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Research Article

Theta Pairing of Hypersurface Rings

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Abstract: In this article, we prove a conjecture on the positive definiteness of the Hochster Theta pairing over a general isolated hypersurface singularity, namely: Let R be an admissible isolated hypersurface singularity of dimension n. If n is odd, then $(-1)^{(n+1)/2}\Theta$ is positive semi-definite on $K_0'(R)_{\mathbb{Q}}$. The conjecture is expected to be true for the polynomial ring over any field. We prove this conjecture over any field of arbitrary characteristic. We also provide two different proofs of the above conjecture over $\mathbb{C}$ using the Hodge theory of isolated hypersurface singularities and structural facts about the category of matrix factorizations. The first proof over $\mathbb{C}$ is a more complete and developed version of a former work of the author. We have extended some of the former results in this article. The second proof over $\mathbb{C}$ is quite direct and uses a former result of the author on Riemann-Hodge bilinear relations for Grothendieck residue pairing of isolated hypersurface singularities.

Keywords: Hochster Theta pairing, matrix factorization, Riemann-Hodge bilinear relations, residue pairing, chern character

MSC: 14A05, 14A10, 14A15, 14C05, 14C15, 14C17

1. Introduction

In this paper we study Theta pairing of Hochster on a hypersurface ring of the form $R := P/(f)$, where P is an arbitrary ring and f is a non-zero divisor. We may assume P is a local ring of dimension $n+1$ by localization. In our case we assume $P = \mathbb{C}\{x_0, \ldots, x_n\}$ and f a holomorphic germ, or $P = \mathbb{C}[x_0, \ldots, x_n]$ and then f would be a polynomial. To study a hypersurface ring R we consider finitely generated modules over that. In our case the element f defines a map $f : \mathbb{C}^{n+1} \to \mathbb{C}$, and we choose a representative for the Milnor fibration as $f : X \to T$, where T is the disc around 0. We assume $0 \in \mathbb{C}^{n+1}$ is the only singularity of f. A fundamental concept related to hypersurface rings is that of the matrix factorization.

Definition 1 A matrix factorization of f in P is a pair of matrices A and B such that $AB = BA = f \cdot \mathrm{Id}$.

Because, in a free resolution of such an R-module M, the depth of the successive syzygies strictly increases, after n-steps we reach to an exact sequence of the form.

$$0 \to M' \to F_{n-1} \to F_{n-2} \to \ldots \to F_0 \to M \to 0 \tag{1}$$

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where the F_i are free R-modules of finite rank and $\operatorname{depth}_R(M') = n$. Then either $M' = 0$, or M' is a maximal Cohen-Macaulay module, that is $\operatorname{depth}_R(M') = n$. It follows that "up to free modules" any R-module can be replaced by a maximal Cohen-Macaulay module in Grothendieck ring of R. If M is a maximal Cohen-Macaulay R-module that is minimally generated by p elements, its resolution as P-module has the form

$$\begin{array}{ccccccccc} 0 & \to & P^p & \stackrel{A}{\to} & P^p & \to & M & \to & 0 \\ & & \downarrow & \stackrel{B}{\swarrow} & \downarrow & & \downarrow 0 & & \\ 0 & \to & P^p & \stackrel{A}{\to} & P^p & \to & M & \to & 0 \end{array}$$

where A is some $p \times p$ matrix with $\det(A) = f^q$. The fact that multiplication by f acts as 0 on M produces a matrix B such that $A \cdot B = B \cdot A = f \cdot \mathrm{Id}$, where I is the identity matrix. This gives a matrix factorization (A, B) of f determined uniquely up to base change in the free module P^p, by M. This leads to the observation, due to D. Eisenbud that, any R-module has a minimal resolution that is eventually 2-periodic. It follows that all the homological invariants like $\operatorname{Tor}_k^R(M, N)$, $\operatorname{Ext}_R^k(M, N)$ are 2-priodic [1, 2]. Based on this fact, M. Hochster defines the following invariant of a hypersurface ring, called Θ-invariant.

Definition 2 (Hochster Theta pairing) Assume R is a hypersurface ring and M and N are finitely generated R-modules. The theta pairing of the two R-modules M and N over R is defined by

$$\Theta(M, N) := l(\operatorname{Tor}_{2k}^R(M, N)) - l(\operatorname{Tor}_{2k+1}^R(M, N)), \quad k \gg 0. \tag{2}$$

The definition makes sense as soon as the lengths appearing are finite. This certainly happens if R has an isolated singular point.

Let $K_0'(R)$ be the Grothendieck group of finitely generated R-modules, i.e. the free abelian group on the finitely generated modules modulo relations obtained by short exact sequences. One can simply show that, the Hochster theta pairing $\Theta(\cdot, \cdot)$ is additive on short exact sequences in each argument, and thus determines a $\mathbb{Z}$-valued pairing on $K_0'(R)$, i.e. we have a well-defined pairing $\Theta : K_0'(R) \times K_0'(R) \to \mathbb{Z}$. One loses no information by tensoring with $\mathbb{Q}$ and often theta is interpreted as a symmetric bilinear form on the rational vector space $K_0'(R)_{\mathbb{Q}}$.

1.1 *Problem statement*

According to the results in [3] one serves evidence to propose the following conjecture about the Θ-pairing in general.

Conjecture 1 [3] Assume R is an admissible isolated hypersurface singularity of dimension n. In case, n is odd, the pairing $(-1)^{(n+1)/2}\Theta$ is positive semi-definite on $K_0'(R)_{\mathbb{Q}}$.

The above conjecture is expected to be true over any field. In [3] it is proven for homogeneous polynomials f over the complex numbers using Riemann-Hodge bilinear relations for the variation of Hodge structure defined by the isolated singularity f. In this text, we propose to prove several results concerning the Conjecture 1 for arbitrary possibly inhomogeneous isolated singularity over $\mathbb{C}$.

