatrix} 0 & 0 & \kappa \\ 0 & 0 & \omega \\ 0 & 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 & \kappa \\ 0 & 0 & \omega \\ 0 & 0 & 0 \end{pmatrix}.$$

With automorphism α, associated skew-derivation d and for all $\kappa, \omega \in \mathbb{R}$, it is evident that $\mathscr{F}$ stands for generalized skew-derivation on $\mathscr{M}_{\mathscr{R}}$.

According to Maksa [14], a function $\mathscr{D}$ from $R \times R$ to R is thought to have symmetry if $\mathscr{D}(p, q) = \mathscr{D}(q, p)$ for each p, q in R. A function from $R \times R$ into R is said to be bi-additive if $\mathscr{D}$ obeys additivity in each of the two slots. The following is an introduction to the bi-derivations theory: The mapping $\mathscr{D}$, additive in each tuple and possessing symmetric property is known as symmetric bi-derivation when the mappings $t \mapsto \mathscr{D}(m, t)$ and the map $m \mapsto \mathscr{D}(m, t)$ are both derivations of R. By a 3-additive mapping, we mean a map ∇: $R^3 \longrightarrow R$ having additivity in all 3 slots. A function $\mathfrak{h}$ on R is termed as a skew 3-derivation if it is 3-additive and satisfy the condition $\mathfrak{h}(p, r) = \mathfrak{h}(p)\alpha(r) + p\mathfrak{h}(r)$, for all $r, p \in R$. A 3-additive function ∇: $R^3 \longrightarrow R$ is called a generalized skew 3-derivation if for each $u \in R$, the mapping $w \longmapsto \nabla(w, u, u)$ is a generalized skew derivation, with a related skew derivation $\mathfrak{h}$ and an automorphism α. The map ∇ will follow the same definition to be generalized skew 3-derivation in all 3 slots. A function ∂ on R is referred to the trace of ∇ for a symmetric 3-additive mapping ∇ when it is written as $\partial(l) = \nabla(l, l, l)$, l in R. The trace ∂ satisfies the following property:

$$\partial(l+t) = \partial(l) + \partial(t) + 3\nabla(l, t, t) + 3\nabla(t, l, l), \quad \text{and}$$

$$\partial(l-t) = \partial(l) - \partial(t) + 3\nabla(l, t, t) - 3\nabla(t, l, l).$$

We begin our investigation with the idea of the author obtained in [7] for bi-derivation and [15] for skew derivations. The idea of permuting skew 3-derivations in rings was presented in [13], where the author also demonstrated that a prime ring having a skew 3-derivation (nonzero and symmetric) must, under certain circumstances, be commutative. Many direct and indirect generalization of this idea has been done by several researchers. A noticeable extension was obtained by [10], in which authors proved the following: Let a prime ring R having $char(R) \neq 2, 3$, $\mathscr{D}$ be a symmetric skew biderivation with automorphism α, and ∇: $R^2 \longrightarrow R$ be a symmetric generalized skew biderivation of R, linked to α and $\mathscr{D}$. If the trace ∂ of ∇ satisfies $\partial_1(l)\partial_2(l) = 0$ for every l in R, then either $\partial_1 = 0$ or $\partial_2 = 0$. A comparable and helpful generalization is found in [16] for semi-biderivations and [17] in case of generalized skew derivations.

A more recent results obtained in [18], which states that: let $m_1, m_2 \neq 0$ be fixed two positive integers, R a prime ring, the right Martindale quotient ring Q, G_1 and G_2 be two generalized skew derivations of R associated with automorphism φ of R. If $G_1(a^{m_1+m_2}) = G_2(a^{m_1})a^{m_2}$, then either R must be commutative or $G_1(a) = G_2(a) = \varsigma a$ for every $a \in R$ and every $\varsigma \in Q$. Our objective is to discover an extension of such previously mentioned findings for generalized skew 3-derivations of prime rings.

2. Prerequisite

In order to establish our main theorem, we fix some preparation results. We begin with the subsequent lemmas:

Lemma 1 ([11]) A mapping π: $Q \to Q$ that is additive and offers $\pi(uw) = \pi(u)w + \gamma(u)\phi(w)$ for all $u, w \in Q$ is a right generalized skew derivation. Here, ϕ represents a skew derivation of R, γ is an automorphism of R, and Q is the right Martindale ring of quotient of R. Furthermore, for any $u \in R$, there exists $\pi(1) = a \in Q$ such that $\pi(u) = au + \phi(u)$.

