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Research Article

On the Fekete-Szegö Inequalities of the Generalized Mittag-Leffler Function Associated with a Lambert Series

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Abstract: This study aims to investigate the Fekete-Szegö problem for the linear operator generated by the convolution (the Hadamard product) involving one of the generalized forms of the Mittag-Leffler function and the well-known Lambert series. The findings will mainly apply on some subclasses of starlike and convex functions.

***Keywords*:** fekete-szegö problem, mittag-leffler function, hadamard product, linear operator, lambert series, starlike and convex functions

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1. Introduction

The Mittag-Leffler function $E_\alpha(z)$ for $\alpha \in \mathbb{C}$, with $\mathscr{R}e(\alpha > 0)$ ([1] and [2]) is defined as;

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \; z \in \mathbb{C}$$

The extended form of the Mittag-Leffler function that depends on two parameters was studied by Wiman [3]. For all $\alpha, \beta \in \mathbb{C}$, with $\mathscr{R}e(\alpha, \beta > 0)$, the two-parameters function $E_{\alpha, \beta}(z)$ is defined as;

$$E_{\alpha, \beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \; z \in \mathbb{C}.$$

Generally, The Mittag-Leffler function and its generalizations have been considered by many researchers (see for instance [4]). In here, we limit our focus to the generalization given by Salah and Darus [5] as follows:

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$$
qF_{\alpha,\ \beta}^{\theta,\ k} = \sum_{n=0}^{\infty} \prod_{j=1}^{q} \frac{(\theta_j)_{k_{jn}}}{(\beta_j)_{\alpha_{jn}}} \cdot \frac{z^n}{n!}, \tag{1}
$$

where $(\theta)_v$ refers to the well-known Pochhammer symbol given by:

$$
(\theta)_v := \frac{\Gamma(\theta+v)}{\Gamma(\theta)} = \begin{cases} 1, & \text{if } v = 0,\ \theta \in \mathbb{C}\backslash\{0\} \\ \theta(\theta+1)\ldots(\theta+n-1), & \text{if } v = n \in \mathscr{N},\ \theta \in \mathbb{C}, \end{cases}
$$

$$
(1)_n = n!,\ n \in \mathscr{N}_0,\ \mathscr{N}_0 = \mathscr{N} \cup \{0\},\ \mathscr{N} = \{1,\ 2,\ 3,\ \ldots\},
$$

and

$$
\left(q \in \mathscr{N},\ j = 1,\ 2,\ 3,\ \ldots q;\ Re\left\{\theta_j,\ \beta_j\right\} > 0,\ \text{ and } Re\alpha_j > \max\left\{0,\ Rek_j - 1;\ Rek_j\right\};\ Rek_j > 0\right).
$$

The Lambert series (see [6–9]), is widely considered in certain problems of the number theory due to its connection to the well-known arithmetic functions such as:

$$
\sum_{n=1}^{\infty} \sigma_0(n) x^n = \sum_{n=1}^{\infty} \frac{x^n}{1-x^n}, \tag{2}
$$

where $\sigma_0(n) = d(n)$ is the number of positive divisors of n.

$$
l(z) = \sum_{n=1}^{\infty} \sigma_\alpha(n) x^n = \sum_{n=1}^{\infty} \frac{n^\alpha x^n}{1-x^n}, \tag{3}
$$

where $\sigma_\alpha(n)$ is the higher-order sum of divisors function of n.

We limit our study to the series provided by (3). More specifically, we write $\sigma_1(n) = \sigma(n)$ when $\alpha = 1$. In this case $\sigma(n)$ is the sum of divisors function that is potentially found in one of the straightforward equivalent statements to the widely recognized Riemann hypothesis.

We first distinguish between the Lambert series and Lambert W function, which naturally arises in the resolution of numerous scientific and engineering problems [10].

Guy Robin [11] demonstrated in 1984 that

$$
\sigma(n) < e^\gamma n \log\log n + \frac{0.6483n}{\log\log n},\ n \geq 3 \tag{4}
$$

Additionally, he proved that the Riemann hypothesis is equivalent to

$$\sigma(n) < e^{\gamma} n \log(\log n),\ n > 5040, \tag{5}$$

here $\gamma = 0.7721\cdots$, refers to the Euler-Mascheroni constant.