1.2 *Contributions of the text*

We prove the Conjecture 1 in general over an arbitrary field k of $\operatorname{char}(k) \neq 2$. The Theorem appears as the last Theorem in the text namely Theorem 8. We generally show that the Hochster Theta paring in the homogeneous case and the non-homogeneous case are related naturally (see Theorem 4 and Corollary 1). The proof of the Theorem 7 which was the main result of [4] is extended in more detail in this version of the article. One reason was to express some details that might raise doubts about the proof. The proof is a technical matter, of how the mixed Hodge structure of isolated singularities in the case of a non-homogeneous polynomial is defined. This article is a developed version of [4–6]. We present more contributions than [4] where the statement of some of them appeared informally in [4]. For example, Lemma 1, Proposition 1, and Proposition 3 are of this form. We also provide a second proof of the Conjecture 1 over $\mathbb{C}$ using Riemann-Hodge bilinear relations for the Grothendieck residue pairing, and a Theorem of Polishchuk-Vaintrob [7]. The difference between this work and [3, 8] is the matter of how to deal with an inhomogeneous polynomial isolated singularity. The results and contributions appear with their proofs in the text.

1.3 *Related works*

A simple and former version of this work can be found in [4–6]. In [3] the authors state and prove the Conjecture 1 for homogeneous polynomials (hypersurfaces) over complex numbers (see [3] Theorem 3.4). However, one open part of the conjecture is open when the isolated singularity is not homogeneous. This is the contribution in [4, 5], where the proof of the Conjecture 1 over $\mathbb{C}$ is given based on [9, 10]. In the current version, we have extended the proof for more details and have added several additional contributions. In the original work of Buchweitz [1] the concept of Theorem 3 is postponed as a concluding remark for a possible future work, namely (Remark 5.4 in [1]). Another recent paper is [8], where a more developed version of the Conjecture 1 is proven using Adams operations on localized chern classes. The paper [8] addresses some results from [11].

2. Theta pairing

When $P = \mathbb{C}[x_0, \ldots, x_n]$ and then f is a polynomial, we shall consider the two cases where f is homogeneous or not separately.

2.1 *Homogeneous case*

When $f \in P$ is a homogeneous polynomial, there is a complete description of the Hochster theta pairing in terms of intersection multiplicities [1, 3]. In fact, using the additivity property of Θ mentioned above we can assume $M = \mathscr{O}_Y = R/I$, $N = \mathscr{O}_Z = R/J$, where $Y, Z \subseteq X_0$ are the sub-varieties defined by the ideals I, J respectively. Then it is simply known $\Theta(\mathscr{O}_Y, \mathscr{O}_Z) = i(0; Y, Z)$ in case that $Y \cap Z = 0$, where $i(0; ., .)$ is the ordinary intersection multiplicity in $\mathbb{C}^{n+1}$, see [1]. Any R-module admits a finite filtration with sub-quotients of the form R/I, knowing $\Theta(\mathscr{O}_Y, \mathscr{O}_Z)$ determines $\Theta(M, N)$ for all modules M, N. One can also formulate the above by the cup product in cohomology.

Theorem 1 [1] Assume $f \in \mathbb{C}[x_1, \ldots, x_{2m+2}]$ is a homogeneous polynomial of degree d, and $X_0 = f^{-1}(0) \in \mathbb{C}^{2m+2}$ and $\overline{X_0} = V(f) \in \mathbb{P}^{2m+1}$ the associated projective cone of degree d. Let Y and Z be also co-dimension m cycles in $\overline{X_0}$. If Y, Z intersect transversely, then

$$\Theta(\mathscr{O}_Y, \mathscr{O}_Z) = -\frac{1}{d}[[Y]] \cdot [[Z]] \tag{3}$$

where $[[Y]] := d[Y] - \deg(Y) \cdot h^m$ is the primitive class of $[Y]$, with $h \in H^1(\overline{X_0})$ the hyperplane class.

The primitive class of a cycle is the projection of its fundamental class into the orthogonal complement to h^m with respect to the intersection pairing. Here $h^{2m} = d = \deg(\overline{X_0})$ and $[Y] \cdot h^m = \deg(Y)$, the description of the primitive class follows. Substituting, the claim can be reformulated as

$$\Theta(\mathscr{O}_Y, \mathscr{O}_Z) = \frac{-1}{d}[[Y]] \cdot [[Z]] = -d[Y] \cdot [Z] + \deg(Y)\deg(Z), \tag{4}$$

The above formula gives a way to calculate Θ when f is homogeneous.

Over $\mathbb{C}$ when $f \in \mathbb{C}[x_0, \ldots, x_n]$ in consideration is a homogeneous polynomial of degree d, such that $X := \mathrm{Proj}(R)$ is a smooth projective $\mathbb{C}$-variety, the Theta pairing is induced, via chern character map, from the pairing on the primitive part of de Rham cohomology. W. Morre et al. show that when n is odd, there is a commutative diagram

$$\begin{array}{ccc} K_0'(R)_{\mathbb{Q}}^{\otimes 2} & \xleftarrow[\cong]{} & \left(\dfrac{K(X)_{\mathbb{Q}}}{\alpha}\right)^{\otimes 2} \\ \Theta\big\downarrow & & \big\downarrow (\mathrm{ch}^{n-1/2})^{\otimes 2} \\ \mathbb{C} & \xleftarrow[\theta]{} & \left(\dfrac{H^{(n-1)/2}(X,\, \mathbb{C})}{\mathbb{C}\cdot h^{(n-1)/2}}\right)^{\otimes 2} \end{array} \tag{5}$$

where h is the class of a hyperplane section, $K(X)$ is the ordinary Grothendieck group of X as a smooth projective variety, and $\alpha = [\mathscr{O}_X] - [\mathscr{O}_X(1)]$. The map θ downstairs is also called the Theta pairing and satisfies the following,

$$\theta: \frac{H^{(n-1)/2}(X,\, \mathbb{C})}{\mathbb{C}\cdot h^{(n-1)/2}} \times \frac{H^{(n-1)/2}(X,\, \mathbb{C})}{\mathbb{C}\cdot h^{(n-1)/2}} \to \mathbb{C} \tag{6}$$

$$(a,\, b) \longmapsto \left(\int_X a \cup h^{(n-1)/2}\right)\left(\int_X a \cup h^{(n-1)/2}\right) - d\left(\int_X a \cup b\right) \tag{7}$$

and Theta would vanish for n even. When $n = 1$ by h^0 we mean $1 \in H^0(X,\, \mathbb{C})$ [3].