Lemma 2 If R is a ring of $char(R) \neq 2$ *and* 3, and ∇_1, ∇_2: $R^3 \longrightarrow R$ are two symmetric 3-additive mappings of R. Assume that either $\partial_1(l) = 0$ or $\partial_2(l) = 0$ for any $l \in R$, where ∂_1 and ∂_2 are the traces of ∇_1 and ∇_2 respectively. Then either $\nabla_1 = 0$ or $\nabla_2 = 0$.

Proof.

Assume that $\partial_1(l) = 0$ for each $l \in R$, then linearization yields that

$$0 = \partial_1(l+\nu) = \partial_1(l) + \partial_1(\nu) + 3\nabla_1(l, l, \nu) + 3\nabla_1(l, \nu, \nu). \tag{1}$$

Given hypothesis allows us to write $\partial_1(l) = 0$ and $\partial_1(\nu) = 0$, we get

$$0 = \partial_1(l+\nu) = 3\nabla_1(l, l, \nu) + 3\nabla_1(l, \nu, \nu). \tag{2}$$

Also, we can have

$$0 = \partial_1(l-\nu) = \partial_1(l) - \partial_1(\nu) - 3\nabla_1(l, l, \nu) + 3\nabla_1(l, \nu, \nu). \tag{3}$$

After simplification, we have

$$0 = \partial_1(l-\nu) = -3\nabla_1(l, l, \nu) + 3\nabla_1(l, \nu, \nu). \tag{4}$$

Comparing (2) and (4) we have $\nabla_1(l, \nu, \nu) = 0$ and $\nabla_1(l, l, \nu) = 0$, for each $l, \nu \in R$, that is $\nabla_1 = 0$. Similarly, if $\partial_2(\nu) = 0$ for every $\nu \in R$, then $\nabla_2 = 0$.

Thus we assume that both $\partial_1 \neq 0$ and $\partial_2 \neq 0$ and find many contradictions as follows. That is, let's say that there are $l, \nu \in R$ such that $\partial_1(l) \neq 0$ and $\partial_2(\nu) \neq 0$ which leads to the conclusion that $\partial_2(l) = 0$ and $\partial_1(\nu) = 0$ by the hypothesis of the Lemma.

Next, we consider two cases:

• Case 1. $\partial_1(l+v)=0$. In this case, if also $\partial_1(l-v)=0$ then by comparing (1) and (3) and using the fact that $\partial_1(v)=0$ we have:

$$\partial_1(l)+3\nabla_1(l, v, v)=0. \tag{5}$$

Replacing v by $2v$ in (5) and using (5) again we have the contradiction $\partial_1(l)=0$. Also if $\partial_1(l+2v)=0$ and $\partial_1(l-2v)=0$, then we get

$$\partial_1(l)+12\nabla_1(l, v, v)=0. \tag{6}$$

Replace v by $2v$ in (6) and using (6)again we have the contradiction $\partial_1(l)=0$.

Thus both $\partial_1(l-v)\neq 0$ and $\partial_1(l-2v)\neq 0$, that is $\partial_2(l-v)=0$ and $\partial_2(l-2v)=0$, which gives, respectively,

$$0=\partial_2(l-v)=-\partial_2(v)-3\nabla_2(l, l, v)+3\nabla_2(l, v, v), \text{ and} \tag{7}$$

$$0=\partial_2(l-2v)=-8\partial_2(v)-6\nabla_2(l, l, v)+12\nabla_2(l, v, v). \tag{8}$$

Comparing (7) and (8), we have

$$6\partial_2(v)-6\nabla_2(l, v, v)=0. \tag{9}$$

Putting $2l$ instead of l in (9) and using (9) again we get the contradiction $\partial_2(v)=0$.

