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Research Article

Finite Sum of Integral Operators from the Fractional Cauchy Spaces to Bloch-Type and Zygmund-Type Spaces

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Abstract: The family of fractional Cauchy transforms, defined on the open unit disc in the complex plane, is of classical and modern interest. Membership of an analytic function in the family is determined by the requirement that the function can be expressed as an integral of a certain kernel against a complex Borel measure on the disc. Such an integral representation imposes a growth condition on the function and its derivatives. This exposes a connection between the families of Cauchy transforms and familiar spaces of analytic functions, such as the Bloch spaces and the Zygmund space. The notion of a composition operator has been a fruitful area of study. More generally, many authors have studied weighted composition operators, the differentiation operator, integral-type operators, and various products of such operators, acting from one normed linear space of analytic functions to another such space. A common theme of such works is to characterize the operator-theoretic notions of boundedness and compactness in terms of the inducing symbols of the operator. We extend these studies to a specific linear transformation which will be defined as the sum of finitely many integral operators. Our conclusions include a complete characterization of boundedness and compactness of the integral sum, acting from the fractional Cauchy spaces to the Bloch-type and Zygmund-type spaces.

Keywords: linear operators, Banach spaces of analytic functions

MSC: 47B33, 46B01, 30E20

1. Introduction

We begin by establishing notation. We let $\mathbb{N}$ denote the set of positive integers. We denote the open unit disc in the complex plane by $\mathbb{D}$ and its boundary by $\partial\mathbb{D}$. We let $H(\mathbb{D})$ be the linear space of holomorphic functions on $\mathbb{D}$. We let $\mathfrak{M}$ denote the set of all complex Borel measures on $\partial\mathbb{D}$, endowed with the total variation norm.

A weight ν is a positive and continuous function on $\mathbb{D}$. The weight ν is called radial if $\nu(z) = \nu(|z|)$ for $z \in \mathbb{D}$. The weight ν is said to be typical if it is radial, non-increasing in the variable $|z|$, and $\lim_{|z|\to 1} \nu(z) = 0$.

For $\alpha > 0$, the family $\mathscr{F}_\alpha(\mathbb{D}) = \mathscr{F}_\alpha$ of Cauchy transforms is the collection of all $f \in H(\mathbb{D})$ which admit a representation of the form

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$$f(z)=\int_{\partial\mathbb{D}}\frac{d\mu(\zeta)}{(1-\overline{\zeta}z)^{\alpha}}\quad(z\in\mathbb{D}) \tag{1}$$

for some $\mu\in\mathfrak{M}$, where the principal branch of the logarithm is used here. The space $\mathscr{F}_\alpha$ is a Banach space with the norm

$$\|f\|_{\mathscr{F}_\alpha}=\inf_{\mu\in\mathfrak{M}}\left\{\|\mu\|:\ f(z)=\int_{\partial\mathbb{D}}\frac{d\mu(\zeta)}{(1-\overline{\zeta}z)^{\alpha}}\right\},$$

where $\|\mu\|$ is the total variation of the measure μ.

Let ν be a typical weight. Then the Bloch-type space $\mathscr{B}_\nu$ and the Zygmund-type space $\mathscr{Z}_\nu$ are defined for $f\in H(\mathbb{D})$, respectively, by

$$f\in\mathscr{B}_\nu\Leftrightarrow\|f\|_{\mathscr{B}_\nu}=|f(0)|+\sup_{z\in\mathbb{D}}\nu(z)|f'(z)|<\infty$$

and

$$f\in\mathscr{Z}_\nu\Leftrightarrow\|f\|_{\mathscr{Z}_\nu}=|f(0)|+|f'(0)|+\sup_{z\in\mathbb{D}}\nu(z)|f''(z)|<\infty.$$

If $\nu(z)=1-|z|^2$, then the spaces $\mathscr{B}_\nu$ and $\mathscr{Z}_\nu$ are the standard Bloch space $\mathscr{B}$ and the standard Zygmund space $\mathscr{Z}$.

Let φ be an analytic self-map of $\mathbb{D}$ and let $g\in H(\mathbb{D})$. Many authors have studied the operators $C_\varphi(f)=f\circ\varphi$, $M_g(f)=gf$ and the differentiation operator $D(f)=f'$, acting between various normed families of analytic functions. More recent work has focused on products of such operators [1–12]. Also of interest is the integral operator $J_{\varphi,g}$ defined for $f\in H(\mathbb{D})$ by

$$(J_{\varphi,g}f)(z)=\int_0^z f'(\varphi(w))g(w)dw.$$

We mention [13–18] although this is only a partial list.

S. Stević was among the first to study the sum of two weighted differentiation composition operators, that is, $\psi_0(f\circ\varphi)+\psi_1(f'\circ\varphi)$ where $\psi_j\in H(\mathbb{D})$. See [19]. The investigation of such operators was continued in [12, 20, 21]. Stević then proposed a study of the operator

$$(T_nf)(z)=\sum_{j=0}^{n}\psi_j(z)f^{(j)}(\varphi(z))$$

for generic natural number n.

We extend these studies to an operator defined as the finite sum of integral operators, acting from the spaces of fractional Cauchy transforms to the Bloch-type or Zygmund-type spaces. Let $n\in\mathbb{N}$ and let φ be an analytic self-map of $\mathbb{D}$. For $k=1,2,\ldots,n$, let $g_k\in H(\mathbb{D})$. For $f\in H(\mathbb{D})$ we define the operator J_n by

$$(J_n f)(z) = \int_0^z f'(\varphi(w))g_1(w)\,dw + \int_0^z f''(\varphi(w))g_2(w)\,dw + \cdots + \int_0^z f^{(n)}(\varphi(w))g_n(w)\,dw.$$

We establish necessary and sufficient conditions for boundedness and compactness of the operator J_n acting from the spaces of fractional Cauchy transforms to the Bloch-type and Zygmund-type spaces.

We present a brief description of the layout of the paper. In Section 2, we gather the necessary background on the structure of the families $\mathscr{F}_\alpha$. In Section 3, we provide necessary and sufficient conditions for boundedness and compactness of the operator J_n: $\mathscr{F}_\alpha \to \mathscr{B}_\nu$. Our method is illustrated by a brief example at the end of Section 3. In Section 4, necessary and sufficient conditions are established for boundedness and compactness of J_n: $\mathscr{F}_\alpha \to \mathscr{Z}_\nu$. The paper closes with an overview of the results and a brief discussion of the distinction between problems involving the sum of two operators and problems involving the sum of n operators for $n \geq 3$.