Theorem 2 [3] For R and X as above and n odd the restriction of the pairing $(-1)^{(n+1)/2}\Theta$ to

$$\mathrm{image}\left(\mathrm{ch}^{\frac{n-1}{2}}\right): \frac{K(X)_{\mathbb{Q}}}{\langle\alpha\rangle} \longrightarrow \frac{H^{(n-1)/2}(X,\, \mathbb{C})}{\mathbb{C}\cdot h^{\frac{n-1}{2}}} \tag{8}$$

is positive definite. i.e. $(-1)^{(n+1)/2}\Theta(v,\, v) \geq 0$ with equality holding if and only if $v = 0$. In this way, θ is semi-definite on $K_0'(R)$.

We outline the proof of the Theorem 2. Let $\mathrm{Hg}^p_{\mathbb{Q}} = H^{n-1}_{\mathbb{Q}} \cap H^{(p,\, p)}(X)$, $p = (n-1)/2$ be the subspace of Hodge classes. The image of the chern class map is generally contained in $\mathrm{Hg}^p_{\mathbb{Q}}$, cf. [12], sec. 19.3.6, page 387. W. Moore et al. define an injection

$$\begin{aligned} &\mathrm{em}\colon \mathrm{Hg}^p_{\mathbb{Q}}/\mathbb{Q}\cdot h^p \hookrightarrow H^{n-1}(X,\, \mathbb{Q}), \\ &\mathrm{em}(a) = a - \frac{\int_X a \cup h^p}{d} h^p, \quad a \in \mathrm{Hg}^p_{\mathbb{Q}}. \end{aligned} \tag{9}$$

The image of em is contained in $H^{p,\, p}(X)$. It is also contained in the primitive part of $H^{n-1}(X,\, \mathbb{Q})$, i.e., in the subspace of elements x with $h \cup x = 0$. Indeed, by the Lefschetz hyperplane theorem the vanishing of $h \cup \mathrm{em}(a)$ follows from the vanishing of $\int_X h^p \cup \mathrm{em}(a)$, which is clear. A straightforward computation verifies that

$$\Theta(a,\, b) = -d \cdot Q(\mathrm{em}(a),\, \mathrm{em}(b)), \quad a,\, b \in \mathrm{Hg}^p_{\mathbb{Q}}, \tag{10}$$

where Q is the Poincare pairing. The Hodge-Riemann bilinear relations ([13], page 160-169) give that $(-1)^{\frac{(n-1)(n-2)}{2}} Q$ is positive definite on the primitive part of $\mathrm{Hg}_{\mathbb{Q}}^{p}$ and hence on the image of e, see [3].

2.2 *Non-homogeneous case*

When the polynomial f is not homogeneous one should modify the above arguments as follows. Assume $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is a germ of isolated singularity. We choose a representative $f_X : X \to T$ over a small disc T according to the Milnor fibration theorem. It is standard (see [5, 9, 10]) to embed the Milnor fibration $f_X : X \to T$ into a compactified (projective) fibration $f_Y : Y \to T$ such that the fiber Y_t sits in $\mathbb{P}^{n+1}$ for $t \neq 0$. The projective fibration f_Y has a unique singularity at $0 \in Y_0$ over $t = 0$. The following theorem defines a polarized MHS of the middle cohomology of Milnor fibration f_X.

Theorem 3 [9] Assume $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is an isolated hypersurface singularity. Then $H^n(X_\infty, \mathbb{Q})$ is equipped with a PMHS of weight n on $H^n(X_\infty, \mathbb{Q})_{\neq 1}$ and a PMHS of weight $n+1$ on $H^n(X_\infty, \mathbb{Q})_1$, where H_1^n and $H_{\neq 1}^n$ are generalized eigenspaces of the monodromy M.

Define $\Omega_f := \Omega_X^{n+1}/df \wedge \Omega_X^n$. The module of relative differentials Ω_f is a $\mathbb{C}$-vector space of rank μ, the Milnor number of the isolated singularity f. One can define a bilinear form

$$\mathrm{res}_{f,\,0} : \Omega_f \times \Omega_f \to \mathbb{C} \tag{11}$$

$$(g_1 dx,\ g_2 dx) \longmapsto \mathrm{Res}_0 \left[\frac{g_1 g_2 dx}{\dfrac{\partial f}{\partial x_0} \cdots \dfrac{\partial f}{\partial x_n}} \right]$$

It is a symmetric bilinear pairing (Grothendieck residue pairing), which is non-degenerate [14]. By [5, 15] the module Ω_f has a graded polarized complex MHS (see [16, 17] for the exact definition). Let $\Omega_f = \bigoplus J^{p,\,q}$ be the Hodge-Deligne decomposition, cf. [18, 19]. Define the operator C on Ω_f by setting $C|_{J^{p,\,q}} = (-1)^p$ and also a pairing on $\Omega_f \times \Omega_f$ by $\psi_f(.,\ .) := \mathrm{res}_{f,\,0}(.,\ C\ .)$.

Theorem 4 ([5] page 117) There is an isomorphism $\phi_f : \Omega_f \to H^n(X_\infty)$ such that the pairing $\psi_f : \Omega_f \times \Omega_f \to \mathbb{C}$ is the pull back of the polarization form $S : H^n(X_\infty) \times H^n(X_\infty) \to \mathbb{C}$ under ϕ_f up to a multiplication by a scalar constant $\gamma \in \mathbb{C}$ as $S(\phi_f\ .,\ \phi_f\ .) = \psi_f(.,\ .) \times \gamma$.

The mixed Hodge structure on Ω_f is defined via ϕ [5, 15]. In case $f = f(z)$ is a quasihomogeneous polynomial, the mixed Hodge structure can be explained via the monomial basis of the Milnor or Jacobi ring. In this case the map ϕ is given by

$$[z^\alpha dz] \longmapsto c_\alpha \cdot \left[\mathrm{res}_{f=1}(z^\alpha dz/(f-1)^{[l(\alpha)]}) \right], \quad l(\alpha) = \sum (\alpha_i + 1) w_i,\ c_\alpha \in \mathbb{C},$$

where w_i is the weight of z_i, and z^α is a monomial basis of Jacobi ring of f. The Hodge structure is the same as pole filtration and can be explained by the degrees $l(\alpha)$ [5, 20, 21].