• Case 2. $\partial_1(l+v)\neq 0$

In this case, we have

$$0=\partial_2(l+v)=\partial_2(v)+3\nabla_2(l, l, v)+3\nabla_2(l, v, v). \tag{10}$$

If also $\partial_2(l-v)=0$, then

$$0=\partial_2(l-v)=-\partial_2(v)-3\nabla_2(l, l, v)+3\nabla_2(l, v, v). \tag{11}$$

Comparing (10) and (11), we obtain

$$\partial_2(v)+3\nabla_2(l, l, v)=0. \tag{12}$$

Replacing l by $2l$ in (12) and using (12) again we get the contradiction $\partial_2(v)=0$. Also, if $\partial_2(l+2v)=0$ and $\partial_2(l-2v)=0$, then we find the equations as below:

$$0 = \partial_2(l+2v) = 2\partial_2(v) + 6\nabla_2(l, l, v) + 12\nabla_2(l, v, v), \tag{13}$$

and

$$0 = \partial_2(l-2v) = -2\partial_2(v) - 6\nabla_2(l, l, v) + 12\nabla_2(l, v, v). \tag{14}$$

Comparing (13) and (14), we have

$$\partial_2(v) + 3\nabla_2(l, l, v) = 0. \tag{15}$$

Putting $2u$ instead of u in (15) and using (15) again we will arrive at the contradiction $\partial_2(v) = 0$. Thus, both $\partial_2(l - v) \neq 0$ and $\partial_2(l-2v) \neq 0$. That is $\partial_1(l-v) = 0$ and $\partial_1(l-2v) = 0$ which gives respectively:

$$0 = \partial_1(l-v) = \partial_1(l) - 3\nabla_1(l, l, v) + 3\nabla_1(l, v, v), \tag{16}$$

and

$$0 = \partial_1(l-2v) = \partial_1(l) - 6\nabla_1(l, l, v) + 12\nabla_1(l, v, v). \tag{17}$$

On comparing the last two equations, we get

$$3\partial_1(l) - 6\nabla_1(l, l, v) = 0. \tag{18}$$

Putting $2v$ instead of v in (18) and reusing (18) yielding a contradiction $\partial_1(l) = 0$. □

3. Main results

We start with the following:

Theorem 1 Let a prime ring R possessing char $R \neq 2, 3, 5$, D be a symmetric skew 3-derivation on R with an automorphism α. If ∇_1, ∇_2: $R^3 \to R$ are symmetric generalized skew 3-derivations with α and associated skew 3-derivations D_1, D_2 respectively such that $\partial_1(\tau)\partial_2(\tau) = 0$ for every $\tau \in R$, then either $\nabla_1 = 0$ or $\nabla_2 = 0$ on R, where ∂_1 and ∂_2 stands for the traces of ∇_1 and ∇_2 respectively.

Proof. Suppose

$$\nabla_1(\tau, \tau, \tau)\nabla_2(\tau, \tau, \tau) = 0 \text{ for each } \tau \in R. \tag{19}$$

Take some fix $x_0 \in R$ such that $\nabla_1(x_0, x_0, x_0) \in Z(R)$. By (19), either $\nabla_1(x_0, x_0, x_0) = 0$ or $\nabla_2(x_0, x_0, x_0) = 0$. Similar argument shows that if $\nabla_2(x_0, x_0, x_0) \in Z(R)$, then either $\nabla_1(x_0, x_0, x_0) = 0$ or $\nabla_2(x_0, x_0, x_0) = 0$. Hence, from the above observation, we arrive at the condition that for any $k \in R$, if either $\nabla_1(k, k, k) \in Z(R)$ or $\nabla_2(k, k, k) \in Z(R)$, then this implies that either $\nabla_1 = 0$ or $\nabla_2 = 0$, and Lemma 2 contributes to the conclusion.

Now, we assume that there is some $v \in R$ such that $0 \neq \nabla_1(v, v, v) \notin Z(R)$ and $0 \neq \nabla_2(v, v, v) \notin Z(R)$. Replace τ by $l+v$ in (19) to find

$$
\begin{aligned}
0 = {} & \partial_1(l)\partial_2(v) + \partial_1(l)\partial_2(l) + 3\partial_1(l)\nabla_2(l, l, v) \\
& +3\partial_1(l)\nabla_2(v, v, l) + \partial_1(v)\partial_2(l) + \partial_1(v)\partial_2(v) + 3\partial_1(v)\nabla_2(l, l, v) \\
& +3\partial_1(l)\nabla_2(v, v, l) + 3\nabla_1(l, l, v)\partial_2(l) + 3\nabla_1(l, l, v)\partial_2(v) \\
& +9\nabla_1(l, l, v)\nabla_2(l, l, v) + 9\nabla_1(l, l, v)\nabla_2(v, v, l) \\
& +3\nabla_1(v, v, l)\partial_2(l) + 3\nabla_1(v, v, l)\partial_2(v) \\
& +9\nabla_1(v, v, l)\nabla_2(l, l, v) + 9\nabla_1(v, v, l)\nabla_2(v, v, l) \text{ for each } l \in R.
\end{aligned}
\tag{20}
$$

