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Research Article

Reliable Computational Method for Systems of Fractional Differential Equations Endowed with ψ-Caputo Fractional Derivative

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Abstract: This study develops a highly convergent computational method, the ψ-Laplace Adomian Decomposition Method (ψ-LADM), for solving coupled systems of ψ-Caputo Fractional Differential Equations (FDEs). The effectiveness of the proposed method has been assessed using various numerical test examples, including a real-world application for atmospheric convection models utilizing Lorenz chaotic dynamical systems. Notably, the method consistently produced solutions that matched the true solutions of the governing models. In the case of the Lorenz chaotic system, the obtained solutions accurately portrayed the characteristic phase portraits of a true chaotic system.

Keywords: ψ-Caputo derivative, generalized Laplace transform, systems of fractional differential equations, Adomian decomposition method, Lorenz chaotic system

MSC: 26A33, 34A08, 44A10

1. Introduction

The Fractional differential equations (FDEs) represent a generalization of traditional ordinary differential equations, where classical-order derivatives are replaced by non-integer-order derivatives. The field of FDEs is rapidly advancing, becoming a fast-growing field of mathematical study [1–3]. The growth of FDEs is driven by their common relevance in modeling various physical phenomena across a variety of scientific and technological spheres, such as material science, viscoelasticity, neural networks, electrical circuits, thermodynamics, control theory, and population dynamics [4–6]. Because of this versatility, there are several methods and approaches for defining fractional derivatives. Some of the most notable ones include the Riemann-Liouville, Caputo, Caputo-Hadamard, Hadamard, Caputo-Erdelyi-Kober, and the Erdelyi-Kober [7, 8]. In all these definitions, a distinctive feature is the inclusion of a specific type of kernel dependency. Hence, to examine non-integer order differential equations in a more general manner, a non-integer order derivative that depends on a function, referred to as the ψ-Caputo derivative has been introduced [9]. This form of differentiation relies on a kernel ψ. By selecting the function ψ in specific ways, one can regain some renowned non-integer derivatives such as the Riemann-Liouville, Caputo, Caputo-Hadamard, or the Caputo-Erdelyi-Kober operators [7, 8, 10, 11]. This flexibility in choosing ψ allows for a certain level of control when modeling physical processes. Almeida et al. [12] established the uniqueness and existence results for solutions to fractional nonlinear systems of differential equations involving ψ-Caputo

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derivatives, employing fixed point theorems. The ψ-Caputo fractional derivatives serve as a highly effective means to model various real-world physical phenomena and reveal hidden characteristics. Achieving high-precision solutions is consistently necessary. Some researchers [13–15] have employed an approximate analytical method for the numerical solution of a one-dimensional moving boundary problem to address various physical problems. This approach allows for the application of a conventional finite difference technique on the unknown domain, treating the boundary velocity as a second dependent variable. Shahrezaee et al. [16] generalized the fractional Bloch equations (FBEs) by using a fractional derivative of a function with respect to another function (ψ-Caputo derivative) and obtain ψ-Caputo FBEs; then, they utilized the generalized Laplace transform method (GLTM) for solving the FBEs analytically.

Since most fractional differential equations lack analytical solutions, approximation and computational techniques are widely used to solve them. As a result, many researchers have expressed an interest in advancing numerical methods for tackling FDEs to extend the applicability of numerical techniques to the study of these equations with ψ-Caputo derivatives, such as the ψ-Haar wavelet method [17] and the operational matrix method using ψ-shifted Legendre polynomials [18] for solving initial value problems (IVPs) of ψ-FDEs.

Besides Boundary-Value Problems (BVPs) of ψ-FDEs, methods like the Daftardar-Gejji and Jafari method (DGJIM) and the Adomian decomposition method (ADM) can be found in [19]. Notably, ADM, pioneered in the 1980s by Adomian, is an effective method for finding computational and closed-form solutions for various FDEs. This technique works efficiently for both IVPs and BVPs, linear and nonlinear models, and even coupled systems of fractional equations [19, 20]. Additionally, it requires no linearization or perturbation. When ADM is coupled with the Laplace Transform (LT), it becomes a powerful approach known as the (LADM) for obtaining approximate solutions to fractional-order mathematical models. For more details on LADM and its applications to various integer and fractional-order differential equations, refer to recent works [21–23].

Motivated by the above-cited works, the present study utilizes LADM to develop a reliable computational method for solving coupled systems of (FDEs) involving the ψ-Caputo derivative. We achieve this by leveraging the generalized Laplace Transform (ψ-LT) discussed in [24, 25] and coupling it with the efficient ADM. This combined approach, referred to as the generalized Laplace Adomian decomposition method (ψ-LADM), offers rapid convergence and can handle both linear and nonlinear dynamical systems for ψ-FDEs. To demonstrate its efficacy, we will examine several numerical test examples, including a real-world application for atmospheric convection using the Lorenz chaotic dynamical system. The paper is organized into five sections, starting with an introduction. The subsequent content is outlined as follows: Section 2 lays the groundwork by introducing essential concepts, definitions, and mathematical tools for our study. Section 3 dives into the heart of the method, constructing a specific formula the ψ-LADM for solving ψ-FDE systems. Section 4 showcases numerical examples to verify and illustrate the method's effectiveness. Section 5 concludes the paper by summarizing the key findings.

2. Preliminaries

In this section, we aim to review principal definitions and some properties of fractional integrals, and derivatives and generalized Laplace transform, which will be used in this paper [9, 24, 25].

2.1 *Basic definitions and properties of the ψ-fractional calculus*

We will go over some defintions of ψ-fractional integral and differential operators in this subsection.

Definition 2.1 If $y : I \to \mathbb{R}$ is an integrable function, where $I = [a, b]$ and $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$ and $\psi(x) \in C^n(I)$ upon which $\psi'(x) \neq 0$ $\forall x \in I$. Then, the fractional integral and fractional derivatives of order $\alpha > 0$ of the function y with regard to another function ψ are defined as follows [9]

$$I_a^{\alpha,\,\psi} y(x) := \{\Gamma(\alpha)\}^{-1} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{-1+\alpha} y(t)dt, \tag{1}$$

and

$$
\begin{aligned}
D_{a^+}^{\alpha,\ \psi} y(x) : &= \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n I_{a^+}^{n-\alpha,\ \psi} y(x), \\
&= \{\Gamma(n-\alpha)\}^{-1} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \int_a^x \psi'(t)(\psi(x)-\psi(t))^{-1-\alpha+n} y(t)dt,
\end{aligned} \tag{2}
$$

sequentially, with $n = 1 + [\alpha]$.

Remark 2.2 If $\psi(x) = x$, and $\psi(x) = \ln(x)$, then the equation (1) is reduced to classical Riemann-Liouville and Hadamard fractional integrals, respectively,

Remark 2.3 If $\psi(x) = x$, and $\psi(x) = \ln(x)$, then the equation (2) is reduced to classical Riemann-Liouville derivative and Hadamard fractional derivative, respectively.

Definition 2.4 Given the interval $I = [a,\ b]$ with $\alpha > 0$, $n \in \mathbb{N}$. The ψ-Caputo fractional derivative of order $\alpha > 0$ of the function y is defined as follows [9]

$$
{}^C D_a^{\alpha,\ \psi} y(x) = \{\Gamma(n-\alpha)\}^{-1} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{n-\alpha-1} y_\psi^{[n]}(t)dt, \tag{3}
$$

where

$$
y_\psi^{[n]} y(x) := \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n y(x).
$$

where both the functions $\psi(x)$ and $y \in C^n(I)$ with the condition that ψ is increasing and $\psi'(x) \neq 0\ \forall x \in I$. Further,

$$
\begin{cases} n = \alpha, & \text{for} \quad \alpha \in \mathbb{N}, \\ n = 1 + [\alpha], & \text{for} \quad \alpha \notin \mathbb{N}. \end{cases}
$$

Remark 2.5 If $\psi(x) = x$, and $\psi(x) = \ln(x)$, then the equation (3) is reduced to classical Caputo derivative and Caputo-Hadamard fractional derivative, respectively.

