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Research Article

Analytical and Computational Results for Neutral Impulsive Fractional System on Time Scales

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Abstract: This paper delves into the existence and uniqueness of neutral fractional integro-differential impulsive dynamic equations across various time scales, enriched by nonlocal initial conditions using the Caputo-Nabla derivative. By leveraging the refined fixed point theorem, the study provides a robust framework for establishing existence. The theoretical findings are elegantly illustrated through detailed graphical representations, enhancing the comprehension and appeal of the results.

***Keywords*:** neutral equations, Caputo-Nabla derivative, time scales, fixed point

MSC: 37C25, 26E70, 34K40, 34N05

Abbreviation

FDE	Fractional Differential Equations
FODE	Fractional Order Differential Equations
EU	Existence and Uniqueness
NFDE	Neutral Fractional Differential Equations
ld	Left-dense
C∇D	Caputo Nabla Derivative

1. Introduction

Fractional calculus is based on classical calculus ideas such as integral and derivative operators, just as fractional exponents develop from integer exponents [1, 2]. Many know that integer-order derivatives and integrals have multiple meanings depending on the geometrical and physical components. But when it comes to fractional-order integration and differentiation, which covers a constantly growing domain in both theory and practical implementations difficulties, this assumption is disproved [3–5]. Fractional differential equations (FDE) have attracted significant interest across disciplines

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such as physics, chemistry, and engineering due to their numerous applications in the fields. Many physical and natural phenomena can be effectively modeled using fractional order differential equations (FODEs), which often yield more accurate results than traditional integer order differential equations. Consequently, FODEs are recognized as a powerful and specialized tool in this field [6].

Dynamic equations in fractional differential equations offer a robust framework for modeling the evolution of complex systems over time using non-integer order derivatives. These innovative models excel at capturing anomalous behavior and long-range effects, which are often challenging to describe with traditional integer-order dynamics. Hence, Dynamic equations provide a richer and more precise depiction of system behavior over time, allowing for a deeper comprehension of complex systems and enhanced predictive accuracy compared to static equations. This makes dynamic analysis an indispensable tool for numerous engineering and scientific applications. Numerical techniques are crucial in the analysis of dynamical models. Recently, innovative numerical methods have been developed and applied specifically for fractional-order operators, enhancing our understanding and capabilities in this field [7]. Currently, the field of fractional differential equations (FDEs) is undergoing intense research, particularly in establishing the existence and uniqueness (EU) of solutions [8].

There may be instances in the real world where neither wholly continuous nor entirely discrete phenomena can properly portray. To sufficiently accommodate both conditions in these scenarios, we need a shared domain. Stefan Hilger proposed the concept of a common entity called the time scale T to integrate continuous and discrete calculus seamlessly [9–11]. The unification of these criteria forms the foundation of this domain. To address this particular model, which integrates both differential and variance equations, we formulated dynamic equations based on time-scale principles [12–14]. Many researchers studied dynamic equations with local initial and boundary conditions. And may be non-linear or linear [15]. Numerous authors have applied fractional calculus in examining dynamic equations due to its precision and the benefits it offers in interpreting physical phenomena [16–18].

There are several real-world scenarios where systems may undergo temporary disruptions, albeit brief in comparison to the overall process duration. In this instance, the resolution of these equations might display abrupt changes at certain time intervals $\iota_1 < \iota_2 < \iota_3 < ...$, given in the form $a(\iota_l^+) - \{a(\iota_l^-) = \mathscr{I}_l(\iota_l,\ a(\iota_l^-))$. Dynamic equations featuring jump discontinuities as solutions are known as impulsive dynamic equations [19–21]. Researchers have recently become interested in dynamical impulsive equations on time scales [22]. On time scales with nonlocal beginning circumstances, however, there is a scarcity of literature exploring impulsive dynamic equations through the lens of fractional calculus [23, 24].

Neutral fractional differential equations (NFDEs) distinguish themselves from conventional fractional differential equations through their incorporation of both the unknown function and its fractional derivative. Unlike regular fractional differential equations that solely involve the unknown function, NFDEs encompass both, rendering them inherently more intricate to scrutinize and analyze. Neutral fractional differential equations (NFDEs) distinguish themselves by employing delayed derivatives, which distinguishes them from retarded differential equations when determining both past and current function values. Neutral-type differential equations on high-speed computers simulate elastic networks with the specific aim of linking switching circuits [25]. Neutral differential equations have become increasingly prominent in applied mathematics due to their practical utility and recent surge in attention [26, 27].

The researchers in [28] explored the dynamics of an impulsive dynamic equation with a nonlocal initial condition. In the study by [29], the authors investigated the fractional impulsive dynamic equation featuring a nonlocal initial condition across time scales.

On the basis of aforementioned research [29], we emphasize the necessity of exploring the neutral fractional impulsive dynamic equation with nonlocal initial condition:

$$\begin{cases} {}^{C}D^{\mathfrak{w}}[\mathfrak{p}(\iota)-\mathfrak{g}(\iota,\ \mathfrak{p},\ N_1(\mathfrak{p}(\iota)))] = \mathscr{L}(\iota,\ \mathfrak{p}(\iota),\ N_2(\mathfrak{p}(\iota)),\ {}^{C}D^{\mathfrak{w}}\mathfrak{p}(\iota)), & \iota \in \mathfrak{I}_{\mathfrak{T}},\ \iota \neq \iota_l \\ \mathfrak{p}(\iota_l^{+}) - \mathfrak{p}(\iota_l^{-}) = \mathscr{I}_l(\iota_l,\ \mathfrak{p}(\iota_l^{-})), \quad l = 1,\ 2,\ \ldots,\ m \\ \mathfrak{p}(0) = \vartheta(\mathfrak{p}), \end{cases} \tag{1}$$

Here,

$$N_1(\mathfrak{p}(\iota)) = \int_0^{\iota} h_1(\iota,\ s,\ \mathfrak{p}(\iota))\nabla s$$

$$N_2(\mathfrak{p}(\iota)) = \int_0^{\iota} h_2(\iota,\ s,\ \mathfrak{p}(\iota))\nabla s$$

where $\iota \in \mathfrak{T}$, $\mathfrak{T} > 0$ and $\mathscr{L} : \mathfrak{I}_{\mathfrak{T}} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ denotes the leftdense (ld) continuous function and ${}^{C}D^{\mathfrak{w}}$ is Caputo-Nabla derivative (C∇D). We assume that $0 < \iota_o < \iota_1 < \iota_2 < \iota_3 < \ldots < \iota_n < \iota_{n+1} = \mathfrak{T}$, expressing the inclination at a specific time, utilizing the phrase $\mathfrak{p}(\iota_l^{+}) = \lim_{d\to 0}\mathfrak{p}(\iota + d)$ and $\mathfrak{p}(\iota_l^{-}) = \lim_{d\to 0}\mathfrak{p}(\iota - d)$ represents the limits from both the function $\mathfrak{p}$'s positive and negative extremes at $\iota = \iota_l$ within time scales. Let $\mathscr{I}_l$ be a function that remains continuous and real-valued across $\mathbb{R}$ for all $l = 1,\ 2,\ \ldots,\ m$ and $\mathscr{I}_l(\iota_l,\ \mathfrak{p}(\iota_l^{-}))$ is impulses interaction within $\mathfrak{I}_{\mathfrak{T}}$.

2. Preliminaries

Definition 2.1 [30] One defines backward jump operator as $\rho : \mathfrak{T} \to \mathbb{R}$, specified as $\rho(\iota) = \{\tau \in \mathfrak{T} : \tau < \iota\}$. ι is called left scattered point on $\mathfrak{T}$ if $\rho(\iota) = \iota - 1$ for any $\iota \in \mathfrak{T}$ and it's often described as left dense when $\rho(\iota) = \iota$. Let $\mathfrak{T}_{\upsilon} = \mathfrak{T}\backslash\{y\}$, else let $\mathfrak{T}$ is min right scattered point $\mathfrak{T}_{\upsilon} = \mathfrak{T}$.

Definition 2.2 [29] If $\mathfrak{x}(\cdot,\ a,\ b)$ exhibits ld continuity for every pair of parameters $(\iota,\ \tau) \in \mathbb{R} \times \mathbb{R}$ on $\mathfrak{T}$, left dense continuous function is $\mathfrak{x} : \mathfrak{T} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and for fixed point $\iota \in \mathfrak{T}$, $\mathfrak{x}(\iota,\ \cdot,\ \cdot)$ which is continuous on $\mathbb{R} \times \mathbb{R}$.

Definition 2.3 [13] Assume $\mathfrak{g} : \mathfrak{T} \to \mathbb{R}$ and $\mathscr{G}_{\nabla}(\iota) = \mathfrak{g}(\iota)$ for all $\iota \in \mathfrak{T}_{\upsilon}$, then

$$\int_a^{\iota} \mathfrak{g}(\mathfrak{x})\nabla\mathfrak{x} = \mathscr{G}(\mathfrak{x}) - \mathscr{G}(a).$$

Proposition 2.4 [9] Presume $\mathfrak{g}$ to be a steadily ascending, uninterrupted function in $[0,\ \mathfrak{T}] \cap \mathfrak{T}$. Let $\mathscr{G}$ adds to $\mathfrak{g}$ within the interval $[0,\ \mathfrak{T}]$, where $\mathfrak{T}$ belongs to the set $\mathbb{R}$, then it is possible to acquire

$$\mathscr{G}(\iota) = \begin{cases} \mathfrak{g}(\iota), & \text{if } \iota \in \mathfrak{T}, \\ \mathfrak{g}(\tau), & \text{if } \iota \in (\iota,\ \rho(\iota)) \notin \mathbb{R}, \end{cases}$$

then

$$\int_{\mathfrak{s}}^{\mathfrak{t}} \mathfrak{g}(\iota)\nabla\iota \leq \int_{\mathfrak{s}}^{\mathfrak{t}} \mathfrak{g}(\iota)d\iota, \tag{2}$$

for $\mathfrak{s},\ \mathfrak{t} \in [0,\ \mathfrak{T}] \cap \mathfrak{T}$, preceding $\mathfrak{s} < \mathfrak{t}$.