Again replacing τ by $v-l$ in (19), we bring out

$$
\begin{aligned}
0 = {} & \partial_1(l)\partial_2(l) - \partial_1(v)\partial_2(l) - 3\partial_1(v)\nabla_2(v, v, l) \\
& +\partial_1(v)\partial_2(v) + 3\partial_1(v)\nabla_2(l, l, v) - \partial_1(l)\partial_2(v) + 3\partial_1(l)\nabla_2(v, v, l) \\
& -3\partial_1(l)\nabla_2(l, l, v) - 3\nabla_1(v, v, l)\partial_2(v) + 3\nabla_1(v, v, l)\partial_2(l) \\
& +9\nabla_1(v, v, l)\nabla_2(v, v, l) - 9\nabla_1(v, v, l)\nabla_2(l, l, v) \\
& +3\nabla_1(l, l, v)\partial_2(v) - 3\nabla_1(l, l, v)\partial_2(l) \\
& -9\nabla_1(l, l, v)\nabla_2(v, v, l) + 9\nabla_1(l, l, v)\nabla_2(l, l, v) \text{ for every } l \in R.
\end{aligned}
\tag{21}
$$

On subtracting (20) and (21), we find

$$
\begin{aligned}
0 = {} & 2\partial_1(l)\partial_2(v) + 2\partial_1(v)\partial_2(l) + 6\partial_1(l)\nabla_2(l, l, v) \\
& +6\partial_1(v)\nabla_2(v, v, l) + 6\nabla_1(l, l, v)\partial_2(l) + 6\nabla_1(v, v, l)\partial_2(v) \\
& +18\nabla_1(v, v, l)\nabla_2(l, l, v) + 18\nabla_1(l, l, v)\nabla_2(v, v, l) \text{ for each } l \in R.
\end{aligned}
\tag{22}
$$

Substitute $l+v$ for l in (22) to get after simplification

$$
\begin{aligned}
0 = {} & 6\partial_1(v)\partial_2(l) + 48\partial_1(v)\nabla_2(l, l, v) + 90\partial_1(v)\nabla_2(l, v, v) \\
& + 6\partial_1(l)\partial_2(v) + 48\nabla_1(l, l, v)\partial_2(v) + 90\nabla_1(l, v, v)\partial_2(v) \\
& + 12\partial_1(l)\nabla_2(l, v, v) + 36\nabla_1(l, l, v)\nabla_2(l, l, v) \\
& + 54\nabla_1(l, v, v)\nabla_2(l, l, v) + 144\nabla_1(l, v, v)\nabla_2(l, v, v) \\
& + 12\nabla_1(l, v, v)\partial_2(l) + 54\nabla_1(l, l, v)\nabla_2(l, v, v) \text{ for each } l \in R.
\end{aligned} \tag{23}
$$

Put $-l$ for l in (23) and subtract from (23) to obtain

$$
\begin{aligned}
0 = {} & 6\partial_1(v)\partial_2(l) + 6\partial_1(l)\partial_2(v) + 90\partial_1(v)\nabla_2(l, v, v) \\
& + 90\nabla_1(l, v, v)\partial_2(v) + 54\nabla_1(l, v, v)\nabla_2(l, l, v) \\
& + 54\nabla_1(l, l, v)\nabla_2(l, v, v) \text{ for every } l \in R.
\end{aligned} \tag{24}
$$

Again substitute $l+v$ for l in (24) and add with (24) to find

$$
\begin{aligned}
0 = {} & 72\partial_1(v)\nabla_2(l, l, v) + 72\nabla_1(l, l, v)\partial_2(v) \\
& + 216\nabla_1(l, v, v)\nabla_2(l, v, v) \\
& + 180\nabla_1(l, v, v)\partial_2(v) + 180\partial_1(v)\nabla_2(l, v, v) \text{ for each } l \in R.
\end{aligned} \tag{25}
$$