Proposition 2.6 In addition, considering $y(x) = (\psi(x) - \psi(a))^{\beta-1}$ where $\beta \in \mathbb{R}$, $\beta > n$, $\alpha > 0$, certain vita features for $\psi(x)$-fractional operators are thus deduced as follows [9]:

- ${}^C D_{a^+}^{\alpha,\ \psi} y(x) = \dfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (\psi(x)-\psi(a))^{\beta-\alpha-1}$,
- $I_a^{\alpha,\ \psi} y(x) = \dfrac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (\psi(x)-\psi(a))^{\beta+\alpha-1}$,
- $I_a^{\alpha,\ \psi} \left({}^C D_a^{\alpha,\ \psi} y(x) \right) = y(x) - \displaystyle\sum_{k=0}^{n-1} \frac{y_\psi^{[k]}(a)}{k!} (\psi(x)-\psi(a))^k$.

Moreover, the last result reduces to $I_a^{\alpha,\ \psi} \left({}^C D_a^{\alpha,\ \psi} y(x) \right) = y(x) - y(a)$, when $\alpha \in (0,\ 1)$.

Furthermore, we need to consider several special functions that are essential for finding the solutions of ψ-FDEs.

Definition 2.7 The Mittag-Leffler function $E_\alpha(z)$ is a famous function in the theory of fractional calculus that was initiated by Gosta Mittag-Leffler and takes the following one-parameter expression [25]

$$E_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k+1)},\ \ \alpha\in\mathbb{C},\ \ Re(\alpha)>0,\ \ z\in\mathbb{C}. \tag{4}$$

In addition, Wiman presented the two-parameter generalisation of the Mittag-Leffler function $E_{\alpha}(z)$ as follows

$$E_{\alpha,\ \beta}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k+\beta)},\ \ \alpha,\ \beta\in\mathbb{C},\ \ Re(\alpha)>0,\ \ Re(\beta)>0,\ \ z\in\mathbb{C}. \tag{5}$$

2.2 *Generalised Laplace transform*

In this subsection, we discuss ψ-LT, ψ-LT of some elementary functions, and ψ-LT for ψ-fractional operators.

Definition 2.8 [24, 25] If $y:[0,\ \infty)\to\mathbb{R}$ is a real-valued function and ψ is an increasing positive function with $\psi(0)=0$. Then, the Laplace transform of the function y with regard to the function ψ is given as follows [24, 25]

$$\mathcal{L}_{\psi}\{y(x)\}=Y(s)=\int_{0}^{\infty}e^{-s\psi(x)}\psi'(x)y(x)dx,\quad\forall\, s\in\mathbb{C}, \tag{6}$$

Theorem 2.9 [24, 25] The generlised Laplace transform may be written as a combination of the classical Laplace transform with the operation of composition with ψ or ψ^{-1}, as follows:

$$\mathcal{L}_{\psi}=\mathcal{L}\circ\varphi_{\psi}^{-1}, \tag{7}$$

where the functional opertor φ_{ψ} is defined by

$$\left(\varphi_{\psi}f\right)(x)=\left(f(\psi(x))\right). \tag{8}$$

Corollary 2.10 [24, 25] The inverse generlised Laplace transform may be written as a combination of the inverse classical Laplace transform with the operation of composition with ψ or ψ^{-1}, as follows:

$$\mathcal{L}_{\psi}^{-1}=\varphi_{\psi}\circ\mathcal{L}^{-1}, \tag{9}$$

or in other word

$$\mathcal{L}_{\psi}^{-1}\left[Y(s)\right]=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{s\psi(t)}Y(s)ds. \tag{10}$$

Upon which the integral converges; see Table 1 for evaluation of the generalized Laplace transform of some elementary functions.

Table 1. ψ-Laplace transform of some functions

ψ-Laplace transform	
$y(x)$	$\mathcal{L}_\psi\{y(x)\} = Y(s)$
1	$\frac{1}{s},\ s > 0$
$e^{a\psi(x)}$	$\frac{1}{s-a}$, for $s > a$
$(\psi(x))^\alpha$	$\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$, for $s > 0$
$E_\alpha(\lambda(\psi(x))^\alpha)$	$\frac{s^{\alpha-1}}{s^\alpha-\lambda}$, for $\mathrm{Re}(\alpha) > 0$, and $\lvert\frac{\lambda}{s^\alpha}\rvert < 1$
$(\psi(x))^{\alpha-1}E_{\alpha,\,\alpha}(\lambda(\psi(x))^\alpha)$	$\frac{1}{s^\alpha-\lambda}$, for $\mathrm{Re}(\alpha) > 0$, and $\lvert\frac{\lambda}{s^\alpha}\rvert < 1$

Theorem 2.11 [25] Let y be a piece-wise continuous function of ψ-exponential order that is defined over each finite interval. Then, for any $\alpha > 0$,

$$\mathcal{L}_\psi\left\{\left(I_0^{\alpha,\ \psi(x)}y\right)(x)\right\} = s^{-\alpha}\mathcal{L}_\psi\{y(x)\}. \tag{11}$$

Theorem 2.12 [25] Assume that $\alpha > 0$, $n = \lfloor\alpha\rfloor + 1$, and y is a function such that $y(x)$, $I_0^{n-\alpha,\ \psi(x)}y(x)$, $D_0^{1,\ \psi(x)}I_0^{n-\alpha,\ \psi(x)}y(x)$, ..., $D_0^{n,\ \psi(x)}I_0^{n-\alpha,\ \psi(x)}y(x)$ are of ψ-exponential order and continuous over the interval $(0,\ \infty)$, while ${}^RD_0^{\alpha,\ \psi(x)}y(x)$ is piece-wise continuous over the interval $[0,\ \infty)$. Then,

$$\mathcal{L}_\psi\left\{\left({}^RD_0^{\alpha,\ \psi(x)}y\right)(x)\right\} = s^\alpha\mathcal{L}_\psi\{y(x)\} - \sum_{i=0}^{n-1}s^{n-1-i}\left(I_0^{n-i-\alpha,\ \psi(x)}y\right)(0). \tag{12}$$

Theorem 2.13 [25] Assume that $\alpha > 0$, $n = \lfloor\alpha\rfloor + 1$, and y is a function such that $y(x)$, ${}^CD_0^{1,\ \psi(x)}y(x)$, ${}^CD_0^{2,\ \psi(x)}y(x)$, ..., ${}^CD_0^{n-1,\ \psi(x)}y(x)$ are of ψ-exponential order and continuous over the interval $[0,\ \infty)$, while ${}^CD_0^{\alpha,\ \psi(x)}y(x)$ is piece-wise continuous over the interval $[0,\ \infty)$. Then,

$$\mathcal{L}_\psi\left\{\left({}^CD_0^{\alpha,\ \psi(x)}y\right)(x)\right\} = s^\alpha\mathcal{L}_\psi\{y(x)\} - \sum_{i=0}^{n-1}s^{\alpha-1-i}\left(D_0^{i,\ \psi(x)}y\right)(0). \tag{13}$$

3. ψ-LADM for systems of ψ-Caputo fractional differential equations

In the present section, we will utilize the ADM with the generalized LT termed as (ψ-LADM) to derive a universal scheme for the solution of coupled systems of IVPs for ψ-FDEs with ψ-Caputo derivatives.

Thus, in presenting the method, we make consideration of the generalized system of IVPs for FDEs with an arbitrary function ψ as follows

$$ {}^{C}D_{a^+}^{\alpha_i,\ \psi} y_i(x) = f(x,\ y_i(x)), \quad x \in [a,\ b], \quad i = 1,\ 2,\ \ldots, \tag{14} $$

together with the following imposed initial data

$$ y_i(a) = y_{i,\ a} \ \text{ and } \ y_{i,\ \psi}^{[k]}(a) = y_{i,\ a}^{k}, \ \ k = 1,\ \ldots,\ n-1, \ \ i = 1,\ 2,\ \ldots, \tag{15} $$

where $y_i(x)$ are the unknown functions, ${}^{C}D_{a^+}^{\alpha_i,\ \psi}$ are the ψ-Caputo fractional derivatives of order α_i, ψ is arbitrary function, while $f(x,\ y_i(x))$ are continuous nonlinear functions.