The Riemann hypothesis and the Robin's inequality (5) are not attempted to be proven or disproved in this article. We recommend interested readers to study the papers mentioned in the references [12–16] for further information.

2. Main results

We recall the class $\mathscr{A}$ of analytic functions given by;

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ z \in \mathscr{U}, \tag{6}$$

where $\mathscr{U}$ is the open unit disk $\mathscr{U} := \{z \in \mathbb{C} : \ |z| < 1\}$. These functions are normalized by the constraints $f(0) = f'(0) - 1 = 0$. The subclass of univalent functions in $\mathscr{U}$, is denoted by S. We also, recall Ω, the class of all analytic functions, w in $\mathscr{U}$ that satisfy the conditions of $w(0) = 0$ and $|w(z)| < 1$.

Robertson [17], introduced the starlike and the convex subclasses of S as follows:

A function A function $f \in \mathscr{A}$ given by (6) is said to be a starlike if and only if

$$\mathfrak{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in U.$$

A function A function $f \in \mathscr{A}$ given by (6) is said to be convex if and only if

$$\mathfrak{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\ z \in \mathscr{U}$$

The above two classes are denoted by S^* and K respectively and they are connected by the Alexander's duality relation [18]

$$f \in K \Leftrightarrow zf'(z) \in S^*.$$

In general, given $0 \leq \gamma < 1$, we obtain the following generalizations;

A function $f \in \mathscr{A}$ given by (6) is said to be a starlike of order γ, if and only if

$$\mathfrak{Re}\left(\frac{zf'(z)}{f(z)}\right) > \gamma,\ z \in \mathscr{U}.$$

A function $f \in \mathscr{A}$ given by (1) is said to be the convex of order γ, $0 \leq \gamma < 1$, if and only if

$$\mathfrak{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\gamma,\ z\in\mathscr{U}.$$

The last two subclasses are denoted by $S^*(\gamma)$ and $K(\gamma)$ respectively.
In particular, $S^*(0)=S^*$ and $K(0)=K$.
The Fekete-Szegö theorem [19] states that for a function f of the form (6), the following sharp inequality holds

$$\left|a_3-\mu a_2^2\right|\leq 1+2\exp\left(\frac{-2\mu}{1-\mu}\right),\ 0\leq\mu<1.$$

For two functions $f,\ g$ of the form (6), the Hadamard product $(*)$ is given by

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n.$$

Now, since $qF_{\alpha,\ \beta}^{\theta,\ k}$ does not belong to the class $\mathscr{A}$, we consider some normalization by introducing:

$$q\mathbb{F}_{\alpha,\ \beta}^{\theta,\ k}=\prod_{j=1}^{q}\frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}}\left(qF_{\alpha,\ \beta}^{\theta,\ k}-1\right)=z+\sum_{n=2}^{\infty}\prod_{j=1}^{q}\frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}}\frac{(\theta_j)_{k_jn}}{(\beta_j)_{\alpha_jn}}\cdot\frac{z^n}{n!} \tag{7}$$

Let $f(z)\in\mathscr{A}$. Denote $q\mathbb{F}_{\alpha,\ \beta,\ \sigma}^{\theta,\ k}(f)(z):\ \mathscr{A}\to\mathscr{A}$ the linear operator is defined by;

$$q\mathbb{F}_{\alpha,\ \beta,\ \sigma}^{\theta,\ k}(f)(z)=q\mathbb{F}_{\alpha,\ \beta}^{\theta,\ k}*l(z),$$

by the Hadamard product the latter becomes

$$q\mathbb{F}_{\alpha,\ \beta,\ \sigma}^{\theta,\ k}(f)(z)=z+\sum_{n=2}^{\infty}\prod_{j=1}^{q}\frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}}\frac{(\theta_j)_{k_jn}}{(\beta_j)_{\alpha_jn}}\cdot\frac{\sigma(n)}{n!}a_nz^n. \tag{8}$$