Put $-l$ for l above and add with (25) using Char $(R) \neq 2, 3$ to have

$$
\begin{aligned}
0 = {} & \partial_1(v)\nabla_2(l, l, v) + \nabla_1(l, l, v)\partial_2(v) \\
& + 3\nabla_1(l, v, v)\nabla_2(l, v, v) \text{ for each } l \in R.
\end{aligned} \tag{26}
$$

Again, replace $v-l$ for l in (26) and simplify by using (26) to obtain

$$
-5\partial_1(v)\nabla_2(l, v, v) - 5\nabla_1(l, v, v)\partial_2(v) = 0 \text{ for every } l \in R. \tag{27}
$$

Using char $R \neq 5$, we have

$$
\partial_1(v)\nabla_2(l, v, v) + \nabla_1(l, v, v)\partial_2(v) = 0 \text{ for each } l \in R. \tag{28}
$$

Define the following mappings:

$$\mathfrak{f}_1(x) = \nabla_1(x, \nu, \nu)$$

and

$$\mathfrak{f}_2(x) = \nabla_2(x, \nu, \nu)$$

with

$$f_1(x) = D_1(x, \nu, \nu) \text{ and } f_2(x) = D_2(x, \nu, \nu) \text{ for every } x \in R. \tag{29}$$

Since $0 \neq \nu \in R$ such that $\partial_1(\nu) \neq 0$ and $\partial_2(\nu) \neq 0$, we may write $a_1 = \partial_1(\nu)$ and $a_2 = \partial_2(\nu)$ are non-central. In the above construction, it is easy to see that $\mathfrak{f}_1$ and $\mathfrak{f}_2$ are generalized skew-3-derivation associated with skew-3-derivation f_1 and f_2 respectively. Applying Lemma 1 in the below reduced from (28) $a_1\mathfrak{f}_1(x) + \mathfrak{f}_2(x)a_2 = 0$ for all $x \in R$, For any $x \in R$, $\alpha(x) = qxq^{-1}$ is the integer representation of an invertible element $q \in Q$. From the following, one is held:

• $\mathfrak{f}_1(x) = [a_1, qxq^{-1}]q$ and $\mathfrak{f}_2(x) = q[a_2, x]$ for each $x \in R$ and for $q^{-1}a_1qa_2 \in C$.

• there exists a $\lambda \in C$ such that $\mathfrak{f}_1(x) = qx + \lambda[a_1, qxq^{-1}]q$ and $\mathfrak{f}_2(x) = qx + \lambda q[a_2, x]$ for every $x \in R$, satisfying the condition $a_1q + qa_2 = 0$ and $\lambda q^{-1}a_1qa_2 - a_2 \in C$.

Case I $\mathfrak{f}_1(x) = [a_1, qxq^{-1}]q$ and $\mathfrak{f}_2(x) = q[a_2, x]$ for all $x \in R$ and for $q^{-1}a_1qa_2 \in C$. For every $x, u \in R$, we can have

$$\begin{aligned} \mathfrak{f}_1(xu) &= [a_1, qxuq^{-1}]q \\ &= a_1qxu - qxuq^{-1}a_1q. \end{aligned} \tag{30}$$

It is also observed that

$$\begin{aligned} \mathfrak{f}_1(xu) &= \mathfrak{f}_1(x)u + qxq^{-1}f_1(u) \\ &= a_1qxu - qxq^{-1}a_1qu + qxq^{-1}f_1(u). \end{aligned} \tag{31}$$

From equation (30) and (31), we find $q^{-1}f_1(u) = [q^{-1}aq, u]$, hence

$$f_1(u) = q[q^{-1}a_1q, u] \text{ for each } u \in R. \tag{32}$$

Also, we see in the one hand

$$\begin{aligned} \mathfrak{f}_2(xu) &= q[a_2, xu] \\ &= q[a_2, x]u + qx[a_2, u], \end{aligned} \tag{33}$$

and in another hand

$$\begin{aligned} \mathfrak{f}_2(xu) &= \mathfrak{f}_2(x)u + qxq^{-1}f_2(x) \\ &= q[a_2, x]u + qxq^{-1}f_2(x). \end{aligned} \tag{34}$$