Then, to solve problem (14) by ψ-LADM, we operate ψ-LT on both sides of the equation to get

$$ \mathcal{L}_{\psi}\{{}^{C}D_{a^+}^{\alpha_i,\ \psi(x)} y_i(x)\} = \mathcal{L}_{\psi}\{f(x,\ y_i(x))\}, \tag{16} $$

and further upon using the result in (13) and the initial conditions one gets

$$ \mathcal{L}_{\psi}\{y_i(x)\} - \sum_{k=0}^{n-1} \frac{1}{s^{k+1}} y_{i,\ \psi}^{[k]}(a) = \frac{1}{s^{\alpha_i}} \mathcal{L}_{\psi}\{f(x,\ y_i(x))\}. \tag{17} $$

Next, the application of the inverse ψ-LT on the latter equation yields

$$ y_i(x) = \mathcal{L}_{\psi}^{-1}\left\{\sum_{k=0}^{n-1} \frac{1}{s^{k+1}} y_{i,\ \psi}^{[k]}(a)\right\} + \mathcal{L}_{\psi}^{-1}\left\{\frac{1}{s^{\alpha_i}} \mathcal{L}_{\psi}\{f(x,\ y_i(x))\}\right\}. \tag{18} $$

Further, the standard ADM defines the solution $y_i(x)$ by the following infinite series

$$ y_i(x) = \sum_{n=0}^{\infty} y_{i,\ n}(x), \tag{19} $$

while the nonlinear term is expressed through the decomposed series of Adomian polynomials $A_{i,\ n}$ as follows

$$ f(x,\ y_i(x)) = \sum_{n=0}^{\infty} A_{i,\ n}, \tag{20} $$

where $A_{i,\ n}$ are unequivocally expressed as follows

$$ A_{i,\ n} = \left[\frac{1}{n!} \frac{d^n}{d\lambda^n} f\left(x,\ \sum_{i=0}^{\infty} \lambda^i y_{i,\ n}\right)_{\lambda=0}\right], \ \ n = 0,\ 1,\ 2,\ \ldots \tag{21} $$

Therefore, with the above-decomposed series, equation (18) is thus re-expressed as follows

$$\sum_{n=0}^{\infty} y_{i,\, n}(x) = \mathcal{L}_{\psi}^{-1}\left\{\sum_{k=0}^{n-1}\frac{1}{s^{k+1}}y_{i,\, \psi}^{[k]}(a)\right\} + \mathcal{L}_{\psi}^{-1}\left\{\frac{1}{s^{\alpha_i}}\mathcal{L}_{\psi}\left\{\sum_{n=0}^{\infty}A_{i,\, n}\right\}\right\}, \tag{22}$$

upon which the formal recursive relation is acquired via the standard ADM procedure as follows

$$\begin{cases} y_{i,\, 0}(x) = \mathcal{L}_{\psi}^{-1}\left\{\sum_{k=0}^{n-1}\frac{1}{s^{k+1}}y_{i,\, \psi}^{[k]}(a)\right\} = f(x), \\ \\ y_{i,\, n}(x) = \mathcal{L}_{\psi}^{-1}\left\{\frac{1}{s^{\alpha_i}}\mathcal{L}_{\psi}\{A_{i,\, n-1}\}\right\}, \quad n \geq 1. \end{cases} \tag{23}$$

Moreover, it is noteworthy that the decomposition method suggests defining the zeroth component $y_{i,\, 0}(x)$ typically based on the function f which is derived from the source term and the given initial conditions, as described above. In this regard, Wazwaz [26] proposed a reliable extension of the standard ADM by further decomposing the zeroth component $y_{i,\, 0}(x)$ into two components as follows

$$y_{i,\, 0} = f_0 + f_1,$$

where the first part is associated with the initial term, and the second part is attached to the subsequent part. Thus, under this assumption, the new modified recursive algorithm from (23) takes the following form

$$\begin{cases} y_{i,\, 0}(x) = f_0(x), \\ \\ y_{i,\, n}(x) = f_1(x) + \mathcal{L}_{\psi}^{-1}\left\{\frac{1}{s^{\alpha_i}}\mathcal{L}_{\psi}\{A_{i,\, n-1}\}\right\}, \; n \geq 1. \end{cases} \tag{24}$$

Certainly, the closed-form solution determined in (24) converges more rapidly than that of the standard ADM; see [26] for various comparative examinations in this regard. However, the modified scheme by Wazwaz is highly dependent on the selection of the functions $f_0(x)$ and $f_1(x)$; this scenario will later be shown in the application section.

4. Applications

The current section assesses the competency of the proposed ψ-LADM scheme on some coupled test ψ-fractional problems. More importantly, the section makes consideration of both nonlinear and linear coupled systems of ψ-FDEs, including a real-life scenario of examining the Lorenz chaotic dynamical system, which has huge relevance in atmospheric convection with wide applications in chemical reaction processes and electrical circuits.

4.1 *Numerical applications*

In this subsection, some interesting coupled systems of ψ-FDEs are beseeched as numerical examples and further analyzed using the ψ-LADM approach. Certainly, both the nonlinear and linear coupled systems of IVPs featuring ψ-fractional derivatives are considered.

Example 4.1 Consider the coupled system of IVPs for ψ-FDEs as follows [27]

$$\begin{cases} {}^{C}D_{0}^{0.9,\ \psi} y_1(x) = \dfrac{2}{\Gamma(2.1)} y_1(x) + \dfrac{2}{\Gamma(2.1)} y_2(x) + g_1(x), \\ \\ {}^{C}D_{0}^{0.6,\ \psi} y_2(x) = \dfrac{6}{\Gamma(3.4)} y_1(x) + \dfrac{6}{\Gamma(3.4)} y_2(x) + g_2(x), \\ \\ y_1(0) = 0,\ y_2(0) = 0, \quad \text{for } x \in [0,\ 1], \end{cases} \tag{25}$$

where

$$g_1(x) = \frac{2(\psi(x))^{1.1}}{\Gamma(2.1)} \left(1 - (\psi(x))^{0.9} - (\psi(x))^{1.9}\right),\ \ g_2(x) = \frac{6(\psi(x))^2}{\Gamma(3.4)} \left((\psi(x))^{0.4} - 1 - (\psi(x))\right),$$

with the ψ-fractional system admitting the following exact solution

$$(y_1(x),\ y_2(x)) = (\psi(x)^2,\ \psi(x)^3).$$

To solve the current problem by means of ψ-LADM, we apply the ψ-LT on both sides of the governing equation in (25), and further make use of (13) to ultimately obtain

$$\begin{cases} \mathcal{L}_{\psi}\{y_1(x)\} = \dfrac{1}{s^{0.9}} \mathcal{L}_{\psi}\left\{\dfrac{2}{\Gamma(2.1)} y_1(x)\right\} + \dfrac{1}{s^{0.9}} \mathcal{L}_{\psi}\left\{\dfrac{2}{\Gamma(2.1)} y_2(x)\right\} + \dfrac{1}{s^{0.9}} \mathcal{L}_{\psi}\{g_1(x)\}, \\ \\ \mathcal{L}_{\psi}\{y_2(x)\} = \dfrac{1}{s^{0.6}} \mathcal{L}_{\psi}\left\{\dfrac{6}{\Gamma(3.4)} y_1(x)\right\} + \dfrac{1}{s^{0.6}} \mathcal{L}_{\psi}\left\{\dfrac{6}{\Gamma(3.4)} y_2(x)\right\} + \dfrac{1}{s^{0.6}} \mathcal{L}_{\psi}\{g_2(x)\}. \end{cases} \tag{26}$$