Definition 2.5 ([30], Higher order nabla derivative) Let's examine $\mathbb{H} : \mathfrak{T}_\upsilon \to \mathbb{R}$ on $\mathfrak{T}$. $\mathbb{H}_\nabla$ demonstrates differentiability across $\mathfrak{T}_\upsilon^{(2)} = \mathfrak{T}_{\upsilon\upsilon}$ with $\mathbb{H}_\nabla^{(2)} = (\mathbb{H})_\nabla : \mathfrak{T}_\upsilon^{(2)} \to \mathbb{R}$ where $\mathbb{H}_{\nabla\nabla} = \mathbb{H}_\nabla^{(2)}$ be second order ∇ derivative. Again, following with n^{th} order results in $\mathbb{H}_\nabla^{(n)} : \mathfrak{T}_\upsilon^{(n)} \to \mathbb{R}$.

Definition 2.6 [30] Let $\mathbb{H} : \mathfrak{T}_\upsilon^{(n)} \to \mathbb{R}$, such that $\mathbb{H}_\nabla^{(n)}(\iota)$ (derivative of order n with respect to nabla) appears. In that case, C∇D becomes

$$^{C}D_a^{\mathfrak{w}}\mathbb{H}(\iota) = \frac{1}{\Gamma(n-\mathfrak{w})}\int_a^{\iota}(\iota-\rho(\tau))^{n-\mathfrak{w}-1}\mathbb{H}_\nabla^{n}(\tau)\nabla\tau,$$

When $\mathfrak{w} \in (0,\ 1)$, the result is

$$^{C}D_a^{\mathfrak{w}}\mathbb{H}(\iota) = \frac{1}{\Gamma(1-\mathfrak{w})}\int_a^{\iota}(\iota-\rho(\tau))^{-\mathfrak{w}}\mathbb{H}_\nabla\nabla\tau.$$

Definition 2.7 [30] Within the domain $\mathfrak{T}_\upsilon$, let $\mathbb{H}$ denote any ld continuous function, so RL∇D is

$$D_{\iota_o}^{\mathfrak{w}}\mathfrak{x}(\mathfrak{t}) = \frac{1}{\Gamma(1-\mathfrak{w})}\left(\int_{\iota_o}^{\iota}(\iota-\rho(\tau))^{-\mathfrak{w}}\mathfrak{x}(\tau)\nabla\tau\right)^{\nabla}.$$

Definition 2.8 [9] Suppose $\mathbb{H} : \mathfrak{I}_{\mathfrak{J}} \to \mathbb{R}$, where expression for the fractional integral of RL ∇ derivative of $\mathbb{H}$ can be formulated as

$$D_{\iota_o}^{\mathfrak{w}}\mathbb{H}(\iota) = \mathbb{I}_{\iota_o}^{\mathfrak{w}}\mathbb{H}(\iota) = \frac{1}{\Gamma(\mathfrak{w})}\int_{\iota_o}^{\iota}(\iota-\rho(\tau))^{\mathfrak{w}-1}\mathbb{H}(\tau)\nabla\tau.$$

The integral with respect to ∇ in the context of RL consistently meets the requirement

$$\mathbb{I}_{\iota_o}^{\mathfrak{w}}\mathbb{I}_{\iota_o}^{\mathfrak{p}}\mathbb{H}(\iota) = \mathbb{I}_{\iota_o}^{w+u}\mathbb{H}(\iota).$$

Lemma 2.9 [9] Suppose $\mathfrak{p}(\iota)$ is given, then

$$\begin{cases} D^{\mathfrak{p}}\mathbb{I}^{\mathfrak{w}}a(\iota) = \mathfrak{p}(\iota) \\ D^{\mathfrak{p}}\mathbb{I}^{\mathfrak{w}}a(\iota) = \mathbb{I}^{w-u}\mathfrak{p}(\iota). \end{cases}$$

Definition 2.10 [12] Let's take $\mathscr{C}$ to be a set that is both closed and convex within the Banach space $\mathfrak{X}$. Consider $\mathfrak{g}: \mathfrak{U} \to \mathscr{C}$ as a mapping that is compact, with $\mathfrak{U}$ being a subset of $\mathscr{C}$ that is relatively open containing the origin. In this case

(i) $\mathfrak{g}$ possesses a point that remains unchanged within $\mathfrak{U}$; alternatively,

(ii) At a certain point $\mathfrak{p}$ within the boundary $\delta\mathfrak{U}$ and for a value of γ in the open interval $(0,\ 1)$, it holds that $\mathfrak{p} = \gamma\mathfrak{g}(\mathfrak{p})$.

Definition 2.11 [28] If $\mathfrak{w}$ belongs to the interval $(0, 1)$ and $\mathfrak{p}$ serves as a result to $\mathscr{L}$, defined as $\mathscr{L}: \mathfrak{I}_{\mathfrak{J}} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, which results in

$$^{C}D^{\mathfrak{w}}\mathfrak{p}(\iota) = \mathscr{L}(\iota,\ \mathfrak{p}(\iota),\ {}^{C}D^{\mathfrak{w}}\mathfrak{p}(\iota)),\ \mathfrak{p}(\iota)|_{\iota=0} = \vartheta(\mathfrak{p}).$$

If and only if $\mathfrak{p}$ represents a solution to the equation

$$\mathfrak{p}(\iota) = \vartheta(\mathfrak{p}) + \frac{1}{\Gamma(\mathfrak{w})}\int_{\iota_o}^{\iota}(\iota - \rho(\mathfrak{x}))^{\mathfrak{w}-1}\mathscr{L}(\mathfrak{x},\ \mathfrak{p}(\mathfrak{x}),\ {}^{C}D^{\mathfrak{w}}\mathfrak{p}(\mathfrak{x}))\nabla\mathfrak{x}. \tag{3}$$

3. Main results

One can utilize a Demographic model exhibiting a stop-start pattern for comparing the dynamic equation (1) with that model. If we consider the adverse effects on this specific species, we can witness how the population varies over time, as indicated by the C∇D ${}^{C}D^{\mathfrak{w}}\mathfrak{p}(\iota)$, during the initial time period, concerning ι within the interval $\mathfrak{I}_{\mathfrak{T}} = [0,\ \mathfrak{T}] \cap \mathfrak{T}$. Next, exploring a particular time frame $\iota_1,\ \iota_2,\ \iota_3,\ \ldots$, such that $0 < \iota_1 < \iota_2 < \iota_3,\ \ldots,\ \iota_{\mathfrak{m}} < \iota_{\mathfrak{m}+1} = \mathfrak{T}$, $\lim_l = \infty$, impulse effects briefly influence individuals, resulting in a temporary increase in the population represented by $u(\iota)$, where $u(\iota_l^+)$ and $u(\iota_l^-)$ indicate the species population before and after the impulse at time ι_l.

Consider a set comprising every ld continuous function $\mathscr{C}(\mathfrak{I}_{\mathfrak{T}},\ \mathbb{R})$. Put $\mathfrak{I}_o = [0,\ \iota_1]$ and $\mathfrak{I}_l = [\iota_l,\ \iota_{k+1}]$ for all $l = 1,\ 2,\ \ldots,\ \mathfrak{m}$.

Let

$$\mathscr{PC}(\mathfrak{I}_{\mathfrak{T}},\ \mathbb{R}) = \{\mathfrak{p}: \mathfrak{I}_l \to \mathbb{R},\ \mathfrak{p} \in \mathscr{C}(\mathfrak{I}_{\mathfrak{T}},\ \mathbb{R}) \text{ and } \mathfrak{p}(\iota_l^+) \text{ and } \mathfrak{p}(\iota_l^-) \text{ exists with } \mathfrak{p}(\iota_l^-) = \mathfrak{p}(\iota_l),\ l = 1,\ 2,\ \ldots,\ m\},$$

and

$$\mathscr{PC}^1(\mathfrak{I}_{\mathfrak{T}},\ \mathbb{R}) = \{\mathfrak{p}: \mathfrak{I}_l \to \mathbb{R},\ \mathfrak{p} \in \mathscr{C}^1(\mathfrak{I}_{\mathfrak{T}},\ \mathbb{R}),\ l = 1,\ 2,\ \ldots,\ m\}.$$

The set $\mathscr{PC}(\mathfrak{I}_{\mathfrak{T}},\ \mathbb{R})$ be Banach space $||\mathfrak{p}||_{\mathscr{PC}} = \sup_{\iota\in\mathfrak{I}_{\mathfrak{T}}}|\mathfrak{p}(\iota)|$.

Definition 3.1 Let $\mathfrak{p} \in \mathscr{PC}^1(\mathfrak{I}_{\mathfrak{T}},\ \mathbb{R})$ constitute a solution to equation (1). If $\mathfrak{p}$ fulfills equation (1) over $\mathfrak{I}_{\mathfrak{T}}$ then $\mathfrak{p}(\iota_l^+) - \mathfrak{p}(\iota_l^-) = \mathscr{I}_l(\iota_l,\ \mathfrak{p}(\iota_l^-))$ and $\mathfrak{p}(0) = \vartheta(\mathfrak{T})$.