Remark 1 Unfortunately, the map ϕ fails to preserve the $\mathbb{Q}$ or even $\mathbb{R}$ structures, cf. [5, 20]. That is, the MHS on Ω_f is just a graded polarized complex MHS, as it is defined in [16, 17]. This notion is weaker than $\mathbb{Q}$-MHS.

Remark 2 The Theorem 4 allows us to define Riemann-Hodge bilinear relations for the residue pairing $\mathrm{res}_{f,\,0}$ on Ω_f.

Remark 3 The constant $\gamma \in \mathbb{C}$ is related to the well-known Γ-class in the cohomology of X_∞. In other words, the difference on the $\mathbb{Z}$ structure on both sides of the map ϕ is measured by the integral of some characteristic class.

Theorem 5 [5] (Riemann-Hodge bilinear relations for Grothendieck residue pairing on Ω_f) Assume $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is a holomorphic germ with an isolated singularity. The 3-tuple $(\Omega_f,\ \phi^{-1}F^\bullet,\ \phi^{-1}W_\bullet)$ defines a graded polarized "complex"

MHS [5, 16, 17, 20] which is polarized by ψ_f in the following sense. Suppose $\mathfrak{f}$ is the corresponding map to $N: H^n(X_\infty) \to H^n(X_\infty)$, via the isomorphism ϕ. Define $P_l = P\mathrm{Gr}_l^W := \ker(\mathfrak{f}^{l+1}: \mathrm{Gr}_l^W \Omega_f \to \mathrm{Gr}_{-l-2}^W \Omega_f)$. Going to W-graded pieces; $\psi_l: P\mathrm{Gr}_l^W \Omega_f \otimes_{\mathbb{C}} P\mathrm{Gr}_l^W \Omega_f \to \mathbb{C}$ is non-degenerate and according to Lefschetz decomposition $\mathrm{Gr}_l^W \Omega_f = \bigoplus_r \mathfrak{f}^r P_{l-2r}$ we obtain a set of non-degenerate bilinear forms, ψ_l on $P\mathrm{Gr}_l^W \Omega_f$ such that the corresponding hermitian form associated to these bilinear forms is positive definite. In other words,

- $\psi_l(x, y) = 0,\ x \in P_r,\ y \in P_s,\ r \neq s.$
- If $x \neq 0$ in P_l, then, $\psi_l\ (C_l x, \mathfrak{f}^l \cdot \bar{x}) > 0$, where C was defined above and C_l is the Weil operator.

Proof. According to the definition, the MHS on $H^n(X_\infty)$ is a graded polarized complex MHS. Using the commutativity of the diagram (14), Ω_f also receives a graded polarized "complex" MHS as follows. By the Mixed Hodge Metric theorem [16, 17], on the Deligne-Hodge decomposition; $\Omega_f = \bigoplus_{p,\ q} J^{p,\ q}$, there exists a unique hermitian form; $\mathscr{R}$ with, $i^{p-q}\mathscr{R}(v, \bar{v}) > 0, v \in J^{p,\ q}$ where the mentioned Deligne decomposition (see Lemma 2.3 [9]) is orthogonal with respect to $\mathscr{R}$. This shows the existence of the unique polarization forms $\{\psi_l\}$ defined in the following standard way. Let $N := \log M_u$ be the logarithm of the unipotent part of the monodromy for the Milnor fibration defined by f. We have the decomposition $\mathrm{Gr}_l^W H^n(X_\infty) = \bigoplus_r N^r P_{l-2r},\ P_l := \ker N^{l+1}: \mathrm{Gr}_l^W H^n \to \mathrm{Gr}_{-l-2}^W H^n$ and the level forms $S_l: P_l \otimes P_l \to \mathbb{C},\ S_l(u, v) := S(u, N^l v)$ which polarize the primitive subspaces P_l. By using the isomorphism ϕ, a similar type of decomposition exists for Ω_f. That is the isomorphic image $P_l' := \phi^{-1} P_l$ satisfies $\mathrm{Gr}_l^W \Omega_f = \bigoplus_r N^r P_{l-2r}',\ P_l' := \ker \mathfrak{f}^{l+1}: \mathrm{Gr}_l^W \Omega_f \to \mathrm{Gr}_{-l-2}^W \Omega_f$ and the level forms $\psi_l: P_l' \otimes P_l' \to \mathbb{C},\ \psi_l := \psi_f(u, \mathfrak{f}^l v)$ polarize the primitive subspaces P_l', where $\mathfrak{f}$ is the map induced from multiplication by f on $\mathrm{Gr}_l^W \Omega_f$. Specifically, this shows that the relations mentioned at the end of the Theorem hold for instance by Lemma 2.1 [9].

Remark 4 Notice that the factor γ does not affect the property mentioned in Theorem 5.

Remark 5 Theorem 5 appeared as one of the author's thesis results in [5, 20]. The expression of Riemann-Hodge bilinear relations for Grothendieck residue pairing is new to the literature.

Given two disjoint n-dimensional cycles $\alpha,\ \beta$ in the Milnor sphere $S = S^{2n+1}$, we can form the linking number $l(\alpha,\ \beta) \in \mathbb{Z}$, which is defined as the intersection number $\Gamma \cdot \beta$ between a chain Γ with $\partial\Gamma = \alpha$ and β. One has $l(\alpha,\ \beta) = (-1)^{n+1} l(\beta,\ \alpha)$, so that linking is symmetric for odd-dimensional cycles.

Definition 3 Consider the Milnor fibration $f: X \to \Delta$ as before. Fix $t \in D$. We define a "half-monodromy map" $h_{1/2}: H_n(X_t) \to H^n(X_{-t})$ by parallel transport along an anticlockwise half-turn from t to $-t$. The Seifert form of the singularity $S: H_n(X_t) \times H_n(X_t) \to \mathbb{Z}$, is given by $(\alpha,\ \beta) \mapsto l(\alpha,\ h_{1/2}(\beta))$. If we restrict the Seifert form S to $H^n(L) \cong H^n(\partial X_t) \subset H^n(X_t)$ we obtain a $(-1)^{n+1}$-symmetric form called the linking form $\mathrm{lk}: H^n(L) \times H^n(L) \to \mathbb{Z}$ of the link, see [22] for instance.