Next, we then apply the inverse ψ-LT on both sides of the latter equation to obtain

$$\begin{cases} y_1(x) = \mathcal{L}_{\psi}^{-1}\left\{\dfrac{1}{s^{0.9}} \mathcal{L}_{\psi}\left\{\dfrac{2}{\Gamma(2.1)} y_1(x)\right\}\right\} + \mathcal{L}_{\psi}^{-1}\left\{\dfrac{1}{s^{0.9}} \mathcal{L}_{\psi}\left\{\dfrac{2}{\Gamma(2.1)} y_2(x)\right\}\right\} + \mathcal{L}_{\psi}^{-1}\left\{\dfrac{1}{s^{0.9}} \mathcal{L}_{\psi}\{g_1(x)\}\right\}, \\ \\ y_2(x) = \mathcal{L}_{\psi}^{-1}\left\{\dfrac{1}{s^{0.6}} \mathcal{L}_{\psi}\left\{\dfrac{6}{\Gamma(3.4)} y_1(x)\right\}\right\} + \mathcal{L}_{\psi}^{-1}\left\{\dfrac{1}{s^{0.6}} \mathcal{L}_{\psi}\left\{\dfrac{6}{\Gamma(3.4)} y_2(x)\right\}\right\} + \mathcal{L}_{\psi}^{-1}\left\{\dfrac{1}{s^{0.6}} \mathcal{L}_{\psi}\{g_2(x)\}\right\}. \end{cases} \tag{27}$$

In addition, upon deploying the use of the ADM, the above equation is further expressed in a series representation as enshrined in the standard ADM procedure as follows

$$
\begin{cases}
\sum_{n=0}^{\infty} y_{1,\,n}(x) = \mathcal{L}_{\psi}^{-1}\left\{\frac{1}{s^{0.9}}\mathcal{L}_{\psi}\left\{\frac{2}{\Gamma(2.1)}\sum_{n=0}^{\infty} y_{1,\,n}(x)\right\}\right\} + \mathcal{L}_{\psi}^{-1}\left\{\frac{1}{s^{0.9}}\mathcal{L}_{\psi}\left\{\frac{2}{\Gamma(2.1)}\sum_{n=0}^{\infty} y_{2,\,n}(x)\right\}\right\} \\
\qquad + (\psi(x))^{2} - 0.721282(\psi(x))^{2.9} - 0.554833(\psi(x))^{3.9}, \\
\sum_{n=0}^{\infty} y_{2,\,n}(x) = \mathcal{L}_{\psi}^{-1}\left\{\frac{1}{s^{0.6}}\mathcal{L}_{\psi}\left\{\frac{6}{\Gamma(3.4)}\sum_{n=0}^{\infty} y_{1,\,n}(x)\right\}\right\} + \mathcal{L}_{\psi}^{-1}\left\{\frac{1}{s^{0.6}}\mathcal{L}_{\psi}\left\{\frac{6}{\Gamma(3.4)}\sum_{n=0}^{\infty} y_{2,\,n}(x)\right\}\right\} \\
\qquad + (\psi(x))^{3} - 1.082913(\psi(x))^{2.6} - 0.9024289(\psi(x))^{3.6},
\end{cases} \tag{28}
$$

which then possess the resulting modified recursive algorithm form (24) for the governing model as follows

$$
\begin{cases}
y_{1,\,0}(x) = (\psi(x))^{2}, \\
y_{1,\,n}(x) = \mathcal{L}_{\psi}^{-1}\left\{\frac{1}{s^{0.9}}\mathcal{L}_{\psi}\left\{\frac{2}{\Gamma(2.1)} y_{1,\,n-1}(x)\right\}\right\} + \mathcal{L}_{\psi}^{-1}\left\{\frac{1}{s^{0.9}}\mathcal{L}_{\psi}\left\{\frac{2}{\Gamma(2.1)} y_{2,\,n-1}(x)\right\}\right\} \\
\qquad -0.721282(\psi(x))^{2.9} - 0.554833(\psi(x))^{3.9}, \ \forall\, n \geq 1,
\end{cases} \tag{29}
$$

and

$$
\begin{cases}
y_{2,\,0}(x) = (\psi(x))^{3}, \\
y_{2,\,n}(x) = \mathcal{L}_{\psi}^{-1}\left\{\frac{1}{s^{0.6}}\mathcal{L}_{\psi}\left\{\frac{6}{\Gamma(3.4)} y_{1,\,n-1}(x)\right\}\right\} + \mathcal{L}_{\psi}^{-1}\left\{\frac{1}{s^{0.6}}\mathcal{L}_{\psi}\left\{\frac{6}{\Gamma(3.4)} y_{2,\,n-1}(x)\right\}\right\} \\
\qquad -1.082913(\psi(x))^{2.6} - 0.9024289(\psi(x))^{3.6}, \ \forall\, n \geq 1.
\end{cases} \tag{30}
$$

Moreover, when the recursive relation above is expressed term-wise, one gets some of the ψ-terms explicitly as follows

$$
\begin{cases}
y_{1,\,0}(x) = (\psi(x))^2, \\
\\
y_{1,\,1}(x) = \mathcal{L}_{\psi}^{-1}\left\{\frac{1}{s^{0.9}}\mathcal{L}_{\psi}\left\{\frac{2}{\Gamma(2.1)}\psi(x)^2\right\}\right\} + \mathcal{L}_{\psi}^{-1}\left\{\frac{1}{s^{0.9}}\mathcal{L}_{\psi}\left\{\frac{2}{\Gamma(2.1)}\psi(x)^3\right\}\right\} \\
\qquad -0.721282(\psi(x))^{2.9} - 0.554833(\psi(x))^{3.9}, \\
\qquad = 0.721282(\psi(x))^{2.9} + 0.554833(\psi(x))^{3.9} - 0.721282(\psi(x))^{2.9} - 0.554833(\psi(x))^{3.9}, \\
\qquad = 0, \\
\\
y_{1,\,2}(x) = 0, \\
\\
y_{1,\,3}(x) = 0, \\
\\
\qquad \vdots
\end{cases}
$$

and

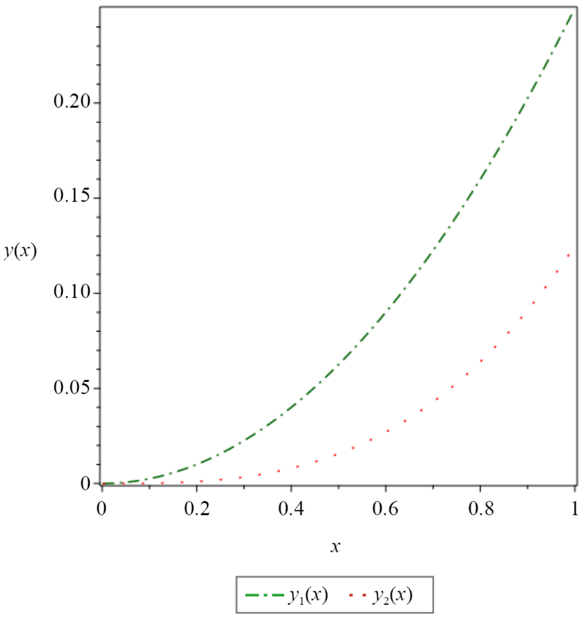
$$
\begin{cases}
y_{2,\,0}(x) = (\psi(x))^3, \\
\\
y_{2,\,1}(x) = \mathcal{L}_{\psi}^{-1}\left\{\frac{1}{s^{0.6}}\mathcal{L}_{\psi}\left\{\frac{6}{\Gamma(3.4)}\psi(x)^3\right\}\right\} + \mathcal{L}_{\psi}^{-1}\left\{\frac{1}{s^{0.6}}\mathcal{L}_{\psi}\left\{\frac{6}{\Gamma(3.4)}\psi(x)^3\right\}\right\} \\
\qquad -1.082913(\psi(x))^{2.6} - 0.9024289(\psi(x))^{3.6}, \\
\qquad = 1.082913(\psi(x))^{2.6} + 0.9024289(\psi(x))^{3.6} - 1.082913(\psi(x))^{2.6} - 0.9024289(\psi(x))^{3.6} \\
\qquad = 0, \\
\\
y_{2,\,2}(x) = 0, \\
\\
y_{2,\,3}(x) = 0, \\
\\
\qquad \vdots
\end{cases}
$$