Definition 1 Let $f(z)\in\mathscr{A}$. Then $f(z)\in qS_{\alpha,\ \beta,\ \sigma}^{\theta,\ k}(\lambda)$ if and only if

$$\mathscr{R}\left\{\frac{z\left[q\mathbb{F}_{\alpha,\ \beta,\ \sigma}^{\theta,\ k}(f)(z)\right]}{q\mathbb{F}_{\alpha,\ \beta,\ \sigma}^{\theta,\ k}(f)(z)}\right\}>\lambda,\ 0\leq\lambda<1,\ z\in\mathscr{U}.$$

Definition 2 Let $f(z)\in\mathscr{A}$. Then $f(z)\in q\mathscr{C}_{\alpha,\ \beta,\ \sigma}^{\theta,\ k}(\lambda)$ if and only if

$$\mathscr{R}\left\{\frac{\left[z\left(q\mathbb{F}^{\theta,\,k}_{\alpha,\,\beta,\,\sigma}(f)(z)\right)'\right]'}{\left(q\mathbb{F}^{\theta,\,k}_{\alpha,\,\beta,\,\sigma}(f)(z)\right)'}\right\} > \lambda,\ 0 \le \lambda < 1,\ z \in \mathscr{U}.$$

Next, we study the general characteristics and the distortion theorems for the function $f(z) \in \mathscr{A}$ that is a member of the new subclasses $qS^{\theta,\,k}_{\alpha,\,\beta,\,\sigma}(\lambda)$ and $q\mathscr{C}^{\theta,\,k}_{\alpha,\,\beta,\,\sigma}(\lambda)$, provided that the coefficient bounds are obtained. In order to compute the sharp upper bounds of a_2 and the Fekete-Szegö inequality, we need the following Lemma (Duren [20]).

Lemma 1 Given that $h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n \in \mathscr{P}$, such that $h(z)$ is analytic in $\mathscr{U}$, and $\mathscr{P}$ is the class of all analytic functions with a positive real part. Then

i. $\left|h_2 - \frac{h_1^2}{2}\right| \le 2 - \frac{\left|h_1^2\right|}{2}$,

ii. $|h_n| \le 2,\ n \in \mathbb{N}$.

Theorem 1 Let $f(z) \in \mathscr{A}$. If

$$\sum_{n=2}^{\infty}(n-\lambda)\,|a_n|\prod_{j=1}^{q}\frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}}\frac{(\theta_j)_{k_j n}}{(\beta_j)_{\alpha_j n}}\cdot\frac{\sigma(n)}{n!} \le 1-\lambda,\ 0 \le \lambda < 1, \tag{9}$$

Then $f(z) \in qS^{\theta,\,k}_{\alpha,\,\beta,\,\sigma}(\lambda)$. The result (9) is sharp.

Proof. For short hand we write $\phi = \prod_{j=1}^{q}\frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}}\frac{(\theta_j)_{k_j n}}{(\beta_j)_{\alpha_j n}}\cdot\frac{\sigma(n)}{n!}$. Let (9) hold true. Since

$$1-\lambda \ge \sum_{n=2}^{\infty}(n-\lambda)\,|a_n|\cdot\phi$$

$$\ge \sum_{n=2}^{\infty}\lambda\,|a_n|\,\phi - \sum_{n=2}^{\infty} n\,|a_n|\,\phi,$$

that is

$$\frac{1+\sum_{n=2}^{\infty} n\,|a_n|\,\phi}{1+\sum_{n=2}^{\infty}|a_n|\,\phi} > \lambda.$$

Hence

$$\mathscr{R}\left\{\frac{\left\{q\mathbb{F}^{\theta,\,k}_{\alpha,\,\beta,\,\sigma}(f)(z)\right]'}{q\mathbb{F}^{\theta,\,k}_{a,\,\beta,\,\sigma}(f)(z)}\right\} > \lambda.$$

The inequality (9) is sharp and the equality holds at the extremal function given by

$$f(z)=z+\sum_{n=2}^{\infty}\frac{(1-\lambda)}{(n-\lambda)}\prod_{j=1}^{q}\frac{(\alpha_j)_{k_j}}{(\beta_j)_{\alpha_j}}\frac{(\beta_j)_{\alpha_j n}}{(\theta_j)_{k_j n}}\frac{n!}{\sigma(n)}z^n.$$

□

Corollary 1 If $f(z)\in qS^{\theta,\,k}_{\alpha,\,\beta,\,\sigma}(\lambda)$. Then

$$|a_n|\le\frac{(1-\lambda)}{(n-\lambda)}\prod_{j=1}^{q}\frac{(\alpha_j)_{k_j}}{(\beta_j)_{\alpha_j}}\frac{(\beta_j)_{\alpha_j n}}{(\theta_j)_{k_j n}}\frac{n!}{\sigma(n)},\ n\ge 2.$$

By utilizing the inequality (4), we write

$$\frac{1}{\sigma(n)}>\frac{\log\log n}{e^{\gamma}n(\log\log n)^2+0.6483n},\ n\ge 3,$$

we derive the following lower bounds inequality.