In viewing of (33) and (34), we notice that $q^{-1}f_2(u) = [a_2, u]$ for all $u \in R$. This implies that

$$f_2(u) = q[a_2, u] \text{ for every } u \in R. \tag{35}$$

From (32) and (35), we have

$$a_1 f_2(x) + f_1(x)a_2 \text{ for each } x \in R. \tag{36}$$

Utilizing Lemma 1, there exists $k_1, k_2 \in Q$ with the condition $\mathfrak{f}_1(x) = k_1x + f_1(x)$ and $\mathfrak{f}_2(x) = k_2x + f_2(x)$, for every $x \in R$. Investigate (36) and (28), we have

$$a_1k_2x + k_1xa_2 = 0 \text{ for every } x \in R.$$

Since $0 \neq a_2 \notin C$, we find $k_1 = a_1k_2 = 0$. By the above observations, we have $\mathfrak{f}_1(x) = f_1(x)$, $\nabla_1(x, \nu, \nu) = D_1(x, \nu, \nu)$ and $d_1(x) = \partial_1(x)$, where d_1 is the trace of D_1. Hence, (28) takes the form

$$a_1D_2(x, \nu, \nu) + D_1(x, \nu, \nu)a_2 \text{ for each } x \in R. \tag{37}$$

Put x for xu in (37) and making the use of (37) to get

$$D_1(x, \nu, \nu)[u, a_2] + [a_1, \alpha(x)]D_2(u, \nu, \nu) = 0 \text{ for every } x, u \in R. \tag{38}$$

Swap wx in place of x in (38) to get after suitable substitution and manipulation

$$D_1(w, \nu, \nu)x[u, a_2] + [a_1, \alpha(w)]\alpha(x)D_2(u, \nu, \nu) = 0 \text{ for every } w, x, u \in R. \tag{39}$$

Put qxq^{-1} for $\alpha(x)$ and xq for x in (39) to find

$$D_1(w, \nu, \nu)xq[u, a_2] + [a_1, qwq^{-1})]qxD_2(u, \nu, \nu) = 0 \text{ for every } x, u, w \in R. \tag{40}$$

By collecting the notations, we are able to write $D_2(u, \nu, \nu) = f_2(u) = q[a_2, u]$ and $D_1(w, \nu, \nu) = f_1(w) = [a_1, qwq^{-1}]q$. With above setting, reword equation (40) as

$$2[a_1, qwq^{-1}]qxq[a_2, u] = 0 \text{ for every } x, u, w \in R.$$

Primeness of R gives that either $[a_1, qwq^{-1}]q = 0$ for each $w \in R$ or $q[a_2, u] = 0$ for each u in R. If we consider first case, then we have $a_1 \in C$. In other way, if we take $q[a_2, u] = 0$, $u \in R$, then we get $a_2 \in C$. In both cases, we arrive at a contradiction.

Case 2 For some $\lambda \in C$, we have $\mathfrak{f}_1(x) = qx + \lambda[a_1, qxq^{-1}]q$ and $\mathfrak{f}_2(x) = qx + \lambda q[a_2, x]$ for every $x \in R$, with the property $a_1q + qa_2 = 0$ and $\lambda q^{-1}a_1qa_2 - a_2 \in C$. Remark that $a_2 \notin C$ gives us $\lambda \neq 0$. Also, notice that $\lambda a_2^2 + a_2 = \lambda' \in C$ and $a_2q^{-1}a_2 = 0$ as $a_1a_2 = 0$. For every x, u in R, we have

$$\begin{aligned} \mathfrak{f}_1(xu) &= qxu + \lambda[a_1, qxuq^{-1}]q \\ &= qxu + \lambda a_1qxu - \lambda qxuq^{-1}a_1q \text{ for every } u, x \in R, \end{aligned} \tag{41}$$

and

$$\begin{aligned} \mathfrak{f}_1(xu) &= \mathfrak{f}_1(x)u + qxq^{-1}f_1(u) \\ &= qxu + \lambda[a_1, qxq^{-1}]qu + qxq^{-1}f_1(u) \text{ for each } x, u \in R. \end{aligned} \tag{42}$$

Comparing (41) and (42), we find $qx(-\lambda uq^{-1}a_1q + \lambda q^{-1}a_1qu - q^{-1}f_1(u)) = 0$. Applying primeness of R, and use of the condition $a_1q = -qa_2$, we get

$$f_1(u) = \lambda q[u, a_2] \text{ for every } u \in R. \tag{43}$$

Next, consider the function $\mathfrak{f}_2(x)$ and optimizing it as below

$$\begin{aligned} \mathfrak{f}_2(xu) &= qxu + \lambda q[a_2, xu] \\ &= qxu + \lambda q[a_2, x]u + \lambda qx[a_2, u]. \end{aligned} \tag{44}$$