Lemma 3.2 The ld continuous function $\mathscr{L}: \mathfrak{I}_{\mathfrak{T}} \to \mathbb{R}$, such that (1) solution is,

$$\begin{cases} {}^{C}D^{\mathfrak{w}}[\mathfrak{p}(\iota) - \mathfrak{g}(\iota,\ \mathfrak{p},\ N_1(\mathfrak{p}(\iota)))] = \mathscr{L}(\iota,\ \mathfrak{p}(\iota),\ N_2(\mathfrak{p}(\iota)),\ {}^{C}D^{\mathfrak{w}}\mathfrak{p}(\iota)), & \iota \in \mathfrak{I}_{\mathfrak{T}},\ \iota \neq \iota_l \\ \mathfrak{p}(\iota_l^+) - \mathfrak{p}(\iota_l^-) = \mathscr{I}_l(\iota_l,\ \mathfrak{p}(\iota_l^-)), & l = 1,\ 2,\ \ldots,\ m \\ \mathfrak{p}(0) = \vartheta(\mathfrak{p}), \end{cases} \tag{4}$$

in which the integral equation delineates

$$\mathfrak{p}(\iota)\begin{cases}\vartheta(\mathfrak{p})+\mathfrak{g}(\iota)+\dfrac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})}\displaystyle\int_0^{\iota}(\iota-\rho(\tau))^{\mathfrak{w}-1}\mathscr{L}(\tau,\ \mathfrak{p}(\tau),\ N_2(\mathfrak{p}(\tau)),\ {}^{C}D^{\mathfrak{w}}\mathfrak{p}(\tau))\nabla\tau,\ \ \iota\in\mathfrak{J}_o\\[2ex] \vartheta(\mathfrak{p})+\mathfrak{g}(\iota)+\dfrac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})}\displaystyle\sum_{i=1}^{l}\int_{\iota_{i-1}}^{\iota_i}(\iota_i-\rho(\tau))^{\mathfrak{w}-1}\mathscr{L}(\tau,\ \mathfrak{p}(\tau),\ N_2(\mathfrak{p}(\tau)),\ {}^{C}D^{\mathfrak{w}}\mathfrak{p}(\tau))\nabla\tau\\[2ex] +\dfrac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})}\displaystyle\int_{\iota_k}^{\iota}(\iota-\rho(\tau))^{\mathfrak{w}-1}\mathscr{L}(\tau,\ \mathfrak{p}(\tau),\ N_2(\mathfrak{p}(\tau)),\ {}^{C}D^{\mathfrak{w}}\mathfrak{p}(\iota))(\tau)\nabla\tau\\[2ex] +\displaystyle\sum_{i=1}^{l}\mathscr{I}_i(\iota_i,\ \mathfrak{p}(\iota_i^-)),\quad \iota\in\mathfrak{J}_l.\end{cases} \tag{5}$$

Proof. Let $\iota\in\mathfrak{J}_o$, in such a case, the solution to equation (4) is articulated as

$$\mathfrak{p}(\iota)=\vartheta(p)+\mathfrak{g}(\iota)+\frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})}\int_0^{\iota}(\iota-\rho(\tau))^{\mathfrak{w}-1}\mathscr{L}(\tau,\ \mathfrak{p}(\tau),\ N_2(\mathfrak{p}(\tau)),\ {}^{C}D^{\mathfrak{w}}\mathfrak{p}(\tau))\nabla\tau. \tag{6}$$

For $\iota\in\mathfrak{J}_1$, the problem

$$\begin{cases}{}^{C}D^{\mathfrak{w}}[\mathfrak{p}(\iota)-\mathfrak{g}(\iota,\ \mathfrak{p},\ N_1(\mathfrak{p}(\iota)))]=\mathfrak{H}(\iota),\\[1ex] \mathfrak{p}(\iota_1^+)-\mathfrak{p}(\iota_1^-)=\mathscr{I}_1(\iota_1,\ \mathfrak{p}(\iota_1^-)),\end{cases}$$

hold the solution

$$\mathfrak{p}(\iota)=\mathfrak{p}(\iota_1^+)+\mathfrak{g}(\iota)+\frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})}\int_{\iota_1}^{\iota}(\iota-\rho(\tau))^{\mathfrak{w}-1}\mathscr{L}(\tau,\ \mathfrak{p}(\tau),\ N_2(\mathfrak{p}(\tau)),\ {}^{C}D^{\mathfrak{w}}\mathfrak{p}(\tau))\nabla\tau. \tag{7}$$

Again,

$$\mathfrak{p}(\iota_1^+)-\mathfrak{p}(\iota_1^-)=\mathscr{I}_1(\iota_1,\ \mathfrak{p}(\iota_1^-)). \tag{8}$$

Utilizing equation (8) within equation (7) leads to

$$\begin{aligned}\mathfrak{p}(\iota)&=\mathfrak{p}(\iota_1^-)+\mathscr{I}_1(\iota_1,\ \mathfrak{p}(\iota_1^-))+\mathfrak{g}(\iota)\\ &\quad+\frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})}\int_{\iota_1}^{\iota}(\iota-\rho(\tau))^{\mathfrak{w}-1}\mathscr{L}(\tau,\ \mathfrak{p}(\tau),\ N_2(\mathfrak{p}(\tau)),\ {}^{C}D^{\mathfrak{w}}\mathfrak{p}(\tau))\nabla\tau,\end{aligned}$$

which follows that

$$\mathfrak{p}(\iota) = \vartheta(p) + \mathscr{I}_1(\iota_1,\ \mathfrak{p}(\iota_1^-)) + \mathfrak{g}(\iota)$$

$$+ \frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})} \int_{\iota_1}^{\iota} (\iota - \rho(\tau))^{\mathfrak{w}-1} \mathscr{L}(\tau,\ \mathfrak{p}(\tau),\ N_2(\mathfrak{p}(\tau)),\ {}^{C}D^{\mathfrak{w}}\mathfrak{p}(\tau)) \nabla\tau$$

$$+ \frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})} \int_{0}^{\iota} (\iota - \rho(\tau))^{\mathfrak{w}-1} \mathscr{L}(\tau,\ \mathfrak{p}(\tau),\ N_2(\mathfrak{p}(\tau)),\ {}^{C}D^{\mathfrak{w}}\mathfrak{p}(\tau)) \nabla\tau, \quad \iota \in \mathfrak{I}_1.$$

By the concept of mathematical induction and extending it to encompass $\iota \in \mathfrak{I}_l$, where $l = 1,\ 2,\ ...,\ \mathfrak{m}$, it becomes possible to assert that,

$$\mathfrak{p}(\iota) = \vartheta(\mathfrak{p}) + \mathfrak{g}(\iota) + \frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})} \int_{0}^{\iota} (\iota - \rho(\tau))^{\mathfrak{w}-1} \mathscr{L}(\tau,\ \mathfrak{p}(\tau),\ N_2(\mathfrak{p}(\tau)),\ {}^{C}D^{\mathfrak{w}}\mathfrak{p}(\tau)) \nabla\tau$$

$$+ \frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})} \sum_{i=1}^{l} \int_{\iota_{i-1}}^{\iota_i} (\iota_i - \rho(\tau))^{\mathfrak{w}-1} \mathscr{L}(\tau,\ \mathfrak{p}(\tau),\ N_2(\mathfrak{p}(\tau)),\ {}^{C}D^{\mathfrak{w}}\mathfrak{p}(\tau)) \nabla\tau$$

$$+ \sum_{i=1}^{l} \mathscr{I}_i(\iota_i,\ \mathfrak{p}(\iota_i)), \quad l = 1,\ 2,\ ...,\ \mathfrak{m}.$$

The subsequent hypotheses are requisite for establishing both the existence and uniqueness result of equation (1):

(A1) $\mathscr{L} : \mathfrak{I}_{\mathfrak{T}} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are the functions which is ld continuous, with $\mathscr{K} > 0$ and $0 < \mathscr{G} < 1$ such that they satisfy

$$|\mathscr{L}(\iota,\ \tau_1,\ \tau_2) - \mathscr{L}(\iota,\ \iota_1,\ \iota_2)| \leq \mathscr{K}|\tau_1 - \iota_1| + \mathscr{G}|\tau_2 - \iota_2|, \ \text{ for all } \ \iota \in \mathbb{I},$$

$\tau_i,\ \iota_i \in \mathbb{R}$ for $\mathfrak{I} = 1,\ 2$.