Corollary 2 If $f(z)\in qS^{\theta,\,k}_{\alpha,\,\beta,\,\sigma}(\lambda)$. Then

$$|a_n|>\frac{(1-\lambda)}{(n-\lambda)}\prod_{j=1}^{q}\frac{(\alpha_j)_{k_j}}{(\beta_j)_{\alpha_j}}\frac{(\beta_j)_{\alpha_j n}}{(\theta_j)_{k_j n}}\frac{(n-1)!\log\log n}{e^{\gamma}(\log\log n)^2+0.6483},\ n\ge 3.$$

Similarly, if we assume the Robin's inequality (5), we deduce

Corollary 3 If $f(z)\in q\delta^{\theta,\,k}_{\alpha,\,\beta,\,\sigma}(\lambda)$, and the Robin's inequality (5) holds true. Then

$$|a_n|>\frac{(1-\lambda)}{(n-\lambda)}\prod_{j=1}^{q}\frac{(\alpha_j)_{k_j}}{(\beta_j)_{\alpha_j}}\frac{(\beta_j)_{\alpha_j n}}{(\theta_j)_{k_j n}}\cdot\frac{(n-1)!}{e^{\gamma}\log\log n},\ n>5040.$$

Using the same method of Theorem 1, we can verify the following result:

Theorem 2 Let $f(z)\in\mathscr{A}$. If

$$\sum_{n=2}^{\infty}n(n-\lambda)\,|a_n|\prod_{j=1}^{q}\frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}}\frac{(\theta_j)_{k_j n}}{(\beta_j)_{\alpha_j n}}\cdot\frac{\sigma(n)}{n!}\le 1-\lambda,\ 0\le\lambda<1 \tag{10}$$

Then $f(z)\in q\mathscr{C}^{\theta,\,k}_{\alpha,\,\beta,\,\sigma}(\lambda)$. The result (10) is sharp.

Corollary 4 If $f(z)\in q\mathscr{C}^{\theta,\,k}_{\alpha,\,\beta,\,\sigma}(\lambda)$. Then

$$|a_n|\le\frac{(1-\lambda)}{(n-\lambda)}\prod_{j=1}^{q}\frac{(\alpha_j)_{k_j}}{(\beta_j)_{\alpha_j}}\frac{(\beta_j)_{\alpha_j n}}{(\theta_j)_{k_j n}}\frac{(n-1)!}{\sigma(n)},\ n\ge 2.$$

Corollary 5 If $f(z) \in q\mathscr{C}^{\theta,\, k}_{\alpha,\, \beta,\, \sigma}(\lambda)$. Then

$$|a_n| > \frac{(1-\lambda)}{n(n-\lambda)} \prod_{j=1}^{q} \frac{(\alpha_j)_{k_j}}{(\beta_j)_{\alpha_j}} \frac{(\beta_j)_{\alpha_j n}}{(\theta_j)_{k_j n}} \frac{(n-1)! \log\log n}{e^{\gamma}(\log\log n)^2 + 0.6483},\ n \geq 3.$$

Corollary 6 If $f(z) \in qS^{\theta,\, k}_{\alpha,\, \beta,\, \sigma}(\lambda)$, and the Robin's inequality (5) holds true. Then

$$|a_n| > \frac{(1-\lambda)}{n(n-\lambda)} \prod_{j=1}^{q} \frac{(\alpha_j)_{k_j}}{(\beta_j)_{\alpha_j}} \frac{(\beta_j)_{\alpha_j n}}{(\theta_j)_{k_j n}} \cdot \frac{(n-1)!}{e^{\gamma} \log\log n},\ n > 5040.$$