Throughout this paper, constants are denoted by C. These constants are positive and not necessarily the same at each occurrence.

2. Methods

In this section, we present several known results that will be used in this work.

Lemmas 1 and 3 appear in [22]. The case $k = 0$ in Lemma 2 appears in [22]. The result generalizes to positive integer k by an argument given in [3].

Lemma 1 Let $\alpha > 0$ and $k \in \mathbb{N}$. Let $f \in H(\mathbb{D})$. Then $f \in \mathscr{F}_\alpha$ if and only if $f^{(k)} \in \mathscr{F}_{\alpha+k}$. Moreover, if $f \in \mathscr{F}_\alpha$, there is a constant C depending only on α and k such that $\| f^{(k)} \|_{\mathscr{F}_{\alpha+k}} \leq C \| f \|_{\mathscr{F}_\alpha}$.

Lemma 2 Let $\alpha > 0$ and let $|w| \leq 1$. Fix a non-negative integer k and define

$$f_{w,k}(z) = \frac{(1 - |w|^2)^k}{(1 - \overline{w}z)^{\alpha+k}}, \ \ (|z| < 1).$$

Then $f_{w,k} \in \mathscr{F}_\alpha$ and there is a constant C independent of w such that $\|f_{w,k}\|_{\mathscr{F}_\alpha} \leq C$.

Lemma 3 Let $z \in \mathbb{D}$ and $\alpha > 0$. Let k be a non-negative integer. Then there is a constant C depending only on α and k such that

$$|f^{(k)}(z)| \leq C \frac{\|f\|_{\mathscr{F}_\alpha}}{(1 - |z|^2)^{\alpha+k}}$$

for every $f \in \mathscr{F}_\alpha$.

3. Results: The operator J_n: $\mathscr{F}_\alpha \to \mathscr{B}_\nu$

In this section we characterize boundedness and compactness of J_n: $\mathscr{F}_\alpha \to \mathscr{B}_\nu$ in terms of the symbols φ, g_k for $k = 1, 2, \ldots, n$ and the weight ν. The following lemma will be useful in the proof of Theorem 1.

Lemma 4 Fix $n \in \mathbb{N}$ and fix $\alpha > 0$. Let φ be a self-map of $\mathbb{D}$ and let $g_k \in H(\mathbb{D})$ for $k = 1, 2, \ldots, n$. Assume that J_n: $\mathscr{F}_\alpha \to \mathscr{B}_\nu$ is bounded. Then

$$\sup_{z\in\mathbb{D}} \nu(z) \mid g_k(z) \mid < \infty \text{ for } k = 1, 2, \ldots, n.$$

Proof. By assumption, there exists a constant C independent of f such that

$$\| J_n f \|_{\mathscr{B}_\nu} \leq C \| f \|_{\mathscr{F}_\alpha} .$$

Note that $f_k(z) = z^k/k! \in \mathscr{F}_\alpha$ for $k = 1, 2, \ldots, n.$ A calculation yields $(J_n f_1)(z) = \int_0^z g_1(w)\, dw$ and therefore

$$\sup_{z\in\mathbb{D}} \nu(z)\,|g_1(z)| = \sup_{z\in\mathbb{D}} \nu(z) \mid (J_n f_1)'(z) \mid \leq C \ \| f_1 \|_{\mathscr{F}_\alpha} .$$

Next, $(J_n f_2)(z) = \int_0^z \varphi(w) g_1(w)\, dw \ + \ \int_0^z g_2(w)\, dw$ and it follows that

$$\sup_{z\in\mathbb{D}} \nu(z) \mid \varphi(z) g_1(z) + g_2(z) \mid = \sup_{z\in\mathbb{D}} \nu(z) \mid (J_n f_2)'(z) \mid \leq C \ \| f_2 \|_{\mathscr{F}_\alpha} .$$

Since

$$\nu(z) \mid g_2(z) \mid \leq \ \nu(z) \mid \varphi(z) g_1(z) + g_2(z) \mid \ + \nu(z) \mid \varphi(z) g_1(z) \mid$$

and since $\mid \varphi(z) \mid < 1$, it follows that

$$\sup_{z\in\mathbb{D}} \nu(z) \mid g_2(z) \mid < C\, (\| f_1 \|_{\mathscr{F}_\alpha} + \| f_2 \|_{\mathscr{F}_\alpha}).$$

To complete the proof, fix $1 \leq k \leq n-1$ and assume

$$\sup_{z\in\mathbb{D}} \nu(z) \mid g_j(z) \mid < \infty \text{ for } j = 1, 2, \ldots, k.$$

By a calculation,

$$(J_n f_{k+1})'(z) = \frac{(\varphi(z))^k}{k!} g_1(z) + \ldots + \varphi(z) g_k(z) + g_{k+1}(z)$$

and an argument using the Triangle Inequality yields

$$\sup_{z\in\mathbb{D}} \nu(z) \mid g_{k+1}(z) \mid < \infty.$$

The proof is complete. $\square$

Theorem 1 Fix $\alpha > 0$ and $n \in \mathbb{N}$. Let ν be a weight function. Let φ be a self-map of $\mathbb{D}$ and let $g_k \in H(\mathbb{D})$ for $k = 1, 2, \ldots, n$. The operator J_n: $\mathscr{F}_\alpha \to \mathscr{B}_\nu$ is bounded if and only if

$$M_k = \sup_{z\in\mathbb{D}} \nu(z) \frac{\mid g_k(z) \mid}{(1-\mid \varphi(z) \mid^2)^{\alpha+k}} < \infty \; (k = 1, 2, \ldots, n). \tag{2}$$