(A2) There exists $\mathbb{A} > 0,\ \mathbb{F} > 0$ and $0 < \mathbb{E} < 1$, such that

$$|\mathscr{L}(\iota,\ \tau,\ \iota)| \leq \mathbb{A} + \mathbb{F}|\tau| + \mathbb{E}|\iota|, \ \text{ for all } \ \tau,\ \iota \in \mathbb{R}.$$

(A3) $\mathscr{I}_l(\iota,\ \mathfrak{p})$ denote a function that remains ld continuous for all $l = 1,\ 2,\ ...,\ \mathfrak{m}$, such that they satisfy:

(i) $\exists$ '+' ve constant $\mathscr{M}_l$ for $l = 1,\ 2,\ ...,\ \mathfrak{m}$, such that

$$|\mathscr{I}_l(\iota,\ \mathfrak{p})| \leq \mathscr{M}_l, \ \text{ for all } \ \iota \in \mathfrak{I}_l,\ \mathfrak{p} \in \mathbb{R}.$$

(ii) $\exists$ '+' ve constants $\mathbb{L}_l$, for $l = 1,\ 2,\ ...,\ \mathfrak{m}$, such that

$$|\mathscr{I}_l(\iota,\ \mathfrak{p}) - \mathscr{I}_l(\iota,\ \mathbb{H})| \leq \mathbb{L}_l|\mathfrak{p} - \mathbb{H}|, \ \text{ for all } \ \iota \in \mathfrak{I}_l,\ \mathfrak{p},\ \mathbb{H} \in \mathbb{R}.$$

(A4) $\exists$ non '$-$' ve increasing function $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|\vartheta(\iota) - \vartheta(\tau)| \leq \mathfrak{H}|\iota - \tau| \text{ for all } \iota \in \mathfrak{I}_{\mathfrak{T}},$$

and a '+' ve constant $\mathfrak{H}$ such that

$$|\vartheta(\iota) - \vartheta(\tau)| \leq \mathfrak{H}|\iota - \tau| \text{ for all } \iota,\ \tau \in \mathfrak{I}_{\mathfrak{T}}.$$

(A5) In a time scale interval, where $\iota \in \mathfrak{I}_o$, suppose $\exists$ a function $\mathfrak{p}(\iota)$ such that

$$\mathfrak{p}(\iota) = \vartheta(p) + \mathfrak{g}(\iota) + \frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})}\int_0^\iota (\iota - \rho(\tau))^{\mathfrak{w}-1}\mathscr{L}(\iota,\ \mathfrak{p}(\iota),\ N_2(\mathfrak{p}(\iota)),\ {}^C D^{\mathfrak{w}}\mathfrak{p}(\iota))\nabla\tau.$$

(A6) The operator $\mathscr{W}_{s_i}^{t_{i+1}} : \mathscr{L}^2(\mathfrak{I},\ \mathbb{R}) \to \mathbb{R}$ defined by

$$\mathscr{W}_{s_i}^{t_{i+1}} u = \frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})}\int_0^\iota (\iota - \rho(\tau))^{\mathfrak{w}-1} Bu(\tau)\Delta\tau,\ \ i = 1,\ 2,\ 3,\ \ldots,\ m,$$

where the bounded invertible operator $\mathscr{W}_{s_i}^{t_{i+1}}$ takes the values in $\mathscr{L}^2(\mathfrak{I},\ \mathbb{R})/\mathrm{Ker}\mathscr{W}_{s_i}^{t_{i+1}}$ in which there exists a positive constant $\mathscr{M}_B$ such that $||B|| \leq \mathscr{M}_B$.

The subsequent theorem relies on the principles established in the Banach contraction theorem.

Theorem 3.3 If conditions (A1) through (A5) and

$$\sum_{i=1}^{\mathfrak{m}} \mathbb{L}_i + \mathfrak{H} + \mathfrak{g}(\iota) + \frac{\mathfrak{g}(0)\mathscr{K}\mathfrak{T}^{\mathfrak{w}}(\mathfrak{m}+1)(N_1(\mathfrak{p}(\iota)) + N_2(\mathfrak{p}(\iota)))}{(1-\mathscr{G})(\mathfrak{w}+1)} < 1,$$

are satisfied, then equation (1) necessitates the presence of a solution within $\mathfrak{I}_{\mathfrak{T}}$.

Proof. Assume ${}^C D^{\mathfrak{w}}[\mathfrak{p}(\iota) - \mathfrak{g}(\iota,\ \mathfrak{p},\ N_1(\mathfrak{p}(\iota)))] = N_1(\mathfrak{p}(\iota))\mathbb{H}(\iota)$. Let $\varXi \subseteq \mathscr{PC}(\mathfrak{I}_l,\ \mathbb{R})$, such that

$$\varXi = \{\mathfrak{p} \in \mathscr{PC}^1(\mathfrak{I}_l,\ \mathbb{R}) : ||\mathfrak{p}||_{\mathscr{PC}} \leq \omega\}$$

and $\Omega : \varXi \to \varXi$ such that

$$(\Omega\mathfrak{p})(\iota) = \vartheta(\mathfrak{p}) + \mathfrak{g}(\iota) + \frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})}\int_0^\iota (\iota - \rho(\tau))^{\mathfrak{w}-1}\mathscr{L}(\iota,\ \mathfrak{p}(\iota),\ N_2(\mathfrak{p}(\iota)),\ {}^C D^{\mathfrak{w}}\mathfrak{p}(\iota))\nabla\tau,$$

for $\iota \in \mathfrak{I}_o$ and

$$(\Omega\mathfrak{p})(\iota) = \vartheta(\mathfrak{p}) + \mathfrak{g}(\iota) + \frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})}\sum_{i=1}^{l}\int_{\iota_{i-1}}^{\iota_i}(\iota-\rho(\tau))^{\mathfrak{w}-1}\mathscr{L}(\iota,\ \mathfrak{p}(\iota),\ N_1(\mathfrak{p}(\iota))\mathbb{H}(\iota))\nabla\tau$$

$$+\sum_{i=1}^{l}\mathscr{I}_i(\iota_i,\ \mathfrak{p}(\iota_i^-)) + \frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})}\int_{\iota_l}^{\iota}(\iota-\rho(\tau))^{\mathfrak{w}-1}\mathscr{L}(\iota,\ \mathfrak{p}(\iota),\ N_2(\mathfrak{p}(\iota)),\ {}^{C}D^{\mathfrak{w}}\mathfrak{p}(\iota))\nabla\tau,$$

for $\iota \in \mathfrak{J}_l$, so $l = 1,\ 2,\ ...,\ \mathfrak{m}$.

Case 1 If $\iota \in \mathfrak{J}_l$ such that $\mathfrak{p} \in \Xi$,

$$|(\Omega\mathfrak{p})(\iota)| = |\vartheta(\mathfrak{p})| + |\mathfrak{g}(\iota)| + |\frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})}\sum_{i=1}^{l}\int_{\iota_{i-1}}^{\iota_i}(\iota-\rho(\tau))^{\mathfrak{w}-1}N_1(\mathfrak{p}(\iota))\mathbb{H}(\tau)\nabla\tau|$$

$$+|\sum_{i=1}^{l}\mathscr{I}_i(\iota_i,\ \mathfrak{p}(\iota_i^-))| + |\frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})}\int_{\iota_l}^{\iota}(\iota-\rho(\tau))^{\mathfrak{w}-1}N_2(\mathfrak{p}(\iota))\mathbb{H}(\tau)\nabla\tau|,$$

here $\mathbb{H} \in \Xi$, $\iota \in \mathfrak{J}_{\mathfrak{T}}$, then equation (1) one can get $\mathbb{H} = \mathscr{L}(\iota,\ \mathfrak{p},\ \mathbb{H})$.

$$\begin{aligned} |\mathbb{H}| &= |\mathscr{L}(\iota,\ \mathfrak{p},\ \mathbb{H})| \\ &\leq \mathbb{A} + \mathbb{F}|\mathfrak{p}(\iota)| + \mathbb{E}|\mathbb{H}(\iota)| \\ &\leq \frac{\mathbb{A}+\mathbb{F}\omega}{1-\mathbb{E}}. \end{aligned} \tag{9}$$

Once more, computing the norm of $\mathscr{PC}(\mathfrak{J}_{\mathfrak{T}},\ \mathbb{R})$, in (9) then,

$$||\mathfrak{p}||_{\mathscr{PC}} \leq \frac{\alpha+\mathbb{F}\omega}{1-\mathbb{E}}$$

here $||\mathbb{A}||_{\mathscr{PC}} = \alpha$.

By applying theme of Case 1 and Proposition 2.4, results in

$$
\begin{aligned}
||\Omega||_{\mathscr{PC}} &= \sup_{\iota\in\mathbb{I}}|\Omega\mathfrak{p}(\iota)| \\
&\leq \nu|\mathfrak{p}| + \mathfrak{g}(\iota) + \sum_{i=1}^{\mathfrak{m}} \mathscr{M}_i \\
&\quad + \frac{\mathfrak{g}(0)[\mathbb{A}+\mathbb{F}|\mathfrak{p}|](N_1(\mathfrak{p}(\iota))+N_2(\mathfrak{p}(\iota)))}{(1-\mathbb{E})\Gamma(\mathfrak{w})}\left[\sum_{i=1}^{\mathfrak{m}}\int_{\iota_i-1}^{\iota_i}(\iota-\tau)^{(\mathfrak{w}-1)}d\tau + \int_{\iota_l}^{\iota}(\iota-\tau)^{(\mathfrak{w}-1)}d\tau\right] \\
&\leq \nu\omega + \mathfrak{g}(\iota) + \sum_{i=1}^{\mathfrak{m}}\mathscr{M}_i + \frac{\mathfrak{g}(0)\mathfrak{T}^{\mathfrak{w}}(\alpha+\mathbb{F}\omega)(\mathfrak{m}+1)(N_1(\mathfrak{p}(\iota))+N_2(\mathfrak{p}(\iota)))}{\Gamma(\mathfrak{w}+1)(1-\mathbb{E})} \\
&\leq \omega,
\end{aligned}
\tag{10}
$$

where

$$
\omega = \frac{\sum_{i=1}^{\mathfrak{m}}\mathscr{M}_i + \mathfrak{g}(\iota) + \frac{\mathfrak{g}(0)(\mathfrak{m}+1)\mathfrak{T}^{\mathfrak{w}}\alpha(N_1(\mathfrak{p}(\iota))+N_2(\mathfrak{p}(\iota)))}{\Gamma(\mathfrak{w}+1)(1-\mathbb{E})}}{1-\nu+\frac{(\mathfrak{m}+1)\mathfrak{T}^{\mathfrak{w}}\mathbb{F}\mathfrak{g}(0)(N_1(\mathfrak{p}(\iota))+N_2(\mathfrak{p}(\iota)))}{\Gamma(\mathfrak{w}+1)(1-\mathbb{E})}}.
$$