Definition 3 Let $\xi(z)$ be univalent starlike function with respect to 1 which maps the unit disk $\mathscr{U}$ onto a region in the right half plane which is symmetric about the real axis, $\xi(0) = 1$ and $\xi'(0) > 0$. An analytic function $f(z)$ is in the class $q\mathbb{F}^{\theta,k}_{\alpha,\beta,\sigma}(\xi)$ if

$$\frac{z\left(q\mathbb{F}^{\theta,k}_{\alpha,\beta,\sigma}(f)(z)\right)'}{q\mathbb{F}^{\theta,\, k}_{\alpha,\, \beta,\, \sigma}(f)(z)} \prec \xi(z). \tag{11}$$

Lemma 2 (see [21]) Let $h_1(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots$ be an analytic function with positive real part in $\mathscr{U}$. Then

$$\left|c_2 - v c_1^2\right| \leq 2\max\{1,\ |2v-1|\}, \tag{12}$$

and the result is sharp for the functions given by

$$h(z) = \frac{1+z}{1-z},\ h(z) = \frac{1+z^2}{1-z^2}. \tag{13}$$

Theorem 3 Let the conditions of Theorem 1 be satisfied. Then

$$|a_2| \leq \frac{4(1-\lambda)}{3(1+\lambda)} \prod_{j=1}^{q} \frac{(\alpha_j)_{k_j}}{(\beta_j)_{\alpha_j}} \frac{(\beta_j)_{2\alpha_j}}{(\theta_j)_{2k_j}},\ 0 \leq \lambda < 1,$$

and for every $\mu \in \mathbb{C}$ the following inequality is sharp

$$\left|a_3 - \mu a_2^2\right| \leq \frac{1-\lambda}{(2+\lambda)B} \max\left\{1,\ \left|B_2 + \frac{1-\lambda}{1+\lambda} B_1^2 - \mu\mu \frac{(1-\lambda)(2+\lambda)B}{(1+\lambda)^2 A^2} B_1^2\right|\right\}$$

Proof. Since $f(z) \in qS^{\theta,\, k}_{\alpha,\, \beta,\, \sigma}(\lambda)$ then the condition

$$\mathscr{R}\left\{z\frac{\left[q\mathbb{F}^{\theta,\,k}_{\alpha,\,\beta,\,\sigma}(f)(z)\right]'}{q\mathbb{F}^{\theta,\,k}_{\alpha,\,\beta,\,\sigma}(f)(z)}\right\} > \lambda,\ 0 \le \lambda < 1,\ z \in \mathscr{U},$$

is equivalent to

$$z\left[q\mathbb{F}^{\theta,\,k}_{\alpha,\,\beta,\,\sigma}(f)(z)\right]' = (1-\lambda)h(z)q\mathbb{F}^{\theta,\,k}_{\alpha,\,\beta,\,\sigma}(f)(z),\ z \in \mathscr{U},$$

for a given function $h \in \mathscr{P}$. We equate the coefficients, and we obtain the values of

$$a_2 = \frac{(1-\lambda)h_1}{(1+\lambda)A} \tag{14}$$

and

$$a_3 = \frac{1-\lambda}{(2+\lambda)B}\left(\frac{1-\lambda}{1+\lambda}h_1^2 + h_2\right). \tag{15}$$

By using equation (14) and Lemma 1, we achieve the required result of $|a_2|$

$$|a_2| \le \frac{4(1-\lambda)}{3(1+\lambda)}\prod_{j=1}^{q}\frac{(\alpha_j)_{k_j}}{(\beta_j)_{\alpha_j}}\frac{(\beta_j)_{2\alpha_j}}{(\theta_j)_{2k_j}}.$$

Where

$$A = \frac{2}{3}\prod_{j=1}^{q}\frac{(\alpha_j)_{k_j}}{(\beta_j)_{\alpha_j}}\frac{(\beta_j)_{2\alpha_j}}{(\theta_j)_{2k_j}} \text{ and } B = \frac{3}{2}\prod_{j=1}^{q}\frac{(\alpha_j)_{k_j}}{(\beta_j)_{\alpha_j}}\frac{(\beta_j)_{3\alpha_j}}{(\theta_j)_{3k_j}}. \tag{16}$$