Notably, the above series solution can be written in a more compact form as follows

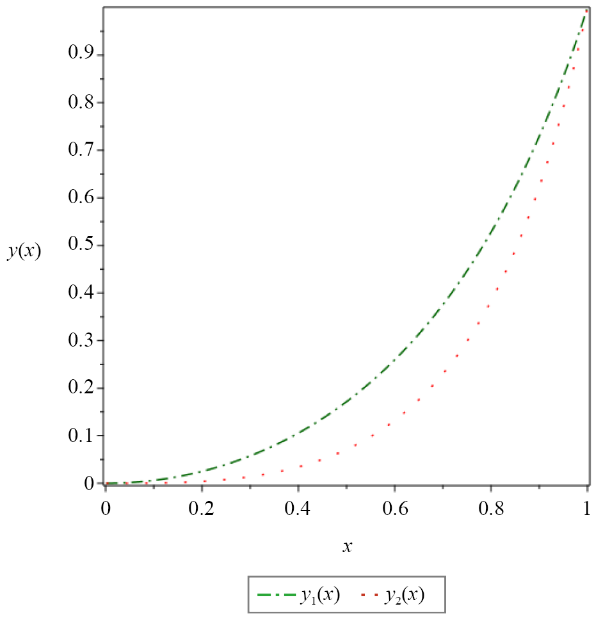
$$
\begin{cases}
y_{1,\,n}(x) = 0, \ \ \forall n \geq 1, \\
\\
y_{2,\,n}(x) = 0, \ \ \forall n \geq 1,
\end{cases}
\tag{31}
$$

upon which the exact solution is arrived as

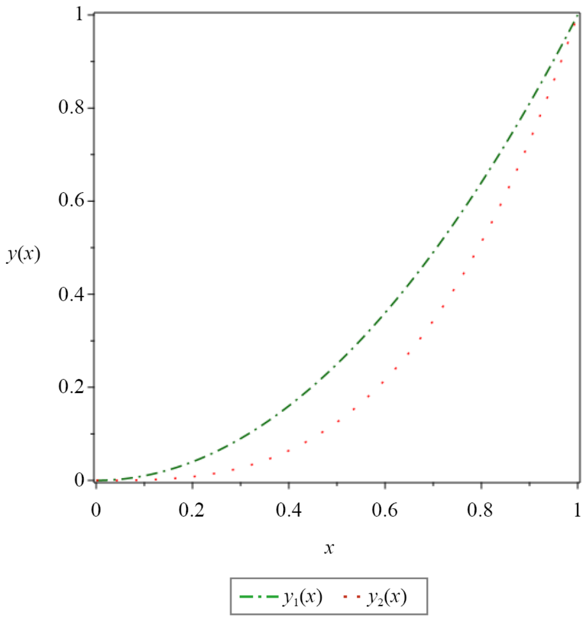
$$
\begin{cases} y_1(x) = (\psi(x))^2, \\ \\ y_2(x) = (\psi(x))^3. \end{cases} \tag{32}
$$



(a) $\psi(x) = \dfrac{x(x+1)}{2}$



(b) $\psi(x) = \tan\left(\dfrac{\pi x}{4}\right)$



(c) $\psi(x) = x$

Figure 1. ψ-LADM solution for (25) with respect to different kernels

In the same vein, Amal et al. [27] equally solved the governing coupled system (25) through the application of the operational matrix approach by introducing ψ-shifted Legendre polynomials and ψ-shifted Jacobi polynomials and obtained approximate solution by calculating 10 terms. However, the ψ-LADM with Wazwaz modification gives an exact solution while using only 2 terms. In addition, Figure 1 displays the acquired ψ-LADM solution for different choices of ψ for (25).

Example 4.2 Consider the coupled system of IVPs for ψ-FDEs as follows [28]

$$\begin{cases} {}^{C}D_0^{0.7,\ \psi} y_1(x) = -5y_1(x) + y_2(x), \quad y_1(0) = 1, \\ {}^{C}D_0^{0.7,\ \psi} y_2(x) = 4y_1(x) - 2y_2(x), \quad y_2(0) = 2, \\ {}^{C}D_0^{0.7,\ \psi} y_3(x) = -3y_3(x) - 9y_4(x), \quad y_3(0) = 3, \\ {}^{C}D_0^{0.7,\ \psi} y_4(x) = -4y_3(x) - 3y_4(x), \quad y_4(0) = 4. \end{cases} \tag{33}$$

To solve (33) by ψ-LADM, we apply the ψ-LT on both sides of the governing system, together with the application of (13) and the initial conditions to ultimately obtain

$$\begin{cases} \mathcal{L}_\psi\{y_1(x)\} = \frac{1}{s^{0.7}} \mathcal{L}_\psi\{-5y_1(x) + y_2(x)\} + \frac{1}{s}, \\ \mathcal{L}_\psi\{y_2(x)\} = \frac{1}{s^{0.7}} \mathcal{L}_\psi\{4y_1(x) - 2y_2(x)\} + \frac{2}{s}, \\ \mathcal{L}_\psi\{y_3(x)\} = \frac{1}{s^{0.7}} \mathcal{L}_\psi\{-3y_3(x) - 9y_4(x)\} + \frac{3}{s}, \\ \mathcal{L}_\psi\{y_4(x)\} = \frac{1}{s^{0.7}} \mathcal{L}_\psi\{-4y_3(x) - 3y_4(x)\} + \frac{4}{s}. \end{cases} \tag{34}$$

Now, on applying the inverse ψ-LT on both sides of the latter equation, one gets

$$\begin{cases} y_1(x) = \mathcal{L}_\psi^{-1}\left\{ \frac{1}{s^{0.7}} \mathcal{L}_\psi\{-5y_1(x) + y_2(x)\} \right\} + 1, \\ y_2(x) = \mathcal{L}_\psi^{-1}\left\{ \frac{1}{s^{0.7}} \mathcal{L}_\psi\{4y_1(x) - 2y_2(x)\} \right\} + 2, \\ y_3(x) = \mathcal{L}_\psi^{-1}\left\{ \frac{1}{s^{0.7}} \mathcal{L}_\psi\{-3y_3(x) - 9y_4(x)\} \right\} + 3, \\ y_4(x) = \mathcal{L}_\psi^{-1}\left\{ \frac{1}{s^{0.7}} \mathcal{L}_\psi\{-4y_3(x) - 3y_4(x)\} \right\} + 4. \end{cases} \tag{35}$$