Case 2 If $\iota \in \mathfrak{I}_o$, in a similar manner, it is possible to obtain

$$
\begin{aligned}
||\Omega_{\mathfrak{p}}||_{\mathscr{PC}} &\leq \nu\omega + \mathfrak{g}(\iota) + \frac{\mathfrak{g}(0)\mathfrak{T}^{\mathfrak{w}}(\alpha+\mathbb{F}\omega)(N_1(\mathfrak{p}(\iota))+N_2(\mathfrak{p}(\iota)))}{\Gamma(\mathfrak{w}+1)} \\
&\leq \omega.
\end{aligned}
\tag{11}
$$

Thus from (11), $||\Omega_{\mathfrak{p}}||_{\mathscr{PC}} \leq \omega$. As a result, $\Omega(\Xi)$ remains bounded. Additionally $\mathfrak{p},\ \mathfrak{q} \in \Xi$,

$$
\begin{aligned}
&||\Omega_{\mathfrak{p}} - \Omega_{\mathfrak{q}}||_{\mathscr{PC}} \\
&= \sup_{\iota\in\mathfrak{J}_l} |(\Omega_{\mathfrak{p}})(\iota) - (\Omega_{\mathfrak{q}})(\iota)| \\
&\leq \sum_{i=1}^{l} |\mathscr{I}_i(\iota_i,\ \mathfrak{p}(\iota_i^-)) - \mathscr{I}_i(\iota_i,\ \mathfrak{q}(\iota_i^-))| + |\mathfrak{g}(\iota)| \\
&\quad + \frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})} \left| \int_{\iota_l}^{\iota} (\iota - \rho(\tau))^{\mathfrak{w}-1} N_1(\mathfrak{p}(\iota))(\mathbb{H}(\tau) - \mathfrak{I}(\tau)) \nabla\tau \right| \\
&\quad + \frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})} \left| \sum_{i=1}^{l} \int_{\iota_{i-1}}^{\iota_i} (\iota_i - \rho(\tau))^{\mathfrak{w}-1} N_2(\mathfrak{p}(\iota))(\mathbb{H}(\tau) - \mathfrak{I}(\tau)) \nabla\tau \right| + |\vartheta(\mathfrak{p}) - \vartheta(\mathfrak{q})|,
\end{aligned}
\tag{12}
$$

here $\mathfrak{I} \in \Xi$, then $\mathfrak{I}(\iota) = \mathscr{L}(\iota,\ \mathfrak{q}(\iota),\ \mathfrak{I}(\iota))$, and for $\iota \in \mathfrak{J}_{\mathfrak{T}}$, results as

$$
\begin{aligned}
|\mathbb{H}(\iota) - \mathfrak{I}(\iota)| &= |\mathscr{L}(\iota,\ \mathfrak{p}(\iota),\ \mathbb{H}(\iota)) - \mathscr{L}(\iota,\ \mathfrak{q}(\iota),\ \mathfrak{I}(\iota))| \\
&\leq \mathscr{K}|\mathfrak{p}(\iota) - \mathfrak{q}(\iota)| + \mathscr{G}|\mathbb{H}(\iota) - \mathfrak{I}(\iota)| \\
&\leq \frac{\mathscr{K}|\mathfrak{p}(\iota) - \mathfrak{q}(\iota)|}{1 - \mathscr{G}}.
\end{aligned}
\tag{13}
$$

Calculating the norm of $\mathscr{PC}(\mathfrak{J}_{\mathfrak{T}},\ \mathbb{R})$, then (13) results in

$$
||\mathbb{H} - \mathfrak{I}||_{\mathscr{PC}} \leq \frac{\mathscr{K}||\mathfrak{p} - \mathfrak{q}||_{\mathscr{PC}}}{1 - \mathscr{G}}. \tag{14}
$$

By employing equation (14) within the context of (12), and subsequently using Proposition 2.4,

$$||\Omega_{\mathfrak{p}}-\Omega_{\mathfrak{q}}||_{\mathscr{PC}} \leq \sum_{i=1}^{\mathfrak{m}} \mathbb{L}_i|\mathfrak{p}(\iota_i^-)-\mathfrak{q}(\iota_i^-)| + \mathfrak{g}(\iota) + \frac{\mathscr{K}\,\mathfrak{g}(0)|\mathfrak{p}(\tau)-\mathfrak{q}(\tau)|(N_1(\mathfrak{p}(\iota)))}{(1-\mathscr{G})\Gamma(\mathfrak{w})}\int_{\iota_l}^{\iota}(\iota-\tau)^{\mathfrak{w}-1}d\tau$$

$$+\frac{\mathscr{K}\,\mathfrak{g}(0)(N_2(\mathfrak{p}(\iota)))|\mathfrak{p}(\tau)-\mathfrak{q}(\tau)|}{(1-\mathscr{G})\Gamma(\mathfrak{w})}\sum_{i=1}^{\mathfrak{m}}\int_{\iota_i-1}^{\iota_i}(\iota-\tau)^{\mathfrak{w}-1}d\tau+\mathfrak{H}|\mathfrak{p}-\mathfrak{q}|$$

$$\leq ||\mathfrak{p}-\mathfrak{q}||_{\mathscr{PC}}\sum_{i=1}^{\mathfrak{m}}\mathbb{L}_i+\mathfrak{g}(\iota)+\frac{\mathscr{K}\mathfrak{T}^{\mathfrak{w}}\mathfrak{g}(0)(N_1(\mathfrak{p}(\iota)))||\mathfrak{p}-\mathfrak{q}||_{\mathscr{PC}}}{(1-\mathscr{G})\Gamma(\mathfrak{w}+1)} \tag{15}$$

$$+\frac{\mathfrak{m}\mathscr{K}\mathfrak{T}^{\mathfrak{w}}\mathfrak{g}(0)(N_2(\mathfrak{p}(\iota)))||\mathfrak{p}-\mathfrak{q}||_{\mathscr{PC}}}{(1-\mathscr{G})\Gamma(\mathfrak{w}+1)}+\mathfrak{H}||\mathfrak{p}-\mathfrak{q}||_{\mathscr{PC}}$$

$$\leq\left(\sum_{i=1}^{\mathfrak{m}}\mathbb{L}_i+\mathfrak{g}(\iota)+\frac{\mathscr{K}\mathfrak{T}^{\mathfrak{w}}\mathfrak{g}(0)(\mathfrak{m}+1)(N_1(\mathfrak{p}(\iota))+N_2(\mathfrak{p}(\iota)))}{(1-\mathscr{G})\Gamma(\mathfrak{w}+1)}+\mathfrak{H}\right)||\mathfrak{p}-\mathfrak{q}||_{\mathscr{PC}}.$$

Similarly for $\iota\in\mathfrak{J}_o$

$$||\Omega_{\mathfrak{p}}-\Omega_{\mathfrak{q}}||_{\mathscr{PC}} \leq \left(\mathfrak{H}+\mathfrak{g}(\iota)+\frac{\mathscr{K}\mathfrak{T}^{\mathfrak{w}}\mathfrak{g}(0)(N_1(\mathfrak{p}(\iota))+N_2(\mathfrak{p}(\iota)))}{(1-\mathscr{G})\Gamma(\mathfrak{w}+1)}\right)||\mathfrak{p}-\mathfrak{q}||_{\mathscr{PC}}. \tag{16}$$

Thus from (15) and (16), we obtain

$$||\Omega_{\mathfrak{p}}-\Omega_{\mathfrak{q}}||_{\mathscr{PC}} \leq \mathfrak{U}||\mathfrak{p}-\mathfrak{q}||_{\mathscr{PC}},$$

here $\mathfrak{U}=\sum_{i=1}^{\mathfrak{m}}\mathbb{L}_i+\mathfrak{g}(\iota)+\frac{\mathscr{K}\mathfrak{T}^{\mathfrak{w}}\mathfrak{g}(0)(\mathfrak{m}+1)(N_1(\mathfrak{p}(\iota))+N_2(\mathfrak{p}(\iota)))}{(1-\mathscr{G})\Gamma(\mathfrak{w}+1)}+\mathfrak{H}$ which is <1. Thus, if $\Omega:\Xi\to\Xi$ acts as a contraction operator, then, it possesses a fixed point by virtue of the Banach contraction theorem. This fixed point serves as the solution to equation (1).

The condition for a solution in Equation (1) relies on a nonlinear alternative to Leray-Schauder's fixed point theorem.

Theorem 3.4 Let conditions (A1) to (A5) hold and *exists* a '+' constant β, results in

$$\nu\beta+\sum_{i=1}^{\mathfrak{m}}\mathscr{M}_i+\mathfrak{g}(\iota)+\frac{(\mathfrak{m}+1)\mathfrak{T}^{\mathfrak{w}}\mathfrak{g}(0)(\mathbb{A}+\mathbb{F}\beta)(N_1(\mathfrak{p}(\iota))+N_2(\mathfrak{p}(\iota)))}{\Gamma(\mathfrak{w}+1)(1-\mathbb{E})}<\beta \tag{17}$$

$\therefore$ equation (1) contains at least one solution within $\mathfrak{J}_{\mathfrak{T}}$.

Proof. Subsequent procedures are employed to establish the proof of the theorem:

Step 1 $\Omega:\Xi\to\Xi$ be continuous.