For the Fekete-Szegö function $\left|a_3 - \mu a_2^2\right|$, consider $\xi(z) = 1 + B_1 z + B_2 z^2 + \ldots$ if $f(z) \in q\mathbb{F}^{\theta,\,k}_{\alpha,\,\beta,\,\sigma}(\xi)$, then

$$h(z) = \xi\left(\frac{h_1(z)-1}{h_1(z)+1}\right). \tag{17}$$

Since $\xi(z)$ is univalent and $h(z) \prec \xi(z)$, then the function below, is analytic and has a positive real part in $\mathscr{U}$.

$$h_1(z) = \frac{1+\xi^{-1}(h(z))}{1-\xi^{-1}(h(z))} = 1 + c_1 z + c_2 z^2 + \ldots \tag{18}$$

Next, by the means of (17) and (18) we find the values of $h_1(z)$ and $h_2(z)$

$$h_1(z) = \frac{1}{2}B_1 c_1,$$

and

$$h_2(z) = \frac{1}{2}\left(c_2 - \frac{1}{2}c_1^2\right) + \frac{1}{4}B_2 c_1^2,$$

so, by using Lemma 2 we find that

$$\begin{aligned} \left|a_3 - \mu a_2^2\right| &\leq \frac{2(1-\lambda)}{(2+\lambda)B} + \left\{\frac{(1-\lambda)^2}{(1+\lambda)(2+\lambda)} - \mu\frac{(1-\lambda)^2}{(1+\lambda)^2 A^2}\right\}|h_1|^2 \\ &\leq W(x) = \frac{2(1-\lambda)}{(2+\lambda)B} + \left\{K - \frac{(1-\lambda)}{2(2+\lambda)B}\right\}x^2,\ x := |h_1|^2. \end{aligned} \tag{19}$$

Consequently, we deduce

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} W(0) = \dfrac{2(1-\lambda)}{(2+\lambda)B} & \text{if } K \leq \dfrac{(1-\lambda)}{2(2+\lambda)B} \\ \\ W(2) = 4K & \text{if } K > \dfrac{(1-\lambda)}{2(2+\lambda)B}, \end{cases}$$

where

$$K := \frac{(1-\lambda)}{2(2+\lambda)B} + \frac{(1-\lambda)^2}{(1+\lambda)(2+\lambda)} - \mu\frac{(1-\lambda)^2}{(1+\lambda)^2 A^2}.$$

The equality is reached by the functions satisfying;

$$z\frac{\left[q\mathbb{F}_{\alpha,\,\beta,\,\sigma}^{\theta,\,k}(f)(z)\right]'}{q\mathbb{F}_{\alpha,\,\beta,\,\sigma}^{\theta,\,k}(f)(z)} = \frac{1+z(1-2\lambda)}{1-z}$$

$$z\frac{\left[q\mathbb{F}_{\alpha,\,\beta,\,\sigma}^{\theta,\,k}(f)(z)\right]'}{q\mathbb{F}_{\alpha,\,\beta,\,\sigma}^{\theta,\,k}(f)(z)} = \frac{1+z^2(1-2\lambda)}{1-z^2}.$$

We also have

$$a_3-\mu a_2^2=\frac{(1-\lambda)}{2(2+\lambda)B}\left(c_2-vc_1^2\right), \tag{20}$$

The values of A and B are given in (16), and the value of v is given by

$$v=\frac{1}{2}\left(1-B_2-\frac{1-\lambda}{1+\lambda}B_1^2+\mu\frac{(1-\lambda)(2+\lambda)B}{(1+\lambda)^2A^2}B_1^2\right),$$

Thus, the result is proved by Lemma 2. □

Corollary 7 Under the assumption of Theorem 3, if $\lambda=0$, then

$$|a_2|\leq\frac{4}{3}\prod_{j=1}^{q}\frac{(\alpha_j)_{k_j}}{(\beta_j)_{\alpha_j}}\frac{(\beta_j)_{2\alpha_j}}{(\theta_j)_{2k_j}},\ 0\leq\lambda<1,$$

and for all $\mu\in\mathbb{C}$

$$\left|a_3-\mu a_2^2\right|\leq\frac{1}{2B}\max\left\{1,\left|B_2+B_1^2-\mu\frac{2B}{A^2}B_1^2\right|\right\}.$$

Next, we prove the result for the class $q\mathscr{C}_{\alpha,\ \beta,\ \sigma}^{\theta,\ k}(\lambda)$.