Theorem 6 [1] If f is not quasi-homogeneous and if n is even then $\Theta(M,\ N) = 0$. If $n = 2m+1$ is odd, then

$$\Theta(\mathscr{O}_Y,\ \mathscr{O}_Z) = \mathrm{lk}(\mathrm{ch}(M),\ \mathrm{ch}(N)) \tag{12}$$

where M and N are the Maximal Cohen-Macaulay representatives of $\mathscr{O}_Y,\ \mathscr{O}_Z$ in the Grothendieck ring of $R/(f)$, respectively. Here $\mathrm{ch}: K_0(L) \to H^{ev}(L)$ is the chern character.

Remark 6 [1] Only the m-th component $\mathrm{ch}^m \in H^{2m}(L,\ \mathbb{Q})$ contributes to linking form.

Lemma 1 Assume $f_X: X \to T$ is a local isolated singularity in $\mathbb{C}^{n+1}$, embedded into the projective fibration $f_Y: Y \to T'$, in the construction of the MHS on vanishing cohomologies. Then the two chern character maps $\mathrm{ch}_X: K_0'(X) \to \Omega_{f_X}$ and $\mathrm{ch}_Y: K_0'(Y) \to \Omega_{f_Y}$ are compatible. Moreover the two residue pairings on res_{f_X} and res_{f_Y} agree via the embedding.

Proof. By the method of establishing the MHS on the vanishing cohomology, cf. the fibration f_X embeds into that of f_Y fiberwise (cf. [5, 9, 10]). We have an open (local) embedding as follows

$$
\begin{array}{ccc} X \hookrightarrow Y & & \mathrm{ch}_Y : K_0'(Y) \longrightarrow \Omega_{f_Y} \\ \downarrow \quad \downarrow & \curvearrowright & \downarrow \quad \downarrow \\ T \hookrightarrow T' & & \mathrm{ch}_X : K_0'(X) \longrightarrow \Omega_{f_X} \end{array} \tag{13}
$$

where the vertical maps in the first diagram are the corresponding fibrations and the vertical maps in the diagram on the right are pullbacks through inclusions. The horizontal maps are Polishchuk-Vaintrob chern characters and take values in the $\mathbb{C}$-module Ω_f (cf. [7] sec. 4, page 37).

Proposition 1 If n is odd, then $(-1)^{n+1}\Theta_{f_Y}$ is positive definite on $K_0'(R)_{\mathbb{Q}}$.

Proof. The equation defining f_Y is given by $\pi_Y = F(z) - tz_{n+1}^d = 0$ where $F(z) = z_{n+1}^d f_X(z_0/z_{n+1}, \ldots, z_n/z_{n+1})$ is the homogenization of f_X and $d = \deg(f)$. The space Y sits in $\mathbb{P}^{n+1} \times T$ and it defines an isolated singularity at 0 whose fibers can be regarded as projectivization of fibers of f_X (see [9] for details). The Hochster Theta pairing for f_Y is that of F, which is homogeneous defining isolated singularity in $\mathbb{P}^{n+1}$. It can be regarded as an isolated singularity at $0 \in \mathbb{C}^{n+2}$. Therefore the proposition is established from the Theorem 1.3 of [3].

The following theorem proves the Conjecture 1 in the case of a non-homogeneous isolated hypersurface singularity $f \in P$.

Theorem 7 [4] Let S be an isolated hypersurface singularity of dimension n. If n is odd, then $(-1)^{(n+1)/2}\Theta$ is positive semi-definite on $K_0'(R)_{\mathbb{Q}}$.

Proof. The theorem is proved in case the isolated singularity f is a homogeneous polynomial. We attempt the non-homogeneous case. By additivity of Θ on each variable, we may replace M, N with maximal Cohen-Macaulay modules. The following diagram is commutative by the functorial properties of chern character.

$$
\begin{array}{ccccc} K_0'(Y_0) & \xrightarrow{\mathrm{ch}_Y} & \Omega_{f_Y} & \xrightarrow{\Phi_Y} & H^n(Y_\infty) \\ \iota^* \downarrow & & \downarrow \iota^* & & \downarrow \iota^* \\ K_0'(X_0) & \xrightarrow[\mathrm{ch}_X]{} & \Omega_{f_X} & \xrightarrow[\Phi_X]{} & H^n(X_\infty). \end{array} \tag{14}
$$

where Y_∞ is the projective completion of X_∞ defined by a homogeneous polynomial f_Y, as explained at the beginning of this section (see also [5, 9, 15]). The MHS on $H^{n-1}(X_\infty)$ decomposes to two pure HS of weights $n-1$ and n which is explained by Theorem 3. Because the chern character is a Hodge cycle, it is of type (p, p) in the Hodge decomposition (see [12] page 387). We have the decomposition $H^{n-1}(X_\infty) = H_{\neq 1} \bigoplus H_1$ into the generalized eigenspaces of the monodromy M of the fibration. If the image of the chern character lies in $H^{n-1}(X_\infty)_{\neq 1}$, then the positivity of Theta pairing follows from the commutativity of diagram (14) and the fact that $H^{n-1}(Y_\infty)_{\neq 1} = H^{n-1}(X_\infty)_{\neq 1}$ and the Theorem 1 (or by Theorem 2). If the image of chern character lies in $H^{n-1}(X_\infty)_1$, we need to modify this argument as follows. According to [9] we have the following short exact sequence

$$
0 \to \ker(N_Y) \to H^{n-1}(Y_t, \mathbb{Q})_1 \xrightarrow{\iota^*} H^{n-1}(X_t, \mathbb{Q})_1 \to 0 \tag{15}
$$

where $\iota : X_t \hookrightarrow Y_t$ is an embedding through the homogenization of the fibration explained in the proof of Lemma 1, see [9, 10]. One can lift the chern character in $H^{n-1}(X_t, \mathbb{Q})_1$ to $H^{n-1}(Y_t, \mathbb{Q})_1$, where we have the positivity by Theorem 2. On the other hand taking $t \to \infty$ in (15), makes it a short exact sequence of MHS. The cup products on $H^{n-1}(X_t, \mathbb{Q})_1$ and $H^{n-1}(Y_t, \mathbb{Q})_1$, coincide up to cup with elements in $\ker N_Y$. By Theorem 4 the same holds for the residue pairings ψ_{f_X} and ψ_{f_Y} when we restrict to the classes corresponding to cohomology classes in $H^{n-1}(X_t, \mathbb{Q})_1$ and $H^{n-1}(Y_t, \mathbb{Q})_1$.