Suppose $\{\mathfrak{p}_n\}$ be a sequence of Ξ such that $\mathfrak{p}_n\to\mathfrak{p}$, results in $\iota\in\mathfrak{J}_l,\ l=1,\ 2,\ \ldots,\ \mathfrak{m}$.

$$||\Omega_{\mathfrak{p}_n}-\Omega_{\mathfrak{q}}||_{\mathscr{PC}}=\sup_{\iota\in\mathfrak{I}_l}|(\Omega_{\mathfrak{p}_n})(\iota)-(\Omega_{\mathfrak{q}})(\iota)|$$

$$\leq\sum_{i=1}^{\mathfrak{m}}|\mathscr{I}_i(\iota_i,\ \mathfrak{p}_n(\iota_i^-))-\mathscr{I}_i(\iota_i,\ \mathfrak{p}(\iota_i^-))|+|\mathfrak{g}(\iota)|\frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})}\left|\int_{\iota_l}^{\iota}(\iota-\tau)^{\mathfrak{w}-1}N_1(\mathfrak{p}(\iota))(\mathbb{H}_n(\tau)-\mathbb{H}(\tau))d\tau\right| \quad (18)$$

$$+\frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})}\left|\sum_{i=1}^{\mathfrak{m}}\int_{\iota_{i-1}}^{\iota_i}(\iota_i-\tau)^{\mathfrak{w}-1}N_2(\mathfrak{p}(\iota))(\mathbb{H}_n(\tau)-\mathbb{H}(\tau))d\tau\right|+|\vartheta(\mathfrak{p}_n)-\vartheta(\mathfrak{p})|,$$

here $\mathbb{H}_n\in\Xi$, such that $\mathbb{H}_n=\mathscr{L}(\iota,\ \mathfrak{p}_n,\ \mathbb{H}_n)$, and for $\iota\in\mathfrak{I}_l$, we get

$$|\mathbb{H}_n-\mathbb{H}|=|\mathscr{L}(\iota,\ \mathfrak{p}_n,\ \mathbb{H}_n)-\mathscr{L}(\iota,\ \mathfrak{p},\ \mathbb{H})|$$

$$\leq\mathscr{K}|\mathfrak{p}_n-\mathfrak{p}|+\mathscr{G}|\mathbb{H}_n-\mathbb{H}| \quad (19)$$

$$\leq\frac{\mathscr{K}|\mathfrak{p}_n-\mathfrak{p}|}{1-\mathscr{G}}.$$

Calculating the norm of $\mathscr{PC}(\mathfrak{I}_{\mathfrak{T}},\ \mathbb{R})$, then (19) becomes

$$||\mathbb{H}_n-\mathbb{H}||_{\mathscr{PC}}\leq\frac{\mathscr{K}||\mathfrak{p}_n-\mathfrak{p}||_{\mathscr{PC}}}{1-\mathscr{G}}. \quad (20)$$

Using (20) in (18), then we obtain

$$||\Omega_{\mathfrak{p}_n}-\Omega_{\mathfrak{q}}||_{\mathscr{PC}}\leq||\mathfrak{p}_n-\mathfrak{p}||_{\mathscr{PC}}$$

$$\leq\left(\sum_{i=1}^{\mathfrak{m}}\mathbb{L}_i+\mathfrak{g}(\iota)+\frac{\mathscr{K}\mathfrak{T}^{\mathfrak{w}}\mathfrak{g}(0)(\mathfrak{m}+1)(N_1(\mathfrak{p}(\iota))+N_2(\mathfrak{p}(\iota)))}{(1-\mathscr{G})\Gamma(\mathfrak{w}+1)}+\mathfrak{H}\right). \quad (21)$$

As $n\to\infty$ let $\mathfrak{p}_n\to\mathfrak{p}$ such that $||\Omega_{\mathfrak{p}_n}-\Omega_{\mathfrak{q}}||_{\mathscr{PC}}\to 0$. Consequently, Ω exhibits continuity.
Similarly for $\iota\in\mathfrak{I}_o$, the proof follows a comparable approach.
Step 2 Let Ω map Ξ to $\mathscr{PC}(\mathfrak{I}_{\mathfrak{T}},\ \mathbb{R})$.
Suppose $\mathfrak{x}_1,\ \mathfrak{x}_2\in\mathfrak{I}_l,\ l=1,\ 2,\ ...,\ \mathfrak{m}$, such that $\mathfrak{x}_1<\mathfrak{x}_2$, one can obtain

$$||\Omega_{\mathfrak{p}}(\mathfrak{x}_2)-\Omega_{\mathfrak{q}}(\mathfrak{x}_1)||_{\mathscr{PC}}=\sup_{\iota\in\mathfrak{I}_l}|(\Omega_{\mathfrak{p}})(\mathfrak{x}_2)-(\Omega_{\mathfrak{q}})(\mathfrak{x}_1)|$$

$$\leq \frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})}\left|\int_{\iota_l}^{\mathfrak{x}_1}(\mathfrak{x}_2-\rho(\tau))^{\mathfrak{w}-1}-(\mathfrak{x}_1-\rho(\tau)^{\mathfrak{w}-1})N_1(\mathfrak{p}(\iota))\mathbb{H}(\tau)\nabla\tau\right|+|\mathfrak{g}(\iota)|$$

$$+\frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})}\left|\int_{\mathfrak{x}_1}^{\mathfrak{x}_2}(\mathfrak{x}_2-\rho(\tau))^{\mathfrak{w}-1}N_2(\mathfrak{p}(\iota))\mathbb{H}(\tau)\nabla\tau\right|+\sum_{0<\iota_l<x_2-x_1}|\mathscr{I}_{\iota_l}(\iota_l,\ \mathfrak{p}(\iota_l^-))|$$

$$\leq \frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})}\left|\int_{\iota_l}^{\mathfrak{x}_1}(\mathfrak{x}_2-(\tau))^{\mathfrak{w}-1}-(\mathfrak{x}_1-(\tau)^{\mathfrak{w}-1})N_1(\mathfrak{p}(\iota))\mathbb{H}(\tau)\nabla\tau\right|+|\mathfrak{g}(\iota)|$$

$$+\frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})}\left|\int_{\mathfrak{x}_1}^{\mathfrak{x}_2}(\mathfrak{x}_2-(\tau))^{\mathfrak{w}-1}N_2(\mathfrak{p}(\iota))\mathbb{H}(\tau)\nabla\tau\right|+\sum_{0<\iota_l<x_2-x_1}|\mathscr{I}_{\iota_l}(\iota_l,\ \mathfrak{p}(\iota_l^-))|$$

$$\leq \frac{(\mathbb{A}+\mathbb{F}\omega)\mathfrak{g}(o)}{(1-\mathbb{E})\Gamma(\mathfrak{w})}\left(\left|\int_{\iota_l}^{\mathfrak{x}_1}(\mathfrak{x}_2-(\tau))^{\mathfrak{w}-1}-(\mathfrak{x}_1-(\tau)^{\mathfrak{w}-1})N_1(\mathfrak{p}(\iota))\mathbb{H}(\tau)\nabla\tau\right|\right.$$

$$\left.+\frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})}\left|\int_{\mathfrak{x}_1}^{\mathfrak{x}_2}(\mathfrak{x}_2-(\tau))^{\mathfrak{w}-1}N_2(\mathfrak{p}(\iota))\mathbb{H}(\tau)\nabla\tau\right|\right)+|\mathfrak{g}(\iota)|+\sum_{0<\iota_l<x_2-x_1}|\mathscr{I}_{\iota_l}(\iota_l,\ \mathfrak{p}(\iota_l^-))|.$$

Since $(\mathfrak{x}-(\tau))^{\mathfrak{w}-1}$ is continuous and if $x_1\to x_2$, so that $||\Omega_{\mathfrak{p}}(\mathfrak{x}_2)-\Omega_{\mathfrak{q}}(\mathfrak{x}_1)||_{\mathscr{PC}}\to 0$. Hence, Ω exhibits equicontinuity within $\mathfrak{I}_l$. As outcome for $\mathfrak{x}_1$ and $\mathfrak{x}_2$ within $\mathfrak{I}_o$ is similar, thus the result is omitted.

Step 3 Allow Ω to assign elements from Ξ to a bounded set of $\mathscr{PC}(\mathfrak{I}_{\mathfrak{T}},\ \mathbb{R})$.

It's evident from equation (10) that $||\Omega(a)||\leq\omega$ for $\omega\in\mathbb{R}$. Upon completing Steps 1 through 3 and employing the Arzela-Ascoli theorem, it becomes evident that Ω exhibits complete continuity.

Step 4 Assume $\gamma\in(0,\ 1)$, $l=\{\mathfrak{p}\in\mathscr{PC}(\mathfrak{I}_l,\ \mathbb{R}):\mathfrak{p}=\gamma\Omega(\mathfrak{p}),\ 0<\gamma<1\}$ be bounded. Again for $\iota\in\mathfrak{I}_l$, $l=1,\ 2,\ \ldots,\ \mathfrak{m}$, results as

$$|\mathfrak{p}(\iota)|=|\gamma\Omega(\mathfrak{p})\iota|=\left|\gamma\left(\vartheta(\mathfrak{p})+\mathfrak{g}(\iota)+\frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})}\sum_{i=1}^{l}\int_{\iota_i-1}^{\iota_i}(\iota-\rho(\tau))^{\mathfrak{w}-1}N_1(\mathfrak{p}(\iota))\mathbb{H}(\tau)\nabla\tau\right.\right.$$

$$\left.\left.+\frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})}\int_{\iota_l}^{\iota}(\iota-\rho(\tau))^{\mathfrak{w}-1}N_2(\mathfrak{p}(\iota))\mathbb{H}(\tau)\nabla\tau+\sum_{i=1}^{l}\mathscr{I}_i(\iota_i,\ \mathfrak{p}(\iota_i^-))\right)\right|$$

$$\leq \nu||\mathfrak{p}||_{\mathscr{PC}}+\sum_{i=1}^{n}\mathscr{M}_i+\mathfrak{g}(\iota)\frac{(\mathbb{A}+\mathbb{F}||\mathfrak{p}||_{\mathscr{PC}})\mathfrak{g}(0)\mathfrak{T}^{\mathfrak{w}}(\mathfrak{m}+1)(N_1(\mathfrak{p}(\iota))+N_2(\mathfrak{p}(\iota)))}{\Gamma(\mathfrak{w}+1)(1-\mathbb{E})}.$$