Theorem 4 If the conditions of Theorem 2 are satisfied, then

$$|a_2|\leq\frac{2(1-\lambda)}{3(1+\lambda)}\prod_{j=1}^{q}\frac{(\alpha_j)_{k_j}}{(\beta_j)_{\alpha_j}}\frac{(\beta_j)_{2\alpha_j}}{(\theta_j)_{2k_j}},\ 0\leq\lambda<1,$$

and for every $\mu\in\mathbb{C}$ the following inequality is sharp

$$\left|a_3-\mu a_2^2\right|\leq\frac{1-\lambda}{3(2+\lambda)B}\max\left\{1,\ \left|B_2+\frac{1-\lambda}{1+\lambda}B_1^2-\mu\frac{3(1-\lambda)(2+\lambda)B}{2(1+\lambda)^2A^2}B_1^2\right|\right\}.$$

Proof. Since $f\in q\mathscr{C}_{\alpha,\ \beta,\ \sigma}^{\theta,\ k}(\lambda)$ then the condition

$$\mathscr{R}\left\{\frac{\left[z\left(q\mathbb{F}_{\alpha,\ \beta,\ \sigma}^{\theta,\ k}(f)(z)\right)'\right]'}{\left(q\mathbb{F}_{\alpha,\ \beta,\ \sigma}^{\theta,\ k}(f)(z)\right)'}\right\}>\lambda,\ 0\leq\lambda<1,$$

is equivalent to

$$\left(q\mathbb{F}_{\alpha,\,\beta,\,\sigma}^{\theta,\,k}(f)(z)\right)' + \left[z\left(q\mathbb{F}_{\alpha,\,\beta,\,\sigma}^{\theta,\,k}(f)(z)\right)'\right]' = (1-\lambda)h(z)\left(q\mathbb{F}_{\alpha,\,\beta,\,\sigma}^{\theta,\,k}(f)(z)\right)',$$

for a given function of positive real part, $h \in \mathscr{P}$. By equating the coefficients, we obtain

$$a_2 = \frac{(1-\lambda)h_1}{2(1+\lambda)A}, \tag{21}$$

and

$$a_3 = \frac{1-\lambda}{3(2+\lambda)B}\left(\frac{1-\lambda}{1+\lambda}h_1^2 + h_2\right) \tag{22}$$

Therefore, from (21) and Lemma 1, the result of $|a_2|$ is obtained.
In addition, we conclude the following;

$$a_3 - \mu a_2^2 = \frac{1-\lambda}{6(2+\lambda)B}\left(c_2 - vc_1^2\right), \tag{23}$$

Again, the values of A and B are given in (16), and the value of v here is given by

$$v = \frac{1}{2}\left(1 - B_2 - \frac{1-\lambda}{1+\lambda}B_1^2 + \mu\frac{3(1-\lambda)(2+\lambda)B}{2(1+\lambda)^2A^2}B_1^2\right),$$

the result then, is proved by using Lemma 2. □

Corollary 8 Under the assumption of Theorem 4, if $\lambda = 0$, then

$$|a_2| \le \frac{2}{3}\prod_{j=1}^{q}\frac{(\alpha_j)_{k_j}}{(\beta_j)_{\alpha_j}}\frac{(\beta_j)_{2\alpha_j}}{(\theta_j)_{2k_j}},\ 0 \le \lambda < 1,$$

and for all $\mu \in \mathbb{C}$.

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{6B}\max\left\{1,\ \left|B_2 + B_1^2 - \mu\frac{2B}{A^2}B_1^2\right|\right\}.$$

3. Conclusions

In this study, we have introduced a linear operator that is the Hadamard product of the most generalized Mittag-Leffler function along with the Lambert series, where the coefficients represent the function of sum of divisors. Hence, we investigated the inclusion conditions of the new operator into the well-known subclasses of starlike and convex functions of order λ. Finally; we have studied the Fekete-Szegö inequality

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Conflict of interest

The author declares there is no conflict of interest at any point with reference to research findings.

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