Assume the chern characters take values in $H^{n-1}(X_t,\ \mathbb{Q})_1 = \iota^* H^{n-1}(Y_t,\ \mathbb{Q})_1$. By the exact sequence (15) a $(p,\ p)$-class α in $H^{n-1}(X_t,\ \mathbb{Q})_1$ can be written as a sum of a $(p,\ p)$-class β in $H^{n-1}(Y_t,\ \mathbb{Q})_1$ and another class $N\gamma$ in $\ker N_Y$ which must be also a $(p,\ p)$-class. One can also write a similar short exact sequence for the corresponding subspace $(\Omega_{f_Y})_1$ and $(\Omega_{f_X})_1$ via the isomorphism ϕ in Theorem 4. If we regard ι as an inclusion, this shows that we can calculate the residue pairing of f_X at $(\alpha,\ \alpha)$ via the residue pairing of f_Y at $(\beta + N_Y\gamma,\ \beta + N_Y\gamma)$. By the positivity of Θ_Y namely 1, then we are done. We can make this argument more precise, as follows.

Back to the proof of the Theorem 3.4 of [3], let $\mathrm{Hg}^p_{\mathbb{Q}}(X_t) \subset H^{n-1}(X_t,\ \mathbb{Q})$ be the subspace of $(p,\ p)$-classes, $p = (n-1)/2$. As said above the image of chern character generally lies in $\mathrm{Hg}^p_{\mathbb{Q}}$. Let h be a hyperplane section of the fibration f_X. By the same method as in [3] (proof of Theorem 3.4) one can embed $\mathrm{em} : \mathrm{Hg}^p_{\mathbb{Q}}(X)/h^p \hookrightarrow H^{2p}(X)$, where here because $\cup h^p$ is an isomorphism by the Lefschetz hyperplane theorem, the image of em is a primitive class (see the argument after Theorem 2). By what we said in above the formula $\Theta_X(a,\ b) = -dQ_X(\mathrm{em}(\mathrm{ch}(a),\ \mathrm{em}(\mathrm{ch}(b)))$, still holds for the fibration X, where Q_X is the Poincaré pairing. This for either $a,\ b \in H_{\neq 1}(X)$ where we can calculate Θ via the cup product Q_Y, or $a,\ b \in H_1(X)$ where this time the above lifting argument of the proof works. By Riemann-Hodge bilinear relations $(-1)^{(n-1)(n-2)/2} Q_X$ is positive definite on $\mathrm{Hg}^p_{\mathbb{Q}}(X)$.

Remark 7 The proof of the Theorem 7 is an extended version of the proofs in [4–6]. Although the proof in the above references points out the same idea, it seemed to us some of the details which were necessary to be mentioned was missed in [4–6]. In this way, the above proof contains one of the main contributions of this version. Especially we can formulate the next corollary as a generalization of the identity (6) to the inhomogeneous case.

Corollary 1 The following formula holds for the Theta pairing of a general isolated hypersurface singularity f_X of degree d (not necessarily homogeneous f),

$$\Theta(a,\ b) = \left(\int_X \mathrm{ch}(a) \cdot h^p\right)\left(\int_X \mathrm{ch}(b) \cdot h^p\right) - d\int_X \mathrm{ch}(a) \cdot \mathrm{ch}(b) \tag{16}$$

where h is a hyperplane section, em was defined in the proof of Theorem 7, and $p = (n-1)/2$ (here n is already assumed to be odd). The formula (16) reads as $\Theta_X(a,\ b) = -dQ_X(\mathrm{em}(\mathrm{ch}(a),\ \mathrm{em}(\mathrm{ch}(b))$, where Q is the Poincaré pairing of $H^{n-1}(X_\infty,\ \mathbb{C})$ and em is defined by the formula (9).

Proof. The formula (16) holds in the quasi-homogeneous case by Theorem 3.1 in [3]. The general case follows from the argument of the proof of Theorem 7.

The Hochster Theta pairing can also be calculated for a general isolated singularity via the Polishchuk-Vaintrob chern character [7]. There is an extension of the chern character map of smooth schemes in algebraic geometry over hypersurface rings, that takes values in the Hochschild homology of $R = P/f$, namely

$$\mathrm{ch} : K_0'(R) \longrightarrow HH_*(R) \tag{17}$$

The chern character of matrix factorizations has been studied in detail in [7] with explicit formulas for calculations in the isolated singularity case. The chern character is the same as Denis trace map on Hochschild homology (cf. [23] Chap. 8, [24]). The above form also agrees with the usual chern character map of schemes in the smooth category ([12] page 387, [23]).

Proposition 2 [7, 25] The Hochster Θ-pairing of two maximal Cohen-Macaulay modules $M,\ N$ is given up to a sign by the local residue of their chern classes as elements in Ω_f, that is

$$\Theta(M,\ N) = (-1)^{n(n-1)/2}\mathrm{res}_{f,\ 0}(\mathrm{ch}(M),\ \mathrm{ch}(N)) \tag{18}$$

only for $M,\ N$ maximal Cohen-Macaulay.

Remark 8 [7] It is not hard to show that $HH_*(R) = \Omega_f$ if f is an isolated singularity. Therefore, the chern character takes values in Ω_f.

Second Proof of Conjecture 1 over $\mathbb{C}$: We can express a second direct proof of the Conjecture 1 over $\mathbb{C}$ using Theorem 2 and Theorem 5.