Proof. First assume (2) and let $f \in \mathscr{F}_\alpha$. Note that $(J_n f)(0) = 0$. For each $k = 1, 2, \ldots, n$, Lemma 3 yields a constant C_k depending only on α such that

$$\mid f^{(k)}(\varphi(z)) \mid \leq \frac{C_k \parallel f \parallel_{\mathscr{F}_\alpha}}{(1-\mid \varphi(z) \mid^2)^{\alpha+k}}$$

for all $k = 1, 2, \ldots, n$ and for $z \in \mathbb{D}$. Therefore

$$\nu(z) \mid (J_n f)'(z) \mid \leq \parallel f \parallel_{\mathscr{F}_\alpha} \sum_{k=1}^{n} \frac{\nu(z)\, C_k \mid g_k(z) \mid}{(1-\mid \varphi(z) \mid^2)^{\alpha+k}} \leq (\sum_{k=1}^{n} C_k M_k) \; \parallel f \parallel_{\mathscr{F}_\alpha}$$

and thus $\sup_{z\in\mathbb{D}} \nu(z) \mid (J_n f)'(z) \mid \leq C \parallel f \parallel_{\mathscr{F}_\alpha}$. Therefore J_n: $\mathscr{F}_\alpha \to \mathscr{B}_\nu$ is bounded.

For the converse, suppose that J_n: $\mathscr{F}_\alpha \to \mathscr{B}_\nu$ is bounded. Lemma 4 yields

$$\sup_{|\varphi(z)| \leq 1/2} \frac{\nu(z) \mid g_k(z) \mid}{(1-\mid \varphi(z) \mid^2)^{\alpha+k}} < \infty \text{ for } k = 1, 2, \ldots, n.$$

In order to complete the argument we define the following test functions

$$f_\lambda(z) = \sum_{j=0}^{n-1} \gamma_j \frac{(1-\mid \varphi(\lambda) \mid^2)^j}{(1-\overline{\varphi(\lambda)}z)^{\alpha+j}} \; (z \in \mathbb{D}) \tag{3}$$

where λ varies over $\mathbb{D}$ and the scalars γ_j depend only on α. By Lemma 2, $\parallel f_\lambda \parallel_{\mathscr{F}_\alpha} \leq C$ for a constant C independent of λ. Therefore $\parallel J_n(f_\lambda) \parallel_{\mathscr{B}_\nu} \leq C$. In particular,

$$\sup_{z\in\mathbb{D}} \nu(z) \mid (J_n f_\lambda)'(z) \mid \leq C$$

for all $\lambda \in \mathbb{D}$.

To obtain $M_1 < \infty$, we will choose the scalars γ_j in (3) so that

$$f'_\lambda(\varphi(\lambda)) = \frac{\overline{\varphi(\lambda)}}{(1-\mid \varphi(\lambda) \mid^2)^{\alpha+1}}$$

and $f_\lambda^{(k)}(\varphi(\lambda)) = 0$ for $k = 2, 3, \ldots, n$. Note that

$$f_\lambda^{(k)}(z) = \overline{\varphi(\lambda)}^k \sum_{j=0}^{n-1} \Big(\prod_{l=j}^{j+k-1} (\alpha+l)\ \gamma_j \frac{(1- \mid \varphi(\lambda) \mid^2)^j}{(1-\overline{\varphi(\lambda)}z)^{\alpha+j+k}} \Big)$$

and

$$f_\lambda^{(k)}(\varphi(\lambda)) \ = \ \frac{\overline{\varphi(\lambda)}^k}{(1- \mid \varphi(\lambda) \mid^2)^{\alpha+k}} \sum_{j=0}^{n-1} \Big(\prod_{l=j}^{j+k-1} (\alpha+l) \Big)\ \gamma_j$$

for $k = 1, 2, \ldots, n$. Therefore the scalars $\gamma_0, \gamma_1, \ldots, \gamma_{n-1}$ are chosen to obey the following n equations.

$$\alpha\ \gamma_0 + (\alpha+1)\ \gamma_1 + \cdots + (\alpha+n-1)\ \gamma_{n-1} = 1$$

$$\alpha(\alpha+1)\ \gamma_0 + (\alpha+1)(\alpha+2)\ \gamma_1 + \cdots + (\alpha+n-1)(\alpha+n)\ \gamma_{n-1} = 0$$

$$\prod_{l=0}^{2}(\alpha+l)\ \gamma_0 + \prod_{l=1}^{3}(\alpha+l)\ \gamma_1 + \cdots + \prod_{l=n-1}^{n+1}(\alpha+l)\ \gamma_{n-1} = 0$$

$$\vdots$$

$$\prod_{l=0}^{n-1}(\alpha+l)\ \gamma_0 + \prod_{l=1}^{n}(\alpha+l)\ \gamma_1 + \cdots + \prod_{l=n-1}^{2n-2}(\alpha+l)\ \gamma_{n-1} = 0. \tag{4}$$

To see that such γ_j exist, let A_n denote the $n \times n$ matrix of the coefficients of the equations at (4). Let Γ be the $n \times 1$ column matrix in which γ_j appears in the $(j+1)$-st row for $j = 0, 1, \ldots, n-1$. Let E_1 be the $n \times 1$ column matrix with entry 1 in the first row and with all other entries 0. By an argument using Lemma 3 [9], the determinant of the matrix A_n is

$$\alpha(\alpha+1)\ldots(\alpha+n-1)\prod_{j=1}^{n-1} j!$$

and thus the γ_j are found as the solution to the equation $A_n\Gamma = E_1$.

With these γ_j in (3), it follows that

$$C \geq \| J_n f_\lambda \|_{\mathscr{B}_\nu} = \sup_{z\in\mathbb{D}} \nu(z) \mid \sum_{k=1}^{n} f_\lambda^{(k)}(\varphi(z)) g_k(z) \mid$$

$$\geq \nu(\lambda) \mid \sum_{k=1}^{n} f_\lambda^{(k)}(\varphi(\lambda)) g_k(\lambda) \mid$$

$$= \nu(\lambda) \frac{\mid \varphi(\lambda) \mid \mid g_1(\lambda) \mid}{(1- \mid \varphi(\lambda) \mid^2)^{\alpha+1}}$$

for all $\lambda \in \mathbb{D}$. We conclude that

$$\sup_{|\varphi(\lambda)|>1/2} \frac{\nu(\lambda) \mid g_1(\lambda) \mid}{(1- \mid \varphi(\lambda) \mid^2)^{\alpha+1}} \leq C.$$

The initial remark using Lemma 4 now yields $M_1 < \infty$.