Thus,

$$\frac{||\mathfrak{p}||_{\mathscr{PC}}}{\nu||\mathfrak{p}||_{\mathscr{PC}}+\sum_{i=1}^{n}\mathscr{M}_i++\mathfrak{g}(\iota)\frac{(\mathbb{A}+\mathbb{F}||\mathfrak{p}||_{\mathscr{PC}})\mathfrak{g}(0)\mathfrak{T}^{\mathfrak{w}}(\mathfrak{m}+1)(N_1(\mathfrak{p}(\iota))+N_2(\mathfrak{p}(\iota)))}{\Gamma(\mathfrak{w}+1)(1-\mathbb{E})}}\leq 1.$$

Equation (17) yields a '+' ve constant β such that $||\mathfrak{p}||_{\mathscr{PC}}\neq\beta$. Suppose $\psi=\{\mathfrak{p}\in\mathscr{PC}(\mathfrak{I}_{\mathfrak{T}},\ \mathbb{R}):||\mathfrak{p}||_{\mathscr{PC}}<\beta\}$ such that $\Omega:\tilde{\psi}\to\mathscr{PC}(\mathfrak{I}_{\mathfrak{T}},\ \mathbb{R})$ exhibits continuity and entirely continuity. So there exists no $\mathfrak{p}\in\partial(\psi)$ such that $\mathfrak{p}=\gamma\Omega(\mathfrak{p})$, $\gamma\in(0,\ 1)$. Therefore, according to the nonlinear alternative of Leray-Schauder's fixed poin theorem, it follows for Ω, solution of equation (1) corresponds to a fixed point.

Result of $\iota\in\mathfrak{I}_o$ is nearly the same; hence, it is excluded.

Following this a numerical example represents the main findings. Whereas in the future we delve into exploring the impact of more complex neutral terms, such as those involving multiple delays or nonlinearities with application in real world problem.

4. Example

Example 4.1 Contemplate an initial condition that spans across nonlocality over a time range within a dynamic equation featuring neutral impulses $\mathfrak{T}=\left[0,\ \frac{1}{5}\right]\cup\left[\frac{1}{4},\ 1\right]$.

$$\begin{cases} {}^{C}D^{\frac{1}{4}}[\mathfrak{p}(\iota)-\mathfrak{g}(\iota,\ \mathfrak{p},\ N_1(\mathfrak{p}(\iota)))]=\dfrac{e^{-5\iota}[4+\mathfrak{g}(0)N_2(\mathfrak{p}(\iota))(|\mathfrak{p}(\iota)|+|{}^{C}D^{\mathfrak{w}}\mathfrak{p}(\iota)|)+\mathfrak{g}(\iota)]}{25e^{2\iota}(1+|\mathfrak{p}(\iota)|)}, \\ \iota\in[0,\ 1]\cap\mathfrak{T},\ \iota\neq\dfrac{1}{5}. \\ \mathfrak{p}\left(\dfrac{1}{5}^{+}\right)-\mathfrak{p}\left(\dfrac{1}{5}^{-}\right)=\dfrac{1+\mathfrak{p}\left(\dfrac{1}{5}\right)}{15},\quad \iota_1=\dfrac{1}{5}. \\ \mathfrak{p}(0)=\dfrac{\mathfrak{p}}{10}. \end{cases} \tag{22}$$

We set

$$\mathscr{L}(\iota,\ \mathfrak{p},\ \mathfrak{q})=\frac{e^{-5\iota}[4+\mathfrak{g}(0)(|\mathfrak{p}(\iota)|+|\mathfrak{q}(\iota)|)+\mathfrak{g}(\iota)]}{25e^{2\iota}(1+|\mathfrak{p}(\iota)|)}. \tag{23}$$

It is clear that the r.h.s of equation (23) exhibits continuity for $\mathfrak{p},\ \mathfrak{q}\in\mathbb{R}$ across the time scale. Consequently for all $\iota\in[0,\ 1]\cap\mathfrak{T}$ and $\mathbb{H},\ \mathfrak{I}\in\mathbb{R}$, one get

$$\begin{aligned} \mathscr{L}(\iota,\ \mathfrak{p},\ \mathfrak{q}) &\leq \frac{4+\mathfrak{g}(0)N_2(\mathfrak{p}(\iota))(|\mathfrak{p}(\iota)|+|\mathfrak{q}(\iota)|)+\mathfrak{g}(\iota)}{25e^2} \\ &\leq \frac{4}{25e^2}+\frac{1}{25e^2}|\mathfrak{p}(\iota)|+\frac{1}{25e^2}|\mathfrak{q}(\iota)|+\frac{2}{25e^2}. \end{aligned} \tag{24}$$

Then we get, $\mathbb{A} = \frac{4}{25e^2}$, $\mathbb{F} = \frac{1}{25e^2}$, $\mathbb{E} = \frac{1}{25e^2}$, $\mathfrak{g}(0) = 1$, $\mathfrak{g}(\iota) = \frac{2}{25e^2}$. Next

$$|\mathscr{L}(\iota,\ \mathfrak{p},\ \mathfrak{q}) - \mathscr{L}(\iota,\ \mathbb{H},\ \mathfrak{I})| \leq \frac{1}{25e^2}|\mathfrak{p} - \mathbb{H}| + \frac{1}{25e^2}|\mathfrak{q} - \mathfrak{I}|,$$

$$|\mathscr{I}_1(\iota,\ \mathfrak{p}) - \mathscr{I}_1(\iota,\ \mathfrak{q})| \leq \frac{1}{15}|\mathfrak{p} - \mathbb{H}|,\ |\vartheta(\mathfrak{p}) - \vartheta(\mathbb{H})| \leq \frac{1}{10}|\mathfrak{p} - \mathbb{H}|,\ |\vartheta(\mathfrak{p})| \leq \frac{1}{10}.$$

Thus one can obtain $\mathscr{K} = \frac{1}{25e^2}$, $\mathscr{G} = \frac{1}{25e^2}$, $\mathbb{L} = \frac{1}{15}$, $\mathfrak{H} = \frac{1}{10}$ and say, $N_1(\mathfrak{p}(\iota)) = \frac{1}{30}$, $N_2(\mathfrak{p}(\iota)) = \frac{1}{20}$. Therefore, based on the provided data, it can be concluded that equation (22) fulfills all the criteria outlined in (A1) through (A5).

Consequently, for $\mathfrak{m} = 1$ one can obtain

$$\mathbb{L} + \mathfrak{g}(\iota) + \frac{\mathscr{K}\mathfrak{T}^{\mathfrak{w}}\mathfrak{g}(0)(\mathfrak{m}+1)(N_1(\mathfrak{p}(\iota)) + N_2(\mathfrak{p}(\iota)))}{(1-\mathscr{G})\Gamma(\mathfrak{w}+1)} + \mathfrak{H} \leq \frac{1}{15} + \frac{1}{10} + \frac{1}{25e^2} + \frac{4\frac{1}{25e^2}}{\left(1 - \frac{1}{25e^2}\right)\Gamma\left(\frac{1}{4}+1\right)}$$

$$\leq 1.$$

$\therefore$ The criteria specified in Theorem 3.3 have been met, leading us to conclude the uniqueness of the solution to equation (22).

Example 4.2

Consider the following fractional dynamic equation with impulses $\mathfrak{T} = \left[0,\ \frac{1}{3}\right] \cup \left[\frac{1}{2},\ 1\right]$.

$$\begin{cases} {}^{C}D^{\frac{1}{2}}[\mathfrak{p}(\iota) - \mathfrak{g}(\iota,\ \mathfrak{p},\ N_1(\mathfrak{p}(\iota)))] = \dfrac{e^{-3\iota}[2 + \mathfrak{g}(0)N_2(\mathfrak{p}(\iota))(|\mathfrak{p}(\iota)| + |{}^{C}D^{\mathfrak{w}}\mathfrak{p}(\iota)|) + \mathfrak{g}(\iota)]}{9e^2\iota(1 + |\mathfrak{p}(\iota)|)}, \\ \iota \in [0,\ 1] \cap \mathfrak{T},\ \iota \neq \dfrac{1}{3}. \\ \mathfrak{p}\left(\dfrac{1}{3}^{+}\right) - \mathfrak{p}\left(\dfrac{1}{3}^{-}\right) = \dfrac{1 + \mathfrak{p}\left(\frac{1}{3}\right)}{9}, \quad \iota_1 = \dfrac{1}{3}. \\ \mathfrak{p}(0) = \dfrac{\mathfrak{p}}{6}. \end{cases} \tag{25}$$

We set

$$\mathscr{L}(\iota,\ \mathfrak{p},\ \mathfrak{q}) = \frac{e^{-3\iota}[2 + \mathfrak{g}(0)(|\mathfrak{p}(\iota)| + |\mathfrak{q}(\iota)|) + \mathfrak{g}(\iota)]}{9e^2\iota(1 + |\mathfrak{p}(\iota)|)}. \tag{26}$$