Proof. (Second Proof of the Conjecture 1 over $\mathbb{C}$) By Theorem 2 working over $\mathbb{C}$ the Conjecture 1 follows from Theorem 5. The Riemann-Hodge bilinear relations for Grothendieck residue paring of the isolated hypersurface singularity f_X proves the Conjecture 1 over $\mathbb{C}$. This method already recovers the proof of Theorem 2 too.

One can also apply the above method to answer vanishing results over $\mathbb{C}$. For instance the following was conjectured in ([19] page 352) which answered in [1] for the case $P = \mathbb{C}[[x_1, \ldots, x_n]]$.

Corollary 2 Let $R = P/(f)$ be a hypersurface ring with isolated singularity, where $P = \mathbb{C}[x_0, \ldots, x_n]$. If n be even, then $\Theta(M, N) = 0$ for any pair of $M, N \in \mathrm{mod}(R)$.

Proof. Assume first f is homogeneous. By using the same argument as in the proof of Theorem 7 with the same diagram as (14) we have

$$\Theta(M, N) = \mathrm{res}_{f_Y}(\mathrm{ch}(M), \mathrm{ch}(N)) = S_Y(\phi \circ \mathrm{ch}(M), \phi \circ \mathrm{ch}(N)) = 0 \tag{19}$$

The Hodge classes $\phi \circ \mathrm{ch}(-)$ are zero in $H^n(Y_\infty)$. In the nonhomogeneous case we also have the same statement by the Proof of 7 and Corollary 1.

Buchweitz et al. [1] mentions a connection between the K. Saito higher residue pairing and the Hochster Theta pairing of a general isolated hypersurface singularity f. This appeared in the above reference in a remark at the end of that paper. Some results about the deformation behavior of Theta appeared in the Appendix there. We recollect the following proposition using all there.

Proposition 3 If the number of variables $n+1$ is even, the Theta pairing is given by the higher residue pairing $K_f : \mathscr{H}_f^{(0)} \times \mathscr{H}_f^{(0)} \to \mathbb{C}[t, t^{-1}]$ of K. Saito, where $\mathscr{H}_f^{(0)} = \Omega^{n+1}/(df \wedge d\Omega^{n-1})$ is the Brieskorn module, i.e.

$$\Theta(M_t, N_t) = K_f(\mathrm{ch}(M_t), \mathrm{ch}(N_t)) \tag{20}$$

Proof. $\Theta_t = \Theta(M_t, N_t)$ is identically zero when $t \in \Delta \subset \mathbb{C}$ is not a singularity, cf. [1] Theorem 6.2. Given a matrix factorization of f namely $AB = BA = f \cdot \mathrm{Id}$, one has

$$\mathrm{ch}(M) = \mathrm{tr}(\exp(dA \wedge dB)) = \sum_i \frac{1}{i!} \mathrm{tr}(dA \wedge dB)^i \tag{21}$$

and $(dA \wedge dB)^i \in \Omega^{2i+1}/(df \wedge d\Omega^{2i-1})$. When $n+1$ is even a top degree form sits in the Brieskorn module, that is a free $\mathbb{C}[[t]]$-module of rank μ ([1] cf. Remark 5.4). The induced pairing on

$$\frac{\mathscr{H}_f^{(0)}}{t \cdot \mathscr{H}_f^{(0)}} \bigotimes \frac{\mathscr{H}_f^{(0)}}{t \cdot \mathscr{H}_f^{(0)}} \longrightarrow \mathbb{C}, \tag{22}$$

is the classical Grothendieck residue, cf. [26]. The higher residue pairing K_f of K. Saito can be regarded as the de Rham realization of the Seifert form of the singularity. This can also be explained by the Theorem 4 as the flat bilinear skew Hermitian form K_f is given via the same formula and is compatible there (cf. [4, 5, 20, 27–30]).

Remark 9 Only in the quasi-homogeneous case the stalk of $\mathscr{H}_f^{(0)}$ at 0, namely $H_f^{(0)}(0) = \Omega^{n+1}/(df \wedge d\Omega^{n-1} + f\Omega^{n+1})$, can be identified with the Jacobian ring $P/J_f \cong \Omega^{n+1}/df \wedge \Omega^n$.

2.3 *The proof of Conjecture 1 over arbitrary field k*

Over an arbitrary field k and especially when $\mathrm{char}(k) \neq 0$ the formalism of the residue pairing $\mathrm{res}_{f,\,0}$ breaks down, and we can not calculate the Theta pairing via Grothendieck residues. Notice that in this case, we consider $f : k^{n+1} \to k$ with isolated singularities. We assume k^{n+1} is equipped with a suitable topology, and consider a representative $f_X : X \to T$ as in the case of complex numbers. We consider the subspace topology on $X \subset k^{n+1}$. For technical computations and to apply standard theorems from theory of isolated hypersurface singularities (see for instance [31]) we will assume $\mathrm{char}(k) \neq 2$. The fibration f_X can be embedded as a local open embedding into a projective fibration denoted by $f_Y : Y \to T$. The fibration defining f_Y is given by $z_{n+1}^d f_X(z_0/z_{n+1},\, \dots,\, z_n/z_{n+1}) - tz_{n+1}^d = 0$ where $d = \deg(f)$. The space Y sits in $\mathbb{P}^{n+1}(k) \times T$ and it defines an isolated singularity at 0 whose fibers can be regarded as projectivization of fibers of f_X.

The vanishing cohomologies of the fibers $H^n(X_t,\, \mathbb{Q})$ are well-defined and are equipped with a cup product Q. By some standard facts on the vanishing cycles ([31] for instance) there exists a distinguished basis for the vanishing cohomology $H^n(X_t,\, \mathbb{Q})$. We can still use the identity (3) in the case when f is homogeneous. For the case that f is not necessarily homogeneous, we argue as follows.