To obtain $M_k < \infty$ for fixed $k = 2, 3, \ldots, n$, let E_k be the $n \times 1$ matrix with entry 0 in every row except the k-th row, where the entry is 1. The scalars γ_j, $j = 0, 1, \ldots, n-1$ are then obtained as the solution of the equation $A_n \Gamma = E_k$. Using these γ_j in (3), we obtain

$$f_\lambda^{(k)}(\varphi(\lambda)) = \frac{\overline{\varphi(\lambda)^k}}{(1- \mid \varphi(\lambda) \mid^2)^{\alpha+k}}$$

and $f_\lambda^{(l)}(\varphi(\lambda)) = 0$ for $l = 1, 2, \ldots, n$ with $l \neq k$. As in the previous argument, it follows that

$$\sup_{|\varphi(\lambda)|>1/2} \frac{\nu(\lambda) \mid g_k(\lambda) \mid}{(1- \mid \varphi(\lambda) \mid^2)^{\alpha+k}} < C.$$

An application of Lemma 4 now yields $M_k < \infty$. We have obtained (2) and the proof is complete. □

The following sequential criterion for compactness of a bounded operator is well known. The criterion is stated here for the operator J_n: $\mathscr{F}_\alpha \to \mathscr{B}_\nu$. In Section 4, the analogous result will be used to characterize compactness of J_n: $\mathscr{F}_\alpha \to \mathscr{Z}_\nu$.

Lemma 5 Assume that J_n: $\mathscr{F}_\alpha \to \mathscr{B}_\nu$ is bounded for fixed $n \in \mathbb{N}$ and for $\alpha > 0$. Then J_n: $\mathscr{F}_\alpha \to \mathscr{B}_\nu$ is compact if and only if for any bounded sequence (f_m) in $\mathscr{F}_\alpha$ such that $f_m \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $m \to \infty$, it follows that $\| J_n(f_m) \|_{\mathscr{B}_\nu} \to 0$ as $m \to \infty$.

Theorem 2 Assume that J_n: $\mathscr{F}_\alpha \to \mathscr{B}_\nu$ is bounded. Then J_n: $\mathscr{F}_\alpha \to \mathscr{B}_\nu$ is compact if and only if

$$\lim_{|\varphi(z)|\to 1} \frac{\nu(z) \mid g_k(z) \mid}{(1- \mid \varphi(z) \mid^2)^{\alpha+k}} = 0 \tag{5}$$

for $k = 1, 2, \ldots, n$.

Proof. First assume that (5) holds. Let (f_m) be a bounded sequence in $\mathscr{F}_\alpha$ with $f_m \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $m \to \infty$. We assume that $\| f_m \|_{\mathscr{F}_\alpha} \leq 1$ for $m = 1, 2, \ldots$. By Lemma 5, we must prove that $\| J_n f_m \|_{\mathscr{B}_\nu} \to 0$ as $m \to \infty$. It is enough to prove that $\sup_{z\in\mathbb{D}} \nu(z) \mid (J_n f_m)'(z) \mid \to 0$ as $m \to \infty$.

Lemma 3 implies that there is a positive constant C such that

$$\mid f_m^{(k)}(\varphi(z)) \mid \leq \frac{C}{(1- \mid \varphi(z) \mid^2)^{\alpha+k}}$$

for $k = 1, 2, \ldots, n$ and for $z \in \mathbb{D}$. Given $\varepsilon > 0$, the hypothesis yields r_0, $0 < r_0 < 1$, such that

$$\frac{\nu(z) \mid g_k(z) \mid}{(1- \mid \varphi(z) \mid^2)^{\alpha+k}} < \frac{\varepsilon}{Cn}$$

for all $k = 1, 2, \ldots, n$, provided $\mid \varphi(z) \mid > r_0$. Therefore

$$\begin{aligned} \sup_{|\varphi(z)|>r_0} \nu(z) \mid (J_n f_m)'(z) \mid &= \sup_{|\varphi(z)|>r_0} \sum_{k=1}^{n} \nu(z) \mid f_m^{(k)}(\varphi(z)) \mid \mid g_k(z) \mid \\ &\leq \sup_{|\varphi(z)|>r_0} \sum_{k=1}^{n} C \frac{\nu(z) \mid g_k(z) \mid}{(1- \mid \varphi(z) \mid^2)^{\alpha+k}} < \varepsilon \end{aligned}$$

for all $m = 1, 2, \ldots$.

Next consider z with $\mid \varphi(z) \mid \leq r_0$. Since J_n: $\mathscr{F}_\alpha \to \mathscr{B}_\nu$ is bounded, Theorem 1 yields

$$\nu(z) \mid (J_n f_m)'(z) \mid \leq \sum_{k=1}^{n} M_k \mid f_m^{(k)}(\varphi(z)) \mid$$

where we assume $M_k > 0$, $k = 1, 2, \ldots, n$. Since $f_m^{(k)} \to 0$ uniformly on the set $\{w\colon \mid w \mid \leq r_0\}$ for each $k = 1, 2, \ldots, n$, there exists $M > 0$ such that

$$\mid f_m^{(k)}(w) \mid < \frac{\varepsilon}{nM_k}$$

for all $\mid w \mid \leq r_0$ and for all $k = 1, 2, \ldots, n$, provided that $m > M$. Therefore

$$\sup_{|\varphi(z)|\leq r_0} \nu(z) \mid (J_n f_m)'(z) \mid < \varepsilon$$

for $m > M$. Since $(J_n f_m)(0) = 0$, we conclude that

$$\| J_n f_m \|_{\mathscr{B}_\nu} = \sup_{z\in\mathbb{D}} \nu(z) \mid (J_n f_m)'(z) \mid \to 0$$

as $m \to \infty$, and the operator is compact.

For the converse, we assume that J_n: $\mathscr{F}_\alpha \to \mathscr{B}_\nu$ is compact. Assume that $|\varphi(z_m)| \to 1$ as $m \to \infty$ for a sequence z_m in $\mathbb{D}$. We note that if no such sequence exists, then the conditions in the theorem are established vacuously.