Further, the above equation is expressed in the standard ADM series form as follows

$$
\begin{cases}
\sum_{n=0}^{\infty} y_{1,\,n}(x) = \mathcal{L}_{\psi}^{-1}\left\{\frac{1}{s^{0.7}}\mathcal{L}_{\psi}\left\{-5\sum_{n=0}^{\infty} y_{1,\,n}(x) + \sum_{n=0}^{\infty} y_{2,\,n}(x)\right\}\right\} + 1, \\
\sum_{n=0}^{\infty} y_{2,\,n}(x) = \mathcal{L}_{\psi}^{-1}\left\{\frac{1}{s^{0.7}}\mathcal{L}_{\psi}\left\{4\sum_{n=0}^{\infty} y_{1,\,n}(x) - 2\sum_{n=0}^{\infty} y_{2,\,n}(x)\right\}\right\} + 2, \\
\sum_{n=0}^{\infty} y_{3,\,n}(x) = \mathcal{L}_{\psi}^{-1}\left\{\frac{1}{s^{0.7}}\mathcal{L}_{\psi}\left\{-3\sum_{n=0}^{\infty} y_{3,\,n}(x) - 9\sum_{n=0}^{\infty} y_{4,\,n}(x)\right\}\right\} + 3, \\
\sum_{n=0}^{\infty} y_{4,\,n}(x) = \mathcal{L}_{\psi}^{-1}\left\{\frac{1}{s^{0.7}}\mathcal{L}_{\psi}\left\{-4\sum_{n=0}^{\infty} y_{3,\,n}(x) - 3\sum_{n=0}^{\infty} y_{4,\,n}(x)\right\}\right\} + 4,
\end{cases}
\tag{36}
$$

that leads to the following recursive scheme

$$
\begin{cases}
y_{1,\,0}(x) = 1, \\
y_{1,\,n}(x) = \mathcal{L}_{\psi}^{-1}\left\{\frac{1}{s^{0.7}}\mathcal{L}_{\psi}\{-5y_{1,\,n-1}(x) + y_{2,\,n-1}(x)\}\right\}, \ \ \forall n \geq 1,
\end{cases}
\tag{37}
$$

$$
\begin{cases}
y_{2,\,0}(x) = 2, \\
y_{2,\,n}(x) = \mathcal{L}_{\psi}^{-1}\left\{\frac{1}{s^{0.7}}\mathcal{L}_{\psi}\{4y_{1,\,n-1}(x) - 2y_{2,\,n-1}(x)\}\right\}, \ \ \forall n \geq 1,
\end{cases}
\tag{38}
$$

$$
\begin{cases}
y_{3,\,0}(x) = 3, \\
y_{3,\,n}(x) = \mathcal{L}_{\psi}^{-1}\left\{\frac{1}{s^{0.7}}\mathcal{L}_{\psi}\{-3y_{3,\,n-1}(x) - 9y_{4,\,n-1}(x)\}\right\}, \ \ \forall n \geq 1,
\end{cases}
\tag{39}
$$

and

$$
\begin{cases}
y_{4,\,0}(x) = 4, \\
y_{4,\,n}(x) = \mathcal{L}_{\psi}^{-1}\left\{\frac{1}{s^{0.7}}\mathcal{L}_{\psi}\{-4y_{3,\,n-1}(x) - 3y_{4,\,n-1}(x)\}\right\}, \ \ \forall n \geq 1.
\end{cases}
\tag{40}
$$

In fact, one may express the above scheme term-wise with respect to each solution pair y_i for $i = 1, 2, 3, 4$, as follows

$$
\begin{cases}
y_{1,\,0}(x) = 1, \\
y_{1,\,1}(x) = \mathcal{L}_{\psi}^{-1}\left\{ \dfrac{1}{s^{0.7}} \mathcal{L}_{\psi}\{-5+2\} \right\} = \dfrac{-3}{\Gamma(1.7)} (\psi(x))^{0.7}, \\
y_{1,\,2}(x) = \mathcal{L}_{\psi}^{-1}\left\{ \dfrac{1}{s^{0.7}} \mathcal{L}_{\psi}\left\{ \dfrac{-3}{\Gamma(1.7)} (\psi(x))^{0.7} \right\} \right\} = \dfrac{15}{\Gamma(2.4)} (\psi(x))^{1.4}, \\
\quad \vdots
\end{cases}
\tag{41}
$$

$$
\begin{cases}
y_{2,\,0}(x) = 2, \\
y_{2,\,1}(x) = \mathcal{L}_{\psi}^{-1}\left\{ \dfrac{1}{s^{0.7}} \mathcal{L}_{\psi}\{4-4\} \right\} = 0, \\
y_{2,\,2}(x) = \mathcal{L}_{\psi}^{-1}\left\{ \dfrac{1}{s^{0.7}} \mathcal{L}_{\psi}\left\{ 4\dfrac{-3}{\Gamma(1.7)} (\psi(x))^{0.7} \right\} \right\} = \dfrac{-12}{\Gamma(2.4)} (\psi(x))^{1.4}, \\
\quad \vdots
\end{cases}
\tag{42}
$$

$$
\begin{cases}
y_{3,\,0}(x) = 3, \\
y_{3,\,1}(x) = \mathcal{L}_{\psi}^{-1}\left\{ \dfrac{1}{s^{0.7}} \mathcal{L}_{\psi}\{-9-36\} \right\} = \dfrac{-45}{\Gamma(1.7)} (\psi(x))^{0.7}, \\
y_{3,\,2}(x) = \mathcal{L}_{\psi}^{-1}\left\{ \dfrac{1}{s^{0.7}} \mathcal{L}_{\psi}\left\{ -3\dfrac{-45}{\Gamma(1.7)} (\psi(x))^{0.7} - 9\dfrac{-24}{\Gamma(1.7)} (\psi(x))^{0.7} \right\} \right\} = \dfrac{351}{\Gamma(2.4)} (\psi(x))^{1.4}, \\
\quad \vdots
\end{cases}
\tag{43}
$$

and lastly

$$
\begin{cases}
y_{4,\,0}(x) = 4, \\
y_{4,\,1}(x) = \mathcal{L}_{\psi}^{-1}\left\{ \dfrac{1}{s^{0.7}} \mathcal{L}_{\psi}\{-12-12\} \right\} = \dfrac{-24}{\Gamma(1.7)} (\psi(x))^{0.7}, \\
y_{4,\,2}(x) = \mathcal{L}_{\psi}^{-1}\left\{ \dfrac{1}{s^{0.7}} \mathcal{L}_{\psi}\{-4\dfrac{-45}{\Gamma(1.7)} (\psi(x))^{0.7} - 3\dfrac{-24}{\Gamma(1.7)} (\psi(x))^{0.7}\} \right\} = \dfrac{252}{\Gamma(2.4)} (\psi(x))^{1.4}, \\
\quad \vdots
\end{cases}
\tag{44}
$$

Finally, when the above scheme is sufficiently expressed, one gets the closed-form exact solution for the prevailing problem in (33) as follows

$$y(x) = \mathbf{C}E_{\alpha,\,1}(\mathbf{A}(\psi(x)-\psi(0))^{\alpha}), \tag{45}$$

where $\alpha = 0.7$, and

$$y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} -5 & 1 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & -4 & -3 \end{bmatrix}, \tag{46}$$

with $\mathbf{C}$ emanating from the imposed initial conditions and $\mathbf{A}$ is the coefficient matrix of the ψ-coupled model (33), while $E_{\alpha,\,1}$ is the two-parameter Mittag-Leffler function earlier defined.

4.2 *Application to Lorenz chaotic system*

In this subsection, we further extend the application of the deployed method to explore a real-world application endowed with the ψ-Caputo fractional derivative. In this regard, we make consideration of the Lorenz chaotic dynamical system, which is a mathematical model discovered in 1963 by Edward Lorenz with vast relevance in the study of atmospheric convection; moreover, its application has been equally established in other fields like in forward osmosis, chemical reactions and in electrical circuits among others. Furthermore, the chaotic behavior has been recently discovered to reveal some salient revelations when the fractional dynamical systems are examined; indeed, this development is attributed to the uniqueness of the fractional-order derivatives-these captivating properties have never been discovered in the corresponding integer-order case [29]. In fact, this finding has received great attention from researchers to study various Lorenz systems with various fractional derivatives.

Here, this study further examines this captivating property associated with the Lorenz dynamical system endowed with the ψ-Caputo fractional derivative by considering the following IVP [30]

$$\begin{cases} {}^{C}D_{1}^{0.98,\ \psi(x)} y_1(x) = 10y_2(x) - 10y_1(x), \\ {}^{C}D_{1}^{0.98,\ \psi(x)} y_2(x) = 123y_1(x) - y_1(x)y_3(x) - y_2(x), \\ {}^{C}D_{1}^{0.98,\ \psi(x)} y_3(x) = y_1(x)y_2(x) - \dfrac{8}{3}y_3(x), \\ y_1(1) = 5,\ y_2(1) = 3,\ y_3(1) = 9,\ \text{for } x \in [1,\ 60], \end{cases} \tag{47}$$

where $\psi(x) = \sqrt{x}$.