It is clear that the r.h.s of equation (25) exhibits continuity for $\mathfrak{p},\ \mathfrak{q} \in \mathbb{R}$ across the time scale. Consequently for all $\iota \in [0,\ 1] \cap \mathfrak{T}$ and $\mathbb{H},\ \mathfrak{I} \in \mathbb{R}$, one get

$$\mathscr{L}(\iota,\ \mathfrak{p},\ \mathfrak{q}) \leq \frac{2+\mathfrak{g}(0)N_2(\mathfrak{p}(\iota))(|\mathfrak{p}(\iota)|+|\mathfrak{q}(\iota)|)+\mathfrak{g}(\iota)}{9e^2}$$
$$\leq \frac{2}{9e^2}+\frac{1}{9e^2}|\mathfrak{p}(\iota)|+\frac{1}{9e^2}|\mathfrak{q}(\iota)|+\frac{1}{9e^2}. \tag{27}$$

Then we get, $\mathbb{A}=\frac{2}{9e^2},\ \mathbb{F}=\frac{1}{9e^2},\ \mathbb{E}=\frac{1}{9e^2},\ \mathfrak{g}(0)=1,\ \mathfrak{g}(\iota)=\frac{1}{9e^2}$. Next

$$|\mathscr{L}(\iota,\ \mathfrak{p},\ \mathfrak{q})-\mathscr{L}(\iota,\ \mathbb{H},\ \mathfrak{I})| \leq \frac{1}{9e^2}|\mathfrak{p}-\mathbb{H}|+\frac{1}{9e^2}|\mathfrak{q}-\mathfrak{I}|,$$

$$|\mathscr{I}_1(\iota,\ \mathfrak{p})-\mathscr{I}_1(\iota,\ \mathfrak{q})| \leq \frac{1}{9}|\mathfrak{p}-\mathbb{H}|,\ |\vartheta(\mathfrak{p})-\vartheta(\mathbb{H})| \leq \frac{1}{5}|\mathfrak{p}-\mathbb{H}|,\ |\vartheta(\mathfrak{p})| \leq \frac{1}{5}.$$

Thus one can obtain $\mathscr{K}=\frac{1}{9e^2},\ \mathscr{G}=\frac{1}{9e^2},\ \mathbb{L}=\frac{1}{9},\ \mathfrak{H}=\frac{1}{5}$ and say, $N_1(\mathfrak{p}(\iota))=\frac{1}{20},\ N_2(\mathfrak{p}(\iota))=\frac{1}{10}$.

Now, let us add a control function in the dynamic equation and suppose when $\mathfrak{T}=\mathbb{R}$, then $[0,\ 3]_{\mathfrak{T}}=[0,\ 3]$. Also we set $\mathfrak{w}=\frac{1}{4},\ t_o=s_o=0,\ \mathfrak{p}(t_1)=2$, and $\mathfrak{p}(T)=3,\ t_1=\frac{2}{5},\ s_1=\frac{3}{5}$. Therefore, the control function $\mathfrak{u}(t)$ is given by,

$$u(t)=\begin{cases} (\mathscr{W}_o^{t_1})^{-1}\left(2+\frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})}\int_0^{t_1}(t_1-\rho(\tau))^{\mathfrak{w}-1}\left(\frac{e^{-3\iota}[2+\mathfrak{g}(0)N_2(\mathfrak{p}(\iota))(|\mathfrak{p}(\iota)|+|^CD^{\mathfrak{w}}\mathfrak{p}(\iota)|)+\mathfrak{g}(\iota)]}{9e^2\iota(1+|\mathfrak{p}(\iota)|)}\right)\Delta\tau\right)(t), \\ t\in[0,\ t_1] \\ (\mathscr{W}_{s_1}^{T})^{-1}\left(3+\frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})}\int_{t_1}^{s_1}(s_1-\rho(\tau))^{\mathfrak{w}-1}\left(\frac{1+\mathfrak{p}\left(\frac{1}{3}\right)}{9}\right)\right. \\ \left.-\frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})}\int_{s_1}^{T}(T-\rho(\tau))^{\mathfrak{w}-1}\left(\frac{e^{-3\iota}[2+\mathfrak{g}(0)N_2(\mathfrak{p}(\iota))(|\mathfrak{p}(\iota)|+|^CD^{\mathfrak{w}}\mathfrak{p}(\iota)|)+\mathfrak{g}(\iota)]}{9e^2\iota(1+|\mathfrak{p}(\iota)|)}\right)\Delta\tau\right)(t),\ \ t\in[s_1,\ T] \end{cases}$$

where,

$$\mathscr{W}_o^{t_1}=\frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})}\int_0^{t_1}(t_1-\rho(\tau))^{\mathfrak{w}-1}\Delta\tau$$

and

$$\mathscr{W}_{s_1}^{T}=\frac{\mathfrak{g}(0)}{\Gamma(\mathfrak{w})}\int_{t_1}^{s_1}(s_1-\rho(\tau))^{\mathfrak{w}-1}\Delta\tau,$$

with $B=1$ in the control system (25) becomes,

$$\begin{cases} {}^{C}D^{\frac{1}{2}}[\mathfrak{p}(t) - \mathfrak{g}(t,\ \mathfrak{p},\ N_1(\mathfrak{p}(t)))] = \dfrac{e^{-3t}[2 + \mathfrak{g}(0)N_2(\mathfrak{p}(t))(|\mathfrak{p}(t)| + |{}^{C}D^{\mathfrak{w}}\mathfrak{p}(t)|) + \mathfrak{g}(t)]}{9e^2 t(1 + |\mathfrak{p}(t)|)} + u(t), \\ t \in [0,\ 1] \cap \mathfrak{T},\ t \neq \dfrac{1}{3}. \\ \mathfrak{p}\left(\dfrac{1}{3}^{+}\right) - \mathfrak{p}\left(\dfrac{1}{3}^{-}\right) = \dfrac{1 + \mathfrak{p}\left(\dfrac{1}{3}\right)}{9}, \quad t_1 = \dfrac{1}{3}. \\ \mathfrak{p}(0) = \dfrac{\mathfrak{p}}{6}. \end{cases} \tag{28}$$

Now we find that,

$$\mathscr{W}_o^{\frac{2}{5}} = 0.812,$$

$$\mathscr{W}_{\frac{3}{5}}^{3} = 1.347.$$

Therefore, based on the provided data, it can be concluded that equation (28) fulfills all the criteria outlined in (A1) through (A6).

Consequently, for $\mathfrak{m} = 1$ one can obtain

$$\left(\mathbb{L} + \mathfrak{g}(t) + \mathfrak{H} + \frac{\mathscr{K}\mathfrak{T}^{\mathfrak{w}}\mathfrak{g}(0)(\mathfrak{m}+1)(N_1(\mathfrak{p}(t)) + N_2(\mathfrak{p}(t)))}{(1-\mathscr{G})\Gamma(\mathfrak{w}+1)}\right)\left(\frac{\mathscr{M}_B\mathscr{M}_{\mathfrak{w}}^{i}}{\Gamma\left(\frac{1}{4}+1\right)}\right)$$

$$\leq \left(\frac{1}{15} + \frac{1}{10} + \frac{1}{25e^2} + \frac{4\frac{1}{25e^2}}{\left(1 - \frac{1}{25e^2}\right)\Gamma\left(\frac{1}{4}+1\right)}\right)\left(\frac{1.347}{\Gamma\left(\frac{1}{4}+1\right)}\right)$$

$$\leq 1.$$

$\therefore$ All the assumptions have been met, concluding that (28) is totally controllable.

Figure 1 displays a robust concurrence between numerical solution and the precise solution over complete range.

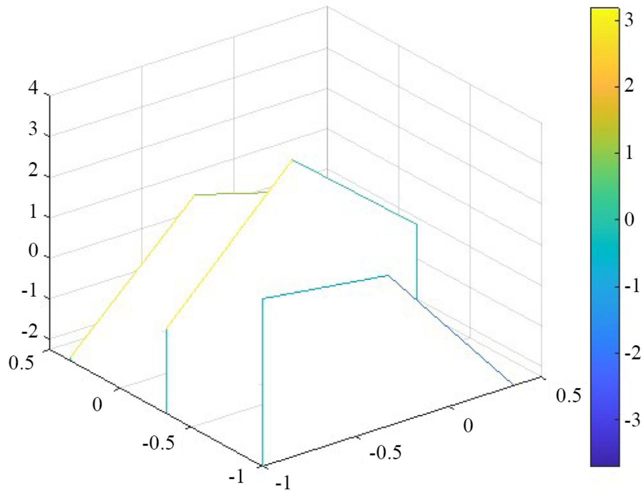

Figure 1. The graph depicting the approximate solution of $\mathfrak{p}(\iota)$

Table 1 below illustrates the numerical method corresponding to the theoretical findings.

Table 1. The fluctuation of $\mathfrak{p}(\iota)$ across various $\mathbb{L}$ and $\mathfrak{g}$ values

$\mathfrak{g}\downarrow$	$\mathbb{L}$ = 1/15	$\mathbb{L}$ = 1/20	$\mathbb{L}$ = 1/25	$\mathbb{L}$ = 1/30	$\mathbb{L}$ = 1/35
1/50	0.6778	0.6611	0.6511	0.6445	0.6397
1/45	0.6800	0.6633	0.6533	0.6467	0.6419
1/40	0.6828	0.6561	0.6447	0.6383	0.6343
1/35	0.6864	0.6697	0.6597	0.6530	0.6483
1/30	0.6911	0.6645	0.6530	0.6467	0.6426

5. Conclusion

This paper explores C∇D through an in-depth analysis across various time scales. It also examines fractional dynamic equations of C∇D, incorporating immediate impulses and a nonlocal initial condition. Furthermore, it includes two illustrative examples showcasing theoretical insights on solution of uniqueness and existence, complemented by a MATLAB-generated graphical representation.

Author's contributions

All author's contributed equally.

Conflict of interest

The authors declares that, they have no conflict of interest.

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