To obtain the limits at (5), we will use test functions f_m ($m = 1, 2, \ldots$) with $\| f_m \|_{\mathscr{F}_\alpha} \leq C$ and $f_m \to 0$ uniformly on compact subsets as $m \to \infty$. By Lemma 5, $\| J_n f_m \|_{\mathscr{B}_\nu} \to 0$ as $m \to \infty$. For $\varepsilon > 0$, there exists $M > 0$ such that

$$\sup_{z\in\mathbb{D}} \nu(z) \mid (J_n f_m)'(z) \mid = \sup_{z\in\mathbb{D}} \nu(z) \mid \sum_{k=1}^{n} f_m^{(k)}(\varphi(z)) g_k(z) \mid < \varepsilon$$

for all $m > M$.

The functions f_m are defined as

$$f_m(z) = \sum_{j=0}^{n-1} \gamma_j \frac{(1- \mid \varphi(z_m) \mid^2)^{j+1}}{(1 - \overline{\varphi(z_m)} z)^{\alpha+j+1}} \tag{6}$$

for various choices of the scalars γ_j, where γ_j depends only on α. Lemma 2 implies $\| f_m \|_{\mathscr{F}_\alpha} \leq C$ for a positive constant C independent of m. It is clear that $f_m \to 0$ uniformly on compact subsets as $m \to \infty$.

A calculation yields

$$f_m^{(k)}(z) = \overline{\varphi(z_m)}^k \sum_{j=0}^{n-1} \prod_{l=j+1}^{j+k} (\alpha + l)\ \gamma_j \frac{(1- \mid \varphi(z_m) \mid^2)^{j+1}}{(1 - \overline{\varphi(z_m)} z)^{\alpha+j+k+1}}$$

and

$$f_m^{(k)}(\varphi(z_m)) \ = \ \frac{\overline{\varphi(z_m)}^k}{(1- \mid \varphi(z_m) \mid^2)^{\alpha+k}} \sum_{j=0}^{n-1} \prod_{l=j+1}^{j+k} (\alpha + l)\ \gamma_j$$

for $k = 1, 2, \ldots$ and for $m = 1, 2, \ldots$.

To obtain the first required limit, we choose the scalars γ_j ($j = 0, 1, \ldots, n-1$) in (6) to obey the system of n equations below.

$$(\alpha+1)\ \gamma_0 + (\alpha+2)\ \gamma_1 + \ldots + (\alpha+n)\ \gamma_{n-1} = 1$$

$$(\alpha+1)(\alpha+2)\ \gamma_0 + (\alpha+2)(\alpha+3)\ \gamma_1 + \ldots + (\alpha+n)(\alpha+n+1)\ \gamma_{n-1} = 0$$

$$\prod_{l=1}^{3}(\alpha+l)\ \gamma_0 + \prod_{l=2}^{4}(\alpha+l)\ \gamma_1 + \ldots + \prod_{l=n}^{n+2}(\alpha+l)\ \gamma_{n-1} = 0$$

$$\vdots$$

$$\prod_{l=1}^{n}(\alpha+l)\ \gamma_0+\prod_{l=2}^{n+1}(\alpha+l)\ \gamma_1+\ldots+\prod_{l=n}^{2n-1}(\alpha+l)\ \gamma_{n-1}=0 \tag{7}$$

Existence of γ_j as described at (7) follows as in the proof of Theorem 1, using the determinant calculation in [9]. With these scalars in (6), we obtain

$$f_m'(\varphi(z_m))=\frac{\overline{\varphi(z_m)}}{(1-\mid\varphi(z_m)\mid^2)^{\alpha+1}}$$

and $f_m^{(k)}(\varphi(z_m))=0$ for $k=2,\,3,\,\ldots,\,n$. Therefore if $m>M$,

$$\nu(z_m)\frac{\mid g_1(z_m)\mid\mid\varphi(z_m)\mid}{(1-\mid\varphi(z_m)\mid^2)^{\alpha+1}}=\nu(z_m)\mid\sum_{k=1}^{n}f_m^{(k)}(\varphi(z_m))g_k(z_m)\mid$$

$$\leq\sup_{z\in\mathbb{D}}\nu(z)\mid\sum_{k=1}^{n}f_m^{(k)}(\varphi(z))g_k(z)\mid$$

$$\leq\|J_nf_m\|_{\mathscr{B}_\nu}<\varepsilon.$$

Since $\mid\varphi(z_m)\mid\to1$, the argument shows that

$$\frac{\nu(z_m)\mid g_1(z_m)\mid}{(1-\mid\varphi(z_m)\mid^2)^{\alpha+1}}\to0$$

as $m\to\infty$. Since z_m is a generic sequence with $\mid\varphi(z_m)\mid\to1$ we conclude that

$$\lim_{|\varphi(z)|\to1}\frac{\nu(z)\mid g_1(z)\mid}{(1-\mid\varphi(z)\mid^2)^{\alpha+1}}=0.$$

The remaining limits are derived in a similar way. For fixed $k=2,\,3,\,\ldots,\,n$, the γ_j are chosen to ensure that

$$f_m^{(k)}(\varphi(z_m))=\frac{\overline{\varphi(z_m)^k}}{(1-\mid\varphi(z_m)\mid^2)^{\alpha+k}}$$

and $f_m^{(l)}(\varphi(z_m))=0$ for $l\neq k$. The argument proceeds as above. We have obtained (5) and the proof is complete. □

We end this section with an example to illustrate the results. Let $\varphi(z)=(z+1)/2$ and let $g_j(z)=(1-z)^\gamma$ for fixed $\gamma>0$ and for $j=1,\,2,\,\ldots,\,n$. Let $\nu(z)=(1-\mid z\mid^2)^\beta$ for fixed $\beta>0$.

First we assume $\beta+\gamma\geq\ \alpha+n$. By a calculation using the Schwarz-Pick Lemma,

$$M_j = \sup_{z\in\mathbb{D}} \frac{\nu(z) \mid g_j(z) \mid}{(1- \mid \varphi(z) \mid^2)^{\alpha+j}} \leq C \sup_{z\in\mathbb{D}} (1- \mid z \mid^2)^{\beta-\alpha-j} \mid 1-z \mid^{\gamma} < \infty$$

for $j = 1, 2, \ldots, n$. By Theorem 1, J_n: $\mathscr{F}_\alpha \to \mathscr{B}_\nu$ is bounded.