Therefore, to solve (47) using the employed ψ-LADM, we apply the ψ-LT on both sides of the equation and further make use of (13) together with the prescribed initial data to ultimately obtain the following

$$
\begin{cases}
\mathcal{L}_\psi\{y_1(x)\} - \dfrac{5}{s} = \dfrac{10}{s^{0.98}}\mathcal{L}_\psi\{y_2(x)\} - \dfrac{10}{s^{0.98}}\mathcal{L}_\psi\{y_1(x)\}, \\[2ex]
\mathcal{L}_\psi\{y_2(x)\} - \dfrac{3}{s} = \dfrac{123}{s^{0.98}}\mathcal{L}_\psi\{y_1(x)\} - \dfrac{1}{s^{0.98}}\mathcal{L}_\psi\{y_1(x)y_3(x)\} - \dfrac{1}{s^{0.98}}\mathcal{L}_\psi\{y_2(x)\}, \\[2ex]
\mathcal{L}_\psi\{y_3(x)\} - \dfrac{9}{s} = \dfrac{1}{s^{0.98}}\mathcal{L}_\psi\{y_1(x)y_2(x)\} - \dfrac{8}{3s^{0.98}}\mathcal{L}_\psi\{y_3(x)\}.
\end{cases} \tag{48}
$$

Now, we proceed by applying the inverse ψ-LT on both sides of the above equation to get a hold of the following expression

$$
\begin{cases}
y_1(x) - \mathcal{L}_\psi^{-1}\left\{\dfrac{5}{s}\right\} = \mathcal{L}_\psi^{-1}\left\{\dfrac{10}{s^{0.98}}\mathcal{L}_\psi\{y_2(x)\}\right\} - \mathcal{L}_\psi^{-1}\left\{\dfrac{10}{s^{0.98}}\mathcal{L}_\psi\{y_1(x)\}\right\}, \\[2ex]
y_2(x) - \mathcal{L}_\psi^{-1}\left\{\dfrac{3}{s}\right\} = \mathcal{L}_\psi^{-1}\left\{\dfrac{123}{s^{0.98}}\mathcal{L}_\psi\{y_1(x)\}\right\} - \mathcal{L}_\psi^{-1}\left\{\dfrac{1}{s^{0.98}}\mathcal{L}_\psi\{y_1(x)y_3(x)\}\right\} - \mathcal{L}_\psi^{-1}\left\{\dfrac{1}{s^{0.98}}\mathcal{L}_\psi\{y_2(x)\}\right\}, \\[2ex]
y_3(x) - \mathcal{L}_\psi^{-1}\left\{\dfrac{9}{s}\right\} = \mathcal{L}_\psi^{-1}\left\{\dfrac{1}{s^{0.98}}\mathcal{L}_\psi\{y_1(x)y_2(x)\}\right\} - \mathcal{L}_\psi^{-1}\left\{\dfrac{8}{3s^{0.98}}\mathcal{L}_\psi\{y_3(x)\}\right\}.
\end{cases} \tag{49}
$$

Next, we employ the ADM procedure on the latter equation to further express it as follows

$$
\begin{cases}
\displaystyle\sum_{n=0}^{\infty} y_{1,\,n}(x) = \mathcal{L}_\psi^{-1}\left\{\dfrac{10}{s^{0.98}}\mathcal{L}_\psi\left\{\sum_{n=0}^{\infty} y_{2,\,n}(x)\right\}\right\} - \mathcal{L}_\psi^{-1}\left\{\dfrac{10}{s^{0.98}}\mathcal{L}_\psi\left\{\sum_{n=0}^{\infty} y_{1,\,n}(x)\right\}\right\} + 5, \\[2ex]
\displaystyle\sum_{n=0}^{\infty} y_{2,\,n}(x) = \mathcal{L}_\psi^{-1}\left\{\dfrac{123}{s^{0.98}}\mathcal{L}_\psi\left\{\sum_{n=0}^{\infty} y_{1,\,n}(x)\right\}\right\} - \mathcal{L}_\psi^{-1}\left\{\dfrac{1}{s^{0.98}}\mathcal{L}_\psi\left\{\sum_{n=0}^{\infty} A_n\right\}\right\} \\
\qquad\qquad\qquad \displaystyle - \mathcal{L}_\psi^{-1}\left\{\dfrac{1}{s^{0.98}}\mathcal{L}_\psi\left\{\sum_{n=0}^{\infty} y_{2,\,n}(x)\right\}\right\} + 3, \\[2ex]
\displaystyle\sum_{n=0}^{\infty} y_{3,\,n}(x) = \mathcal{L}_\psi^{-1}\left\{\dfrac{1}{s^{0.98}}\mathcal{L}_\psi\left\{\sum_{n=0}^{\infty} B_n\right\}\right\} - \mathcal{L}_\psi^{-1}\left\{\dfrac{8}{3s^{0.98}}\mathcal{L}_\psi\left\{\sum_{n=0}^{\infty} y_{3,\,n}(x)\right\}\right\} + 9,
\end{cases} \tag{50}
$$

where A_n and B_n are Adomian polynomials corresponding to the nonlinear terms $y_1(x)y_3(x)$ and $y_1(x)y_2(x)$ defined respectively as follows

$$
\begin{cases}
A_0 = y_{1,\,0}\, y_{3,\,0}, \\[2ex]
A_1 = y_{1,\,0}\, y_{3,\,1} + y_{1,\,1}\, y_{3,\,0}, \\[2ex]
\quad \vdots
\end{cases} \tag{51}
$$

$$\begin{cases} B_0 = y_{1,\,0}\, y_{2,\,0}, \\ B_1 = y_{1,\,0}\, y_{2,\,1} + y_{1,\,1}\, y_{2,\,0}, \\ \quad\vdots \end{cases} \tag{52}$$

Further, the above series equation now based on the standard ADM procedure leads to the resulting recursive scheme of the governing fractional IVP for the chaotic dynamical system as follows

$$\begin{cases} y_{1,\,0} = 5, \\ y_{1,\,n}(x) = \mathcal{L}_{\psi}^{-1}\left\{\dfrac{10}{s^{0.98}}\mathcal{L}_{\psi}\{y_{2,\,n-1}(x)\}\right\} - \mathcal{L}_{\psi}^{-1}\left\{\dfrac{10}{s^{0.98}}\mathcal{L}_{\psi}\{y_{1,\,n-1}(x)\}\right\},\ \ \forall n \geq 1, \end{cases} \tag{53}$$

$$\begin{cases} y_{2,\,0} = 3, \\ y_{2,\,n}(x) = \mathcal{L}_{\psi}^{-1}\left\{\dfrac{123}{s^{0.98}}\mathcal{L}_{\psi}\{y_{1,\,n-1}(x)\}\right\} - \mathcal{L}_{\psi}^{-1}\left\{\dfrac{1}{s^{0.98}}\mathcal{L}_{\psi}\{A_{n-1}\}\right\} \\ \qquad - \mathcal{L}_{\psi}^{-1}\left\{\dfrac{1}{s^{0.98}}\mathcal{L}_{\psi}\{y_{2,\,n-1}(x)\}\right\},\ \ \forall n \geq 1, \end{cases} \tag{54}$$

and

$$\begin{cases} y_{3,\,0} = 9, \\ y_{3,\,n}(x) = \mathcal{L}_{\psi}^{-1}\left\{\dfrac{1}{s^{0.98}}\mathcal{L}_{\psi}\{B_{n-1}\}\right\} - \mathcal{L}_{\psi}^{-1}\left\{\dfrac{8}{3s^{0.98}}\mathcal{L}_{\psi}\{y_{3,\,n-1}(x)\}\right\},\ \ \forall n \geq 1. \end{cases} \tag{55}$$