Simplifying again using the definition of $\mathfrak{f}_2(x)$,

$$\begin{aligned} \mathfrak{f}_2(xu) &= \mathfrak{f}_2(x)u + \alpha(x)f_2(u) \\ &= \mathfrak{f}_2(x)u + qxq^{-1}f_2(u) \\ &= (qx + \lambda q[a_2, x])u + qxq^{-1}f_2(u). \end{aligned} \tag{45}$$

Evaluating (44) and (45), we find

$$qx(\lambda[a_2, u] - q^{-1}f_2(u)) = 0 \text{ for every } u \in R.$$

Since $q \neq 0$, then we get $\lambda[a_2, u] = q^{-1}f_2(u)$ for each $u \in R$. This implies that

$$f_2(u) = \lambda q[a_2, u] \text{ for every } u \in R. \tag{46}$$

In view of (43) and (46), we establish that $f_1 = -f_2$. Repeating the same argument as the above, we may have $k_1 \in Q$ satisfies

$$\begin{aligned} \mathfrak{f}_1(x) &= k_1x + f_1(x) \\ &= k_1x + \lambda q[x, a_2] \text{ for each } x \in R. \end{aligned}$$

Hence, we have $k_1x + \lambda q[x, a_2] = \mathfrak{f}_1(x) = qx + \lambda[a_1, qxq^{-1}]q$. Compare the two sides of $\mathfrak{f}_1(x)$, we get $q = k_1$. Now,

$$\begin{aligned} \mathfrak{f}_2(x) &= qx - f_1(x) \\ &= qx - \lambda q[x, a_2] \text{ for every } x \in R. \end{aligned}$$

By (29), we can find $D_2(x, v, v) = -D_1(x, v, v)$ for every $x \in R$. Now, consider (28),

$$a_1\nabla_2(x, v, v) + \nabla_1(x, v, v)a_2 = 0 \text{ for every } x \in R.$$

Put $x = xw$ and

$$\begin{aligned} &a_1\nabla_2(x, v, v)w - a_1qxq^{-1}D_1(w, v, v) \\ &+\nabla_1(x, v, v)wa_2 + qxq^{-1}D_1(w, v, v)a_2 = 0 \text{ for every } w, x \in R. \end{aligned} \tag{47}$$

From (28) and (47), we obtain

$$\begin{aligned} &-a_1qxq^{-1}D_1(w, v, v) + \nabla_1(x, v, v)[w, a_2] \\ &+qxq^{-1}D_1(w, v, v)a_2 = 0 \text{ for every } x, \ w \in R. \end{aligned} \tag{48}$$

Multiplying $q^{-1}a_2$ from right and left multiplying by a_2q^{-1} and using the fact that $a_1q = -qa_2$ and $a_2q^{-1}a_2 = 0$, we get

$$a_2q^{-2}\nabla_1(x,\,\nu,\,\nu)a_2wq^{-1}a_2 = 0 \text{ for every } w,\, x\in R. \tag{49}$$

From (49) and impose the primeness of R, we have the following observations:

• either $q^{-1}a_2 = 0$, implies that $a_2 = 0$, a contradiction yields.

• or $a_2q^{-2}\nabla_1(x,\,\nu,\,\nu)a_2 = 0$ for all $x\in R$, we notice that $0 = a_2q^{-2}(qx+\lambda q[x,\,a_2])a_2wq^{-1}a_2 = \lambda a_2q^{-1}xa_2^2$ for $x\in R$. This implies that $a_2^2 = 0$. Next, multiplying (28) by a_2 from the right side, we obtain $a_1\nabla_2(x,\,\nu,\,\nu)a_2+\nabla_1(x,\,\nu,\,\nu)a_2^2 = 0$. That is, $a_1\nabla_2(x,\,\nu,\,\nu)a_2 = 0$ for every $x\in R$. That is, $a_1(qx-\lambda q[x,\,a_2])a_2 = 0$. That implies that $-qa_2xa_2 = 0$. Hence, we have $a_2xa_2 = 0$ for each $x\in R$. This results in a contradiction that $a_2 = 0$.

By Case I and Case II, We've achieved the intended result.