If on the other hand $\beta + \gamma < \alpha + j$ for some j, $1 \leq j \leq n$, then

$$M_j \geq 4^{\alpha+j} \sup_{0\leq x<1} \frac{(1-x^2)^{\beta}(1-x)^{\gamma}}{(x+3)^{\alpha+j}(1-x)^{\alpha+j}} = \infty$$

and J_n: $\mathscr{F}_\alpha \to \mathscr{B}_\nu$ is not bounded.

Next assume $\beta + \gamma = \alpha + n$. Note that $\mid \varphi(z) \mid \to 1 \Rightarrow z \to 1$ and

$$\lim_{x\to 1} \frac{(1-x^2)^{\beta}(1-x)^{\gamma}}{(x+3)^{\alpha+n}(1-x)^{\alpha+n}} \neq 0.$$

Thus J_n: $\mathscr{F}_\alpha \to \mathscr{B}_\nu$ is bounded and is not compact.

Finally assume $\beta + \gamma > \alpha + n$. An argument similar to the above establishes the limit conditions at (5) and thus J_n: $\mathscr{F}_\alpha \to \mathscr{B}_\nu$ is compact, by Theorem 2.

4. Further results: The operator J_n: $\mathscr{F}_\alpha \to \mathscr{Z}_\nu$

In this section we characterize boundedness and compactness of the operator J_n: $\mathscr{F}_\alpha \to \mathscr{Z}_\nu$.

We first establish notation. Let $f \in H(\mathbb{D})$. Then

$$(J_n f)''(z) = \sum_{k=1}^{n+1} f^{(k)}(\varphi(z))\Psi_k(z) \tag{8}$$

where $\Psi_1(z) = g_1'(z)$, $\Psi_k(z) = \varphi'(z)g_{k-1}(z) + g_k'(z)$ for $k = 2, 3, \ldots, n$ and $\Psi_{n+1}(z) = \varphi'(z)g_n(z)$. The characterizations for boundedness and compactness will be given in terms of the Ψ_k, $k = 1, 2, \ldots, n+1$.

Lemma 6 is similar to Lemma 4. Brief details will be provided.

Lemma 6 Assume that J_n: $\mathscr{F}_\alpha \to \mathscr{Z}_\nu$ is bounded. Then

$$\sup_{z\in\mathbb{D}} \nu(z) \mid \Psi_k(z) \mid < \infty \text{ for } k = 1, 2, \ldots, n+1.$$

Proof. By assumption there exists a constant C independent of $f \in \mathscr{F}_\alpha$ such that $\| J_n f \|_{\mathscr{Z}_\nu} \leq C \| f \|_{\mathscr{F}_\alpha}$. In particular

$$\sup_{z\in\mathbb{D}} \nu(z) \mid (J_n f)''(z) \mid \leq C \| f \|_{\mathscr{F}_\alpha}.$$

With $f_1(z) = z$, we obtain $(J_n f_1)''(z) = g_1'(z) = \Psi_1(z)$ and thus

$$\sup_{z\in\mathbb{D}} \nu(z) \mid \Psi_1(z) \mid < C \parallel f_1 \parallel_{\mathscr{F}_\alpha} .$$

With $f_2(z) = z^2/2$, we obtain $(J_n f_2)''(z) = \varphi(z)\Psi_1(z) + \Psi_2(z)$. For $z \in \mathbb{D}$,

$$\nu(z) \mid \Psi_2(z) \mid \le \nu(z) \mid \varphi(z)\Psi_1(z) + \Psi_2(z) \mid + \nu(z) \mid \Psi_1(z) \mid$$

$$\le C\,(\parallel f_1 \parallel_{\mathscr{F}_1} + \parallel f_2 \parallel_{\mathscr{F}_2})$$

and thus

$$\sup_{z\in\mathbb{D}} \nu(z) \mid \Psi_2(z) \mid < \infty.$$

An inductive argument similar to the proof in Lemma 4 now completes the proof. □

Theorem 3 Fix $\alpha > 0$ and $n \in \mathbb{N}$. Let φ be a self-map of $\mathbb{D}$ and let $g_k \in H(\mathbb{D})$ for $k = 1, 2, \dots, n$. The operator J_n: $\mathscr{F}_\alpha \to \mathscr{Z}_\nu$ is bounded if and only if

$$N_k = \sup_{z\in\mathbb{D}} \frac{\nu(z) \mid \Psi_k(z) \mid}{(1- \mid \varphi(z) \mid^2)^{\alpha+k}} < \infty \text{ for } k = 1, 2, \dots, n+1. \tag{9}$$

Proof. First assume (9) holds and let $f \in \mathscr{F}_\alpha$. Since $(J_n f)(0) = 0$ and since Lemma 3 yields $\mid (J_n f)'(0) \mid \le C \parallel f \parallel_{\mathscr{F}_\alpha}$, it is enough to prove

$$\sup_{z\in\mathbb{D}} \nu(z) \mid (J_n f)''(z) \mid \le C \parallel f \parallel_{\mathscr{F}_\alpha} .$$

By Lemma 3 and (8),

$$\nu(z) \mid (J_n f)''(z) \mid \le C \parallel f \parallel_{\mathscr{F}_\alpha} \sum_{k=1}^{n+1} \frac{\nu(z) \mid \Psi_k(z) \mid}{(1- \mid \varphi(z) \mid^2)^{\alpha+k}}$$

$$\le C\,(\sum_{k=1}^{n+1} N_k) \parallel f \parallel_{\mathscr{F}_\alpha}$$

and thus the operator is bounded.