In fact, one may express the above scheme term-wise with respect to each solution pair y_i for $i = 1, 2, 3$, as follows

$$
\begin{cases}
y_{1,\,0} = 5, \\
y_{1,\,1} = \mathcal{L}_{\psi}^{-1}\left\{\dfrac{1}{s^{0.98}}\mathcal{L}_{\psi}\{10(3)\}\right\} - \mathcal{L}_{\psi}^{-1}\left\{\dfrac{1}{s^{0.98}}\mathcal{L}_{\psi}\{10(5)\}\right\} = \dfrac{-20}{\Gamma(1.98)}(\psi(x))^{0.98}, \\
y_{1,\,2} = \dfrac{5870}{\Gamma(2.96)}(\psi(x))^{1.96}, \\
\quad\vdots
\end{cases}
\tag{56}
$$

$$
\begin{cases}
y_{2,\,0} = 3, \\
y_{2,\,1} = \mathcal{L}_{\psi}^{-1}\left\{\dfrac{1}{s^{0.98}}\mathcal{L}_{\psi}\{615\}\right\} - \mathcal{L}_{\psi}^{-1}\left\{\dfrac{1}{s^{0.98}}\mathcal{L}_{\psi}\{45\}\right\} - \mathcal{L}_{\psi}^{-1}\left\{\dfrac{1}{s^{0.98}}\mathcal{L}_{\psi}\{3\}\right\} = \dfrac{567}{\Gamma(1.98)}(\psi(x))^{0.98}, \\
y_{2,\,2} = \dfrac{-2802}{\Gamma(2.96)}(\psi(x))^{1.96}, \\
\quad\vdots
\end{cases}
\tag{57}
$$

and

$$
\begin{cases}
y_{3,\,0} = 9, \\
y_{3,\,1} = \mathcal{L}_{\psi}^{-1}\left\{\dfrac{1}{s^{0.98}}\mathcal{L}_{\psi}\{15\}\right\} - \mathcal{L}_{\psi}^{-1}\left\{\dfrac{1}{s^{0.98}}\mathcal{L}_{\psi}\{24\}\right\} = \dfrac{-9}{\Gamma(1.98)}(\psi(x))^{0.98}, \\
y_{3,\,2} = \dfrac{2299}{\Gamma(2.96)}(\psi(x))^{1.96}, \\
\quad\vdots
\end{cases}
\tag{58}
$$

So, when the above iterative solution is summed up, one gets the following series solution

$$
y_1(x) = 5 - \frac{20}{\Gamma(1.98)}(\psi(x))^{0.98} + \frac{5870}{\Gamma(2.96)}(\psi(x))^{1.96} + \ldots, \tag{59}
$$

$$
y_2(x) = 3 + \frac{567}{\Gamma(1.98)}(\psi(x))^{0.98} - \frac{2802}{\Gamma(2.96)}(\psi(x))^{1.96} + \ldots, \tag{60}
$$

$$y_3(x) = 9 - \frac{9}{\Gamma(1.98)}(\psi(x))^{0.98} + \frac{2299}{\Gamma(2.96)}(\psi(x))^{1.96} + \ldots. \tag{61}$$

Moreover, when the numerical simulation of the above result is carried out, the results are depicted in Figures 2 and 3, where the phase portraits are displayed for the ψ-LADM solution with regard to the governing for ψ-fractional Lorenz dynamical system (47) when $\alpha = 0.98$, where the time span $[1,\ 60]$.

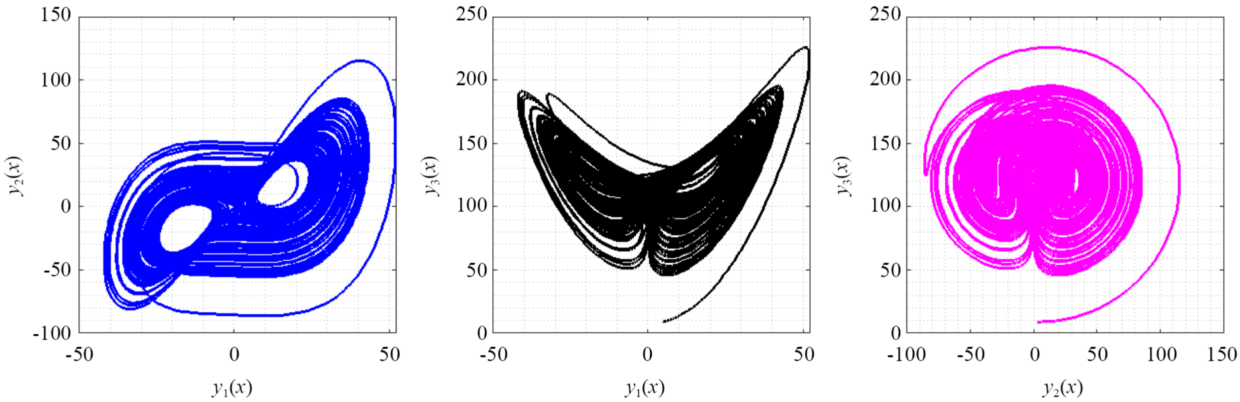
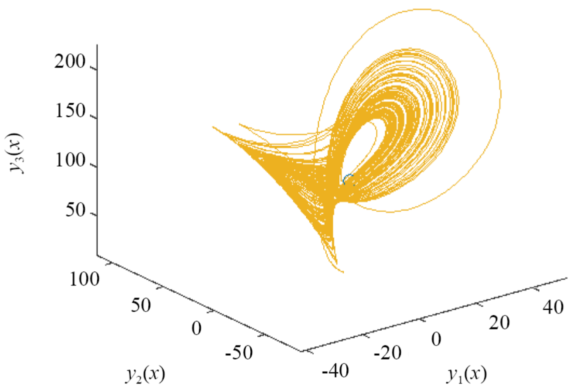
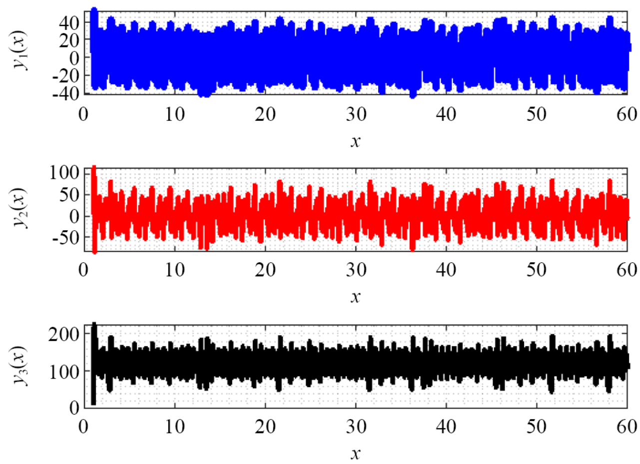


Figure 2. The phase portraits of $y_1(x)$, $y_2(x)$, and $y_3(x)$ for Lorenz system (47) at $\alpha = 0.98$, $\psi(x) = \sqrt{x}$ where the time span $[1,\ 60]$



(a) The phase portraits for Lorenz system (47)



(b) ψ-LADM solution for Lorenz system (47)

Figure 3. The phase portraits of $y_1(x)$, $y_2(x)$, $y_3(x)$ and ψ-LADM solution for Lorenz system (47) at $\alpha = 0.98$, $\psi(x) = \sqrt{x}$ where the time span $[1,\ 60]$

5. Conclusion

In conclusion, this study proposes the ψ-LADM, a reliable computational method for solving coupled systems of ψ-FDEs. ψ-Caputo derivatives offer additional flexibility in physical models, revealing characteristics beyond those captured by classical derivatives. We derived a generalized, rapidly convergent recursive relation applicable to both linear and nonlinear dynamical systems for ψ-FDEs. This method was successfully applied to numerical test examples, including a real-world atmospheric convection model utilizing the Lorenz chaotic system. The obtained solutions exactly matched the governing models, and in the case of Lorenz, accurately portrayed the system's chaotic behavior. We strongly recommend applying the ψ-LADM to other complex dynamical systems with highly nonlinear terms and diverse fractional orders.

Conflict of interest

There is no conflict of interest for this study.

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