We observe the following immediate consequences:

Corollary 1 Let R be a prime ring with char $R\neq 2,\,3$ and $\nabla_1,\,\nabla_2$: $R^3\to R$ be symmetric generalized 3-derivations with respect to symmetric 3-derivations $d_1,\,d_2$ respectively. If $\nabla_1(p,\,p,\,p)\nabla_2(p,\,p,\,p) = 0$ for each $p\in R$, then either $\nabla_1 = 0$ or $\nabla_2 = 0$ on R.

Proof. Application of Theorem 1 with automorphism $\alpha = I$ (an identity map) completes the proof. □

Corollary 2 [10] Let a ring R be prime with char $R\neq 2,\,3$ and $\nabla_1,\,\nabla_2$: $R^2\to R$ be symmetric generalized skew bi-derivations with respect to symmetric skew bi-derivations $d_1,\,d_2$ respectively and an automorphism α. If $\nabla_1(p,\,q)\nabla_2(p,\,q) = 0$ for each $p,\,q\in R$, then either $\nabla_1 = 0$ or $\nabla_2 = 0$ on R.

Proof. Define a map $\Im_i$: $R^3\longrightarrow R$ such that $\Im_i(k,\,l,\,w) = \nabla_i(k,\,l)$ for $i = 1,\,2$ and each $l,\,k\in R$. Application of the Theorem 1 with map $\Im$ completes the proof. □

Corollary 3 [7] Let a ring R be prime having char $R\neq 2,\,3$ and $\nabla_1,\,\nabla_2$: $R^2\to R$ be symmetric generalized bi-derivations associated with the bi-derivations $d_1,\,d_2$ with trace $\partial_1,\,\partial_2$ respectively such that $\partial_1(p)\partial_2(p) = 0$ for each $p\in R$, then either $\partial_1 = 0$ or $\partial_2 = 0$ on R.

Example 1 Let R be a ring such that the product of 4 elements is zero but product of 3 elements is non-zero. Consider the ring $\mathscr{R} = \left\{\begin{pmatrix} m & 0\\ 0 & n\end{pmatrix} : m,\,n\in R\right\}$. Let's define $\nabla_1,\,\nabla_2$: $\mathscr{R}^3\longrightarrow\mathscr{R}$ as

$$\nabla_1,\,\nabla_2(\gamma_1,\,\gamma_2,\,\gamma_3) = \left(\begin{pmatrix} m_1 & 0\\ 0 & n_1\end{pmatrix},\begin{pmatrix} m_2 & 0\\ 0 & n_2\end{pmatrix},\begin{pmatrix} m_3 & 0\\ 0 & n_3\end{pmatrix}\right) = \begin{pmatrix} m_1m_2m_3 & 0\\ 0 & 0\end{pmatrix}.$$

Consider an automorphism α: $\mathscr{R}\longrightarrow\mathscr{R}$ is defined on $\mathscr{R}$ as

$$\alpha\begin{pmatrix} m & 0\\ 0 & n\end{pmatrix} = \begin{pmatrix} n & 0\\ 0 & m\end{pmatrix}.$$

Suppose that D: $\mathscr{R}^3\longrightarrow\mathscr{R}$ is described as

$$D(\gamma_1,\,\gamma_2,\,\gamma_3) = \begin{pmatrix} 0 & 0\\ 0 & n_1n_2n_3\end{pmatrix}.$$

Then, direct calculations implies that $\nabla_1,\,\nabla_2$ are generalized skew 3-derivations on R with associated skew 3-derivation D and automorphism α such that $\nabla_1\nabla_2 = 0$ but both $\nabla_1\neq 0$ and $\nabla_2\neq 0$. Hence the primeness condition is crucial in hypothesis. That's justify our main theorem.

4. Conclusion

In the end, we determine that, for two iterates of generalized skew-3-derivations on prime rings, we have proven our claim. Our proof relies heavily on the automorphism. Our study has significant potential, as evidenced by the numerous corollaries we have presented as outcomes of our primary theorems. Furthermore, our research's future scope might be visualized in two ways:

First: The reader can consider generalized skew-n-derivations on rings, as well as several common subsets of rings, as well as n-iterations or n-derivations.

Second: The reader gets encouraged by automorphisms to consider other mappings such as additive, surjective, epimorphism, etc. in place of automorphisms.

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Conflict of interest

The authors declare no competing financial interest.

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