Next we assume that J_n: $\mathscr{F}_\alpha \to \mathscr{Z}_\nu$ is bounded. Lemma 6 yields

$$\sup_{|\varphi(z)|\le 1/2} \frac{\nu(z) \mid \Psi_k(z) \mid}{(1- \mid \varphi(z) \mid^2)^{\alpha+k}} < \infty \text{ for } k = 1, 2, \dots, n+1.$$

To complete the argument we will use test functions defined by

$$f_\lambda(z) = \sum_{j=0}^{n} \gamma_j \frac{(1-|\varphi(\lambda)|^2)^j}{(1-\overline{\varphi(\lambda)}z)^{\alpha+j}} \tag{10}$$

where $\lambda \in \mathbb{D}$ and the scalars γ_j depend only on α. Lemma 2 implies that $\| f_\lambda \|_{\mathscr{F}_\alpha} \leq C$ and therefore $\| J_n f_\lambda \|_{\mathscr{Z}_\nu} \leq C$ for all $\lambda \in \mathbb{D}$. In particular

$$\sup_{z\in\mathbb{D}} \nu(z) \,|\, \sum_{k=1}^{n+1} f_\lambda^{(k)}(\varphi(z))\Psi_k(z) \,|\, \leq C$$

for all λ.

By a calculation,

$$f_\lambda^{(k)}(\varphi(\lambda)) = \frac{\overline{\varphi(\lambda)}^k}{(1-|\varphi(\lambda)|^2)^{\alpha+k}} \sum_{j=0}^{n} \prod_{l=j}^{j+k-1} (\alpha+l)\, \gamma_j.$$

To obtain $N_1 < \infty$ the γ_j are chosen to obey the system of $n+1$ equations below.

$$\alpha\, \gamma_0 + (\alpha+1)\, \gamma_1 + \ldots + (\alpha+n)\, \gamma_n = 1$$

$$\alpha(\alpha+1)\, \gamma_0 + (\alpha+1)(\alpha+2)\, \gamma_1 + \ldots + (\alpha+n)(\alpha+n+1)\, \gamma_n = 0$$

$$\prod_{l=0}^{2}(\alpha+l)\, \gamma_0 + \prod_{l=1}^{3}(\alpha+l)\, \gamma_1 + \ldots + \prod_{l=n}^{n+2}(\alpha+l)\, \gamma_n = 0$$

$$\vdots$$

$$\prod_{l=0}^{n}(\alpha+l)\, \gamma_0 + \prod_{l=1}^{n+1}(\alpha+l)\, \gamma_1 + \ldots + \prod_{l=n}^{2n}(\alpha+l)\, \gamma_n = 0. \tag{11}$$

Let A_{n+1} be the $(n+1)\times(n+1)$ matrix of the coefficients of the system at (11). A calculation using [9] shows that the determinant of A_{n+1} is non-zero. Let Γ be the $(n+1)\times 1$ matrix with entry γ_j in the $(j+1)$-st row for $j = 0, 1, 2, \ldots, n$. Let E_1 be the $(n+1)\times 1$ matrix with entry 1 in the first row and with all other entries 0. We obtain the scalars γ_j as the solution of the equation $A_{n+1}\Gamma = E_1$. Substitution of these scalars into (10) yields

$$f'_\lambda(\varphi(\lambda)) = \frac{\overline{\varphi(\lambda)}}{(1-|\varphi(\lambda)|^2)^{\alpha+1}}$$

and $f_\lambda^{(k)}(\varphi(\lambda)) = 0$ for $k = 2, 3, \ldots, n+1$. Therefore

$$C \ \geq \ \| \ J_n f_\lambda \ \|_{\mathscr{Z}_\nu} \geq \ \sup_{z\in\mathbb{D}} \nu(z) \mid \sum_{k=1}^{n+1} f_\lambda^{(k)}(\varphi(z)) \ \Psi_k(z) \mid$$

$$\geq \nu(\lambda) \mid f'_\lambda(\varphi(\lambda)) \ \Psi_1(\lambda) \mid$$

$$= \nu(\lambda) \frac{\mid \varphi(\lambda) \ \Psi_1(\lambda) \mid}{(1- \mid \varphi(\lambda) \mid^2)^{\alpha+1}}.$$

Thus

$$\sup_{|\varphi(\lambda)|>1/2} \nu(\lambda) \frac{\mid \Psi_1(\lambda) \mid}{(1- \mid \varphi(\lambda) \mid^2)^{\alpha+1}} \ < \ \infty$$

and an application of Lemma 6 yields $N_1 < \infty$.

To obtain $N_k < \infty$ for fixed $k = 2, 3, \ldots, n+1$, we let E_k be the $(n+1) \times 1$ matrix with entry 1 in the k-th row and with entry 0 in every other row. The desired scalars γ_j are the solution to the equation $A_{n+1}\Gamma = E_k$.

With these scalars in (10), the test function f_λ obeys

$$f_\lambda^{(k)}(\varphi(\lambda)) = \frac{\overline{\varphi(\lambda)}^k}{(1- \mid \varphi(\lambda) \mid^2)^{\alpha+k}}$$

and $f_\lambda^{(l)}(\varphi(\lambda)) = 0$ for $l = 1, \ldots, n+1$ with $l \neq k$. It follows that

$$\sup_{|\varphi(\lambda)|>1/2} \frac{\nu(\lambda) \mid \Psi_k(\lambda) \mid}{(1- \mid \varphi(\lambda) \mid^2)^{\alpha+k}} \ < \ \infty$$

and an application of Lemma 6 then yields $N_k < \infty$. We have obtained (9) and the proof is complete. □

Theorem 4 Assume that J_n: $\mathscr{F}_\alpha \to \mathscr{Z}_\nu$ is bounded. Then J_n: $\mathscr{F}_\alpha \to \mathscr{Z}_\nu$ is compact if and only if

$$\lim_{|\varphi(z|\to 1} \frac{\nu(z) \mid \Psi_k(z) \mid}{(1- \mid \varphi(z) \mid^2)^{\alpha+k}} = 0 \tag{12}$$

for $k = 1, 2, \ldots, n+1$.

Proof. First assume that (12) holds. Let (f_m) be a bounded sequence in $\mathscr{F}_\alpha$ with $f_m \to 0$ uniformly on compact subsets as $m \to \infty$. We may assume $\| \ f_m \ \|_{\mathscr{F}_\alpha} \leq 1$ for $m = 1, 2, \ldots$. As shown in Lemma 5, it is enough to show that $\| \ J_n f_m \ \|_{\mathscr{Z}_\nu} \to 0$ as $m \to \infty$. First note that the assumption of uniform convergence implies that $\mid (J_n f_m)'(0) \mid \to 0$ as $m \to \infty$. It remains to prove that $\sup_{z\in\mathbb{D}} \nu(z) \mid (J_n f_m)''(z) \mid \to 0$ as $m \to \infty$.