Proposition 4 The embedding of the fibration f_X into the fibration f_Y as in Proposition 1 makes R_X an R_Y-module where the following relation holds for the associated Theta pairings

$$\Theta_Y(M,\, N) = \Theta_X(R_X \otimes_{R_Y} M,\, R_X \otimes_{R_Y} N), \quad M,\, N \in R_Y\text{-Mod}. \tag{23}$$

Proof. The proof of the identity (3) is based on a formula on Poincaré series, cf. [1]

$$H(M \otimes_R^L N) = H(M \otimes_R N) + \mathrm{polyn} + \frac{H(\mathrm{Tor}_R^{\mathrm{ev}}(M,\, N)) - H(\mathrm{Tor}_R^{\mathrm{odd}}(M,\, N))}{1 - t^d}. \tag{24}$$

where polyn means a polynomial. Then one calculates the residues at both sides of the identity (24). Notice that

$$\mathrm{res}_{t=1} H(M \otimes_R^L N) = \frac{1}{d} \deg(Y) \deg(Z),$$

$$\mathrm{res}_{t=1} H(M \otimes_R N) = \mathrm{res}_{t=1} H(M \otimes_P^L N) = [Y] \cdot [Z], \tag{25}$$

$$\mathrm{res}_{t=1} \frac{H(\mathrm{Tor}_R^{\mathrm{ev}}(M,\, N)) - H(\mathrm{Tor}_R^{\mathrm{odd}}(M,\, N))}{1 - t^d} = \frac{1}{d} \Theta(\mathscr{O}_Y,\, \mathscr{O}_Z).$$

All the formulas in (24) and (25) hold also in the affine case and when f is not homogeneous, except the one involving the degrees of Y and Z.

The formal embedding $X \times k \hookrightarrow Y \times k$ which is also mentioned explicitly in the proof of The Proposition 1 in general applies in any characteristics and over any field k. The local open embedding of the fibration f_X into f_Y corresponds to a surjection $R_Y \to R_X$, which makes R_X an R_Y-module. Then the maximal Cohen-Macauley (MCM) modules over R_X are obtained by the MCM modules over R_Y. Because all the terms in the identity (24) are invariant under the base change, it follows that one can equally compute the Theta pairings via f_X or f_Y. It follows that the Theta pairing Θ_X can be computed as Θ_Y on the extended modules via the base change. In other words, (23) holds.

Remark 10 We obtained a similar result in Corollary 1 in the cohomological context over $\mathbb{C}$. In fact Proposition 4 verifies the formula (16).

Theorem 8 Assume k is an arbitrary field of $\mathrm{char}(k) \neq 2$, and d is an integer that is invertible in k, i.e. $(d,\ \mathrm{char}(k)) = 1$. Assume $f : k^{n+1} \to k$ is a polynomial fibration of degree d with isolated singularity so that $R = P/(f)$ is an admissible isolated hypersurface singularity ring of dimension n. In case, n is odd, the pairing $(-1)^{(n+1)/2}\Theta$ is positive semi-definite on $K_0'(R)_{\mathbb{Q}}$.

Proof. By Proposition 4, it suffices to prove the Conjecture 1 just for the homogeneous fibration Θ_Y, over an arbitrary field. In this case we can use the formula (3) over the field k. If we assume d is invertible in the field k. Then the argument to define the map "em" of the proof of the Theorem 1 is valid also over k. Thus we can apply the formula (10). Thus we are reduced to prove that $(-1)^{\frac{(n-1)}{2}} Q$ is positive definite on the primitive part of $\mathrm{Hg}^p_{\mathbb{Q}},\ p = (n-1)/2$ for an isolated hypersurface singularity defined over the field k. Note that Q takes values in $\mathbb{Q}$. For the positive definiteness of $(-1)^{\frac{(n-1)}{2}} Q(\mathrm{em} \otimes \mathrm{em})$ we argue as follows. When $\mathrm{char}(k) \neq 2$ by the standard topological methods, cf. [31], there exists a distinguished basis of the vanishing cohomology $H^n(X_t)$, namely $\delta_1,\ \ldots,\ \delta_\mu$ (μ is the Milnor number of f) which satisfy $Q(\delta_i,\ \delta_i) = (-1)^{\frac{n(n-1)}{2}}(1 + (-1)^n)$, cf. [31]. Now when n is odd Q is antisymmetric and we can calculate

$$Q(\mathrm{em}(\delta),\ \mathrm{em}(\delta)) = Q\left(\delta - \frac{\int_X \delta \cup h^p}{d} h^p,\ \delta - \frac{\int_X \delta \cup h^p}{d} h^p\right) = \frac{(-1)^p}{d}\left(\int_X \delta \cup h^p\right)^2$$

for $\delta = \delta_j,\ j = 1,\ \ldots,\ \mu$. Here we have used $Q(\delta,\ \delta) = 0$ because n is odd, and $Q(h^p,\ h^p) = (-1)^p h^{2p} = (-1)^p d$. The other two terms involving $Q(h^p,\ \delta)$ and $Q(\delta,\ h^p)$ cancel out, for the $(-1)^n$-symmetry of Q. The numbers $\int_X \delta \cup h^p$ are analog of degree numbers for the classes δ and are real numbers. This proves the claim.

Remark 11 The argument of [13] pages 160-169 on the positive definiteness of $(-1)^{\binom{n}{2}} Q$ on primitive classes, also works here as in the proof of Theorem 2. This is due to the integral properties of the form Q on primitive classes. In this case, one can still define a complex structure on $H^n(X_t,\ \mathbb{C})$ through the coefficients. Then we can deduce the above claim using the positive definiteness of the form $H(.,\ .) = i^n Q(.,\ \bar{.})$ on $H^n(X_t)$ and the integral property of Q.

Remark 12 The conjecture 1 is also proved in [8] by some other methods in relation with the positivity of Euler pairing on a derived category. On the $\mathbb{Z}/2$-graded differential category (dg-category) of matrix factorizations $MF_R(f)$, one has the following Euler pairing

$$\chi(C_1,\ C_2) := \sum_j (-1)^j \dim_k H^j \underline{\mathrm{Hom}}(C_1,\ C_2) \tag{26}$$

where $\underline{\mathrm{Hom}}$ is taken in the derived category. The main result of [8] namely Theorem 1.6 stresses that the pairing (26) is positive definite. The Euler pairing is connected to Hochster Theta pairing via the local chern character.

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Conflict of interest

The authors declare no competing interest.

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