As in previous arguments, there is a positive constant C with

$$| f_m^{(k)}(\varphi(z)) | \leq \frac{C}{(1- | \varphi(z) |^2)^{\alpha+k}}$$

for $k = 1, 2, \ldots, n+1$ and for $z \in \mathbb{D}$.

Given $\varepsilon > 0$, there exists r_0, $0 < r_0 < 1$, such that

$$\frac{\nu(z) | \Psi_k(z) |}{(1- | \varphi(z) |^2)^{\alpha+k}} < \frac{\varepsilon}{C(n+1)}$$

for all $k = 1, 2, \ldots, n+1$, provided that $| \varphi(z) | > r_0$. For such z,

$$\nu(z) | (J_n f_m)''(z) | \leq C \sum_{k=1}^{n+1} \nu(z) \frac{| \Psi_k(z) |}{(1- | \varphi(z) |^2)^{\alpha+k}} < \varepsilon$$

and we obtain

$$\sup_{|\varphi(z)|>r_0} \nu(z) | (J_n f_m)''(z) | < \varepsilon$$

for $m = 1, 2, \ldots$.

Next consider z with $| \varphi(z) | \leq r_0$. Since $J_n \colon \mathscr{F}_\alpha \to \mathscr{Z}_\nu$ is bounded, Theorem 3 gives

$$\nu(z) | (J_n f_m)''(z) | \leq \sum_{k=1}^{n+1} N_k | f_m^{(k)}(\varphi(z)) |$$

where we assume $N_k > 0$. Since $f_m^{(k)} \to 0$ on the set $\{w\colon | w | \leq r_0\}$, there exists N such that

$$| f_m^{(k)}(w) | < \frac{\varepsilon}{(n+1)N_k}$$

for all $| w | \leq r_0$, for all $m > N$ and for $k = 1, 2, \ldots, n+1$. We obtain

$$\sup_{|\varphi(z)|\leq r_0} \nu(z) | (J_n f_m)''(z) | < \varepsilon$$

for all $m > N$.

Since $f_m \to 0$ uniformly on compact subsets, it follows easily that $| (J_n f_m)'(0) | \to 0$ as $m \to \infty$. We conclude that

$$\| J_n f_m \|_{\mathscr{Z}_\nu} \to 0 \text{ as } m \to \infty$$

and therefore $J_n\colon \mathscr{F}_\alpha \to \mathscr{Z}_\nu$ is compact.

Finally we assume that $J_n\colon \mathscr{F}_\alpha \to \mathscr{Z}_\nu$ is compact. As in the proof of Theorem 2, we consider a sequence z_m in $\mathbb{D}$ with $|\varphi(z_m)| \to 1$ as $m \to \infty$. We define a sequence of test functions f_m $(m = 1, 2, \ldots)$ as

$$f_m(z) = \sum_{j=0}^{n} \gamma_j \frac{(1-|\varphi(z_m)|^2)^{j+1}}{(1-\overline{\varphi(z_m)}z)^{\alpha+j+1}} \tag{13}$$

where, as before, the γ_j depend only on α. Therefore $\| f_m \|_{\mathscr{F}_\alpha} \leq C$ and $f_m \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $m \to \infty$. By Lemma 5,

$$\sup_{z\in\mathbb{D}} \nu(z) \,|\, (J_n f_m)''(z) \,| = \sup_{z\in\mathbb{D}} \nu(z) \,\Big|\, \sum_{k=1}^{n+1} f_m^{(k)}(\varphi(z))\Psi_k(z) \,\Big| \to 0$$

as $m \to \infty$.

For example, to obtain the first limit at (12), the scalars $\gamma_0, \ldots, \gamma_n$ in (13) are chosen so that

$$f_m'(\varphi(z_m)) = \frac{\overline{\varphi(z_m)}}{(1-|\varphi(z_m)|^2)^{\alpha+1}}$$

and $f_m^{(k)}(\varphi(z_m)) = 0$ for $k = 2, 3, \ldots, n+1$. We then obtain

$$\nu(z_m)\frac{|\,\varphi(z_m)\,\Psi_1(z_m)\,|}{(1-|\varphi(z_m)|^2)^{\alpha+1}} \leq \sup_{z\in\mathbb{D}} \nu(z) \,|\, (J_n f_m)''(z) \,| \to 0$$

as $m \to \infty$. Since $|\varphi(z_m)| \to 1$ as $m \to \infty$, we obtain the first required limit.

Existence of γ_j as described follows as in Theorem 3, using the invertible matrix A_{n+1}.

A similar argument yields the remaining limits at (12). The proof is complete. $\square$

5. Conclusions

In this work we have characterized the self-maps φ, the holomorphic functions g_k $(k = 1, 2, \ldots, n)$ and the weight ν for which the integral operator J_n defined for $f \in H(\mathbb{D})$ by

$$(J_n f)(z) = \int_0^z f'(\varphi(w))g_1(w)\,dw + \int_0^z f''(\varphi(w))g_2(w)\,dw \ldots + \int_0^z f^{(n)}(\varphi(w))g_n(w)\,dw$$

is bounded or compact, acting from the spaces of Cauchy transforms to the Bloch-type space and the Zygmund-type space.

Our method is applicable to various finite sums of three or more operators. Because of known growth conditions on $f \in \mathscr{F}_\alpha$, it is relatively straight-forward to find conditions sufficient for boundedness or compactness of such an operator. The key here is the construction of suitable test functions, in order to prove necessity of our conditions. In the case of the sum of two operators, it is possible to solve explicitly for the coefficients γ_j of suitable test functions f_w $(w \in \mathbb{D})$ defined as

$$f_w(z) = \sum_{j=0}^{n-1} \gamma_j \frac{(1-|w|^2)^j}{(1-\overline{w}z)^{\alpha+j}} \; (z \in \mathbb{D}).$$

However, in the case of generic $n \geq 3$, the approach here is to avoid calculation by proving the existence of the appropriate scalars γ_j as described in a matrix equation. Probably the first to take this approach was S. Stević [9]. Indeed, in this paper the current authors make use of a slightly modified calculation of the determinant of a matrix that first appeared in [9].

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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